IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received November 17, 2020, accepted December 6, 2020, date of publication December 11, 2020, date of current version January 8, 2021.

Digital Object Identifier 10.1109/ACCESS.2020.3044156

The Marshall-Olkin Odd Burr III-G Family: Theory, Estimation, and Engineering Applications

AHMED Z. AFIFY[1], GAUSS M. CORDEIRO[2], NOOR AKMA IBRAHIM[3], FARRUKH JAMAL[4], MOHAMED ELGARHY[5], AND MOHAMED ARSLAN NASIR[6]

[1]Department of Statistics, Mathematics and Insurance, Benha University, Benha 13511, Egypt
[2]Departamento de Estatística, Universidade Federal de Pernambuco, Recife 50710-165, Brazil
[3]Institute of Mathematical Research, University Putra Malaysia, Seri Kembangan 43400, Malaysia
[4]Department of Statistics, The Islamia University Bahawalpur, Bahawalpur 63100, Pakistan
[5]Higher Institute of Commercial Sciences, El Mahalla El Kubra 31951, Egypt
[6]Department of Statistics, Government S. E. College, Bahawalpur 63100, Pakistan

Corresponding author: Ahmed Z. Afify (ahmed.afify@fcom.bu.edu.eg)

ABSTRACT We propose a new flexible class called the Marshall-Olkin odd Burr III family for generating continuous distributions and derive some of its statistical properties. We provide three special models which accommodate symmetrical, right-skewed and left-skewed shaped densities as well as bathtub, decreasing, increasing, reversed-J shaped and upside-down bathtub failure rate functions. The parameters are estimated by maximum likelihood, least squares and a percentile method. Some simulations investigate the accuracy of the three methods. We illustrate the utility of a special model through three applications to engineering field.

INDEX TERMS Burr III distribution, engineering data, exponential distribution, parameter estimation, maximum likelihood, stochastic ordering.

I. INTRODUCTION

There has be an increasing motivation for constructing new generated families of continuous distributions by adding shape parameters to a baseline distribution due to desirable properties of the generated models. Some well-known generated families were introduced recently such as the Marshall-Olkin-G (Marshall and Olkin, 1997), exponentiated-G (Gupta *et al.*, 1998), Kumaraswamy-G (Cordeiro and de Castro, 2011), Lomax-G (Cordeiro *et al.*, 2014), Kumaraswamy Marshall-Olkin-G (Alizadeh *et al.*, 2015), odd Burr generalized-G (Alizadeh *et al.*, 2016), generalized odd log-logistic-G (Cordeiro *et al.*, 2017), generalized tan family (Al-Mofleh, 2018), generalized odd Lindley-G (Afify *et al.*, 2019), odd Lomax-G (Cordeiro *et al.*, 2019) and odd Dagum-G (Afify and Alizadeh, 2020) among others.

Jamal *et al.* (2017) proposed the *odd Burr-III-G* (OBIII-G for short) class, based on the Burr III distribution, by introducing two extra parameters to a baseline G distribution to obtain a more flexible class. Let $G(x;\delta)$ be the parent cumulative distribution function (cdf) with parameter vector δ, and $\bar{G}(x;\delta) = 1 - G(x;\delta)$. The OBIII-G family cdf follows by integrating the Burr III density with positive shape parameters b and c, namely

$$W_{\mathrm{OBIII}}(x;\delta) = b\,c \int_0^{\frac{G(x;\delta)}{\bar{G}(x;\delta)}} w^{-c-1}\,(1+w^{-c})^{-b-1}dw$$
$$= \left\{1 + \left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}, \quad x > 0. \quad (1)$$

By differentiating (1), we obtain its probability density function (pdf)

$$w_{\mathrm{OBIII}}(x;\delta) = \frac{c\,b\,g(x;\delta)\,\bar{G};\,\delta^{c+1}(x;\delta)}{G^{c-1}(x;\delta)}\left\{1 + \left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b-1}. \quad (2)$$

Marshall and Olkin (1997) proposed a general method to construct new models by adding a shape parameter to a specified distribution. Let $W(x;\delta)$ and $w(x;\delta)$ be the cdf and pdf of a specified distribution G. The cdf and pdf of the *Marshall-Olkin*-G (MO-G) class have the forms

$$F_{\mathrm{MO}}(x;\alpha,\delta) = \frac{W(x;\delta)}{\phi + (1-\phi)W(x;\delta)}, \quad \phi > 0, \quad (3)$$

The associate editor coordinating the review of this manuscript and approving it for publication was Yanbo Chen.

and

$$f_{\mathrm{MO}}(x) = \frac{\phi\, w(x;\delta)}{[\phi + (1-\phi)W(x;\delta)]^2}, \tag{4}$$

respectively. Clearly, $F_{\mathrm{MO}}(x;\phi,\delta)$ reduces to $W(x;\delta)$ when $\phi = 1$. For different values of ϕ, $F_{\mathrm{MO}}(x)$ can have more flexibility than $W(x;\delta)$.

We propose and study a new generator called the *Marshall-Olkin odd Burr III-G* (MOOB-G) family by taking (1) as the baseline cdf in Equation (3). Thus, the cdf of the MOOB-G family has the form

$$F_{\mathrm{MOOB}}(x;\phi,c,b,\delta) = \frac{\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}}{1-(1-\phi)\left(1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}\right)},\ x>0, \tag{5}$$

where $\phi > 0$, $b > 0$ and $c > 0$ are the shape parameters and δ is the baseline parameter vector.

The cdf (5) can be explained by combining the Burr III distribution to generate $W(x)$ with the distribution of an unknown geometric number N of independent risk factors or components generated by the baseline odds ratio. Let T be a random variable (rv) having cdf $G(t;\delta)$ describing a stochastic system and Z be a rv representing the *odds ratio*. The risk that a system having the lifetime T will be not working at time z is $G(z;\delta)/\bar{G}(z;\delta)$. If the randomness of the odds ratio Z is modeled by the Burr III distribution given by Equation (1), the cdf of Z can be written as $Pr(Z \leq z) = H(z;b,c,\delta)$. Further, consider a sequence of independent and identically distributed (iid) odds ratio rvs $Z_1, Z_2, \cdots$ with associated risks $H(z;b,c,\delta)$. We define the minimum odds ratio $X = \min\{Z_1,\ldots,Z_N\}$, where N is an unknown number described by a discrete geometric random variable (with support in $\{1,2,\ldots\}$ and probability parameter ϕ) independent of the Z_i's. The probability generating function (pgf) of N takes the form $E(s^N) = \tau(s;\phi) = \phi\, s\,[1-(1-\phi)s]^{-1}$. Under this set-up and $0 < \phi < 1$, the cdf of X is given by (5). For the case $\phi > 1$, the cdf of X is equal to (5) if N has a geometric rv with probability $1/\phi$.

Furthermore, the sub-models of the MOOB-G family can provide symmetrical, left-skewed, symmetrical, right-skewed, and reversed-J densities, and upside-down bathtub, increasing, decreasing, bathtub, and reversed-J shaped hazard rates. These sub-models are also capable of modeling different shapes of aging and failure criteria. Hence, the MOOB-G family can be a useful alternative to many classes for modeling skewed data in real-life applications.

The pdf corresponding to (5) has the form

$$f_{\mathrm{MOOB}}(x;\phi,c,b,\delta) = \frac{\phi\, c\, b\, g(x;\delta)\,\bar{G}^{c-1}(x;\delta)\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b-1}}{G^{c+1}(x;\delta)\left[1-(1-\phi)\left(1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}\right)\right]^2}. \tag{6}$$

For $c = b = 1$, we obtain as a special case the MO-G family (Marshall and Olkin, 1997). The MOOB-G family is identical to the OBIII-G class (Jamal *et al.*, 2017) when $\phi = 1$ and it reduces to the baseline G distribution when $\phi = c = b = 1$. The MOOB-G family can be quite effective for real data analysis.

Henceforth, let $X \sim$MOOB-G(ϕ,b,c,δ) be a rv with density (6). The hazard rate function (hrf) of X is (7), as shown at the bottom of the next page.

The quantile function (qf) of X (for $0 < u < 1$) follows by inverting (5) as

$$Q_{\mathrm{MOOB}}(u) = F^{-1}_{\mathrm{MOOB}}(u) = G^{-1}\left\{\frac{\left[\left(\frac{u\phi}{1-\bar{\phi}u}\right)^{-\frac{1}{b}}-1\right]^{-\frac{1}{c}}}{1+\left[\left(\frac{u\phi}{1-\bar{\phi}u}\right)^{-\frac{1}{b}}-1\right]^{-\frac{1}{c}}}\right\}. \tag{8}$$

We organize the article as follows. In Section 2, we define three special models in the new family. In Section 3, we obtain a useful linear representation for the family density. Some of its mathematical properties are reported in Section 4. In Section 5, we discuss the estimation of the unknown parameters using three methods (maximum likelihood, least squares and percentile estimation). In Section 6, some simulation results validate the proposed methods. The usefulness of the new family is illustrated in Section 7 using three applications. Finally, some conclusions are addressed in Section 8.

II. THREE SPECIAL DISTRIBUTIONS

In this section, we present three special distributions of the MOOB-G family by choosing some baseline distributions commonly used in lifetime data analysis along with their density and hazard rate plots.

A. MOOB-EXPONENTIAL (MOOB-EX)

Consider the exponential (Ex) distribution with parameter $a > 0$ and cdf $G(x) = 1 - \exp(-ax)$, $x > 0$. Then, the MOOB-Ex density follows from (6) as f(x), shown at the bottom of the next page.

The MOOB-Ex distribution includes the OBIII-Ex distribution (Jamal *et al.*, 2017) when $\phi = 1$. For $\phi = b = c = 1$, we obtain the Ex distribution. The MO-Ex distribution (Marshall and Olkin, 1997) follows when $b = c = 1$. Some plots of the density and hrf of the MOOB-Ex distribution are shown in Figure 1. They show that the MOOB-Ex density can be reversed J-shape, right-skewed, left-skewed and concave down, whereas the hrf can be decreasing, increasing or bathtub.

B. MOOB-LOMAX (MOOB-LX)

The cdf of the Lomax (Lx) distribution is $G(x) = 1 - (1 + x/d)^{-a}$, where $a > 0$ and $d > 0$. Then, the MOOB-Lx density can be expressed from (6) as shown at the bottom of the next page.

Three special cases of the MOOB-Lx distribution are: the MO-Lx distribution (Ghitany *et al.*, 2007)

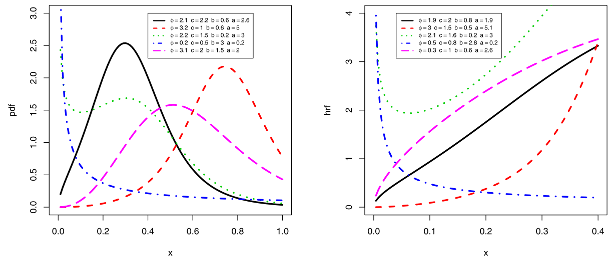

FIGURE 1. Density and hrf plots of the MOOB-Ex distribution.

when $b = c = 1$; the OBIII-Lx distribution (Jamal *et al.*, 2017) when $\phi = 1$; the Lx distribution when $\phi = b = c = 1$. Some plots of the density and hrf of this distribution are given in Figure 2. They show that the pdf can be unimodal, reversed J-shape and right-skewed, and the hrf can be increasing, decreasing, bathtub or unimodal.

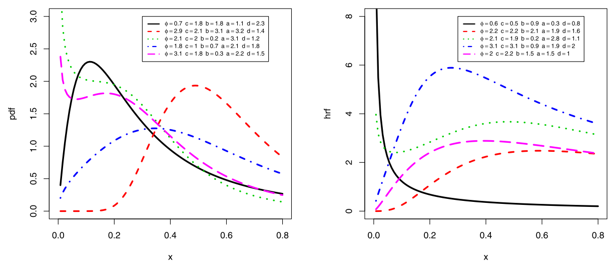

FIGURE 2. Density and hrf plots of the MOOB-Lx distribution.

C. MOOB-LINDLEY (MOOB-LI)

Consider the cdf $G(x) = 1 - \frac{1+a+a\,x}{1+a}\exp(-a\,x)$, $a > 0$, of the Lindley (Li) distribution. Then, the pdf of the MOOB-Li density follows from (6) as f(x), shown at the bottom of the next page.

Three special cases of the MOOB-Li distribution are: the MO-Li distribution (Ghitany *et al.*, 2012) when $b = c = 1$; the OBIII-Li distribution (Jamal *et al.*, 2017) when $\phi = 1$; the Li distribution when $\phi = b = c = 1$. Some plots of the density and hrf of this distribution are given in Figure 3 for some scenarios. They show that the density can be unimodal, bimodal and reversed J-shape, and that the hrf can be decreasing, increasing, or bathtub.

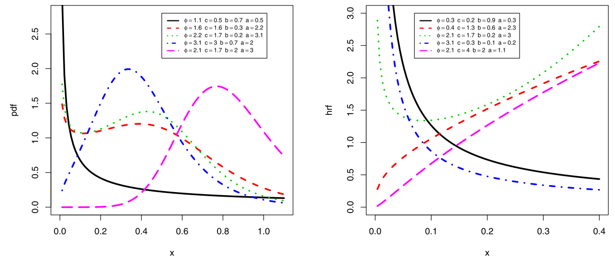

FIGURE 3. Density and hrf plots of the MOOB-Li distribution.

III. LINEAR REPRESENTATION

Following Cordeiro *et al.* (2014), we obtain a linear representation for the MO-G density

$$f_{\mathrm{MO}}(x) = \begin{cases} \sum_{j=0}^{\infty} p_j\, h_{j+1}(x) & \text{if } \phi \in (0,1)\,, \\ \sum_{j=0}^{\infty} q_j\, h_{j+1}(x) & \text{if } \phi > 1, \end{cases}$$

where (for $j = 0, 1, 2, \ldots$)

$$\begin{cases} p_j = \dfrac{\phi\,(-1)^j}{j+1} \sum_{l=j}^{\infty} \dbinom{l}{j} (l+1)\,\overline{\phi}^{\,l} & \text{if } \phi \in (0,1)\,, \\ q_j = \phi^{-1}\left(1-\phi^{-1}\right)^j & \text{if } \phi > 1, \end{cases}$$

and

$$h_{j+1}(x) = (j+1)\,c\,b\,\frac{g(x;\delta)G^{1-c}(x;\delta)}{[1-G(x;\delta)]^{-c-1}}\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-(j+1)b-1}. \quad (9)$$

$$h_{\mathrm{MOOB}}(x;\phi,c,b,\delta) = \frac{\phi c b g(x;\delta)\bar{G}^{c-1}(x;\delta)G^{-c-1}(x;\delta)\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b-1}}{\left(1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}\right)\left[1-(1-\phi)\left(1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}\right)\right]}. \quad (7)$$

$$f(x) = \frac{\phi c b\, a\, \exp(-ac\,x)\left[1-\exp(-a\,x)\right]^{-c-1}\left\{1+\left[\frac{1-\exp(-a\,x)}{\exp(-a\,x)}\right]^{-c}\right\}^{-b-1}}{\left[1-(1-\phi)\left(1-\left\{1+\left[\frac{1-\exp(-a\,x)}{\exp(-a\,x)}\right]^{-c}\right\}^{-b}\right)\right]^2}.$$

$$f(x) = \frac{\phi\, c\, b\, \frac{a}{d}\left(1+\frac{x}{d}\right)^{-a-1}\left[\left(1+\frac{x}{d}\right)^{-a}\right]^{c-1}\left\{1+\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-c}\right\}^{-b-1}}{\left[1-\left(1+\frac{x}{d}\right)^{-a}\right]^{c+1}\left[1-(1-\phi)\left(1-\left\{1+\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{-c}\right\}^{-b}\right)\right]^2}.$$

Consider the power series (which converges everywhere)

$$(1+z)^{-d}=\sum_{i=0}^{\infty}\binom{-d}{i}z^{i}. \tag{10}$$

Applying (10) to (9) leads to

$$h_{j+1}(x) = (j+1)\,c\,b\,g(x;\delta) \times\sum_{i=0}^{\infty}\binom{-(j+1)\,b-1}{i}G^{1-(i+1)c}(x;\delta)\,[1-G(x;\delta)]^{(i+1)c+1}.$$

By expanding the binomial term in power series, we can write

$$h_{j+1}(x)=g(x;\delta)\sum_{i,r=0}^{\infty}A_{i,r}^{(j+1)}\,G^{r+1-(i+1)c}(x;\delta), \tag{11}$$

where (for $i, r \geq 0$)

$$A_{i,r}^{(j+1)}=(j+1)\,c\,b\,(-1)^{r}\binom{-(j+1)\,b-1}{i}\binom{(i+1)c+1}{r}.$$

For any real a, the following power series (converges everywhere) holds

$$G(x;\delta)^{a}=\sum_{m=0}^{\infty}s_{m}(a)\,G(x;\delta)^{m}, \tag{12}$$

where

$$s_{m}(a)=\sum_{n=m}^{\infty}(-1)^{n+m}\binom{a}{n}\binom{n}{m}.$$

Applying (12) to equation (11) and changing the order of the sums, we can write

$$h_{j+1}(x)=\sum_{m=0}^{\infty}B_{m}^{(j+1)}\,\pi_{m+1}(x),$$

where $\pi_{m+1}(x) = (m+1)\,g(x;\delta)\,G(x;\delta)^{m}$ is the exponentiated-G (exp-G) density with power parameter $m+1$ (for $m \geq 0$) and

$$B_{m}^{(j+1)}=(m+1)^{-1}\sum_{i,r=0}^{\infty}A_{i,r}^{(j+1)}\,s_{m}(r+1-(i+1)c).$$

Hence, the density of X is rewritten as

$$f_{\mathrm{MOOB}}(x)=\begin{cases}\displaystyle\sum_{m=0}^{\infty}v_{m}\,\pi_{m+1}(x) & \text{if } \phi\in(0,1),\\ \displaystyle\sum_{m=0}^{\infty}w_{m}\,\pi_{m+1}(x) & \text{if } \phi>1,\end{cases} \tag{13}$$

where

$$v_{m}=\sum_{m=0}^{\infty}p_{j}\,B_{m}^{(j+1)}$$

and

$$w_{m}=\sum_{m=0}^{\infty}q_{j}\,B_{m}^{(j+1)}.$$

Equation (13) proves that the density of X is a linear combination of exp-G densities and then some MOOB-G properties (see Section 4.2) can follow from those of the exp-G distribution. We can adopt at most ten terms for each sum in (13) to provide accurate results in most analytical platforms.

IV. THE MOOB-G PROPERTIES

A. SHAPES AND ASYMPTOTICS

We can determine, numerically, the critical points of the density and hazard rate functions of X. For the density, they are the roots of the equation:

$$\frac{g'(x;\delta)}{g(x;\delta)}-(c-1)\frac{g(x;\delta)}{1-G(x;\delta)}-(c+1)\frac{g(x;\delta)}{G(x;\delta)} -(b+1)\frac{c\,g(x;\delta)\,\bar{G}^{c-1}(x;\delta)}{G^{c+1}(x)\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}} +\frac{2\,c\,b\,\bar{\phi}\,g(x;\delta)\,\bar{G}^{c-1}(x;\delta)\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b-1}}{\left[1-(1-\phi)\left(1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}\right)\right]}+$$

For the hazard rate, they follow from the equation

$$\frac{g'(x;\delta)}{g(x;\delta)}-(c-1)\frac{g(x;\delta)}{1-G(x;\delta)}-(c+1)\frac{g(x;\delta)}{G(x;\delta)} -(b+1)\frac{c\,g(x;\delta)\,\bar{G}^{c-1}(x;\delta)}{G^{c+1}(x;\delta)\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}} \times\frac{bc^{2}\,g(x;\delta)\,\bar{G}^{c-1}(x;\delta)\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b-1}}{G^{c+1}(x;\delta)\left(1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}\right)}$$

$$f(x)=\frac{\phi cb\frac{a^{2}}{a+1}(1+x)\exp(-a\,x)\left[\frac{1+a+a\,x}{1+a}\exp(-a\,x)\right]^{c-1}\left\{1+\left[\frac{1-\frac{1+a+a\,x}{1+a}\exp(-a\,x)}{\frac{1+a+a\,x}{1+a}\exp(-a\,x)}\right]^{-c}\right\}^{-b-1}}{\left[1-\frac{1+a+a\,x}{1+a}\exp(-a\,x)\right]^{c+1}\left[1-(1-\phi)\left(1-\left\{1+\left[\frac{1-\frac{1+a+a\,x}{1+a}\exp(-a\,x)}{\frac{1+a+a\,x}{1+a}\exp(-a\,x)}\right]^{-c}\right\}^{-b}\right)\right]^{2}}.$$

$$+\frac{\phi\, bc^2\, g(x;\delta)\, \bar{G}^{c-1}(x;\delta)\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b-1}}{G^{c+1}(x;\delta)\left[1-(1-\phi)\left(1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}\right)\right]}.$$

We can examine the critical points of the pdf and hrf of X from the last two equations by using many easy-to-use programming environments including Matlab, Mathematica, Maple and Ox.

The tail behaviors of the cdf, pdf and hrf of X are (when $G(x;\delta) \to 0$)

$$F(x) \sim \left(\frac{1}{\phi}\right) G(x;\delta)^{bc},$$
$$f(x) \sim \left(\frac{1}{\phi}\right) b\, c\, g(x;\delta)[G(x;\delta)]^{b\,c-1},$$
$$h(x) \sim \left(\frac{1}{\phi}\right) b\, c\, g(x;\delta)[G(x;\delta)]^{b\,c-1}.$$

If $x \to \infty$,

$$F(x) \sim 1-\phi\, b\,[1-G(x;\delta)]^{c},$$
$$f(x) \sim \phi\, b\, c\, g(x;\delta)\,[1-G(x;\delta)]^{c-1},$$
$$h(x) \sim \frac{c\, g(x;\delta)}{1-G(x;\delta)}.$$

B. MOMENTS

Let Y_{m+1} be the exp-G rv with power parameter $m+1$. The rth moment of X follows from (13) as

$$\mu'_r = E\left(X^r\right)\begin{cases}\sum_{m=0}^{\infty} v_m\, E\left(Y_{m+1}^r\right) & \text{if } \phi \in (0,1)\,,\\ \sum_{m=0}^{\infty} w_m\, E\left(Y_{m+1}^r\right) & \text{if } \phi > 1.\end{cases}$$

Similarly, the rth incomplete moment of X can be expressed as

$$\varphi_r(t) = \begin{cases}\sum_{m=0}^{\infty} v_m \int_{-\infty}^{t} x^r\, \pi_{m+1}(x) & \text{if } \phi \in (0,1)\,,\\ \sum_{m=0}^{\infty} w_m \int_{-\infty}^{t} x^r\, \pi_{m+1}(x) & \text{if } \phi > 1.\end{cases}$$

The integral in the last two equations represents the rth incomplete moment of Y_{m+1}.

The moment generating function (mgf) of X comes from (13) as

$$M_X(t) = \begin{cases}\sum_{m=0}^{\infty} v_m\, M_{m+1}(t) & \text{if } \phi \in (0,1)\,,\\ \sum_{m=0}^{\infty} w_m\, M_{m+1}(t) & \text{if } \phi > 1,\end{cases}$$

where $M_{m+1}(t)$ is the mgf of Y_{m+1} (for $m \geq 0$). Then, $M_X(t)$ can be obtained from the exp-G generating function.

We provide some numerical values of the first four moments denoted by μ'_r $(r = 1, 2, 3, 4)$, variance σ^2, skewness (Sk), and kurtosis (Kur) of the MOOB-Ex distribution for different parameter values. Table 1 lists these numerical values.

TABLE 1. The Numerical Values of the First Four Moments, σ^2, Sk, and Kur of the MOOB-Ex Distribution for Some Parametric Values.

(ϕ,c,b,a)	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	Sk	Kur
(0.5,0.5,0.5,0.5)	1.01372	7.59493	92.21433	1492.68605	6.56726	4.23045	27.08413
(1.5,0.5,0.5,0.5)	2.39992	20.55147	262.73193	4359.96201	14.791845	2.50333	10.20705
(3.0,0.5,0.5,0.5)	3.80713	36.54753	493.08842	8422.16504	22.05318	1.79623	4.89005
(6.0,0.5,0.5,0.5)	5.63804	61.65663	893.12122	15906.37333	29.86875	1.27843	0.98814
(0.5,2.0,0.5,0.5)	0.81273	1.24493	2.90042	9.44915	0.58443	2.10122	21.66604
(0.5,6.0,0.5,0.5)	1.075824	1.27863	1.65062	2.29803	0.12123	0.33354	508.17355
(6.0,6.0,0.5,0.5)	1.568364	2.59465	4.50321	8.19189	0.13489	0.22435	1238.51904
(0.5,0.5,2.0,0.5)	3.56625	29.48014	365.49743	5952.57802	16.76208	2.05188	6.60155
(6.0,6.0,6.0,0.5)	2.12327	4.63435	10.41455	24.14698	0.12608	0.88032	3506.16206
(0.5,2.0,2.0,2.0)	0.43354	0.23535	0.16176	0.14166	0.04738	1.80974	258.12465
(6.0,6.0,6.0,6.0)	0.17699	0.03213	0.00603	0.00112	0.00081	0.87983	44376.80374

C. STOCHASTIC ORDERING

The stress strength model has been widely used in applications of engineering and physics. Consider two independent rvs X_1 ~MOOB-G(ϕ_1, c, b, δ) and X_2 ~MOOB-G(ϕ_2, c, b, δ). The stress strength model can be expressed as

$$f(x) = \frac{\phi_1\, c\, b\, g(x;\delta)\, \bar{G}^{c-1}(x;\delta)\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b-1}}{G^{c+1}(x;\delta)\left[1-(1-\phi_1)\left(1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}\right)\right]^2}$$

and

$$g(x) = \frac{\phi_2\, c\, b\, g(x;\delta)\, \bar{G}^{c-1}(x;\delta)\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b-1}}{G^{c+1}(x;\delta)\left[1-(1-\phi_2)\left(1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}\right)\right]^2}.$$

Then, the ratio $\frac{f(x)}{g(x)}$ takes the form

$$\frac{f(x)}{g(x)} = \frac{\phi_1}{\phi_2}\left(\frac{1-\bar{\phi}_1 z}{1-\bar{\phi}_2 z}\right)^2,$$

where $z = 1-\left\{1+\left[\frac{G(x;\delta)}{1-G(x;\delta)}\right]^{-c}\right\}^{-b}$.

By differentiating the last equation in relation to x, we have

$$\frac{d}{d\,x}\frac{f(x)}{g(x)} = \frac{\phi_1}{\phi_2}\, 2\left(\frac{1-\bar{\phi}_1 z}{1-\bar{\phi}_2 z}\right)\frac{(\bar{\phi}_1-\bar{\phi}_2)\, z'}{(1-\bar{\phi}_2 z)^2}.$$

After some algebra, we obtain

$$\frac{d}{d\,x}\frac{f(x)}{g(x)} = 2\,\frac{\phi_1}{\phi_2}\,(\bar{\phi}_1-\bar{\phi}_2)\, z\left[\frac{1-\bar{\phi}_1 z'}{\left(1-\bar{\phi}_2 z\right)^3}\right],$$

where $\frac{d\,z}{d\,x} = z'$. Finally, if $\phi_2 < \phi_1$ then $\frac{d}{d\,x}\frac{f(x)}{g(x)} < 0$ and the likelihood ratio exists $X <_{lr} Y$.

V. ESTIMATION METHODS

Let $x_1, \ldots, x_n$ be the observations from the MOOB-G family with vector of parameters $\theta = (\phi, c, b, \delta^T)^T$. This section is devoted to address the maximum likelihood, least squares method and a percentile method to estimate θ from complete samples only. The performances of the three methods are investigated through Monte Carlo simulations using **Mathematica** software.

A. MAXIMUM LIKELIHOOD

By maximizing the total log-likelihood function

$$\begin{aligned}\ell(\theta) = n\log(\phi\, c\, b) &+ \sum_{i=1}^{n}\log g(x_i;\delta)\\ &+ (c-1)\sum_{i=1}^{n}\log\bar{G}(x_i;\delta)\\ &- (c+1)\sum_{i=1}^{n}\log G(x_i;\delta)\\ &- (b+1)\sum_{i=1}^{n}\log(z_i)\\ &- 2\sum_{i=1}^{n}\log\left[1-(1-\phi)(1-z_i^{-b})\right], \end{aligned} \quad (14)$$

where $z_i = 1 + \left[\frac{G(x_i;\delta)}{\bar{G}(x_i;\delta)}\right]^{-c}$, we can find the maximum likelihood estimates (MLEs) of the parameters in θ.

Equation (14) can be maximized either directly under statistical computing environment such as Ox (sub-routine MaxBFGS), SAS (PROC NLMIXED), R, Maple or Mathematica or by solving the nonlinear likelihood equations by differentiating (14). The distributions in the MOOB-G family can also be fitted to real data using the *AdequacyModel* package (see `https://www.r-project.org/`). We can compute the MLEs, their standard errors and goodness-of-fit statistics from this package by defining only the pdf and cdf of the distribution under study.

The score functions for the model parameters are

$$U_\phi = \frac{n}{\phi} - 2\sum_{i=1}^{n}\left[\frac{1-z_i^{-b}}{1-(1-\phi)\left(1-z_i^{-b}\right)}\right],$$

$$\begin{aligned}U_c = \frac{n}{c} &+ \sum_{i=1}^{n}\log\bar{G}(x_i;\delta) - \sum_{i=1}^{n}\log G(x_i;\delta)\\ &+ (b+1)\sum_{i=1}^{n}\frac{\log\left[\frac{G(x_i;\delta)}{\bar{G}(x_i;\delta)}\right]}{z_i\left[\frac{G(x_i;\delta)}{\bar{G}(x_i;\delta)}\right]^{c}}\\ &+ 2b(1-\phi)\sum_{i=1}^{n}\frac{\left[\frac{G(x_i;\delta)}{\bar{G}(x_i;\delta)}\right]^{-c}\log\left[\frac{G(x_i;\delta)}{\bar{G}(x_i;\delta)}\right]}{z_i^{b+1}\left[1-(1-\phi)\left(1-z_i^{-b}\right)\right]},\end{aligned}$$

$$U_b = \frac{n}{b} - \sum_{i=1}^{n}\log(z_i) + 2(1-\phi)\sum_{i=1}^{n}\frac{z_i^{-b}\log(z_i)}{1-(1-\phi)\left(1-z_i^{-b}\right)}$$

and U_δ, as shown at the bottom of the next page, where the vectors

$$g'(x_i;\delta) = \frac{\partial g(x_i;\delta)}{\partial\delta},\quad G'(x_i;\delta) = \frac{\partial G(x_i;\delta)}{\partial\delta},$$
$$\bar{G}'(x_i;\delta) = \frac{\partial\bar{G}(x_i;\delta)}{\partial\delta}$$

have the same dimension of δ. These equations can be solved numerically from the statistical software above.

B. LEAST SQUARES

Suppose that $x_{1:n}, \ldots, x_{n:n}$ are the ordered observations from $F(x;\phi, c, b, \delta)$ given by (5). Note that $E[F(X_{i:n}] = \frac{i}{n+1}$.

Hence, The least square estimates (LSEs) can be determined by minimizing the function

$$S(\theta) = \sum_{i=1}^{n}\left[F(x_{i:n};\phi, c, b, \delta) - \frac{i}{n+1}\right]^2,$$

with respect to the model parameters. The LSEs of ϕ, c, b and δ can be found by solving the nonlinear equations

$$\frac{\partial S(\theta)}{\partial\phi} = 2\sum_{i=1}^{n}\left[F(x_{i:n};\phi, c, b, \delta) - \frac{i}{n+1}\right] F'(x_{i:n};\phi, c, b, \delta)_\phi = 0,$$

$$\frac{\partial S(\theta)}{\partial c} = 2\sum_{i=1}^{n}\left[F(x_{i:n};\phi, c, b, \delta) - \frac{i}{n+1}\right] F'(x_{i:n};\phi, c, b, \delta)_c = 0,$$

$$\frac{\partial S(\theta)}{\partial b} = 2\sum_{i=1}^{n}\left[F(x_{i:n};\phi, c, b, \delta) - \frac{i}{n+1}\right] F'(x_{i:n};\phi, c, b, \delta)_b = 0,$$

$$\frac{\partial S(\theta)}{\partial\delta} = 2\sum_{i=1}^{n}\left[F(x_{i:n};\phi, c, b, \delta) - \frac{i}{n+1}\right] F'(x_{i:n};\phi, c, b, \delta)_\delta = 0,$$

where $F'(x_{i:n};\theta)_\eta = \frac{\partial F(x_{i:n};\theta)}{\partial\eta}$ for any component η of $\theta = (\phi, c, b, \delta^T)^T$. We can solve these equations numerically to obtain the estimates $\hat{\phi}_{LSE}, \hat{c}_{LSE}, b_{LSE}$ and $\hat{\delta}_{LSE}$.

C. PERCENTILE ESTIMATION

Let $p_i = i/(n+1)$ be an estimate of $F(x_{i:n};\phi, c, b, \delta)$ for $i = 1, \ldots, n$. Let $Q_{p_i}(\phi, c, b, \delta)$ be the qf obtained from (8) by setting $u = p_i = i/(n+1)$.

Then, the percentile estimates (PCEs) of the unknown parameters ϕ, c, b and δ can be derived by minimizing

$$M(\theta) = \sum_{i=1}^{n}\left[x_{i:n} - Q_{p_i}(\phi, c, b, \delta)\right]^2,$$

with respect to the model parameters or by solving the nonlinear equations

$$\frac{\partial M(\theta)}{\partial\phi} = 2\sum_{i=1}^{n}\left[x_{i:n} - Q_{p_i}(\phi, c, b, \delta)\right] Q'_{p_i}(\phi, c, b, \delta)_\phi = 0,$$

TABLE 2. MSEs and AEs for the MOOB-Ex Distribution.

n	Parameters	PCE MSEs	PCE AEs	LSE MSEs	LSE AEs	MLE MSEs	MLE AEs
50	$\phi=0.8$	0.02365	0.75625	0.00017	0.80106	0.00434	0.80357
	$c=1.5$	0.25885	1.23699	0.00022	1.49488	0.07658	1.54477
	$b=1.5$	0.71613	1.35755	0.00141	1.50163	0.05512	1.52599
	$a=0.1$	0.84259	0.11118	0.00002	0.09983	0.00068	0.10578
100	$\phi=0.8$	0.01321	0.78389	0.00008	0.80075	0.00217	.80243
	$c=1.5$	0.16421	1.34225	0.00010	1.49735	0.03669	1.52211
	$b=1.5$	0.48057	1.44864	0.00068	1.50144	0.02620	1.51526
	$a=0.1$	0.21325	0.12854	8.0×10^{-6}	0.09985	0.00029	0.10257
150	$\phi=0.8$	0.00887	0.78580	0.00006	0.80035	0.00146	0.80162
	$c=1.5$	0.13359	1.36533	0.00007	1.49780	0.02210	1.50604
	$b=1.5$	0.27976	1.42857	0.00049	1.50037	0.01713	1.50409
	$a=0.1$	0.00572	0.12464	5.7×10^{-6}	0.09994	0.00018	0.10173
50	$\phi=1.0$	0.04069	0.96196	0.00026	1.00020	0.00683	1.00309
	$c=1.5$	0.23737	1.24107	0.00030	1.49469	0.08615	1.55098
	$b=1.5$	0.70676	1.40265	0.00125	1.49882	0.04643	1.52109
	$a=0.1$	3.72914	0.17824	0.00002	0.10010	0.00078	0.10653
100	$\phi=1.0$	0.02105	0.96477	0.00013	1.00065	0.00360	1.00476
	$c=1.5$	0.16594	1.28907	0.00016	1.49715	0.03910	1.51671
	$b=1.5$	0.34082	1.35769	0.00060	1.50069	0.02483	1.51584
	$a=0.1$	0.96345	0.18612	8.3×10^{-6}	0.09991	0.00031	0.10224
150	$\phi=1.0$	0.01473	0.98425	0.00009	0.99984	0.00208	1.00137
	$c=1.5$	0.11425	1.37234	0.00011	1.49878	0.02417	1.52061
	$b=1.5$	0.25914	1.43886	0.00041	1.49927	0.01389	1.51116
	$a=0.1$	0.01012	0.12758	5.8×10^{-6}	0.10009	0.00017	0.10170
50	$\phi=0.8$	0.02536	0.77259	0.00016	0.799946	0.004175	0.801616
	$c=1.2$	0.16312	1.01736	0.00013	1.19658	0.051372	1.24173
	$b=1.5$	0.94427	1.45910	0.00133	1.49845	0.056156	1.53026
	$a=0.1$	1.03099	0.10954	0.00001	0.100097	0.000439	0.104232
100	$\phi=0.8$	0.01358	0.77540	0.00008	0.799989	0.002182	0.805079
	$c=1.2$	0.11915	1.05245	0.00006	1.19798	0.023637	1.22275
	$b=1.5$	0.47563	1.40783	0.00068	1.49914	0.028053	1.52768
	$a=0.1$	0.03915	0.12168	5.5×10^{-6}	0.100039	0.000172	0.101079
150	$\phi=0.8$	0.00960	0.78691	0.00006	0.800086	0.001586	0.802496
	$c=1.2$	0.08960	1.09502	0.00004	1.19861	0.015888	1.2125
	$b=1.5$	0.31271	1.44252	0.00049	1.49971	0.018424	1.51332
	$a=0.1$	0.00555	0.11745	3.9×10^{-6}	0.100006	0.000135	0.101127

TABLE 3. MSEs and AEs for the MOOB-Ex Distribution.

n	Parameters	PCE MSEs	PCE AEs	LSE MSEs	LSE AEs	MLE MSEs	MLE AEs
50	$\phi=0.8$	0.02536	0.77259	0.00016	0.799946	0.004175	0.801616
	$c=1.2$	0.16312	1.01736	0.00013	1.19658	0.051372	1.24173
	$b=1.5$	0.94427	1.45910	0.00133	1.49845	0.056156	1.53026
	$a=0.1$	1.03099	0.10954	0.00001	0.100097	0.000439	0.104232
100	$\phi=0.8$	0.01358	0.77540	0.00008	0.799989	0.002182	0.805079
	$c=1.2$	0.11915	1.05245	0.00006	1.19798	0.023637	1.22275
	$b=1.5$	0.47563	1.40783	0.00068	1.49914	0.028053	1.52768
	$a=0.1$	0.03915	0.12168	5.5×10^{-6}	0.100039	0.000172	0.101079
150	$\phi=0.8$	0.00960	0.78691	0.00006	0.800086	0.001586	0.802496
	$c=1.2$	0.08960	1.09502	0.00004	1.19861	0.015888	1.2125
	$b=1.5$	0.31271	1.44252	0.00049	1.49971	0.018424	1.51332
	$a=0.1$	0.00555	0.11745	3.9×10^{-6}	0.100006	0.000135	0.101127
50	$\phi=0.8$	0.02432	0.76639	0.00015	0.800011	0.00465	0.804949
	$c=0.5$	0.02824	0.41767	0.00002	0.498403	0.00821	0.512498
	$b=1.5$	0.84333	1.41064	0.00124	1.49842	0.05688	1.53309
	$a=0.1$	0.00060	0.10965	2.6×10^{-6}	0.099916	0.00016	0.101204
100	$\phi=0.8$	0.01349	0.78085	0.00009	0.799835	0.00212	0.801308
	$c=0.5$	0.02093	0.44062	0.00001	0.499353	0.00411	0.507379
	$b=1.5$	0.46569	1.42331	0.00075	1.49887	0.02445	1.50984
	$a=0.1$	0.00020	0.10484	1.5×10^{-6}	0.099994	0.00008	0.101055
150	$\phi=0.8$	0.00944	0.78068	0.00005	0.800283	0.00147	0.801575
	$c=0.5$	0.01567	0.44618	7.0×10^{-6}	0.499417	0.00250	0.50372
	$b=1.5$	0.28886	1.39817	0.00045	1.50033	0.01760	1.51024
	$a=0.1$	0.00013	0.10386	9.4×10^{-7}	0.099934	0.00005	0.100361
50	$\phi=0.8$	0.02432	0.76639	0.00015	0.800011	0.00465	0.804949
	$c=0.5$	0.02824	0.41767	0.00002	0.498403	0.00821	0.512498
	$b=1.5$	0.84333	1.41064	0.00124	1.49842	0.05688	1.53309
	$a=0.1$	0.00060	0.10965	2.6×10^{-6}	0.099916	0.00016	0.101204
100	$\phi=0.8$	0.01349	0.78085	0.00009	0.799835	0.00212	0.801308
	$c=0.5$	0.02093	0.44062	0.00001	0.499353	0.00411	0.507379
	$b=1.5$	0.46569	1.42331	0.00075	1.49887	0.02445	1.50984
	$a=0.1$	0.00020	0.10484	1.5×10^{-6}	0.099994	0.00008	0.101055
150	$\phi=0.8$	0.00944	0.78068	0.00005	0.800283	0.00147	0.801575
	$c=0.5$	0.01567	0.44618	7.0×10^{-6}	0.499417	0.00250	0.50372
	$b=1.5$	0.28886	1.39817	0.00045	1.50033	0.01760	1.51024
	$a=0.1$	0.00013	0.10386	9.4×10^{-7}	0.099934	0.00005	0.100361

$$\frac{\partial M(\theta)}{\partial c}=2\sum_{i=1}^{n}\left[x_{i:n}-Q_{p_i}(\phi,c,b,\delta)\right]Q'_{p_i}(\phi,c,b,\delta)_c=0,$$

$$\frac{\partial M(\theta)}{\partial b}=2\sum_{i=1}^{n}\left[x_{i:n}-Q_{p_i}(\phi,c,b,\delta)\right]Q'_{p_i}(\phi,c,b,\delta)_b=0,$$

$$\frac{\partial M(\theta)}{\partial \delta}=2\sum_{i=1}^{n}\left[x_{i:n}-Q_{p_i}(\phi,c,b,\delta)\right]Q'_{p_i}(\phi,c,b,\delta)_\delta=0,$$

where $Q'_{p_i}(\theta)_\eta=\frac{\partial Q_{p_i}(\theta)}{\partial\eta}$, and η denotes any component of $\theta=(\phi,c,b,\delta^T)^T$. The software mentioned before can produce the PCEs $\hat{\phi}_{PCE}$, $\hat{c}_{PCE}$, $\hat{b}_{PCE}$ and $\hat{\delta}_{PCE}$.

VI. SIMULATION STUDY

We perform a detailed simulation study to compare the precision of the estimators of the unknown parameters for the MOOB-Ex and MOOB-Li distributions. The adequacy of these estimators is based on the mean squared

$$\begin{aligned}U_\delta=&\sum_{i=1}^{n}\frac{g'(x_i;\delta)}{g(x_i;\delta)}+(c-1)\sum_{i=1}^{n}\frac{\bar{G}'(x_i;\delta)}{\bar{G}(x_i;\delta)}-(c+1)\sum_{i=1}^{n}\frac{G'(x_i;\delta)}{G(x_i;\delta)}\\&-2bc(1-\phi)\sum_{i=1}^{n}\frac{\left[\bar{G}(x_i;\delta)G'(x_i;\delta)-G(x_i;\delta)\bar{G}'(x_i;\delta)\right]\left[\frac{G(x_i;\delta)}{\bar{G}(x_i;\delta)}\right]^{-c-1}}{z_i^{b+1}\,\bar{G}^2(x_i;\delta)\left[1-(1-\phi)\left(1-z_i^{-b}\right)\right]}\\&+c(b+1)\sum_{i=1}^{n}\frac{\left[\bar{G}(x_i;\delta)G'(x_i;\delta)-G(x_i;\delta)\bar{G}'(x_i;\delta)\right]\left[\frac{G(x_i;\delta)}{\bar{G}(x_i;\delta)}\right]^{-c-1}}{z_i^{-b-1}\,\bar{G}^2(x_i;\delta)},\end{aligned}$$

TABLE 4. MSEs and AEs for the MOOB-Li Distribution.

n	Parameters	PCE		LSE		MLE	
		MSEs	AEs	MSEs	AEs	MSEs	AEs
	$\phi = 0.8$	0.01253	0.78171	0.000095	0.80014	0.00199	0.80171
100	$c = 1.2$	0.10817	1.06704	0.000073	1.19807	0.02433	1.22297
	$b = 1.5$	0.45130	1.43071	0.000770	1.49966	0.02378	1.51413
	$a = 0.2$	2.84033	0.32391	0.000062	0.20018	0.00037	0.20441
	$\phi = 0.8$	0.00572	0.78111	0.000035	0.79908	0.00109	0.80122
200	$c = 1.2$	0.06754	1.07868	0.000026	1.19964	0.01069	1.20482
	$b = 1.5$	0.16729	1.37291	0.000296	1.49702	0.01312	1.50620
	$a = 0.2$	0.05805	0.23124	0.000023	0.20083	0.00019	0.20326
	$\phi = 0.8$	0.00338	0.80643	0.000033	0.79962	0.00073	0.80051
300	$c = 1.2$	0.04194	1.16965	0.000019	1.19947	0.00751	1.20957
	$b = 1.5$	0.12041	1.51267	0.000273	1.49856	0.00847	1.50422
	$a = 0.2$	0.00432	0.21070	0.000021	0.20042	0.00014	0.20334
	$\phi = 0.8$	0.01235	0.78462	0.000092	0.80037	0.00212	0.80048
100	$c = 1.2$	0.10346	1.09470	0.000072	1.19889	0.02330	1.20695
	$b = 1.5$	0.34968	1.45164	0.000756	1.50072	0.02655	1.50455
	$a = 0.4$	9.79763	0.29146	0.000211	0.40007	0.00142	0.40450
	$\phi = 0.8$	0.00782	0.78891	0.000039	0.79965	0.00114	0.80177
200	$c = 1.2$	0.08537	1.09970	0.000030	1.19896	0.01078	1.20280
	$b = 1.5$	0.26358	1.43599	0.000329	1.49838	0.01347	1.50732
	$a = 0.4$	0.73616	0.39329	0.000089	0.40075	0.00070	0.40123
	$\phi = 0.8$	0.00487	0.79675	0.000033	0.79959	0.00070	0.80018
300	$c = 1.2$	0.04968	1.13262	0.000019	1.19940	0.00742	1.20812
	$b = 1.5$	0.15738	1.46643	0.000285	1.49844	0.00850	1.50393
	$a = 0.4$	0.02012	0.44780	0.000075	0.40081	0.00045	0.40180
	$\phi = 1.0$	0.02260	0.96963	0.000110	1.00246	0.00345	1.00839
100	$c = 1.2$	0.13029	1.05072	0.000067	1.19751	0.02130	1.21385
	$b = 1.2$	0.24491	1.11514	0.000321	1.20374	0.01484	1.21295
	$a = 0.2$	0.17225	0.29510	0.000039	0.19866	0.00034	0.20346
	$\phi = 1.0$	0.01196	0.97503	0.000059	0.99981	0.00177	1.00297
200	$c = 1.2$	0.07520	1.10694	0.000028	1.19886	0.01033	1.21157
	$b = 1.2$	0.14519	1.13300	0.000180	1.19930	0.00740	1.20903
	$a = 0.2$	0.01558	0.25013	0.000020	0.20018	0.00018	0.20338
	$\phi = 1.0$	0.00700	0.98475	0.000049	1.00067	0.00093	0.99932
300	$c = 1.2$	0.05078	1.11803	0.000018	1.19875	0.00851	1.20972
	$b = 1.2$	0.07234	1.13296	0.000150	1.20085	0.00490	1.20132
	$a = 0.2$	0.00543	0.22710	0.000017	0.19963	0.00010	0.20367

TABLE 5. MSEs and AEs for the MOOB-Li Distribution.

n	Parameters	PCE		LSE		MLE	
		MSEs	AEs	MSEs	AEs	MSEs	AEs
	$\phi = 1.0$	0.02272	0.95928	0.000112	0.99882	0.00345	1.00314
100	$c = 1.2$	0.12127	1.01203	0.000082	1.19912	0.02367	1.21599
	$b = 1.5$	0.42624	1.33600	0.000555	1.49673	0.02402	1.51442
	$a = 0.4$	1.50032	0.60642	0.000192	0.40181	0.00187	0.41363
	$\phi = 1.0$	0.00807	0.99090	0.000089	0.99933	0.00163	1.00299
200	$c = 1.2$	0.05484	1.09804	0.000046	1.19959	0.01204	1.21115
	$b = 1.5$	0.13158	1.41557	0.000443	1.49819	0.01131	1.51026
	$a = 0.4$	0.42930	0.52371	0.000154	0.40131	0.00085	0.41112
	$\phi = 1.0$	0.00736	0.99045	0.000041	1.00101	0.00112	1.00175
300	$c = 1.2$	0.03898	1.11307	0.000025	1.19878	0.00756	1.20942
	$b = 1.5$	0.11452	1.42561	0.000203	1.50205	0.00846	1.50871
	$a = 0.4$	0.02820	0.45983	0.000068	0.39882	0.00061	0.41046
	$\phi = 1.0$	0.01752	0.98113	0.000149	1.00032	0.00344	1.00215
100	$c = 1.2$	0.10692	1.07976	0.000116	1.19808	0.02024	1.20220
	$b = 1.5$	0.33369	1.42923	0.000702	1.50009	0.02270	1.50538
	$a = 0.2$	0.58235	0.29580	0.000062	0.20012	0.00045	0.20678
	$\phi = 1.0$	0.01549	0.99755	0.000061	1.00027	0.00144	1.00414
200	$c = 1.2$	0.07721	1.11020	0.000043	1.19886	0.00977	1.21151
	$b = 1.5$	0.31670	1.48354	0.000296	1.50018	0.00981	1.51313
	$a = 0.2$	0.05843	0.25663	0.000026	0.19992	0.00018	0.20463
	$\phi = 1.0$	0.00723	0.99707	0.000058	1.00007	0.00095	1.00175
300	$c = 1.2$	0.04531	1.13246	0.000044	1.19929	0.00737	1.20519
	$b = 1.5$	0.19654	1.46842	0.000274	1.49991	0.00625	1.50616
	$a = 0.2$	0.00716	0.22467	0.000026	0.20008	0.00014	0.20478
	$\phi = 1.0$	0.02080	0.94871	0.000132	1.00032	0.00378	1.00260
100	$c = 1.2$	0.09971	1.02132	0.000186	1.19857	0.02372	1.22277
	$b = 1.8$	0.46374	1.57681	0.000896	1.80027	0.03382	1.80983
	$a = 0.2$	0.32599	0.20543	0.000071	0.20021	0.00083	0.21135
	$\phi = 1.0$	0.01101	0.97917	0.000053	0.99984	0.00162	1.00139
200	$c = 1.2$	0.04883	1.11342	0.000077	1.19948	0.01095	1.21745
	$b = 1.8$	0.23897	1.70458	0.000365	1.79920	0.01556	1.81480
	$a = 0.2$	0.04207	0.25224	0.000029	0.20025	0.00033	0.20855
	$\phi = 1.0$	0.00716	0.98468	0.000045	0.99965	0.00128	1.00125
300	$c = 1.2$	0.04786	1.08239	0.000064	1.19978	0.00796	1.20972
	$b = 1.8$	0.19775	1.67096	0.000303	1.79874	0.01161	1.80970
	$a = 0.2$	0.01421	0.24958	0.000024	0.20034	0.00024	0.20811

errors (MSEs). All calculations are done automatically using `Mathematica`. We generate 1,000 samples of the MOOB-Ex and MOOB-Li distributions. We take $n = 50$, 100, 150 for the MOOB-Ex model and $n = 100$, 200, 300 for the MOOB-Li model. The average estimates (AEs) and MSEs of the MLEs, LSEs and PCEs for the MOOB-Ex distribution are given in Tables 2-3. Further, the AEs and MSEs of the MLEs, LSEs and PCEs for the MOOB-Li distribution are listed in Tables 4-5. We note that all the estimators reveal the consistency property. Also, the figures in these tables indicate that the LSEs produce the best results for estimating the parameters of the MOOB-Ex and MOOB-Li distributions in terms of their MSEs in most cases.

VII. DATA ANALYSIS

We prove the flexibility of the MOOB-Ex distribution by comparing with some competitive distributions given in Table 6 by means of three real data sets. The Cramér-Von Mises (CVM), Anderson-Darling (AD) and the Kolmogorov-Smirnov (KS) statistics and the KS p-values of the last one are calculated for the fitted distributions using the `R` software.

The first data set refers to 63 observations of the strengths of 1.5 cm glass fibres (see, Smith and Naylor, 1987) analyzed by Alizadeh *et al.* (2020). The second data set represents 74 observations of the gauge lengths of 20 mm (see, Kundu and Raqab, 2009) analyzed by Afify and Mohamed (2020). The third data set consists of 100 observations of breaking stress of carbon fibres (in Gba) (see, Nichols and Padgett, 2006) analyzed by Cordeiro *et al.* (2017).

Tables 7-9 report the values of the goodness-of-fit measures, the MLEs and associated standard errors (SEs) (in parentheses) for the MOOB-Ex model and other fitted distributions. Some of the fitted distributions in Tables 7-9 have very large SEs compared with their MLEs, wheres all MLEs for the fitted MOOB-Ex distribution are accurate.

Figures 4, 6 and 8 display some plots of the fitted densities for these sets. Further, the corresponding PP plots for the fitted distributions are given in Figures 5, 7 and 9. These plots

TABLE 6. Some Competitive Models for the MOOB-Ex Distribution.

Distribution	Author(s)
Exponential (Ex)	
Marshall-Olkin Nadarajah-Haghighi (MONH)	Lemonte et al. (2016)
Odd exponentiated half-logistic Ex (OEHLEx)	Afify et al. (2018)
Beta generalized Ex (BGEx)	Barreto-Souza et al. (2010)
Kumaraswamy transmuted Ex (KTEx)	Afify et al. (2016)
Kumaraswamy Ex (KEx)	–
Marshall-Olkin Ex (MOEx)	Marshall and Olkin (1997)
Beta Ex (BEx)	Nadarajah and Kotz (2006)

TABLE 7. Goodness-of-Fit Measures, MLEs and (SEs) for Data Set I.

Model	CVM	AD	KS	KS p-value	MLEs			
MOOB-Ex	0.0407	0.2512	0.0758	0.8612	4.1455	12.9915	0.1652	0.4177
(ϕ, c, b, a)					(3.8866)	(2.7502)	(0.0834)	(0.0211)
MONH	0.1015	0.5750	0.0955	0.6132	8.5199	0.1516	162.103	
(ϕ, λ, θ)					(17.310)	(0.3563)	(173.39)	
OEHLEx	0.1356	0.7698	0.1227	0.2982	1.5776	0.0345	2.4614	
(ϕ, λ, a)					(0.5047)	(0.0334)	(0.4798)	
BGEx	0.2021	1.1211	0.1469	0.1314	0.5583	224589.2	0.1810	9.5989
(a, b, λ, ϕ)					(0.2513)	(72373.3)	(0.0955)	(3.6562)
KTEx	0.2496	1.3697	0.1574	0.0879	4.8796	2145.5	0.2910	-0.7206
(a, b, λ, ϕ)					(2.0384)	(4196.05)	(0.1283)	(0.4561)
KEx	0.2687	1.4724	0.1614	0.0749	6.8785	2553.2	0.2373	
(a, b, λ)					(1.1075)	(5216.1)	(0.1228)	
MOEx	0.2596	1.4304	0.1254	0.2750	8247.4	5.8503		
(ϕ, λ)					(7982.2)	(0.6033)		
BEx	0.5686	3.1184	0.2163	0.0054	17.4699	154.884	0.0711	
(a, b, λ)					(3.0852)	(242.09)	(0.1054)	
Ex	0.5702	3.1270	0.4179	0.0000	0.6636			
(a)					(0.0836)			

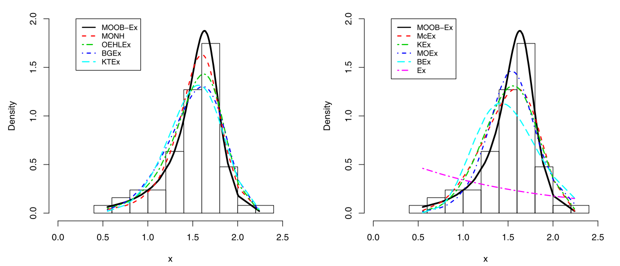

FIGURE 4. The fitted MOOB-Ex density and other densities for data set I.

TABLE 8. Goodness-of-Fit Measures, MLEs and (SEs) for Data Set II.

Model	CVM	AD	KS	KS p-value	MLEs			
MOOB-Ex	0.0233	0.1738	0.0496	0.9932	0.3270	11.5848	0.4421	0.2269
(ϕ, c, b, a)					(0.4850)	(4.5771)	(0.1682)	(0.0286)
MONH	0.0336	0.2310	0.0569	0.9698	2.1710	0.5784	389.87	
(ϕ, η, θ)					(1.4589)	(0.6743)	(540.85)	
OEHLEx	0.0264	0.1995	0.0545	0.9802	3.8721	0.3004	0.8837	
(ϕ, η, a)					(1.9472)	(0.2850)	(0.2609)	
BGEx	0.0267	0.2132	0.0576	0.9663	0.5689	29.513	0.6648	21.846
(a, b, η, ϕ)					(0.9764)	(87.479)	(0.9665)	(59.050)
KTEx	0.0267	0.2097	0.0575	0.9673	8.8699	112.545	0.3562	-0.1036
(a, b, η, ϕ)					(16.810)	(627.38)	(0.9223)	(4.9790)
KEx	0.0267	0.2104	0.0574	0.9674	9.1401	127.95	0.3340	
(a, b, η)					(4.7605)	(457.29)	(0.3569)	
MOEx	0.0443	0.2723	0.0598	0.9537	8104.8	3.6184		
(ϕ, η)					(7212.5)	(0.3468)		
BEx	0.0874	0.5737	0.0682	0.8809	24.317	92.491	0.0947	
(a, b, η)					(3.9884)	(154.90)	(0.1426)	
Ex	0.0875	0.5749	0.4495	0.0000	0.4037			
(a)					(0.0469)			

also reveal that the MOOB-Ex distribution yields the best fit to all data sets.

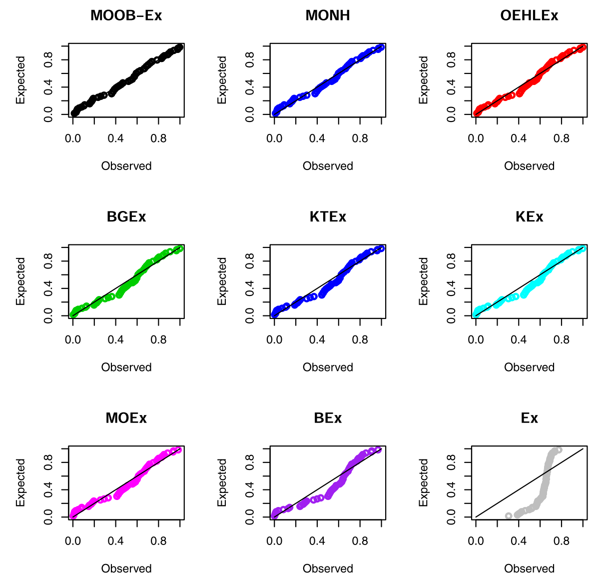

FIGURE 5. PP plots of the MOOB-Ex distribution and other models for data set I.

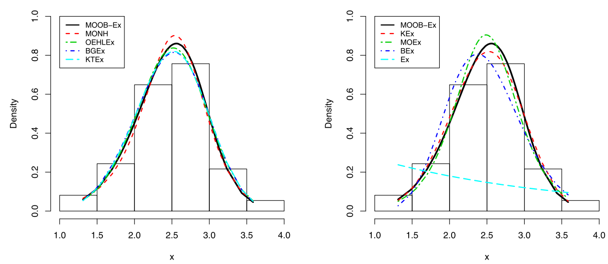

FIGURE 6. The fitted MOOB-Ex density and other densities for data set II.

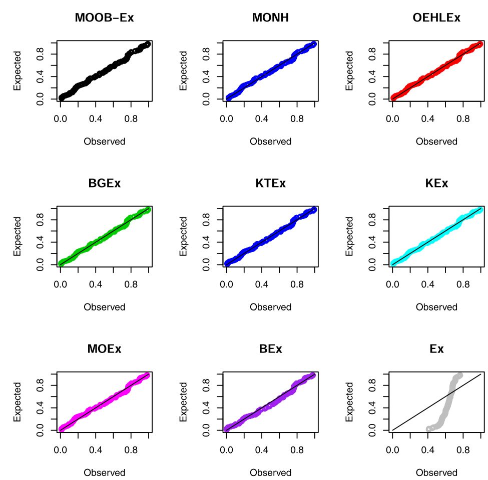

FIGURE 7. PP plots of the MOOB-Ex distribution and other models for data set II.

The hrf plots of the MOOB-Ex model and the TTT plots for the three data sets are, respectively, displayed in Figures 8-10. The TTT plots are concave indicating that the three data sets have increasing hazard rates as shown from the hrf plots of the three data sets.

TABLE 9. Goodness-of-Fit Measures, MLEs and (SEs) for Data Set III.

Model	CVM	AD	KS	KS p-value	MLEs			
MOOB-Ex	0.0515	0.3223	0.0575	0.8945	0.5399	5.2672	0.4555	0.1955
(ϕ, c, b, a)					(0.6185)	(1.5128)	(0.1408)	(0.0390)
MONH	0.0730	0.4305	0.0639	0.8084	0.8254	3.0008	145.189	
(ϕ, η, θ)					(0.4280)	(5.2241)	(300.701)	
OEHLEx	0.0682	0.4011	0.0639	0.8087	2.7130	2.1000	0.2669	
(ϕ, λ, a)					(0.7321)	(1.4127)	(0.1041)	
BGEx	0.0678	0.4007	0.0641	0.8064	0.5169	10.7682	0.3452	7.1016
(a, b, λ, ϕ)					(0.8818)	(27.8992)	(0.5755)	(14.7598)
KTEx	0.0685	0.4007	0.0645	0.7995	2.6891	19.4945	-0.6237	0.2270
(a, b, η, ϕ)					(3.2607)	(38.5474)	(1.7109)	(0.1943)
KEx	0.0699	0.4091	0.0646	0.7967	3.2780	63.2256	0.1134	
(a, b, η)					(0.8372)	(217.67)	(0.1823)	
MOEx	0.0631	0.4102	0.0592	0.8746	75.6861	1.6757		
(ϕ, η)					(33.4913)	(0.1534)		
BEx	0.1483	0.7589	0.0935	0.3461	5.9605	34.5462	0.0615	
(a, b, η)					(0.8218)	(61.1416)	(0.1021)	
Ex	0.1493	0.7643	0.3206	0.0000	0.3815			
(a)					(0.0381)			

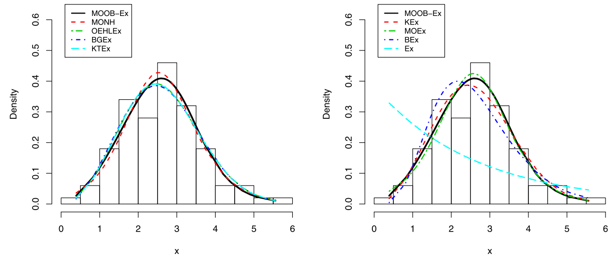

FIGURE 8. **The fitted MOOB-Ex density and other densities for data set III.**

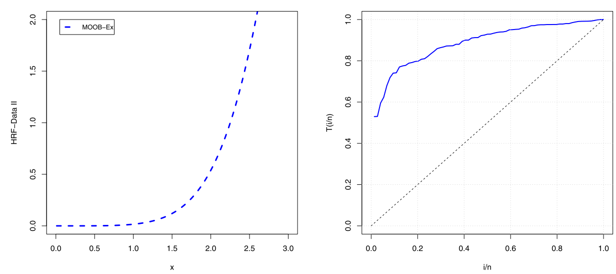

FIGURE 9. **The hrf plot of the MOOB-Ex model and TTT plot for data set II.**

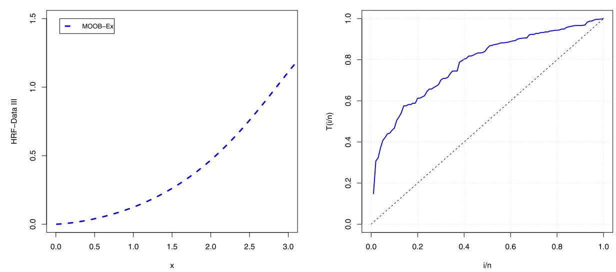

FIGURE 10. **The hrf plot of the MOOB-Ex model and TTT plot for data set III.**

TABLE 10. The Estimates of the MOOB-Ex Parameters for the Three Data Sets.

Method	$\widehat{\phi}$	$\widehat{c}$	$\widehat{b}$	$\widehat{a}$	$-\widehat{\ell}$	CVM	AD	KS	KS p-value
Strengths of 1.5 cm glass fibres data									
MLE	4.1455	12.992	0.1652	0.4177	10.295	0.0407	0.2512	0.0758	0.8612
LSE	28.816	13.791	0.0322	0.4252	11.817	0.0352	0.2803	0.0607	0.9744
PCE	4.3044	11.583	0.1577	0.4149	10.873	0.05024	0.3003	0.0928	0.6497
Gauge lengths of 20 mm data									
MLE	0.3270	11.5848	0.4421	0.2269	51.165	0.0233	0.1738	0.0496	0.9932
LSE	0.3809	12.7094	0.3674	0.2258	51.327	0.0231	0.1867	0.0546	0.9800
PCE	0.7130	9.1599	0.4843	0.2433	51.458	0.02759	0.1883	0.0533	0.9847
Breaking stress of carbon fibres data									
MLE	0.5399	5.2672	0.4555	0.1955	141.096	0.0515	0.3223	0.0575	0.8945
LSE	0.8400	6.8485	0.2748	0.19407	142.517	0.0557	0.4884	0.0413	0.9956
PCE	0.9149	4.3532	0.4955	0.2176	141.385	0.0605	0.3518	0.0605	0.8572

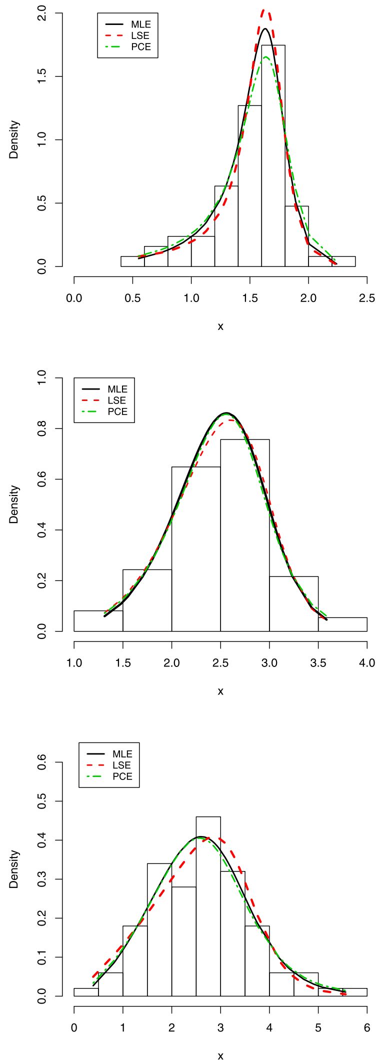

FIGURE 11. **The fitted MOOB-Ex density for data set I (left), for data II (middle) and for data III (right).**

Further, we adopt the three estimation methods discussed in Section 5 to estimate the MOOB-Ex parameters from the three data sets. Table 10 provides the estimates of the MOOB-Ex parameters for these data sets, and the maximized log-likelihoods values ($-\widehat{\ell}$) and the CVM, AD and KS statistics and p-values. Based on the KS and p-values in Table 10, we note that the LSE provides the best estimates of the MOOB-Ex parameters for the first and third data sets, whereas the ML method is recommended to estimate the MOOB-Ex parameters for the second data set. However, all estimation methods perform very well for the three data sets.

The histograms of the three data sets and the fitted MOOB-Ex densities using the three estimation methods, for the three data sets, are displayed in Figure 11.

VIII. CONCLUSION

We propose a new class of distributions with three additional shape parameters, called Marshall-Olkin odd Burr III-G (MOOB-G) family, from any baseline continuous distribution G. The new MOOB-G family extends the Marshall-Olkin-G and odd Burr III-G classes and distributions and three of its special models, called MOOB-exponential, MOOB-Lindley, and MOOB-Lomax distributions are discussed. The family density can be expressed as linear combination of exponentiated-G densities. We explore some mathematical properties of the MOOB-G family. The parameters of the proposed family are estimated by three approaches, called maximum likelihood, least squares and percentile methods. The performance of the three methods are assessed via simulation results obtained for the MOOB-exponential and MOOB-Lindley distributions. Three real data examples show empirically that the MOOB-exponential distribution provides better fits to these data than other known extensions of the exponential model.

REFERENCES

[1] A. Afify and M. Alizadeh, "The odd Dagum family of distributions: Properties and applications," *J. Appl. Probab. Stat.*, vol. 15, no. 1, pp. 45–72, 2020.

[2] A. Z. Afify, G. M. Cordeiro, M. E. Maed, M. Alizadeh, H. Al-Mofleh, and Z. M. Nofal, "The generalized odd Lindley-G family: Properties and applications," *Anais da Academia Brasileira de Ciêcncias*, vol. 91, no. 3, pp. 1–22, 2019.

[3] A. Z. Afify, G. M. Cordeiro, H. M. Yousof, Z. M. Nofal, and A. Alzaatreh, "The Kumaraswamy transmuted-G family of distributions: Properties and applications," *J. Data Sci.*, vol. 14, pp. 245–270, Apr. 2016.

[4] A. Z. Afify and O. A. Mohamed, "A new three-parameter exponential distribution with variable shapes for the hazard rate: Estimation and applications," *Mathematics*, vol. 8, pp. 1–17, Jan. 2020.

[5] A. Z. Afify, M. Zayed, and M. Ahsanullah, "The extended exponential distribution and its applications," *J. Stat. Appl.*, vol. 17, pp. 213–229, Jun. 2018.

[6] M. Alizadeh, A. Z. Afify, M. S. Eliwa, and S. Ali, "The odd log-logistic Lindley-G family of distributions: Properties, Bayesian and non-Bayesian estimation with applications," *Comput. Statist.*, vol. 35, no. 1, pp. 281–308, Mar. 2020.

[7] H. Al-Mofleh, "On generating a new family of distributions using the tangent function," *Pakistan J. Statist. Oper. Res.*, vol. 14, pp. 471–499, Sep. 2018.

[8] W. Barreto-Souza, A. H. S. Santos, and G. M. Cordeiro, "The beta generalized exponential distribution," *J. Stat. Comput. Simul.*, vol. 80, no. 2, pp. 159–172, Feb. 2010.

[9] G. M. Cordeiro, A. Z. Afify, E. M. M. Ortega, A. K. Suzuki, and M. E. Mead, "The odd Lomax generator of distributions: Properties, estimation and applications," *J. Comput. Appl. Math.*, vol. 347, pp. 222–237, Feb. 2019.

[10] G. M. Cordeiro, A. Z. Afify, H. M. Yousof, R. R. Pescim, and G. R. Aryal, "The exponentiated Weibull-H family of distributions: Theory and applications," *Medit. J. Math.*, vol. 14, no. 4, pp. 1–22, Aug. 2017.

[11] G. M. Cordeiro, E. M. Hashimoto, and E. M. M. Ortega, "The McDonald Weibull model," *Statistics*, vol. 48, no. 2, pp. 256–278, Mar. 2014.

[12] G. M. Cordeiro, A. J. Lemonte, and E. M. M. Ortega, "The Marshall-Olkin family of distributions: Mathematical properties and new models," *J. Stat. Theory Pract.*, vol. 8, pp. 343–366, Jun. 2014.

[13] M. E. Ghitany, F. A. Al-Awadhi, and L. A. Alkhalfan, "Marshall–Olkin extended Lomax distribution and its application to censored data," *Commun. Stat. Theory Methods*, vol. 36, pp. 1855–1866, Aug. 2007.

[14] F. Jamal, M. A. Nasir, M. H. Tahir, and N. H. Montazeri, "The odd burr-III family of distributions," *J. Statist. Appl. Probab.*, vol. 6, no. 1, pp. 105–122, Mar. 2017.

[15] D. Kundu and M. Z. Raqab, "Estimation of $R = P(Y < X)$ for three parameter Weibull distribution," *Statist. Probab. Lett.*, vol. 79, pp. 1839–1846, Sep. 2009.

[16] A. J. Lemonte, G. M. Cordeiro, and G. Moreno-Arenas, "A new useful three-parameter extension of the exponential distribution," *Statistics*, vol. 50, pp. 312–337, Mar. 2016.

[17] A. Marshall, "A new method for adding a parameter to a family of distributions with application to the exponential and weibull families," *Biometrika*, vol. 84, no. 3, pp. 641–652, Sep. 1997.

[18] S. Nadarajah and S. Kotz, "The beta exponential distribution," *Rel. Eng. Syst. Saf.*, vol. 91, no. 6, pp. 689–697, Jun. 2006.

[19] M. D. Nichols and W. J. Padgett, "A bootstrap control chart for Weibull percentiles," *Qual. Rel. Eng. Int.*, vol. 22, no. 2, pp. 141–151, 2006.

[20] R. L. Smith and J. C. Naylor, "A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution," *Appl. Statist.*, vol. 36, pp. 358–369, Nov. 1987.

AHMED Z. AFIFY received the Ph.D. degree in statistics from Benha University, in 2018, under the supervision of Prof. Gauss M. Cordeiro. He is currently an Assistant Professor of statistics with the Department of Statistics, Mathematics, and Insurance, Benha University, Benha, Egypt. He has published more than 100 research articles in reputable international journals. His research interests include distribution theory, life testing, reliability analysis, and statistical inference. He serves as a reviewer and an editorial board member of some international journals.

GAUSS M. CORDEIRO received the Ph.D. degree in statistics from Imperial College London, U.K., in 1982. He is currently a Brazilian Statistician who has made significant contributions to the theory of statistical inference, mainly through asymptotic theory and applied probability. He is also a Class A Researcher of the Brazilian Research Council (CNPq) and also a Full Professor of the Universidade Federal de Pernambuco, Brazil, where he is also a member of the Post-Graduate Program in statistics. He has published more than 440 research articles in international scientific journals with referee practice, in 2020, and supervised more than 65 M.Sc. dissertations and D.Sc. theses.

NOOR AKMA IBRAHIM is currently a Professor of statistics with the Department of Mathematics, Faculty of Science, University Putra Malaysia. She has published works in survival analysis, influence diagnostics, modeling, and distribution theory. She is in the Editorial Board of *Malaysian Journal of Mathematical Sciences* (MJMS) and a referee of various journals, locals, and abroad.

FARRUKH JAMAL received the M.Sc. and M.Phil. degrees in statistics from the Islamia University of Bahawalpur (IU), Pakistan, in 2003 and 2006, respectively, and the Ph.D. degree from IUB under the supervision of Dr. M. H. Tahir. He was a Lecturer with the Government S. A. Postgraduate College, from 2012 to 2020, and also a Statistical Officer with the Agriculture Department, from 2007 to 2012. He is currently an Assistant Professor with the Department of Statistics, The Islamia University of Bahawalpur (IUB), Pakistan. He has 130 publications in his credit.

MOHAMED ELGARHY received the M.Sc. and Ph.D. degrees in statistics from the Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt, in 2014 and 2018, respectively. He is currently an Assistant Professor of statistics with the Higher Institute of Commercial Sciences, El Mahalla El Kubra, Egypt. He has more than 80 international publications in his credit. His current research interest includes the generalized classes of distributions and their special models.

MOHAMED ARSLAN NASIR received the Ph.D. degree from IUB under the supervision of Dr. M. H. Tahir. He is currently an Assistant Professor with the Department of Statistics, Government S. E. College, Bahawalpur, Pakistan. He has 40 publications in his credit.