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Inertial-Based Derivative-Free Method for System of Monotone Nonlinear Equations and Application

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ABSTRACT Iterative methods for solving nonlinear problems are of great importance due to their appearance in various areas of applications. In this paper, based on the inertial effect, we propose two projection derivative-free iterative methods for solving system of nonlinear equations. For the purpose of improving the numerical performance, the two methods incorporated the inertial step into the modified Barzilai and Borwein (BB) spectral parameters to generate the sequence of their respective search directions. The two spectral parameters are shown to be well-defined. For each method, the sequence of the search direction is bounded and satisfies the sufficient descent property. We establish the convergence analysis of the two methods based on the assumption that the underlying mapping is Lipschitzian and monotone. We demonstrate the efficiencies of the two methods on some collection of monotone system of nonlinear equations test problems. Finally, we apply the two methods to solve motion control problem involving a two planar robot.

INDEX TERMS Inertial effect, line search, nonlinear monotone equations, nonlinear problems, numerical algorithms, projection method, spectral parameters.

I. INTRODUCTION

Let Λ be a nonempty closed and convex subset of an $n-$dimensional Euclidean space $\mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, denote the inner product and Euclidean norm in $\mathbb{R}^n$. Recall the following definitions:

Definition 1: A mapping $Q : \mathbb{R}^n \to \mathbb{R}^n$ is monotone if for all $x, y \in \mathbb{R}^n$,

$$\langle Q(x) - Q(y),\ x - y \rangle \geq 0.$$

Definition 2: Let $x, y \in \mathbb{R}^n$. A mapping $Q : \mathbb{R}^n \to \mathbb{R}^n$ is said to be L-Lipschitz continuous with constant $L > 0$ if

$$\|Q(x) - Q(y)\| \leq L\|x - y\|, \quad \textit{for all} \ \ x, y \in \mathbb{R}^n.$$

In this paper, we are concerned with the problem of finding a vector, $x^* \in \Lambda \subset \mathbb{R}^n$, such that

$$Q(x^*) = 0, \tag{1}$$

where the mapping $Q : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous. We call problem (1) a nonlinear system of equations with convex constraints. Various problems arising from different areas of applications such as optimization, differential equations and variational inequalities problems, can be converted

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into nonlinear system of equations [1] and [2]. Moreover, by splitting $x \in \mathbb{R}^n$ into two positive and negative parts, the ℓ_1- norm regularized optimization problems can be reformulated as nonlinear system of equations [3]–[5].

Let $x_0 \in \Lambda \subset \mathbb{R}^n$ be an initial guess and $P_\Lambda(x)$ be the projection of $x \in \mathbb{R}^n$ onto the closed and convex set Λ. The hyperplane projection algorithm proposed by Solodov and Svaiter [6] has provoked many iterative algorithms that update their sequences of iterates using the following formula

$$x_{k+1} = P_\Lambda \left[x_k - \frac{\langle Q(z_k),\ x_k - z_k \rangle}{\|Q(z_k)\|^2} Q(z_k) \right],$$

where $z_k = x_k + \alpha_k d_k$ and the scalar $\alpha_k > 0$ is a step length obtained via suitable line search techniques. The search direction $d_k \in \mathbb{R}^n$ is usually generated using conjugate gradient approach, spectral gradient approach or their combinations (see, [7]–[11]).

The classical spectral gradient algorithm, first proposed by Barzilai and Borwein (BB) [12], is a matrix-free algorithm that defines its search direction as

$$d_k = \begin{cases} -Q(x_k), & \text{if } k = 0, \\ -\mu_k Q(x_k), & \text{if } k \geq 1, \end{cases} \tag{2}$$

where the function Q is a vector-valued and μ_k is a scalar, known as spectral parameter, which is updated in every iteration. The BB spectral parameters are updated using the following rules

$$\mu_k^{BB1} = \frac{\|s_{k-1}\|^2}{\langle y_{k-1},\ s_{k-1} \rangle} \quad \text{and} \quad \mu_k^{BB2} = \frac{\langle y_{k-1},\ s_{k-1} \rangle}{\|y_{k-1}\|^2}, \tag{3}$$

where $s_{k-1} = x_k - x_{k-1}$ and $y_{k-1} = Q(x_k) - Q(x_{k-1})$.

In [13], Zhang and Zhou [13] incorporated the spectral gradient method into the projection technique of Solodov and Svaiter [6] and proposed an interesting projection spectral method for solving monotone system of nonlinear equations. In addition, they presented a modified line search strategy that does not use any merit function but takes the monotonicity of Q into consideration. The global convergence of their method was proved under suitable assumptions. Subsequently, Yu *et al.* [14] extended the projection method of Zhang and Zhou [13] to solve monotone system of nonlinear equations with convex constraints. Their method was shown to be globally convergent under monotonicity and Lipschitzian assumptions. The preliminary numerical experiment presented showed that their method competes favourably with the projection method in [10]. Recently, a hybrid projection spectral method for monotone nonlinear equations was proposed in [15] based on the modified spectral parameters in [12] and [16]. The numerical experiments on some monotone system of nonlinear equations showed that the method outperforms the method of Yu *et al.* [14].

On the other hand, consider the inertial-based algorithms which are based on the heavy ball methods of the second-order time dynamic system. The first known inertial method was proposed by Polyak [17] to solve a smooth convex minimization problem. The inertial effect is usually incorporated into an algorithm for the purpose of speeding up the iteration process. Several studies have shown that iterative algorithms for solving nonlinear problems such as variational inequality problems, split feasibility problems, split variational inclusion problems and equilibrium problems that incorporated the inertial step have better numerical performance in terms of number of iterations and CPU time compared to their counterparts without the inertial effect. This impressive advantage motivated some researchers to developed different kind of inertial-type iterative methods (see, for example, [18]–[22] and the references therein). However, to the best of our knowledge, the effect of the inertial step has not been investigated on spectral gradient algorithms for solving nonlinear system of equations.

Question: Can the inertial effect speeds up the numerical performance of the derivative-free spectral iterative algorithm for system of nonlinear equations?

In this paper, we provide answer to the above question. We propose two inertial-based spectral algorithms for solving system of monotone nonlinear equations with convex constraints based on the projection technique. Based on the modified BB1 and BB2 spectral parameters, we present two search directions that are sufficiently descent and bounded. Furthermore, we apply the two proposed algorithms to solve motion control of a two planar robot. The remaining part of this paper is organized as follows. In Section 2, we describe the proposed methods and their global convergence. We report numerical experiments to show the efficiency of the algorithms in Section 3. We give the conclusion in Section 4.

II. INERTIAL-BASED DERIVATIVE-FREE ALGORITHMS AND THEIR CONVERGENCE ANALYSIS

Let x_0 and x_{-1} be the given two starting points and let $w_k = x_k + \alpha_k(x_k - x_{k-1})$ be an inertial step where $\alpha_k \in (0, 1)$ such that $\lim_{k\to\infty} \alpha_k = 0$. We begin this section by stating the following assumptions which are vital in the convergence analysis of the proposed algorithms.

Assumption 1: A. *The solution set of problem* (1) *is not empty.*

B. *The function* $Q : \mathbb{R}^n \to \mathbb{R}^n$ *is monotone and Lipschitz continuous.*

Let $\beta_k > 0$ be a step length and $d_k \in \mathbb{R}^n$ be the search direction. Given a starting point, say x_0, the classical iterative method for solving system of nonlinear equations usually computes its next iterates using the following rule

$$x_{k+1} = x_k + \beta_k d_k,$$

where d_k supposed to satisfy the following descent condition

$$Q(x_k)^T d_k \leq -\psi \|Q(x_k)\|, \quad \psi > 0. \tag{4}$$

It is worth mentioning that the inequality (4) is very important for an iterative algorithm to be globally convergence.

In order to state the proposed algorithms, the following projection operator is of great importance. Given any point $x \in \mathbb{R}^n$, its projection onto the feasible set $\Lambda \subset \mathbb{R}^n$ is defined as

$$P_\Lambda(x) = \text{argmin}\{\|x - y\| : y \in \Lambda\}. \quad (5)$$

The projection defined by (5) satisfies the following properties

$$\|P_\Lambda(x) - P_\Lambda(y)\| \leq \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n.$$

$$\|P_\Lambda(x) - y\| \leq \|x - y\|, \quad \forall y \in \Lambda.$$

We now state the details of the proposed algorithms.

Algorithm 1 Derivative-free Algorithm with Inertia Step (DAIS1)

Inputs: *Given* $x_0, x_{-1} \in \Lambda$, $\kappa, r > 0$, $\sigma, \varrho \in (0, 1)$ *and* $\{\alpha_k\} \in (0, 1)$ *such that* $\lim_{k\to\infty} \alpha_k = 0$. *Set* $k = 0$.

Step 1: *If* $\|Q(x_k)\| \leq Tol$, *then* x_k *is a solution and the iteration process stops.*

Step 2: *Compute* $w_k = x_k + \alpha_k(x_k - x_{k-1})$ *and*

$$d_k = -\frac{\|x_k - w_k\|^2}{\langle Q(x_k) - Q(w_k) + r(x_k - w_k),\ x_k - w_k\rangle} Q(x_k), \quad (6)$$

with $x_k \neq w_k$.

Step 3: *Set* $\upsilon_k = x_k + \beta_k d_k$, $\beta_k = \kappa\varrho^i$ *where* i *is the smallest nonegative integer such that*

$$-\langle Q(x_k + \kappa\varrho^i d_k),\ d_k\rangle \geq \sigma\kappa\varrho^i\|d_k\|^2 \min\left\{1,\ \|Q(x_k + \kappa\varrho^i d_k)\|^{1/c}\right\}, \quad c \geq 1. \quad (7)$$

Step 4: *If* $\|Q(\upsilon_k)\| = 0$, *stop. Else, compute the next iterate using the following*

$$x_{k+1} = P_\Lambda\left[x_k - \frac{\langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle}{\|Q(\upsilon_k)\|^2} Q(\upsilon_k)\right]. \quad (8)$$

Step 5: *Set* $k := k + 1$ *and go to Step 1.*

Algorithm 2 Derivative-free Algorithm with Inertia Step (DAIS2)

Inputs: *Use the same inputs as in Algorithm 1 and replace Step 2 with the following step*

Step 2: *Compute* $w_k = x_k + \alpha_k(x_k - x_{k-1})$ *and*

$$d_k = -\frac{\langle Q(x_k) - Q(w_k) + r(x_k - w_k),\ x_k - w_k\rangle}{\|Q(x_k) - Q(w_k) + r(x_k - w_k)\|^2} Q(x_k), \quad (9)$$

where $x_k \neq w_k$.

Remark 1: The definition of the search directions (6) *and* (9) *are different from those used in [14] and [23]. Unlike the search directions in [14] and [23] that use the difference between two successive iterates* x_k *and* x_{k-1} *as well as their their images* $F(x_k)$ *and* $F(x_{k-1})$, *the two proposed search directions* (6) *and* (9) *use the difference between the iterate* x_k *and the inertial step* w_k *together with their images.*

Remark 2: Motivated by the work of Solodov and Svaiter [6], the step length β_k *in [13] is computed as* $\beta_k = \kappa\varrho^i$ *where* i *is the smallest nonegative integer for which*

$$-\langle Q(x_k + \kappa\varrho^i d_k),\ d_k\rangle \geq \sigma\kappa\varrho^i\|d_k\|^2. \quad (10)$$

Also, Cheng [24] computed the step length $\beta_k = \kappa\varrho^i$ *where* i *is the smallest nonegative integer for which*

$$-\langle Q(x_k + \kappa\varrho^i d_k),\ d_k\rangle \geq \sigma\kappa\varrho^i\|d_k\|^2\|Q(x_k + \kappa\varrho^i d_k)\|. \quad (11)$$

Numerical experiments have shown that the line search strategy (10) *and* (11) *may behave differently on some test problems. We can observe that when* x_k *is far from* x^*, *that is the solutions of problem* (1), *then* $\|Q(x_k + \kappa\varrho^i d_k)\|$ *will be very big and consequently causes the right hand side of* (11) *to be very large. This may result in the increase of the computational costs of the line search. By this observation, Awwal et al. [25] proposed the following modification*

$$-\langle Q(x_k + \kappa\varrho^i d_k),\ d_k\rangle \geq \sigma\kappa\varrho^i\|d_k\|^2\|Q(x_k + \kappa\varrho^i d_k)\|^{1/c}, \quad c \geq 1. \quad (12)$$

If c *is large enough, then* (12) *reduces to* (10) *and if* $c = 1$, *then* (12) *becomes* (11). *The use of the power* $1/c$ *was aimed to reduce the value of* $\|Q(x_k + \kappa\varrho^i d_k)\|$. *However, if* $\|Q(x_k + \kappa\varrho^i d_k)\|$ *is very large, then the corresponding* $\|Q(x_k + \kappa\varrho^i d_k)\|^{1/c}$ *might not be as small as desired.*

Therefore, we consider the line search procedure (7) *to compute the step lengths for Algorithms 1 and* (2). *Please observe that the line search* (7) *contains* (10) *and* (12) *as special cases.*

Lemma 1: Suppose that Assumption 1 holds and the search direction $\{d_k\}$ *is generated by Algorithm 1. Then we have the followings*

(i) *The search direction is well-defined.*

(ii) *The search direction is descent, that is*

$$\langle Q(x_k),\ d_k\rangle \leq -p_1\|Q(x_k)\|^2, \quad p_1 > 0. \quad (13)$$

(iii) *The search direction satisfies*

$$\|d_k\| \leq p_2\|Q(x_k)\|, \quad p_2 > 0. \quad (14)$$

Proof: The monotonicity of Q gives $\langle Q(x_k) - Q(w_k),\ x_k - w_k\rangle \geq 0$. It follows we have

$$\begin{aligned}&\langle Q(x_k) - Q(w_k) + r(x_k - w_k),\ x_k - w_k\rangle\\ &\quad = \langle Q(x_k) - Q(w_k),\ x_k - w_k\rangle + r\|x_k - w_k\|^2\\ &\quad \geq r\|x_k - w_k\|^2. \end{aligned} \quad (15)$$

On the other hand, by the Lipschitz continuity and Cauchy Schwarz inequality, we have

$$\begin{aligned}&\langle Q(x_k) - Q(w_k) + r(x_k - w_k),\ x_k - w_k\rangle\\ &= \langle Q(x_k) - Q(w_k),\ x_k - w_k\rangle + r\|x_k - w_k\|^2\\ &\quad \leq (L + r)\|x_k - w_k\|^2. \end{aligned} \quad (16)$$

Therefore, since $x_k \neq w_k$, then by (15) and (16) we have

$$\frac{1}{L+r} \leq \frac{\|x_k - w_k\|^2}{\langle Q(x_k) - Q(w_k) + r(x_k - w_k),\ x_k - w_k\rangle} \leq \frac{1}{r}. \tag{17}$$

Hence, the search direction defined by (6) is well-defined.

(ii) Now, from (6) and (17), it holds

$$\langle Q(x_k),\ d_k\rangle \leq -\frac{1}{L+r}\|Q(x_k)\|^2.$$

(iii) Similarly, from (6) and (17), we have

$$\|d_k\| \leq \frac{1}{r}\|Q(x_k)\|.$$

□

Lemma 2: Suppose that Assumption 1 holds, and the search direction $\{d_k\}$ is generated by Algorithm 2. Then we have the followings

(i) *The search direction is well-defined.*

(ii) *The search direction is descent, that is*

$$\langle Q(x_k),\ d_k\rangle \leq -p_3\|Q(x_k)\|^2, \quad p_3 > 0. \tag{18}$$

(iii) *The search direction satisfies*

$$\|d_k\| \leq p_4\|Q(x_k)\|, \quad p_4 > 0. \tag{19}$$

Proof: Using the Lipschitz continuity, we have

$$\|Q(x_k) - Q(w_k) + r(x_k - w_k)\|^2 \leq (L+r)^2\|x_k - w_k\|^2. \tag{20}$$

Also, by the monotonicity of Q, we have

$$\begin{aligned}
&\|Q(x_k) - Q(w_k) + r(x_k - w_k)\|^2 \\
&\quad = \langle Q(x_k) - Q(w_k) + r(x_k - w_k),\ Q(x_k) - Q(w_k) \\
&\qquad + r(x_k - w_k)\rangle \\
&\quad = \|Q(x_k) - Q(w_k)\|^2 + 2r\langle Q(x_k) - Q(w_k),\ (x_k - w_k)\rangle \\
&\qquad + r^2\|x_k - w_k\|^2 \\
&\quad \geq \|Q(x_k) - Q(w_k)\|^2 + r^2\|x_k - w_k\|^2 \\
&\quad \geq r^2\|x_k - w_k\|^2.
\end{aligned} \tag{21}$$

This together with (20) gives

$$\begin{aligned}
r^2\|x_k - w_k\|^2 &\leq \|Q(x_k) - Q(w_k) + r(x_k - w_k)\|^2 \\
&\leq (L+r)^2\|x_k - w_k\|^2.
\end{aligned} \tag{22}$$

Therefore, since $x_k \neq w_k$, then by (15), (16) and (22) we have

$$\begin{aligned}
\frac{r}{(L+r)^2} &\leq \frac{\langle Q(x_k) - Q(w_k) + r(x_k - w_k),\ x_k - w_k\rangle}{\|Q(x_k) - Q(w_k) + r(x_k - w_k)\|^2} \\
&\leq \frac{r+L}{r^2}.
\end{aligned} \tag{23}$$

Hence, the search direction defined by (9) is well-defined. Moreover, similar arguments in Lemma 1 show that (ii) and (iii) hold. □

Lemma 3: Suppose the Assumption 1 holds. If the sequences of iterates $\{x_k\}$ and the search direction $\{d_k\}$ are generated by Algorithm 1 then the followings hold

(i) *$\{x_k\}$ and $\{w_k\}$ are bounded*

(ii) *$\lim_{k\to\infty} \|x_k - x^*\|$ exists.*

(iii) *$\lim_{k\to\infty} \beta_k\|d_k\| = 0.$*

(iv) *$\lim_{k\to\infty} \|w_k - x_k\| = 0.$*

Proof: If the point x^* satisfies $Q(x^*) = 0$, then the monotonicity of Q gives

$$\langle Q(\upsilon_k),\ \upsilon_k - x^*\rangle \geq \langle Q(x^*),\ \upsilon_k - x^*\rangle.$$

Furthermore,

$$\begin{aligned}
\langle Q(\upsilon_k),\ x_k - x^*\rangle &= \langle Q(\upsilon_k),\ x_k - \upsilon_k + \upsilon_k - x^*\rangle \\
&= \langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle + \langle Q(\upsilon_k),\ \upsilon_k - x^*\rangle \\
&\geq \langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle + \langle Q(x^*),\ \upsilon_k - x^*\rangle \\
&= \langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle.
\end{aligned} \tag{24}$$

By the definition of x_{k+1}, and (24) we have

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
&\quad = \left\| P_\Lambda\left[x_k - \frac{\langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle}{\|Q(\upsilon_k)\|^2}Q(\upsilon_k)\right] - x^*\right\|^2 \\
&\quad \leq \left\| x_k - x^* - \frac{\langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle}{\|Q(\upsilon_k)\|^2}Q(\upsilon_k)\right\|^2 \\
&\quad = \|x_k - x^*\|^2 - 2\frac{\langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle}{\|Q(\upsilon_k)\|^2}\langle Q(\upsilon_k),\ x_k - x^*\rangle \\
&\qquad + \frac{\langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle^2}{\|Q(\upsilon_k)\|^2} \\
&\quad \leq \|x_k - x^*\|^2 - 2\frac{\langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle}{\|Q(\upsilon_k)\|^2}\langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle \\
&\qquad + \frac{\langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle^2}{\|Q(\upsilon_k)\|^2} \\
&\quad = \|x_k - x^*\|^2 - \frac{\langle Q(\upsilon_k),\ x_k - \upsilon_k\rangle^2}{\|Q(\upsilon_k)\|^2} \\
&\quad \leq \|x_k - x^*\|^2.
\end{aligned} \tag{25}$$

This implies that $\|x_k - x^*\| \leq \|x_0 - x^*\|$ for all k, and therefore the sequence $\{x_k\}$ is bounded and $\lim_{k\to\infty} \|x_k - x^*\|$ exists. Since $\alpha_k \in (0, 1)$ and $\{x_k\}$ is bounded, then it implies that $\{w_k\}$ is also bounded. Hence, (i) and (ii) hold.

From the line search (7), let the $\min\left\{1,\ \|Q(x_k + \kappa\varrho^i d_k)\|^{1/c}\right\} = \|Q(x_k + \kappa\varrho^i d_k)\|^{1/c}$, then we have

$$\sigma^2\beta_k^4\|d_k\|^4\|Q(\upsilon_k)\|^{2/c} \leq \langle Q(\upsilon_k),\ \beta_k d_k\rangle^2. \tag{26}$$

Also, from (25), we have

$$\langle Q(\upsilon_k),\ \beta_k d_k\rangle^2 \leq \|Q(\upsilon_k)\|^2(\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2). \tag{27}$$

Now, combining (26) and (27) gives

$$\begin{aligned}
&\sigma^2\beta_k^4\|d_k\|^4\|Q(\upsilon_k)\|^{2/c} \\
&\qquad \leq \|Q(\upsilon_k)\|^2(\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2).
\end{aligned} \tag{28}$$

Since Q is Lipschitz continuous and the sequence $\{x_k\}$ is bounded, then there exists some constant, $p_5 > 0$, for which

$$\|Q(x_k)\| \leq p_5, \tag{29}$$

which in turns, by (14), gives

$$\|d_k\| \leq p_6, \quad p_6 > 0. \tag{30}$$

Furthermore, by the boundedness of $\{x_k\}$, (30) and the definition of υ_k in Step 3 of Algorithm 1, it holds that $\{\upsilon_k\}$ is also bounded. Similarly, by the Lipschitz continuity of Q, we can find some constants, say p_7 such that

$$\|Q(\upsilon_k)\| \leq p_7. \tag{31}$$

Since $\lim_{k\to\infty} \|x_k - x^*\|$ exists and (31) holds, taking limit on both sides of (28), as $k \to \infty$, we have

$$\sigma^2 \lim_{k\to\infty} \beta_k^4 \|d_k\|^4 \|Q(\upsilon_k)\|^{2/c} = 0. \tag{32}$$

Since $\|Q(\upsilon_k)\| \neq 0$ (see Step 5 of Algorithm 1), we conclude that (iii) holds, that is

$$\lim_{k\to\infty} \beta_k \|d_k\| = 0. \tag{33}$$

Lastly, since $\lim_{k\to\infty} \alpha_k = 0$ then by the boundedness of the $\{x_k\}$, we have $\lim_{k\to\infty} \|w_k - x_k\| = 0$. □

Remark 3: If the $\min\left\{1, \|Q(x_k + \kappa\varrho^i d_k)\|^{1/c}\right\} = 1$, *then* (28) *and* (32) *respectively become*

$$\sigma^2 \beta_k^4 \|d_k\|^4 \leq \|Q(\upsilon_k)\|^2 (\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2), \tag{34}$$

$$\sigma^2 \lim_{k\to\infty} \beta_k^4 \|d_k\|^4 = 0, \tag{35}$$

and thus all the results of Lemma 3 hold.

Remark 4: Suppose that the sequence of iterates $\{x_k\}$ and the search direction $\{d_k\}$ generated by Algorithm 2, then by Lemma 2, the results in Lemma 3 also hold.

Theorem 1: Suppose that Assumption 1 holds, then the sequence of iterates $\{x_k\}$ generated by Algorithm 1 converges to a point x^* such that $Q(x^*) = 0$.

Proof: We claim that the sequence of iterates $\{x_k\}$ generated by Algorithm 1 satisfies

$$\liminf_{k\to\infty} \|Q(x_k)\| = 0. \tag{36}$$

We prove the above claim by contradiction. Let us assume that (36) does not hold, then there exists some constants, say $p_8 > 0$ for which

$$\|Q(x_k)\| \geq p_8, \quad \forall\, k \geq 0. \tag{37}$$

Let the $\min\left\{1, \|Q(x_k + \kappa\varrho^i d_k)\|^{1/c}\right\} = \|Q(x_k + \kappa\varrho^i d_k)\|^{1/c}$. If $\beta_k \neq \kappa$, then $\varrho^{-1}\beta_k$ does not satisfy (7), that is,

$$-\langle Q(x_k + \varrho^{-1}\beta_k d_k),\ d_k\rangle < \sigma\varrho^{-1}\beta_k \|d_k\|^2 \|Q(x_k + \varrho^{-1}\beta_k d_k)\|^{1/c}. \tag{38}$$

Now, from the sufficient descent condition (13), we have

$$\begin{aligned}
p_1\|Q(x_k)\|^2 &\leq -\langle d_k,\ Q(x_k)\rangle \\
&= -\langle d_k,\ Q(x_k) - Q(x_k + \varrho^{-1}\beta_k d_k) + Q(x_k + \varrho^{-1}\beta_k d_k)\rangle \\
&= -\langle d_k,\ Q(x_k) - Q(x_k + \varrho^{-1}\beta_k d_k)\rangle \\
&\quad - \langle d_k,\ Q(x_k + \varrho^{-1}\beta_k d_k)\rangle \\
&< -\langle d_k,\ Q(x_k) - Q(x_k + \varrho^{-1}\beta_k d_k)\rangle \\
&\quad + \sigma\varrho^{-1}\beta_k\|d_k\|^2 \|Q(x_k + \varrho^{-1}\beta_k d_k)\|^{1/c} \\
&\leq \|d_k\| \|Q(x_k + \varrho^{-1}\beta_k d_k) - Q(x_k)\| \\
&\quad + \sigma\varrho^{-1}\beta_k\|d_k\|^2 \|Q(x_k + \varrho^{-1}\beta_k d_k)\|^{1/c} \\
&\leq L\|d_k\| \|x_k + \varrho^{-1}\beta_k d_k - x_k\| \\
&\quad + \sigma\varrho^{-1}\beta_k\|d_k\|^2 \|Q(x_k + \varrho^{-1}\beta_k d_k)\|^{1/c} \\
&\leq L\varrho^{-1}\beta_k\|d_k\|^2 + \sigma\varrho^{-1}\beta_k\|d_k\|^2 \|Q(x_k + \varrho^{-1}\beta_k d_k)\|^{1/c},
\end{aligned} \tag{39}$$

where the second and third inequalities follow from (38) and Cauchy Schwarz inequality, respectively. In addition, the fourth inequality follows from the Lipschitz continuity of Q.

By the Lipschitz continuity of the function Q,

$$\begin{aligned}
\|Q(x_k + \varrho^{-1}\beta_k d_k)\| &= \|Q(x_k + \varrho^{-1}\beta_k d_k) - Q(x^*)\| \\
&\leq L(\|x_k - x^*\| + \varrho^{-1}\beta_k\|d_k\|),
\end{aligned}$$

Since $\beta_k \in (0,\ 1)$, using the boundedness of $\{x_k\}$ and $\{d_k\}$, we can find some constants $p_9 > 0$ such that

$$\|Q(x_k + \varrho^{-1}\beta_k d_k)\| \leq p_9. \tag{40}$$

Applying the Cauchy Schwarz inequality on (13) and using the inequality (37), it holds that $\|d_k\| \geq p_1 p_8$. Therefore, substituting (40) into (39) and rearranging, we have

$$\beta_k\|d_k\| > \frac{\varrho p_1}{(L + \sigma p_9^{1/c})} \frac{\|Q(x_k)\|^2}{\|d_k\|} \geq \frac{\varrho p_1}{(L + \sigma p_9^{1/c})} \frac{p_8^2}{p_6},$$

where the last inequality follows from (30) and (37). Taking limit on both sides as $k \to \infty$, we have

$$\lim_{k\to\infty} \beta_k\|d_k\| > 0, \tag{41}$$

which contradicts (33) and hence (36) must hold. Now, since Q is continuous and the sequence $\{x_k\}$ is bounded, then there is some accumulation point of $\{x_k\}$ say x^* for which $\|Q(x^*)\| = 0$. By the boundedness of $\{x_k\}$, we can find subsequence $\{x_{k_j}\}$ of $\{x_k\}$ for which $\lim_{j\to\infty} \|x_{k_j} - x^*\| = 0$. Since it holds that $\{\|x_k - x^*\|\}$ converges, we can conclude that $\lim_{k\to\infty} \|x_k - x^*\| = 0$ and the proof is complete. □

Remark 5: If the $\min\left\{1, \|Q(x_k + \kappa\varrho^i d_k)\|^{1/c}\right\} = 1$, *then* (38) *and* (39) *respectively become*

$$-\langle Q(x_k + \varrho^{-1}\beta_k d_k),\ d_k\rangle < \sigma\varrho^{-1}\beta_k\|d_k\|^2, \tag{42}$$

$$p_1\|Q(x_k)\|^2 \leq L\varrho^{-1}\beta_k\|d_k\|^2 + \sigma\varrho^{-1}\beta_k\|d_k\|^2, \tag{43}$$

and therefore the proof of the Theorem 1 follows.

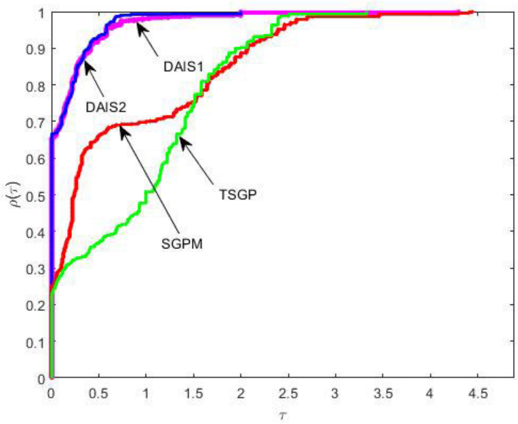

FIGURE 1. Performance profile on number of iterations.

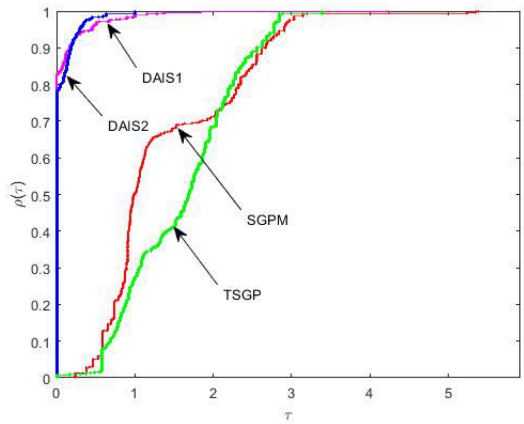

FIGURE 2. Performance profile on number of function evaluations.

III. NUMERICAL RESULTS AND COMPARISON

In this section, we demonstrate the numerical performance as well as the computational advantages of Algorithms 1 and 2. We implement these algorithms to solve a collection of monotone system of nonlinear equations, see Test Problems 1, and motion control model, see Test Problem 2. The parameters used for the implementation of Algorithm 1 (DAIS1) and Algorithm 2 (DAIS2) are $c = 2$, $\alpha_k = \frac{1}{(1+k)^2}$, $r = 0.01$, $\sigma = 0.01$, $\kappa = 1$ and $\rho = 0.5$.

For the Test Problems 1, we compare the two proposed algorithms with two existing competitors namely: (i) "Spectral gradient projection method for monotone nonlinear equations with convex constraints" proposed by Yu et al. [14] and (ii) Algorithm 2.1 of "Two spectral gradient projection methods for constrained equations and their linear convergence rate" proposed by Liu and Duan [23]. For convenience, we denote the methods in [14] and [23] by SGPM and TSGP, respectively. For the Test Problems 1, we use fourteen different starting points, see Table 1, for all the algorithms considered in this experiment. However, since our proposed Algorithms 1 and 2 use two starting points, we initially set $x_{-1} = \{x_0^1 + \iota, x_0^2 + \iota, \ldots, x_0^n + \iota\}$, $\iota \geq 0$, and then update them with different points subsequently. Moreover, we solve each problem using five different dimensions, i.e. 1000, 5000, 10000, 50000 and 100000. This gives a total of seven hundred and seventy (770) test problems. Details of the numerical results are tabulated and can be found in the following link https://github.com/aliyumagsu/Numerical-results-of-DAIS1-and-DAIS-2/blob/main/Tables_DAIS.xlsx.

The numerical performance of each of DAIS1, DAIS2, SGPM and TSGP are depicted in Figures 1 and 2 with the aid of the popular Dolan and Moré performance profile [26]. Figure 1 is plotted based on ITER (number of iterations), needed by each algorithm to obtain the solution of a particular problem, while Figure 2 is based on FVAL (number of function evaluations). From Figure 1, we see that DAIS1 and DAIS2 are very competitive and clearly outperform their main existing competitors, that is SGPM and TSGP. Specifically, DAIS1 and DAIS2 recorded least ITER in about 70% of the entire experiment. On the other hand, we see from Figure 2 that based on the FVAL, Algorithms 1 (DAIS1) slightly performs better than 2 (DAIS2). Interestingly, DAIS1 and DAIS2 clearly perform better than both SGPM and TSGP with DAIS1 and DAIS2 solving about 80% of the test problems with least FVAL. This means we may conclude that the inertial step incorporated into the Algorithms 1 and 2 impacted positively on their numerical performances.

TABLE 1. The initial points used for the problems in Test Problems 1.

Initial Points (IP)	Values
x_1	$(1, 1, 1, \ldots, 1)^T$
x_2	$(0.1, 0.1, 0.1, \ldots, 0.1)^T$
x_3	$(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n})^T$
x_4	$(1 - \frac{1}{n}, 1 - \frac{2}{n}, 1 - \frac{3}{n}, \ldots, 0)^T$
x_5	$(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n})^T$
x_6	$(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n})^T$
x_7	$(\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}, \ldots, 0)^T$
x_8	$(\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, 1)^T$
x_9	rand(0, 1)
x_{10}	$(1.5, 1.5, 1.5, \ldots, 1.5)^T$
x_{11}	$(2, 2, 2, \ldots, 2)^T$
x_{12}	$(0.5, 0.5, 0.5, \ldots, 0.5)^T$
x_{13}	$5\min(ih,\ 1 - ih),\ 1 \leq i \leq n\ , h = 1/(n+1)$
x_{14}	$(-1)^i \left(\frac{i}{i+3}\right),\ 1 \leq i \leq n$

Test Problems 1: *The followings are the list of test problems used in our experiments where* $Q(x) = (q_1(x), q_2(x), \ldots, q_n(x))^T$, *and* $x = (x_1, x_2, \ldots, x_n)^T$*:*

Problem 1 *[27]*

$$q_1(x) = e^{x_1} - 1$$
$$q_i(x) = e^{x_i} + x_{i-1} - 1, \quad i = 1, 2, \ldots, n-1, \quad \text{where } \Lambda = \mathbb{R}^n_+.$$

Problem 2 *[27]*

$$q_i(x_i) = \log(x_i + 1) - \frac{x_i}{n}, \quad i = 1, 2, \ldots, n,$$

$$\text{where } \Lambda = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq n,\ x_i > -1,\ i = 1, 2, \ldots, n\}.$$

Problem 3 *[27]*

$$q_i(x) = 2x_i - \sin|x_i|, \quad i = 1, 2, \ldots, n, \quad \text{where } \Lambda = \mathbb{R}^n_+.$$

Problem 4 *[14]*

$$q_i(x) = e^{x_i} - 1, \quad i = 1, 2, \ldots, n, \quad \text{where } \Lambda = \mathbb{R}^n_+.$$

***Problem 5** [27]*

$$q_1(x) = x_1 - e^{\cos(h(x_1+x_2))}$$
$$q_i(x) = x_i - e^{\cos(h(x_{i-1}+x_i+x_{i+1}))}, \quad i = 2, \ldots, n-1,$$
$$q_n(x) = x_n - e^{\cos(h(x_{n-1}+x_n))}, \text{ where } h = \frac{1}{n+1}, \quad \Lambda = \mathbb{R}^n_+.$$

***Problem 6** [13]*

$$q_i(x) = x_i - \sin(|x_i - 1|), \quad i = 1, 2, \ldots, n-1,$$

where $\Lambda = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq n, \; x_i \geq -1, \; i = 1, 2, \ldots, n\}$.

***Problem 7** [28]*

$$q_i(x) = e^{x_i} + \frac{3}{2}\sin(2x_i) - 1, \; i = 1, 2, \ldots, n, \text{ where } \Lambda = \mathbb{R}^n_+.$$

Problem 8

$$q_1(x) = \frac{5}{2}x_1 + x_2 - 1,$$
$$q_i(x) = x_{i-1} + \frac{5}{2}x_i + x_{i+1} - 1, \; i =, 2, \ldots, n-1,$$
$$q_n(x) = x_{n-1} + \frac{5}{2}x_n - 1, \text{ where } \Lambda = \mathbb{R}^n_+.$$

***Problem 9** [11]*

$$q_1(x) = 2x_1 - x_2 + e^{x_1} - 1,$$
$$q_i(x) = -x_{i-1} + 2x_i - x_{i+1} + e^{x_i} - 1, \quad i =, 2, \ldots, n-1,$$
$$q_n(x) = -x_{n-1} + 2x_n + e^{x_n} - 1, \quad \text{where } \Lambda = \mathbb{R}^n_+.$$

Problem 10

$$q_1(x) = x_1 + \sin(x_1) - 1,$$
$$q_i(x) = -x_{i-1} + 2x_i + \sin(x_i) - 1, \quad i =, 2, \ldots, n-1,$$
$$q_n(x) = x_n + \sin(x_n) - 1, \text{ where } \Lambda = \mathbb{R}^n_+.$$

***Problem 11** [27]*

$$q_i(x) = \frac{i}{n}e^{x_i} - 1, \quad i = 1, 2, \ldots, n, \quad \text{where } \Lambda = \mathbb{R}^n_+.$$

***Test Problems 2:** Consider the motion control of a two-joint planar robotic manipulator. We implement Algorithms 1 and 2 to solve the following motion control model. As described in [29], the discrete-time kinematics equation of two-joint planar robot manipulator at a position level is given as*

$$q(\theta_k) = z_k. \tag{44}$$

The function $q(\cdot)$ *is the kinematics mapping with the following structure*

$$q(\theta) = \begin{bmatrix} \ell_1 c_1 + \ell_2 c_2 \\ \ell_1 s_1 + \ell_2 s_2 \end{bmatrix}, \tag{45}$$

where ℓ_j, $(j = 1, 2)$, *is the length of the* j^{th} *rod,* $c_1 = \cos(\theta_1)$, $s_1 = \sin(\theta_1)$, $c_2 = \cos(\theta_1 + \theta_2)$ *and* $s_2 = \sin(\theta_1 + \theta_2)$. *The vectors* $\theta_k \in \mathbb{R}^2$ *and* $z_k \in \mathbb{R}^2$ *represent the joint angle vector and the end effector position vector, respectively. In view of robotic control, we need to solve following minimization problem at each computational time interval* $t_k \in [0, \; t_f]$

$$\min_{q_k \in \mathbb{R}^2} \frac{1}{2} \|q_k - q_{dk}\|^2, \tag{46}$$

where q_{dk} *is the end effector controlled track and* t_f *denotes the end of task duration,.*

Following the approach in [30], in the course of the motion control experiment, we take the length of the rod $\ell_1 = \ell_2 = 1$ and the end effector is controlled to track a Lissajous curve,

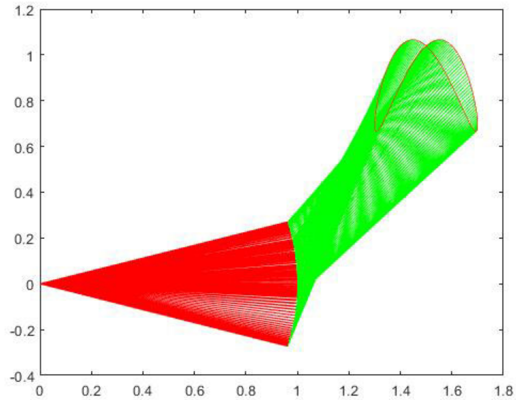

FIGURE 3. DAIS1: Manipulator trajectories.

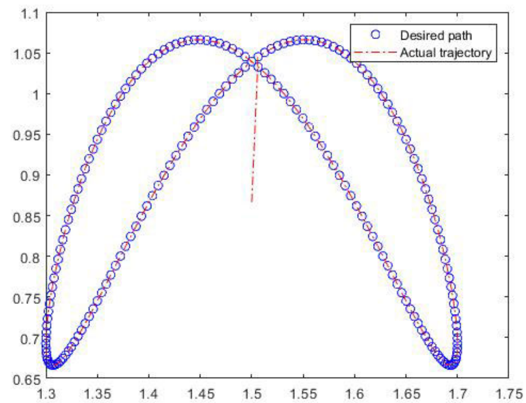

FIGURE 4. DAIS1: End effector trajectory and desired path.

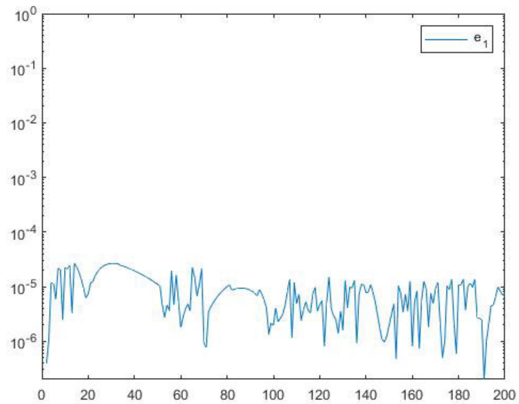

FIGURE 5. DAIS1: Tracking errors on the horizontal x-axis.

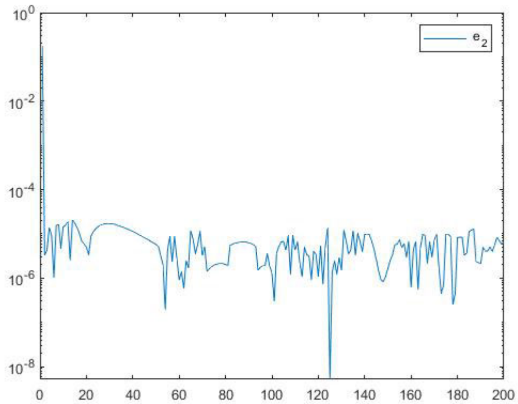

FIGURE 6. DAIS1: Tracking errors on the vertical y-axis.

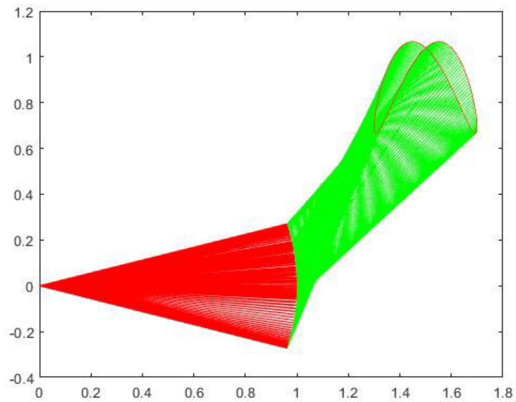

FIGURE 7. DAIS2: Manipulator trajectories.

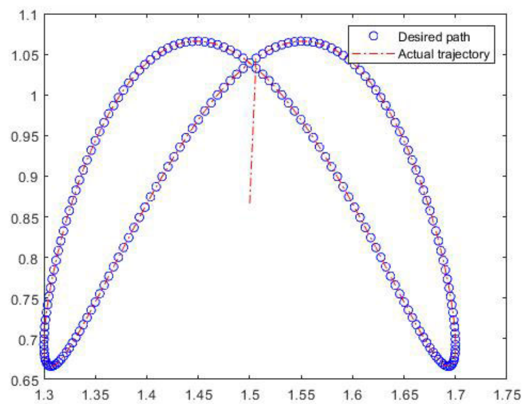

FIGURE 8. DAIS2: End effector trajectory and desired path.

which is expressed as

$$q_{dk} = \begin{bmatrix} 1.5 + 0.2\sin(\frac{\pi t_k}{5}) \\ \frac{\sqrt{3}}{2} + 0.2\sin(\frac{2\pi t_k}{5} + \frac{\pi}{3}) \end{bmatrix}, \tag{47}$$

For both Algorithms 1 and 2, we set $t_f = 10$ seconds and use the parameters $\rho = 0.2$ and $\sigma = 0.08$. We set the initial point $\theta_0 = [0,\ \frac{\pi}{3}]^T$ and divide the task duration [0, 10] into 200 equal parts. The numerical results generated by Algorithms 1 and 2 are depicted in Figures 3–10. Figures 3 and 7 show robot trajectories synthesized by Algorithms 1 and 2, respectively. Figures 4 and 8 plot end effector trajectory and desired path by Algorithms 1 and 2, respectively. Moreover, Figures 5 and 6 show the error recorded by Algorithm 1 on $x-$axis and $y-$axis, respectively. Lastly, Figures 9 and 10 show the error recorded by Algorithm 2 on $x-$axis and $y-$axis, respectively. It can be easily seen from Figures 3, 4, 7 and 8 that both Algorithms 1 and 2 completes the task at hand, successfully. In addition, we see from Figures 5, 6, 9 and 10 that the error recorded by each algorithm is as low as 10^{-5}. Hence, Algorithms 1 and 2 can be successfully implemented to solve real world problems.

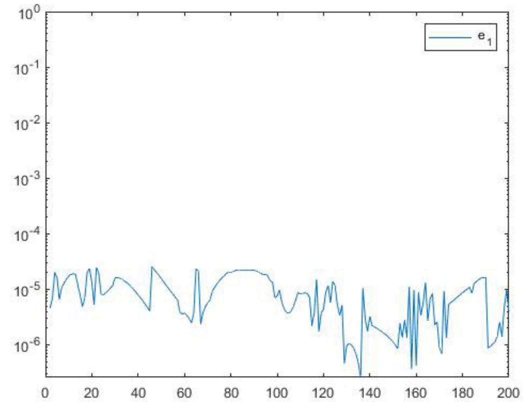

FIGURE 9. DAIS2: Tracking errors on the horizontal x-axis.

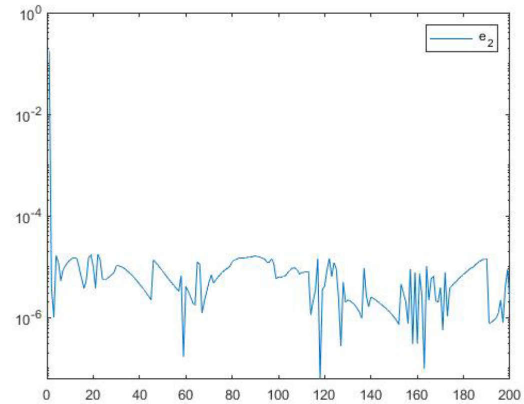

FIGURE 10. DAIS2: Tracking errors on the vertical y-axis.

IV. CONCLUSION

This paper presented two derivative-free methods that utilized inertial step based on projection techniques. Under the monotonicity and Lipschitz continuity assumptions on the underlying mapping, the convergence analyses of the two methods have been established. The two proposed algorithms have been successfully implemented on a collection of system of nonlinear equations as well as motion control problem. Considering the fact that the main differences between the proposed Algorithm 1 and SGPM as well as Algorithm 2 and TSGP is the inertial step, the good numerical performance recorded by both the Algorithms 1 and 2, on the collection of system of nonlinear equations, may be attributed to the inertial effect. Finally, the experiment on the motion control problem have shown that the Algorithms 1 and 2 can be applied to solve practical problems. Future work include applying the proposed algorithms to solve time-varying nonlinear equations.

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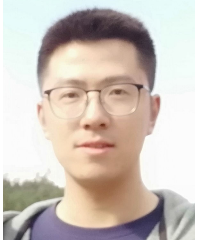

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