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A Unified Mean Square Consensus Criterion for Stochastic Multi-Agent Systems With ROUs and RONs Under Impulse Time Windows

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ABSTRACT This paper probes into the leader-following mean square consensus problem of stochastic multi-agent systems with randomly occurring uncertainties (ROUs) and randomly occurring nonlinearities (RONs) under impulse time windows (ITWs). A new concept named average left endpoint interval (ALEI) related to ITWs is proposed, which is inspired by the average impulsive interval. Based on ALEI and Lyapunov stability theory, a unified mean square consensus criterion for both stabilizing and interference impulses is attained. The introduction of ALEI allows for a larger upper bound of left endpoint interval and smaller lower bound of left endpoint interval, which makes the preset of ITWs more flexible and less conservative. The validity of the theoretical results is verified by the numerical simulation.

INDEX TERMS Stochastic multi-agent systems, randomly occurring uncertainties, randomly occurring nonlinearities, impulse time windows, leader-following mean square consensus.

I. INTRODUCTION

Currently, based on the distributed collaborative control, multi-agent systems (MASs) has been widely used in social life by virtue of the powerful functions brought by cluster effect, such as formation control of robots or aircraft [1], commercial finance [2], sociology [3], traffic management and control [4], epidemiological research [5], etc. Consensus means that all agents tend to a common state eventually under the control protocol, as a central issue in the field of distributed cooperative control, which has attracted extensive attention of researchers in recent years. One or more agents in MASs are selected as the leader to achieve more convenient control and ensuring that their followers can achieve the leader's state under the action of controller, which leads to the study of leader-following consensus of linear or nonlinear MASs [6]–[10].

We know that MASs may be affected by a variety of internal or external environmental factors in practical application, which should be reflected in the construction of the system model as much as possible. Specifically, the unpredictable disturbances and changes of these factors may lead to the random occurrence of nonlinear intrinsic dynamics and parameter uncertainties in a probabilistic way with certain types and intensity. Researchers have proposed concepts randomly occurring uncertainties (ROUs) in [11]–[13] and randomly occurring nonlinearities (RONs) in [14]–[16] to describe the objective phenomenon. In general, the stochastic variable is assumed to conform Bernoulli distribution. On the other hand, the dynamic of each agent may be affected by the stochastic disturbances caused by environmental fluctuation and other factors, which are mainly in the form of noise. If all agents' dynamics are subject to different noises, in this case, the noise is called the heterogeneous one. Otherwise, it is the same noise. When some practical systems or applications, such as financial model, smart grid, unmanned aerial vehicle or robot formation, integrated navigation system, the determination and estimation of aircraft attitude, are disturbed by such noise, which can be modeled by using *Itô* formula. In a word, the researchers are mainly by using *Itô* formula and Lyapunov stability theory to solve the consensus problem of stochastic MASs (SMASs) at present. It is undoubtedly necessary and valuable to study the leader-following consensus of MASs under the case

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where take one or more of ROUs, RONs and stochastic disturbances into account simultaneously in [17], [18], etc. Furthermore, on the basis of [17], the authors studied the leader-following mean square consensus of delayed SMASs by means of comparison principle, Lyapunov stability theory and average impulsive interval (AII) in [19]. In [20], the exponential consensus of SMASs with ROUs and RONs has been studied via pinning control. Recently, many new research fields have been combined with the consensus problem of SMASs, such as decentralised probabilistic consensus control [21], sliding mode control [22], event-triggered control [23], etc.

Actually, reducing the communication traffic in the working process of MASs can save the communication cost and reduce the communication risk (packet dropouts and disorder, wireless channel congestion, delays, etc.) effectively, so it is suitable for the actual situation where the communication bandwidth or communication guarantee level is limited. As an effective control mechanism to reduce communication traffic and control costs, impulsive control with strong adaptability has been introduced into the consensus research of MASs, and abundant results have been achieved [17], [19], [20]. Many literatures assume that the impulsive controller acts on the systems at a series of fixed discrete instants. However, in practical applications such as aircraft formation control, satellite attitude control, and smart grid, the impulsive control signal is a series of discrete digital signals generated by the controller. Affected by random or uncertain factors in the external environment and the limitations of physical equipment, digital signals may skew or jitter. The so-called jitter and skew refer to the short-term or long-term deviation of the position of the digital signal at the actual control time t_k from the ideal time position r_k of the signal respectively, and the deviation value is usually bounded or known. In fact, the value of t_k may be less than, equal to, or greater than r_k. The concept named impulse time window (ITW) is proposed to describe this reality in [24], which stipulates that one impulse appears in a known time window randomly. Based on ITW, the uniform stability of impulsive delayed linear systems has been studied in [25]. In [26], the global exponential stability of memristive neural networks with ITW and time-varying delays has been researched. In [27], the consensus problem of periodically multiple state-jumps impulsive control systems with ITWs has been discussed in detail. In [28], the authors focused on the global exponential stability problem of delayed impulsive functional differential systems with ITWs, and so on [29]–[34]. For the consensus or synchronization criteria derived from the above literatures, there is always an upper bound for the width of ITW. This also makes the impulsive interval has an upper bound, and it will cause the appearance of impulse too frequently in a period of time if the bound is small enough. Without affecting the control demands and effects, this phenomenon may increase some unnecessary control costs. Therefore, how to get a larger upper bound of impulsive interval than the existing results about ITW (i.e., less conservative) is a practical problem that is both meaningful and urgent to be solved. It is worth learning from the authors' work in [35], although it does not involve ITW. Specifically, the unified synchronization criterion of impulsive dynamical networks is given by the proposed AII, which expands the upper bound of impulsive interval and narrows the lower bound of impulsive interval for synchronizing and desynchronizing impulses respectively. In [17], the authors introduced AII into the research on the leader-following mean square consensus of SMASs with ROUs, and similar results were obtained.

Based on the above discussions, the leader-following mean square consensus problem of SMASs with ROUs, RONs and ITWs is studied in this paper. The main contributions are as follows.

- The concept of AII is improved to obtain a new one named average left endpoint interval (ALEI) which related to ITWs. Based on ALEI, a unified mean square consensus criterion that allow for larger upper bound of impulsive interval are given. Compared with the existing literatures with ITWs, such as [24]–[34], the problem of how to get a larger impulsive interval is solved. This also means that the frequency of impulsive control for unstable MASs can be effectively reduced in a period of time, thus saving costs and enhancing adaptability (i.e., the preset of ITWs is more flexible).
- Compared with the existing literatures without ITWs or AII, the researched system model with ROUs, RONs, stochastic disturbances and ITWs in this paper is more general, so our results may be less conservative and more suitable for practical situations. In particular, references [35] and [17] can be the special cases of this paper under some limited conditions.
- For a kind of stable MASs affected by interference impulses, if these impulses appear in a series of known ITWs, a smaller lower bound of impulsive interval can be obtained by ALEI. This means that the interference impulses should not occur too frequently in a given time.

The rest of this text is organised as follows. In section II, we introduce the constructions of system model and control protocol as well as annotate related details. In section III, compare the theoretical results which are derived from Lyapunov stability theory with the existing literatures. In section IV, our results are verified by one numerical simulation example. Section V is the summary of the article.

Nonations: In this paper, $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the real numbers, the n-dimensional Euclidean space and the set of $m \times n$ matrices respectively. $\mathbb{N}^+$ is the positive integers set. $|\cdot|$ and $\|\cdot\|$ represent the absolute value and the Euclidean norm (2-norm) respectively. I_{Nn} is an identity matrix with order Nn. $E(\cdot)$ is the expectation operator. $\otimes$ is the Kronecker product. $(\cdot)^T$ denotes the matrix's transpose. $\lambda_{max}(\cdot)$ represents the matrix's maximal eigenvalue. $\mathcal{L}$ is the Kolmogorov operator. Pr represents the occurrence probability of a random variable.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. ALGEBRAIC GRAPH THEORY

The communication links between agent and their neighbors in MASs can be represented by the topology graph. When all information transmission directions are bidirectional, the graph is called undirected. Otherwise, it is a digraph. For a N order weighted digraph $G = (\tilde{\Omega}, \Xi, A)$ without self-circulation, where $\tilde{\Omega} = \left\{\tilde{\Omega}_1, \ldots, \tilde{\Omega}_N\right\}$ is the node set, $\Xi = \{(\tilde{\Omega}_j, \tilde{\Omega}_i) : i, j = 1, \ldots, N\} \subset \tilde{\Omega} \times \tilde{\Omega}$ is the edge set, $A = [a_{ij}]$ denotes the weighted adjacency matrix, $(\tilde{\Omega}_j, \tilde{\Omega}_i)$ represents the information transmission from $\tilde{\Omega}_j$ to $\tilde{\Omega}_i$, the weight of $(\tilde{\Omega}_j, \tilde{\Omega}_i)$ is represented by a_{ij}, $a_{ij} > 0$ means that $\tilde{\Omega}_i$ can receive the information of its neighbor $\tilde{\Omega}_j$. In order to more conveniently and intuitively express the communication relationship between any two agents, without losing generality, we choose the weight to be 1 or 0 (i.e., receiving is 1, otherwise, it is 0). Let $D = \mathrm{diag}(d_i, i = 1, \ldots, N)$ be the degree matrix, where $d_i = d_{in}(\tilde{\Omega}_i) = \sum_{j=1, j\neq i}^{N} a_{ij}$. Then the Laplacian matrix can be defined as $L = D - A = [l_{ij}]$, where $l_{ij} = \begin{cases} -a_{ij}, i \neq j, \\ -\sum_{j=1, j\neq i}^{N} l_{ij}, i = j. \end{cases}$ If there exists one leader in MASs, let $B = \mathrm{diag}(b_1, \ldots, b_N)$ denote the connection matrix between followers and their leader. If node $\tilde{\Omega}_i$ receives the leader's information, for convenience, let $b_i = 1$. Otherwise, $b_i = 0$.

B. PROBLEM DESCRIPTION AND PROTOCOL CONSTRUCTION

A first-order SMASs consisting of N agents labeled by $i = 1, 2, \ldots, N$ and one leader is considered. The dynamical equation of i-th agent with ROUs and RONs can be described by:

$$dx_i(t) = [A(t)x_i(t) + \varepsilon(t)f\,(t, x_i(t)) + u_i(t)]\,dt + \Phi\,(t, x_i(t))\,dw(t), \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$ is the state of agent i, $A(t) = \tilde{A} + \theta(t)\tilde{\Theta}\Lambda(t)\Theta \in \mathbb{R}^{n\times n}$ is a time-varying matrix, $\tilde{A}$, $\tilde{\Theta}$ and Θ are known matrices with appropriate dimensions, $\Lambda(t)$ is an unknown time-varying matrix and satisfies $\Lambda^T(t)\Lambda(t) \leq I$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear function, $u_i(t)$ is the control input, $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ denotes the matrix-valued function of noise intensity. $w(t)$ is an m-dimensional Wiener process defined on the complete probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfies the usual conditions (i.e., $\mathcal{F}_0$ contains all P-null sets and $\mathcal{F}_t$ is right continuous), $w_i(t)$ and $w_j(t)$ are independent of each other for $i \neq j$.

Remark 1: $\theta(t)\tilde{\Theta}\Lambda(t)\Theta$ and $\varepsilon(t)f(\cdot)$ are used to model the behavior of ROUs and RONs, respectively. The stochastic variables $\theta(t) \in (0, 1)$ and $\varepsilon(t) \in (0, 1)$ are Bernoulli-distribution (or 0-1 distribution) sequences, which satisfy the following assumptions:

$$\begin{cases} \Pr(\theta(t) = 1) = \hat{\theta}, & \Pr(\theta(t) = 0) = 1 - \hat{\theta}, \\ \Pr(\varepsilon(t) = 1) = \hat{\varepsilon}, & \Pr(\varepsilon(t) = 0) = 1 - \hat{\varepsilon}. \end{cases}$$

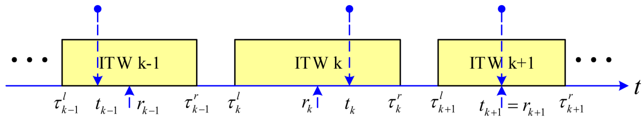

FIGURE 1. The diagram of impulse time windows.

For any appropriate dimensional matrix $\tilde{F}$, we have equations $E((\theta(t) - \hat{\theta})\tilde{F}) = 0$ and $E((\varepsilon(t) - \hat{\varepsilon})\tilde{F}) = 0$. Meanwhile, $\theta(t)$, $\mathcal{E}(t)$ and $w(t)$ are mutually independent.

Remark 2: In [24], [34], although the influence causing by ITWs were researched, the existence of RONs, ROUs and stochastic disturbances is not considered, so the derived results may be conservative. Comparing with [17], [19], etc., the existence of ITWs is considered in this paper, so our results are more suitable for practical applications.

Assumption 1: There is at least one directed spanning tree with leader as the root node in the fixed topology of SMASs.

The dynamics of the leader is governed by:

$$dx_0(t) = [A(t)x_0(t) + \varepsilon(t)f\,(t, x_0(t))]\,dt + \Phi\,(t, x_0(t))\,dw(t). \quad (2)$$

Let $\{r_k\}$ denote the ideal impulse control sequence preset in advance. In practical applications, the impulsive signal generated by the controller appears in the form of a digital signal. Due to the digital signal's jitter or skew, we can get a real control sequence $\{t_k\}$, which satisfies $t_0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to+\infty} t_k = +\infty$, when the impulses act on the SMASs. As mentioned in the introduction, the impulsive signal can be seen as randomly appearing in the corresponding ITW.

Note that for signal skew, the deviation value T_k is usually fixed and knowable (i.e., $T_k = |r_k - t_k| = \tilde{\tau}, \tilde{\tau} > 0$). For the deterministic jitter or system jitter of the signal, T_k is usually bounded and repeatable and predictable (i.e., $\tilde{\sigma} = \max\{T_k, k \in \mathbb{N}\} > 0$). Therefore, it is an effective operation to design the ITW by taking the ideal time r_k as the center point of the window and making the window's width greater than or equal to twice the maximum deviation value (i.e., $\left|\tau_k^r - \tau_k^l\right| \geq 2\tilde{\sigma}$), as shown in FIGURE1.

Assumption 2: We assume that $\tau_{k-1}^l < t_{k-1} < \tau_{k-1}^r < \tau_k^l < t_k < \tau_k^r < \tau_{k+1}^l < t_{k+1} < \tau_{k+1}^r$ holds for any adjacent ITWs, where τ_k^l, τ_k^r and r_k denote the left endpoint, right endpoint and center point of an ITW respectively, t_k and r_k are real and ideal time point when the impulsive control signal acts on the system respectively.

It must be pointed out that, when the signal has random jitter that satisfies the Gaussian distribution, the deviation value will be unbounded theoretically if the test time is long enough. In this case, the ITWs cannot be preset in advance. Restricted by current research techniques and methods, ITWs with unknown widths will be difficult or even impossible to handle in system modeling and theoretical analysis. Therefore, the following Assumption 3 is given.

Assumption 3: All impulsive signals in the control process will not appear random jitter that obeys the Gaussian distribution.

In summary, based on $\{t_k\}$, the impulsive control protocol is designed as follows:

$$u_i(t) = \sum_{k=1}^{+\infty} \delta(t - t_k)(B_k \alpha \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)) + \beta(B_k - I_n)b_i(x_i(t) - x_0(t))), \quad (3)$$

where $\delta(\cdot)$ is the Dirac function, $B_k \in \mathbb{R}^{n\times n}$ denotes the impulsive gain matrix, $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ are the coupling strengths, $j \in N_i$ represents the set of neighbors of agent i. We assume that the state of i-th agent is right-hand continuous at each impulse instant, that is $x_i(t_k) = \lim_{h\to 0^+} x_i(t_k + h)$.

By (1) and (3), we can get the following system model:

$$\begin{cases} dx_i(t) = [A(t)x_i(t) + \varepsilon(t)f(t, x_i(t))]dt \\ +\Phi(t, x_i(t))dw(t), t \neq t_k, \\ \Delta x_i(t) = x_i(t) - x_i(t^-) \\ = (B_k\alpha \sum_{j\in N_i} a_{ij}(x_j(t^-) - x_i(t^-)) \\ +\beta(B_k - I_n)b_i(x_i(t^-) - x_0(t^-))), t = t_k. \end{cases} \quad (4)$$

Let $e_i(t) = x_i(t) - x_0(t)$, $\Delta e_i(t_k) = e_i(t_k) - e_i(t_k^-)$, $\overline{f}(t, e_i(t)) = f(t, x_i(t)) - f(t, x_0(t))$ and $\overline{\Phi}(t, e_i(t)) = \Phi(t, x_i(t)) - \Phi(t, x_0(t))$. Note that the leader is uncontrolled at any time, i.e., $\Delta x_0(t_k) = x_0(t_k) - x_0(t_k^-) = 0$. System (4) is transformed into the following error system (5):

$$\begin{cases} de_i(t) = [A(t)e_i(t) + \varepsilon(t)\overline{f}(t, e_i(t))]dt \\ + \overline{\Phi}(t, e_i(t))dw(t), t \neq t_k, \\ \Delta e_i(t) = B_k\alpha \sum_{j\in N_i} a_{ij}(e_j(t^-) - e_i(t^-)) \\ + \beta(B_k - I_n)b_ie_i(t^-), t = t_k. \end{cases} \quad (5)$$

Let $e(t) = (e_1^T(t), \ldots, e_N^T(t))^T$, $\overline{F}(t, e(t)) = (\overline{f}^T(t, e_1(t)), \ldots, \overline{f}^T(t, e_N(t)))^T$ and $\overline{\Phi}(t, e(t)) = (\overline{\Phi}^T(t, e_1(t)), \ldots, \overline{\Phi}^T(t, e_N(t)))^T$. By using the Kronecker product, system (6) can be obtained as:

$$\begin{cases} de(t) = \left[(I_N \otimes A(t))\, e(t) + \varepsilon(t)\overline{F}(t, e(t))\right] dt \\ \qquad +\overline{\Phi}(t, e(t))dw(t), t \neq t_k, \\ e(t) = Z_k e\left(t^-\right), t = t_k, \end{cases} \quad (6)$$

where $B = \operatorname{diag}(b_1, \cdots, b_N)$ is the connection matrix, $Z_k = I_{N\times n} - (\alpha L) \otimes B_k + (\beta B) \otimes (B_k - I_n) \in \mathbb{R}^{Nn\times Nn}$.

By (6), for $t = t_k$, it yields:

$$\|e(t)\| = \left\|Z_k e\left(t^-\right)\right\| \leq \|Z_k\| \, \|e\left(t^-\right)\|. \quad (7)$$

Remark 3: With different matrix B_k, the following three cases may occur at each time t_k.

Case 1. If $\|e(t)\| < \left\|e\left(t^-\right)\right\|$, the k-th impulse is beneficial to the error system's convergence. Namely, the impulse is stabilising.

Case 2. If $\|e(t)\| = \left\|e\left(t^-\right)\right\|$, the k-th impulse is invalid for the controlled system.

Case 3. If $\|e(t)\| > \left\|e\left(t^-\right)\right\|$, the k-th impulse is adverse to the error system's convergence. In other words, the impulse is destabilising.

Remark 4: Let $\rho = \sup_{k\in\mathbb{N}^+} \lambda_{\max}(Z_k^T Z_k) = \sup_{k\in\mathbb{N}^+} \|Z_k\|^2$. Since the graph of SMASs is fixed, the value of $\|Z_k\|$ only depends on the selection of B_k. For any $k \in \mathbb{N}^+$, the following three cases are discussed.

a. When $\rho \in (0, 1)$, it means that $\|Z_k\| < 1$, and thus all impulses are stabilizing. Specifically, the authors supposed that $Z_k = \mu I_{Nn}$ in [35], and it yields $0 < \rho < 1$ while $|\mu| < 1$, so the impulses are called synchronizing ones. It is worth noting that the object system can be stable or unstable.

b. When $\rho = 1$, one has $\|Z_k\| = 1$, and so the presence of one or more impulses is invalid. Meanwhile, the object system can be stable or unstable. Similarly, $|\mu| = 1$ was given in [35].

c. When $\rho > 1$, it can infer that one or more impulses are destabilizing. In other words, stabilizing and destabilizing impulses may exist simultaneously and the number of destabilizing ones cannot be predicted. In this case, the impulses generated by the controller can be equivalently regarded as an interference source. Note that interference impulses are not always destructive, and some of them may contribute to the stability of the system. When interference impulses are considered, if the object system is required to achieve the consensus, it is usually assumed that the system is stable. Unlike this case, all impulses are desynchronizing while $|\mu| > 1$ in [35].

Some related definitions, assumption and lemmas needed in Section III are given.

Definition 1: For an impulse sequence $\{t_k\}$ in time period (t, T), and one impulse appears randomly in a known ITW, if the value of ALEI is equal to τ_a (positive number), then there exists a positive integer N_0 such that

$$\frac{T - t}{\tau_a} - N_0 \leq N_s(t, T) \leq \frac{T - t}{\tau_a} + N_0, \quad (8)$$

where $N_s(t, T)$ represents the number of left endpoints for ITWs in the sequence within the time period.

Remark 5: When the ITW is not considered, the concept of AII was used to obtain a larger upper bound of impulsive interval for a given non-uniform impulse sequence in [35]. Then, when the window exists, how to get a larger upper bound of impulsive interval becomes a practical problem. Note that the width of all the windows may not be fixed and the instant when the impulse appears is random. However, for a given series of windows, the position of the left endpoints is fixed. We know that the value of AII is related to the interval between any two adjacent fixed impulse instants, that is, $\inf\{t_{k+1} - t_k\} < T_a < \sup\{t_{k+1} - t_k\}$, where T_a is the value of AII. Therefore, motivated by the design of AII, we propose the concept named ALEI which related to the left endpoint interval between any two adjacent fixed windows, and its value is subject to the maximum and minimum value of the left endpoint interval. Based on ALEI, we can obtain a larger upper bound of the interval between the left endpoints of the windows, which means a larger width of ITW can

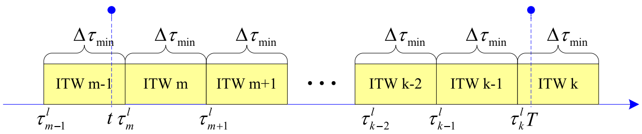

FIGURE 2. Diagram of the maximum number of left endpoints in time period (t, T).

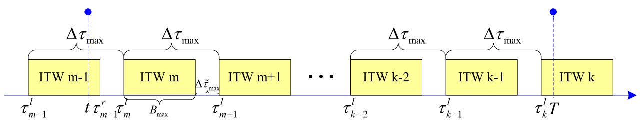

FIGURE 3. Diagram of the minimum number of left endpoints in time period (t, T).

be obtained. This further means that a larger upper bound of impulsive interval can be derived, and the forementioned problem is solved effectively.

Lemma 1: Under Assumption 2, for an impulse sequence $\{t_k\}$ in time period (t, T), there exist some positive numbers $\Delta\tau_{\min}$, $\Delta\tau_{\max}$, $\Delta\tilde{\tau}_{\min}$, $\Delta\tilde{\tau}_{\max}$, $B_{\min}$ and $B_{\max}$ such that

$$\frac{T-t}{\Delta\tau_{\max}} - 1 \le N_s(t, T) \le \frac{T-t}{\Delta\tau_{\min}} + 1, \tag{9}$$

where $B_{\min}$ and $B_{\max}$ are the minimum and maximum widths of ITW respectively, $\Delta\tilde{\tau}_{\min} = \inf_{k\in\mathbb{N}^+}\left\{\tau^l_{k+1} - \tau^r_k\right\} \ge 0$, $\Delta\tilde{\tau}_{\max} = \sup_{k\in\mathbb{N}^+}\left\{\tau^l_{k+1} - \tau^r_k\right\}$, $\Delta\tau_{\min} = \inf_{k\in\mathbb{N}^+}\left\{\tau^l_{k+1} - \tau^l_k\right\} = B_{\min} + \Delta\tilde{\tau}_{\min} = B_{\min}$ and $\Delta\tau_{\max} = \sup_{k\in\mathbb{N}^+}\left\{\tau^l_{k+1} - \tau^l_k\right\} = B_{\max} + \Delta\tilde{\tau}_{\max}$.

Proof: a. By FIGURE2, we have $N_s(t, T) = k - m + 1$. Therefore, it can be attained easily from T-t $\ge \Delta\tau_{\min}(k - m)$ that $N_s(t, T) \le \frac{T-t}{\Delta\tau_{\min}} + 1$.

b. By FIGURE3, $N_s(t, T) = k - m$ can be derived. Then, it can get from T-t $\le (k - m - 1)\Delta\tau_{\max} + 2\Delta\tau_{\max}$ that $N_s(t, T) \ge \frac{T-t}{\Delta\tau_{\max}} - 1$.

To sum up, we have inequality (9). The proof is completed.

Remark 6: Obviously, if the ITWs is not considered in SMASs, then we have $\tau^l_{k+1} - \tau^l_k = t_{k+1} - t_k$. At this time, Definition 1 in [35] and Definition 1 in this paper are equivalent (i.e., $T_a = \tau_a$), and $\frac{T-t}{\sup\{t_{k+1}-t_k\}} - 1 \le N_s(t, T) \le \frac{T-t}{\inf\{t_{k+1}-t_k\}} + 1$ can also be obtained from Lemma 1 in this paper, where $N_s(t, T)$ is the number of impulses in the period.

In order to ensure that the subsequent proof process can be carried out smoothly, we need the following assumption.

Assumption 4: For all $i = 1, \ldots, N$ and $x_i, x_0 \in \mathbb{R}^n$, there exist non-negative constants ϕ and φ such that $\|f(t, x_i) - f(t, x_0)\| \le \phi\|x_i - x_0\|$ and $\|\Phi(t, x_i) - \Phi(t, x_0)\| \le \varphi\|x_i - x_0\|$.

Lemma 2 [36]: For all vectors $x, y \in \mathbb{R}^n$ and constant $\varpi > 0$, we have $x^T y + y^T x \le \varpi x^T x + \varpi^{-1} y^T y$.

Definition 2: The leader-following mean square consensus of SMASs is said to be achieved via impulsive control if $\lim_{t\to\infty} E(\|x_i(t) - x_0(t)\|^2) = 0$ holds for all agents.

III. MAIN RESULTS

Theorem 1: Under the above statements, if there exist constants τ_a and ϖ such that $\ln(\rho)/\tau_a + \lambda_{\max}(\Xi) < 0$, then the leader-following mean square consensus of system (4) with ITWs can be achieved via impulsive control (3), where $\rho = \sup_{k\in\mathbb{N}^+}\|Z_k\|^2$ and $\Xi = \tilde{A} + \tilde{A}^T + \hat{\theta}^2\varpi^{-1}\tilde{\Theta}\tilde{\Theta}^T + \varpi\Theta^T\Theta + \left(2\hat{\varepsilon}\phi + \varphi^2\right) I_n$.

Proof. Construct the Lyapunov function candidate

$$V(t, e(t)) = e^T(t)e(t). \tag{10}$$

The stochastic derivative of (10) is derived by the *Itô* formula along the trajectory of system (6) as follows.

$$dV(t, e(t)) = \mathcal{L}V(t, e(t)) + 2\, e^T(t)\overline{\Phi}(t, e(t))dw(t), \tag{11}$$

$$\begin{aligned}\mathcal{L}V(t, e(t)) = {} & 2e^T(t)[(I_N \otimes A(t))e(t) + \varepsilon(t)\overline{F}(t, e(t))] \\ & + \text{trace}[\overline{\Phi}^T(t, e(t))\overline{\Phi}(t, e(t))].\end{aligned} \tag{12}$$

By Lemma 2, we have

$$\begin{aligned}& 2\, e^T(t)(I_N \otimes A(t))e(t) \\ & \quad \le e^T(t)(I_N \otimes (\tilde{A} + \tilde{A}^T + \hat{\theta}^2\varpi^{-1}\tilde{\Theta}\tilde{\Theta}^T + \varpi\Theta^T\Theta))e(t) \\ & \qquad + 2(\theta(t) - \hat{\theta})e^T(t)(I_N \otimes (\tilde{\Theta}\Lambda(t)\Theta))e(t).\end{aligned}$$

Next, it can be obtained by Assumption 4 that

$$\text{trace}(\overline{\Theta}^T(t, e(t))\overline{\Theta}(t, e(t))) \le \varphi^2 e^T(t)e(t)$$

and

$$\begin{aligned}& 2\varepsilon(t)e^T(t)\overline{F}(t, e(t)) \\ & \quad = 2\hat{\varepsilon}e^T(t)\overline{F}(t, e(t)) + 2(\varepsilon(t) - \hat{\varepsilon})e^T(t)\overline{F}(t, e(t)) \\ & \quad \le 2\hat{\varepsilon}\phi e^T(t)e(t) + 2(\varepsilon(t) - \hat{\varepsilon})e^T(t)\overline{F}(t, e(t)).\end{aligned}$$

To sum up, according to (12), we can get

$$\begin{aligned}E\mathcal{L}V(t, e(t)) \le {} & e^T(t)\left(I_N \otimes \left(\tilde{A} + \tilde{A}^T + \hat{\theta}^2\varpi^{-1}\tilde{\Theta}\tilde{\Theta}^T\right.\right. \\ & \left.\left. + \varpi\Theta^T\Theta + \left(2\hat{\varepsilon}\phi + \varphi^2\right) I_n\right)\right) e(t) \\ \le {} & \lambda_{\max}(\Xi)EV(t, e(t)).\end{aligned}$$

For $t \in [t_{k-1}, t_k)$, suppose that positive constant Δt be small enough such that $t + \Delta t \in (t_{k-1}, t_k)$, then we have

$$\begin{aligned}& EV(t + \Delta t, e(t + \Delta t)) - EV(t, e(t)) \\ & \quad = \int_t^{t+\Delta t} E\mathcal{L}V(s, e(s))ds\end{aligned}$$

Furthermore, one has

$$D^+EV(t, e(t)) = E\mathcal{L}V(t, e(t)) \le \lambda_{\max}(\Xi)V(t, e(t)). \tag{13}$$

When $t \in \left[t_0, \tau^l_1\right)$, it can be obtained from (13) that

$$EV(\tau^l_1, e(\tau^l_1)) \le EV(t_0, e(t_0))\exp(\lambda_{\max}(\Xi)(\tau^l_1 - t_0)). \tag{14}$$

When $t \in \left[\tau^l_k, t_k\right)$ and $k \ge 1$, we can get from (13) that

$$EV(t^-_k, e(t^-_k)) \le EV(\tau^l_k, e(\tau^l_k))\exp(\lambda_{\max}(\Xi)(t_k - \tau^l_k)). \tag{15}$$

When $t = t_k$ and $k \ge 1$, it follows that

$$\begin{aligned}EV(t_k, e(t_k)) &= E(e^T(t^-_k)Z^T_k Z_k e(t^-_k)) \\ &\le \|Z_k\|^2 EV(t^-_k, e(t^-_k)).\end{aligned} \tag{16}$$

When $t \in \left[t_k, \tau^r_k\right)$ and $k \ge 1$, it can be derived by (13) that

$$EV(\tau^r_k, e(\tau^r_k)) \le EV(t_k, e(t_k))\exp(\lambda_{\max}(\Xi)(\tau^r_k - t_k)). \tag{17}$$

When $t \in [\tau_{k-1}^r, \tau_k^l)$ and $k \geq 2$, by (13), one has

$$EV(\tau_k^l, e(\tau_k^l)) \leq EV(\tau_{k-1}^r, e(\tau_{k-1}^r)) \exp(\lambda_{\max}(\Xi)(\tau_k^l - \tau_{k-1}^r)). \tag{18}$$

Therefore, for $t \in \left[\tau_1^l, t_1\right)$ and $k = 1$, it can be obtained from (14) and (15) that

$$\begin{aligned} EV(t_1^-, e(t_1^-)) &\leq EV(\tau_1^l, e(\tau_1^l)) \exp(\lambda_{\max}(\Xi)(t_1 - \tau_1^l)) \\ &\leq EV(t_0, e(t_0)) \exp(\lambda_{\max}(\Xi)(t_1 - t_0)). \end{aligned} \tag{19}$$

When $t = t_1$, according to (16) and (19), we have

$$\begin{aligned} EV(t_1, e(t_1)) &\leq \|Z_1\|^2 EV(t_1^-, e(t_1^-)) \\ &\leq \|Z_1\|^2 EV(t_0, e(t_0)) \exp(\lambda_{\max}(\Xi)(t_1 - t_0)). \end{aligned} \tag{20}$$

For $t \in \left[t_1, \tau_1^r\right)$, according to (17) and (20), we can get

$$\begin{aligned} EV\left(\tau_1^r, e\left(\tau_1^r\right)\right) &\leq EV\left(t_1, e\left(t_1\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(\tau_1^r - t_1\right)\right) \\ &\leq \|Z_1\|^2 EV\left(t_0, e\left(t_0\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(\tau_1^r - t_0\right)\right). \end{aligned} \tag{21}$$

When $k = 2$, by (18) and (21), it yields

$$\begin{aligned} EV\left(\tau_2^l, e\left(\tau_2^l\right)\right) &\leq EV\left(\tau_1^r, e\left(\tau_1^r\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(\tau_2^l - \tau_1^r\right)\right) \\ &\leq \|Z_1\|^2 EV\left(t_0, e\left(t_0\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(\tau_2^l - t_0\right)\right). \end{aligned} \tag{22}$$

For $t \in \left[\tau_2^l, t_2\right)$, it follows from (15) and (22) that

$$\begin{aligned} EV\left(t_2^-, e\left(t_2^-\right)\right) &\leq EV\left(\tau_2^l, e\left(\tau_2^l\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(t_2 - \tau_2^l\right)\right) \\ &\leq \|Z_1\|^2 EV\left(t_0, e\left(t_0\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(t_2 - t_0\right)\right). \end{aligned} \tag{23}$$

When $t = t_2$, by from (16) and (23), one can attain

$$\begin{aligned} EV\left(t_2, e\left(t_2\right)\right) &\leq \|Z_2\|^2 EV\left(t_2^-, e\left(t_2^-\right)\right) \\ &\leq \|Z_1\|^2 \|Z_2\|^2 EV\left(t_0, e\left(t_0\right)\right) \exp(\lambda_{\max}(\Xi)(t_2 - t_0)). \end{aligned} \tag{24}$$

For $t \in \left[t_2, \tau_2^r\right)$, according to (17) and (24), we can obtain

$$\begin{aligned} EV\left(\tau_2^r, e\left(\tau_2^r\right)\right) &\leq EV\left(t_2, e\left(t_2\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(\tau_2^r - t_2\right)\right) \\ &\leq \leq \|Z_1\|^2 \|Z_2\|^2 EV(t_0, e(t_0)) \exp(\lambda_{\max}(\Xi)(\tau_2^r - t_0)). \end{aligned} \tag{25}$$

By the simple induction, for $t \in \left[t_k, \tau_k^r\right)$, we have

$$\begin{aligned} EV(t, e(t)) &\leq \|Z_1\|^2 \cdots \|Z_k\|^2 EV(t_0, e(t_0)) \exp(\lambda_{\max}(\Xi)(t - t_0)) \\ &\leq \rho^k EV\left(t_0, e\left(t_0\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(t - t_0\right)\right) \\ &= \rho^{N_s(t,T)} EV\left(t_0, e\left(t_0\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(t - t_0\right)\right). \end{aligned} \tag{26}$$

a. When $\rho \in (0, 1)$, it can be seen from Remark 4 that all impulses are stabilising, by (26) we have

$$\begin{aligned} EV(t, e(t)) &\leq \rho^{N_s(t,T)} EV\left(t_0, e\left(t_0\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(t - t_0\right)\right) \\ &\leq \rho^{((t-t_0)/\tau_a - N_0)} EV\left(t_0, e\left(t_0\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(t - t_0\right)\right) \\ &= \rho^{-N_0} EV(t_0, e(t_0)) \exp((\ln(\rho)/\tau_a + \lambda_{\max}(\Xi))(t - t_0)). \end{aligned} \tag{27}$$

b. When $\rho = 1$, we can get $\rho^{-N_0} = 1$ and $\ln(\rho) = 0$, Remark 4 shows that one or more impulses are invalid. It follows from (26) that

$$EV(t, e(t)) \leq \rho^{-N_0} EV(t_0, e(t_0)) \exp((\ln(\rho)/\tau_a + \lambda_{\max}(\Xi))(t - t_0)). \tag{28}$$

c. When $\rho > 1$, it can be seen from Remark 4 that the impulses are an interference source. It yields from (26) that

$$\begin{aligned} EV(t, e(t)) &\leq \rho^{N_s(t,T)} EV\left(t_0, e\left(t_0\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(t - t_0\right)\right) \\ &\leq \rho^{((t-t_0)/\tau_a + N_0)} EV\left(t_0, e\left(t_0\right)\right) \exp\left(\lambda_{\max}(\Xi)\left(t - t_0\right)\right) \\ &= \rho^{N_0} EV(t_0, e(t_0)) \exp((\ln(\rho)/\tau_a + \lambda_{\max}(\Xi))(t - t_0)). \end{aligned} \tag{29}$$

Let $\overline{\mathrm{H}} = \max\{\rho^{N_0}, 1, \rho^{-N_0}\}$. In summary, according to (27-29), then we have

$$EV(t, e(t)) \leq \overline{\mathrm{H}} EV(t_0, e(t_0)) \exp((\ln(\rho)/\tau_a + \lambda_{\max}(\Xi))(t - t_0)). \tag{30}$$

Consequently, we can get $e(t) \to 0$ when $t \to +\infty$ as $k \to +\infty$. It means that system (6) is globally exponentially stable in the mean square sense. In other words, the leader-following mean square consensus of system (4) can be achieved via impulsive control (3). This completes the proof.

Remark 7: By Remark 6, if $\tau_k^r = \tau_k^l$, one has $\tau_a = T_a$. At this time, let $\Pr(\varepsilon(t) = 1) = 1$, then we can get Theorem 1 of [17]. In addition, let $\|Z_k\| = \mu I_{Nn}$ and $\tau_k^r = \tau_k^l$, we have $\rho = |\mu|^2$. Then, consensus criterion $\frac{2\ln(|\mu|)}{T_a} + \lambda_{\max}(\Xi) < 0$ can be obtained by Theorem 1 in our paper, which is similar to Theorem 1 in [35].

Remark 8: We know that parameters ρ and $\lambda_{\max}(\Xi)$ are related to the selection of B_k, and different matrices may lead to different results. By Theorem 1 in this paper, the following two cases are discussed:

a. When $\lambda_{\max}(\Xi) \geq 0$, the premise for unstable system (4) to achieve the consensus eventually is that $\rho < e^{-\tau_a \lambda_{\max}(\Xi)} < 1$ holds for all parameters. By Remark 4, we know that all impulses are stabilizing.

b. When $\lambda_{\max}(\Xi) < 0$, for $1 < \rho < e^{-\tau_a \lambda_{\max}(\Xi)}$, the consensus of stable system (4) can be achieved. By Remark 4, the impulses are equivalent to a series of interference ones.

If $\rho \leq 1$, all impulses are stabilizing and the consensus criterion can be satisfied always for the given conditions.

By Remark 8, our consensus criterion is suitable to judge whether the stable or unstable systems can achieve the consensus under the effects of stabilizing or interference impulses. Therefore, the obtained criterion in this paper is a unified one.

Remark 9: By substituting Lemma 1 into (26), the following conclusions are given.

(a1) When $\rho \in (0, 1)$, on the one hand, one has $\ln(\rho)/\Delta\tau_{\max} + \lambda_{\max}(\Xi) < 0$. Since $\tau_a < \Delta\tau_{\max}$, it meas that the obtained criterion by Theorem 1 can be satisfied easily under the same parameters. On the other hand, $\ln(\rho^*)/(\tau_{k+1}^l - \tau_k^l) + \lambda_{\max}(\Xi^*) < 0$ is further obtained while ROUs, RONs and stochastic disturbances are not considered, which is similar to that obtained in [24]–[34]. For the convenience of comparison, it is assumed that the values of $\ln(\rho^*)$ and $\lambda_{\max}(\Xi^*)$ are the same in all references. Then, we have $\tau_a < -\ln(\rho^*)/\lambda_{\max}(\Xi^*)$ and $\tau_a < \Delta\tau_{\max}$, which means that the upper bound of τ_a is $-\ln(\rho^*)/\lambda_{\max}(\Xi^*)$. Since τ_a is always less than $\Delta\tau_{\max}$ objectively, thus by introducing the concept of ALEI into this paper, $\Delta\tau_{\max}$ can be bigger than $-\ln(\rho^*)/\lambda_{\max}(\Xi^*)$. This means that the allowable upper bound of the width of ITWs increases synchronously. In other words, a larger impulsive interval can be derived, which is less conservative for the presetting of ITWs.

(a2) When $\rho > 1$, it yields $\ln(\rho)/\Delta\tau_{\min} + \lambda_{\max}(\Xi) < 0$. Due to $\tau_a > \Delta\tau_{\min}$, thus the same conclusion with that in part (a1) can be derived.

Obviously, the above statements in Remark 9 reflect the superiority of the proposed concept of ALEI in this paper, which applies for a wider range of applications.

Remark 10: The relationship between related parameters in ALEI and impulsive properties is discussed as follows.

(aa1) When all impulses are stabilising, the more numbers of impulses are applied to control the systems, the better the control effect is. In this case, we can increase the upper bound of $N_s(t, T)$ in Definition 1. Moreover, part (a1) in Remark 9 shows that the smaller the value of τ_a is selected, the easier the systems to achieve the consensus is. In this regard, it is a feasible choice to decrease the value of τ_a and keep the value of N_0 unchanged or increased.

(aa2) When the impulses exist as an interference source, it means that the numbers of impulses should be as small as possible. Part (a2) in Remark 9 indicates that the bigger the value of τ_a is selected, the closer the requirement meets. Therefore, we can increase the value of τ_a and keep the value of N_0 unchanged or decreased.

In addition, although the value of N_0 has no influence on whether system (4) can achieve the consensus eventually, it can affect the convergence rate of the system by determining the upper and lower bounds of the numbers of ITWs on the interval (t, T). In other words, the ideal control effect can be achieved by adjusting the value of N_0 according to actual demands.

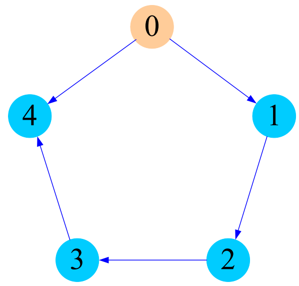

FIGURE 4. The topology of stochastic multi-agent systems.

IV. NUMERICAL SIMULATION

In this section, one numerical simulation example is given to validate the correctness of our results.

Example 1: Consider a class of SMASs composed by four followers and one leader, and the system dynamics (Chua's oscillator [37]) is described by ($i = 0, 1, 2, 3, 4$):

$$\begin{cases} \dot{x}_{i1}(t) = -p_1 x_{i1}(t) + p_1 x_{i2}(t) - p_1 g\left(x_{i1}(t)\right), \\ \dot{x}_{i2}(t) = x_{i1}(t) - x_{i2}(t) + x_{i3}(t), \\ \dot{x}_{i3}(t) = -p_2 x_{i2}(t), \end{cases}$$

where $g(x_{i1}(t)) = m_2 x_{i1}(t) + 0.5(m_1 - m_2)(|x_{i1}(t) + 1| - |x_{i1}(t) - 1|)$, $m_1 < m_2 < 0$, $p_1 = 9.21$, $p_2 = 15.995$, $m_1 = -1.25$ and $m_2 = -0.758$,

$$f(t, x_i(t)) = \begin{bmatrix} -0.5\left(m_1 - m_2\right)\left(|x_{i1}(t)+1| - |x_{i1}(t)-1|\right) \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{A} = \begin{bmatrix} -p_1\left(1 + m_2\right) & p_1 & 0 \\ 1 & -1 & 1 \\ 0 & -p_2 & 0 \end{bmatrix}.$$

Let $\Phi(t, x_i(t)) = [0.3\sin^2(t)x_{i1}(t), 0.3\sin^2(t)x_{i2}(t), 0.3\sin^2(t)x_{i3}(t)]^T$, $\tilde{\Theta} = \mathrm{diag}(0.6, -0.2, 0.6)$, $\Theta = \mathrm{diag}(0.16, 0.6, -0.26)$, and $\Lambda(t) = \mathrm{diag}(-\sin(t), \cos(t), -\sin(t))$. Then, we can get $\phi = 11.5125$ and $\varphi = 0.3$. By FIGURE4, we have

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and $B = \mathrm{diag}(1, 0, 0, 1)$. Let $\hat{\theta} = 0.8$, $\hat{\varepsilon} = 0.4$, $\alpha = 0.6$, $\beta = 0.8$ and $B_k = \mathrm{diag}(0.6, 0.6, 0.6, 0.6)$, the initial values of all agents are $x_0(t) = (0.2, -6, 3.6)^T$, $x_1(t) = (-7.6, 1, 6)^T$, $x_2(t) = (8, -1.6, -3)^T$, $x_3(t) = (8, 0, -3.9)^T$ and $x_4(t) = (-7, 1, -1)^T$ respectively. To sum up, it can be attained by calculation that $\rho = 0.8835$, $\lambda_{\max}(\Xi) = 26.1863$ and $\tau_a < 0.00473$. Let $\tau_a = 0.004$.

For convenience, all ITWs are preseted as shown in FIGURE5, where $\{\Delta\tilde{\tau}_{\min}, \Delta\tau_{\min} + \Delta\tilde{\tau}_{\min}, \ldots, (N_0 - 1)\Delta\tau_{\min} + \Delta\tilde{\tau}_{\min}, N_0\tau_a, N_0\tau_a + \Delta\tau_{\min}, \ldots, 2N_0\tau_a, \ldots\}$

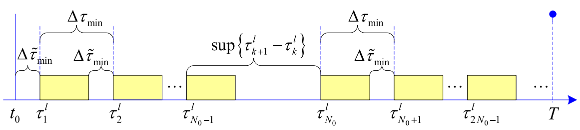

FIGURE 5. Impulse time windows in time period (t_0, T).

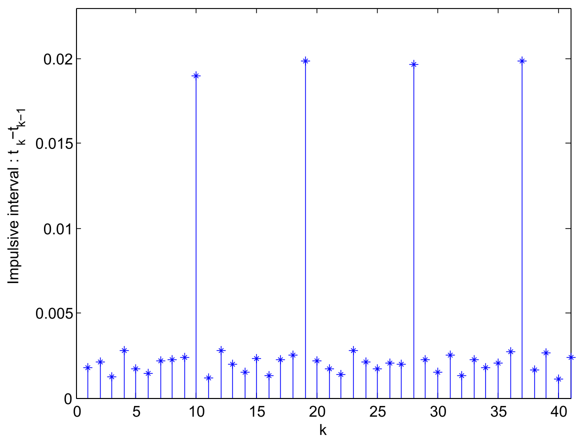

FIGURE 6. The impulsive interval.

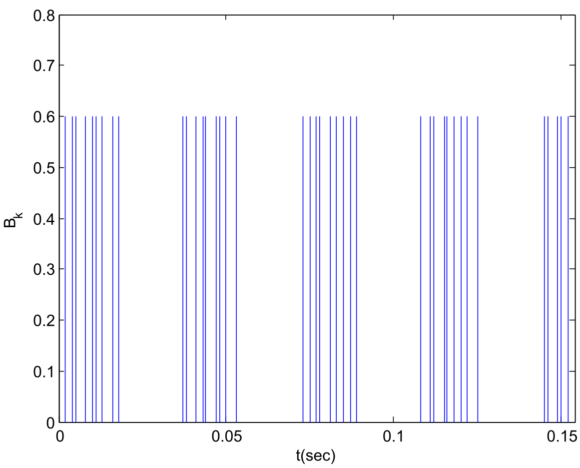

FIGURE 7. Schematic diagram of the impulse sequence.

and

$$\tau^l_k - \tau^l_{k-1} = \begin{cases} \Delta\tilde{\tau}_{\min}, & k = 1, \\ \Delta\tau_{\min}, & k \geq 2 \\ \text{and mod}\,(k, N_0) \neq 0, & \\ N_0(\tau_a - \Delta\tau_{\min}) + \Delta\tau_{\min} + \Delta\tilde{\tau}_{\min}, & k \geq 2 \\ \text{and mod}\,(k, N_0) = 0. & \end{cases}$$

are the left endpoint sequence and the left endpoint interval of the ITWs respectively. Obviously, we have $\inf_{k\in\mathbb{N}^+}\left(\tau^l_k - \tau^l_{k-1}\right) = \Delta\tilde{\tau}_{\min}$ and $\sup_{k\in\mathbb{N}^+}\left(\tau^l_k - \tau^l_{k-1}\right) = N_0\left(\tau_a - \Delta\tau_{\min}\right) + \Delta\tau_{\min} + \Delta\tilde{\tau}_{\min}$. Let $\Delta\tilde{\tau}_{\min} = 0.001$, $\Delta\tau_{\min} = 0.002$ and $N_0 = 10$. Then, simulation Figures 6-9 are given.

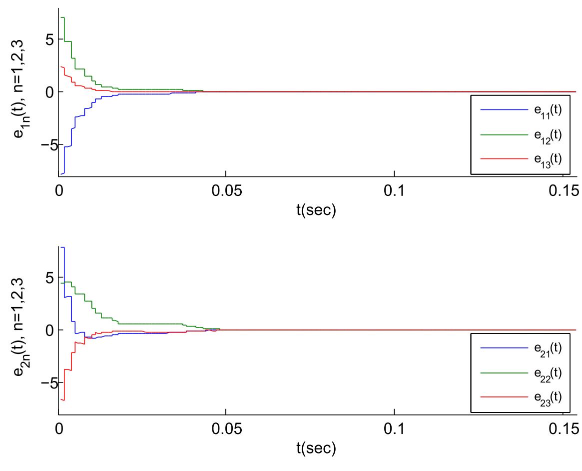

FIGURE 8. State errors of Agents 1 and 2.

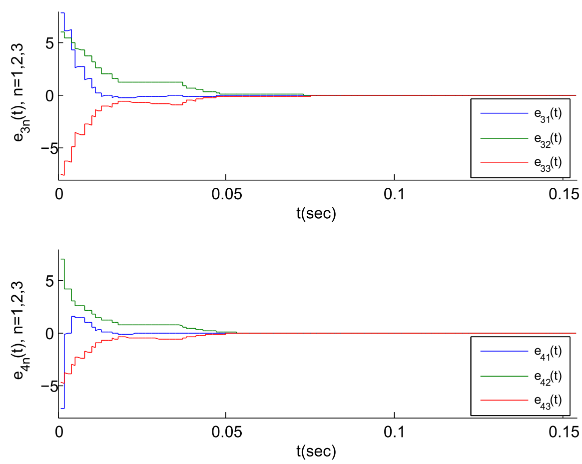

FIGURE 9. State errors of Agents 3 and 4.

By FIGURE6 and FIGURE7, we have $(\Delta t_k)_{\max} = t_{19} - t_{18} = 0.0199$ and $(\Delta t_k)_{\min} = t_{40} - t_{39} = 0.0011$. Based on the same parameters, it can be obtained from Theorem 1 in [38] that $t_{k+1} - t_k \leq 0.00946$. Therefore, Theorem 1 in [38] does not apply to the impulse sequence shown in FIGURE7. However, the results in this paper allow for a larger bound of impulsive interval in the impulse sequence, which is less conservative than that in [38], etc. Moreover, FIGURE8 and FIGURE9 show that the error trajectories of all agents can converge to 0 approximately before 0.1s, which means that the leader-following mean square consensus of SMASs can be achieved eventually.

V. CONCLUSION

In this article, the influence of factors RONs, ROUs and stochastic disturbances is considered comprehensively in the MASs' model. The leader-following consensus of SMASs in the mean square sense is studied under ITWs. In order to obtain a larger upper bound of the impulsive interval, we extend AII to the general case where the ITW is objective existence and propose the new concept named ALEI. Based on ALEI, we derive a unified consensus criterion that can be used to analyse the consensus problem of different types of SMASs with stabilizing or interference impulses. By comparing with the existing results related to ITWs, our results have been proved by Section IV to be less conservative from the perspective of allowing a larger upper bound of impulsive interval. In the future, we will try to introduce the finite-time or fixed-time control protocol into this paper to obtain a faster convergence rate.

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