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General Third-Order-Accuracy Formulas for Time Discretization Applied to Time-Varying Optimization

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ABSTRACT Time discretization is an important part of time-varying problems solving that determines convergence, real-time performance and accuracy for solution models. It is a challenging work compared with relatively simple derivative approximation due to unknown future information and stability constraint. To the best of the authors's knowledge, no effective time discretization was developed other than Euler finite difference formula before recently development of ZeaD formulas. Existing work presents some ZeaD formulas including specific time-discretization formulas having third order accuracy. In this work, N-instant general third-order-accuracy formula is proposed, and it leads to different general third-order-accuracy formulas when different instant number N is considered. Stability and convergence are analyzed, and effective domains for parameters in 5-instant and 6-instant general third-order-accuracy formula are given to guarantee effective time discretization. Furthermore, N-instant general third-order-accuracy formula is employed to solve time-varying optimization, and N-instant general solution model is proposed. Finally, comparative experimental results are presented to substantiate the effectiveness and superiority of proposed general formulas and models.

INDEX TERMS Time discretization, general third-order-accuracy formulas, time-varying optimization, 0-stability.

I. INTRODUCTION

Time discretization plays an important role in many areas of science, especially time-varying problems solving, which makes a transition from continuous time to discrete time [1], [2]. The first effective time discretization formula is Euler forward finite difference formula expressed as

$$\dot{x}(t_k) = \frac{x(t_{k+1}) - x(t_k)}{g} + O(g),$$

where g is the sampling gap and $O(g)$ is the truncation error [3]. It is termed Euler formula in this paper for convenience. Euler formula seems to be only an approximation of first-order derivative formally. However, it exists as the unique effective time discretization formula for decades. It is because that time discretization formulas have to meet some constraints in addition to derivative approximation [4], [5].

Firstly, time discretization formulas are to deal with time-dependent data in real time, and thus they have to be formally one-step-ahead [6], [7]. Specifically, one-step-ahead formulas have the form of

$$\dot{x}(t_k) = \frac{\alpha_1 x(t_{k+1}) + \alpha_2 x(t_k) + \alpha_3 x(t_{k-1}) + \ldots}{g} + O(g^p),$$

where $\alpha_1, \alpha_2 \ldots$ are constants and $O(g^p)$ is truncation error. Constant α_1 is nonzero and it can be rewritten as

$$x(t_{k+1}) = \frac{g}{\alpha_1}\dot{x}(t_k) - \frac{\alpha_2}{\alpha_1}x(t_k) - \frac{\alpha_3}{\alpha_1}x(t_{k-1}) - \ldots + O(g^{p+1}). \quad (1)$$

It is observed from (1) that the left is about information at t_{k+1}, i.e., future information, and the right is

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about information at and before t_k, i.e., current and past information. It means that future information is calculated predictively at each time instant t_k so that time consumption of computation has no infection on real-time performance [8], [9].

Secondly, time discretization formulas have to meet 0-stability constraint [10], [11]. It is a tough requirement that most existing derivative approximation formulas can not satisfy. A series of Lagrange-type formulas were presented in [8], which are all one step ahead. Their accuracy is quite high. However, all these formulas cannot lead to stable models.

A series of formulas have been developed for time discretization with the deep study on time-varying problems in recent years, which have second-order, third-order and even higher order accuracy [12]–[17]. For example, in [12], a second-order-accuracy formula was developed with four instants utilized. In [13], a third-order-accuracy formula was developed with five instants utilized. In [14], a fourth-order-accuracy formula was developed with eight instants utilized. These research results are about specific time discretization formulas, which are difficult to be compared together. Thus, researchers study their general forms in recent years. For example, in [18], a general three-step time discretization formula was proposed, which utilizes four instants and have second order accuracy. In [19], a general four-step time discretization formula was proposed, which utilizes five instants and have third order accuracy. Besides, some other general formulas have been developed and investigated. However, each existing work corresponds to only one general formula, which have fixed instants for time discretization. In this work, different general time discretization formulas are developed by proposing an N-instant general third-order-accuracy (TOA) formula. Each values of instant number N corresponds to a general formula. Recent main study about time discretization formulas compared with this work are listed in Table 1. Theoretical analyses guarantee the existence of effective domain for parameters in N-instant general TOA formula when instant number $N \geq 5$.

Optimization is widely encountered in various science and engineering fields showing its fundamental importance, such that many efforts have been devoted to such problems [4], [19], [21]–[29]. Most existing researches are about time-invariant optimization, and it may be challenging for them to describe some real-time problems in reality such as real-time tracking control of robot manipulator [4], [19]. The proposed N-instant general TOA formula is employed to solve time-varying optimization in this work and N-instant general solution model is proposed.

The remainder of this paper is organized into four sections. In section II, different general TOA formulas are developed by proposing N-instant general TOA formula. The stability and convergence analyses of general formulas are given. In section III, time-varying optimization is solved by the continuous-time solution model and its time discretization realized via N-instant general TOA formula. The corresponding N-instant general solution model is proposed. In section IV, numerous numerical results are presented to substantiate the effectiveness and superiority of proposed N-instant general TOA formula and N-instant general solution model. Section V concludes this paper with final remarks. The main contributions of this work are as follows.

1) Different general TOA formulas with different instants are proposed to approximate and discretize the first-order derivative.
2) Theoretical analyses guarantee the existence of effective domain for parameters in N-instant general TOA formula when instant number $N \geq 5$.
3) N-instant general solution model is proposed to solve time-varying optimization via utilizing N-instant general TOA formula to discretize continuous-time solution model.

TABLE 1. Recent main study about time discretization formulas and their comparison.

Paper	General (Yes/No)	Truncation error	Instant number
[12]	No	$O(g^2)$	4
[13]	No	$O(g^3)$	5
[14]	No	$O(g^4)$	8
[18]	Yes	$O(g^2)$	4
[19]	Yes	$O(g^3)$	5
[20]	Yes	$O(g^4)$	7
This paper	Yes	$O(g^3)$	$N(N \geq 5)$

II. GENERAL TOA FORMULAS

In this section, general TOA formulas are developed by proposing an N-instant general TOA formula. Different values of instant number N lead to different general TOA formulas. Theoretical analyses of N-instant general TOA formula are presented to investigate its stability and convergence.

A. N-INSTANT GENERAL TOA FORMULA

In this subsection, N-instant general TOA formula is proposed on the basis of Taylor expansion and high-order derivatives elimination.

Theorem 1: If $x(t_{k+1})$, $x(t_{k-i})$, $i = 0, 1, \ldots, N-2$, are evenly-spaced sampling points of $x(t)$ with the sampling gap $g \in (0, 1)$ and $x(t)$ has bounded fourth-order derivative, then the following N-instant general TOA formula has third order accuracy:

$$\begin{aligned}\dot{x}(t_k) = &\frac{1}{\tilde{a}g}x(t_{k+1})\\ &+\left(\frac{3}{2g\tilde{a}}-\sum_{i=3}^{N-2}\frac{3i^3-7i^2+4}{4\tilde{a}g}a_i\right)x(t_k)\\ &-\left(\frac{3}{\tilde{a}g}-\sum_{i=3}^{N-2}\frac{i^3-2i^2}{\tilde{a}g}a_i\right)x(t_{k-1})\end{aligned}$$

$$+\left(\frac{1}{2g\tilde{a}}-\sum_{i=3}^{N-2}\frac{i^3-i^2}{4g\tilde{a}}a_i\right)x(t_{k-2})$$
$$+\sum_{i=3}^{N-2}\frac{a_i x(t_{k-i})}{\tilde{a}g}+O(g^3), \quad (2)$$

where

$$\tilde{a}=3+\sum_{i=3}^{N-2}\frac{3i^2-i^3-2i}{2}a_i,$$

and $O(g^3)$ *denotes the truncation error. Different values of instant number* N *lead to different general TOA formulas.*

Proof: Based on Taylor expansion, the expansions of $x(t_{k-i}), i=-1,1,2,\dots,N-2$ are obtained as $x(t)$ has bounded fourth-order derivative:

$$x(t_{k+1})=x(t_k)+g\dot{x}(t_k)+\frac{g^2}{2!}\ddot{x}(t_k)+\frac{g^3}{3!}x^{(3)}(t_k)+\frac{g^4}{4!}x^{(4)}(c_1),$$
$$x(t_{k-1})=x(t_k)-g\dot{x}(t_k)+\frac{(-g)^2}{2!}\ddot{x}(t_k)+\frac{(-g)^3}{3!}x^{(3)}(t_k)+\frac{(-g)^4}{4!}x^{(4)}(c_2),$$
$$x(t_{k-2})=x(t_k)-2g\dot{x}(t_k)+\frac{(-2g)^2}{2!}\ddot{x}(t_k)+\frac{(-2g)^3}{3!}x^{(3)}(t_k)+\frac{(-2g)^4}{4!}x^{(4)}(c_3),$$
$$\vdots$$
$$x(t_{k-N+2})=x(t_k)-(N-2)g\dot{x}(t_k)+\frac{((N-2)g)^2}{2!}\ddot{x}(t_k)+\frac{(-(N-2)g)^3}{3!}x^{(3)}(t_k)+\frac{(-(N-2)g)^4}{4!}x^{(4)}(c_{N-1}),$$

where $\ddot{x}(t_k)$ denotes the second-order derivative of $x(t)$ with respect to t at time instant t_k; $x^{(3)}(t_k)$ and $x^{(4)}(t_k)$ denote the third-order and fourth-order derivatives; symbol ! denotes the factorial operator; $c_1, c_2, c_3, \dots, c_{N-1}$ lie in (t_k, t_{k+1}), (t_{k-1}, t_k), $(t_{k-2}, t_k), \dots, (t_{k-(N-2)}, t_k)$, respectively.

Let the expansion of $x(t_{k+1})$ multiply 1, and let the expansions of $x(t_{k-i})$ multiply a_i with $i=1,2,\dots,N-2$. Adding together with the results yields

$$\begin{aligned}&x(t_{k+1})-(1+a_1+a_2+\dots+a_{N-2})x(t_k)\\&\quad+a_1x(t_{k-1})+a_2x(t_{k-2})+\dots+a_{N-2}x(t_{k-(N-2)})\\&=(1-a_1-2a_2-\dots-(N-2)a_{N-2})g\dot{x}(t_k)\\&\quad+\frac{(1+a_1+4a_2+\dots+(N-2)^2a_{N-2})g^2}{2!}\ddot{x}(t_k)\\&\quad+\frac{(1-a_1-8a_2-\dots-(N-2)^3a_{N-2})g^3}{3!}x^{(3)}(t_k)\\&\quad+O(g^4),\end{aligned} \quad (3)$$

where $O(g^4)$ is the error term absorbing

$$\frac{g^4x^{(4)}(c_1)}{4!}, \frac{a_1(-g)^4x^{(4)}(c_2)}{4!}, \frac{a_2(-2g)^4x^{(4)}(c_3)}{4!}, \dots, \frac{a_{N-2}(-(N-2)g)^4x^{(4)}(c_{N-1})}{4!}.$$

General TOA formula should only include the first-order derivative, and thus the terms of second-order and third-order derivatives should be zero, i.e.,

$$\frac{(1+a_1+4a_2+\dots+(N-2)^2a_{N-2})g^2}{2!}=0,$$
$$\frac{(1-a_1-8a_2-\dots-(N-2)^3a_{N-2})g^3}{3!}=0,$$

which is rewritten as below with sampling gap $g>0$:

$$\begin{bmatrix}1 & 1 & 4 & \dots & (N-2)^2\\ 1 & -1 & -8 & \dots & -(N-2)^3\end{bmatrix}\begin{bmatrix}1\\a_1\\a_2\\\vdots\\a_{N-2}\end{bmatrix}=0,$$

i.e.,

$$\begin{bmatrix}1 & 4\\-1 & -8\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}=-\begin{bmatrix}1 & 9\dots(N-2)^2\\1 & -27\dots-(N-2)^3\end{bmatrix}\begin{bmatrix}1\\a_3\\\vdots\\a_{N-2}\end{bmatrix}.$$

Thus, a_1 and a_2 are substituted by other parameters as below so that the second and third order derivatives could be eliminated:

$$\begin{cases}a_1=-3+\sum_{i=3}^{N-2}\left(i^3-2i^2\right)a_i,\\ a_2=\frac{1}{2}-\sum_{i=3}^{N-2}\frac{i^3-i^2}{4}a_i.\end{cases} \quad (4)$$

We plug (4) into equation (3), and have

$$\begin{aligned}&x(t_{k+1})+\left(\frac{3}{2}-\sum_{i=3}^{N-2}\frac{3i^3-7i^2+4}{4}a_i\right)x(t_k)\\&-\left(3-\sum_{i=3}^{N-2}\left(i^3-2i^2\right)a_i\right)x(t_{k-1})\\&+\left(\frac{1}{2}-\sum_{i=3}^{N-2}\frac{i^3-i^2}{4}a_i\right)x(t_{k-2})+\sum_{i=3}^{N-2}a_ix(t_{k-i})\\&=\left(3+\sum_{i=3}^{N-2}\frac{3i^2-i^3-2i}{2}a_i\right)g\dot{x}(t_k)+O(g^4).\end{aligned} \quad (5)$$

Let equation (5) divide $\left(3+\sum_{i=3}^{N-2}(3i^2-i^3-2i)a_i/2\right)g$, and then, the error term $O(g^4)$ becomes $O(g^3)$. Finally, the N-instant general TOA formula (2) is obtained and it has third order accuracy, i.e., $O(g^3)$. The proof is thus completed. □

TABLE 2. ***N*-instant general TOA formulas with different values of *N*.**

Instant number	General TOA formula
N	$\dot{x}(t_k)=\frac{x(t_{k+1})}{\tilde{a}g}+\left(\frac{3}{2g\tilde{a}}-\sum_{i=3}^{N-2}\frac{3i^3-7i^2+4}{4\tilde{a}g}a_i\right)x(t_k)-\left(\frac{3}{\tilde{a}g}-\sum_{i=3}^{N-2}\frac{i^3-2i^2}{\tilde{a}g}a_i\right)x(t_{k-1})$ $+\left(\frac{1}{2g\tilde{a}}-\sum_{i=3}^{N-2}\frac{i^3-i^2}{4g\tilde{a}}a_i\right)x(t_{k-2})+\sum_{i=3}^{N-2}\frac{a_i}{\tilde{a}g}x(t_{k-i})+O(g^3)$
5	$\dot{x}(t_k)=\frac{x(t_{k+1})}{(3-3a_3)g}+\frac{(3-11a_3)x(t_k)}{(6-6a_3)g}+\frac{(3a_3-1)x(t_{k-1})}{(1-a_3)g}+\frac{(1-9a_3)x(t_{k-2})}{(6-6a_3)g}+\frac{a_3x(t_{k-3})}{(3-3a_3)g}+O(g^3)$
6	$\dot{x}(t_k)=\frac{x(t_{k+1})}{(3-3a_3-12a_4)g}+\frac{(3-11a_3-42a_4)x(t_k)}{(6-6a_3-24a_4)g}+\frac{(9a_3+32a_4-3)x(t_{k-1})}{(3-3a_3-12a_4)g}+\frac{(1-9a_3-24a_4)x(t_{k-2})}{(6-6a_3-24a_4)g}$ $+\frac{a_3x(t_{k-3})}{(3-3a_3-12a_4)g}+\frac{a_4x(t_{k-4})}{(3-3a_3-12a_4)g}+O(g^3)$
7	$\dot{x}(t_k)=\frac{x(t_{k+1})}{(3-3a_3-12a_4-30a_5)g}+\frac{(3-11a_3-42a_4-102a_5)x(t_k)}{(6-6a_3-24a_4-60a_5)g}+\frac{(9a_3+32a_4+75a_5-3)x(t_{k-1})}{(3-3a_3-12a_4-30a_5)g}+\frac{(1-9a_3-24a_4-50a_5)x(t_{k-2})}{(6-6a_3-24a_4-60a_5)g}$ $+\frac{a_3x(t_{k-3})}{(3-3a_3-12a_4-30a_5)g}+\frac{a_4x(t_{k-4})}{(3-3a_3-12a_4-30a_5)g}+\frac{a_5x(t_{k-5})}{(3-3a_3-12a_4-30a_5)g}+O(g^3)$

B. STABILITY AND CONVERGENCE ANALYSES

To analyze and guarantee the stability and convergence of N-instant general TOA formula (2), the following theorem is given.

Theorem 2: If instant number N is no less than five, i.e., $N \geq 5$, then N-instant general TOA formula (2) must have effective domains for parameters a_i, $i = 3, \ldots, N-2$, which lead to stable and convergent time discretization. Besides, when instant number $N = 5$, 5-instant general TOA formula expressed as

$$\dot{x}(t_k)=\frac{x(t_{k+1})}{(3-3a_3)g}+\frac{(3-11a_3)x(t_k)}{(6-6a_3)g}+\frac{(3a_3-1)x(t_{k-1})}{(1-a_3)g}+\frac{(1-9a_3)x(t_{k-2})}{(6-6a_3)g}+\frac{a_3x(t_{k-3})}{(3-3a_3)g}+O(g^3) \quad (6)$$

has an effective domain $1/5 < a_3 < 1/3$ for the parameter a_3.

Proof: The proof is divided into four parts, i.e., $N < 4$, $N = 4$, $N = 5$ and $N > 5$.

Part 1: When the instant number $N < 4$, we have to use three instants at most, i.e., $x(t_{k+1})$, $x(t_k)$ and $x(t_{k-1})$, to discretize the first-order derivative $\dot{x}(t_k)$. It means that two Taylor expansions of $x(t_{k+1})$ and $x(t_{k-1})$ at most can be utilized. To eliminate the second and third order derivatives by the two expansion equations, the following equation have to be satisfied:

$$\begin{cases}\frac{(1+a_1)g^2}{2!}=0,\\ \frac{(1-a_1)g^3}{3!}=0,\end{cases}$$

which is unsatisfiable. Thus, N-instant general TOA formula does not exist when $N < 4$.

Part 2: When the instant number $N = 4$, we have to use four instants, i.e., $x(t_{k+1})$, $x(t_k)$, $x(t_{k-1})$ and $x(t_{k-2})$, to discretize the first-order derivative $\dot{x}(t_k)$. It means that only three Taylor expansions of $x(t_{k+1})$, $x(t_{k-1})$ and $x(t_{k-2})$ can be utilized. To eliminate the second and third order derivatives by the three expansion equations, the following equation have to be satisfied:

$$\begin{cases}\frac{(1+a_1+4a_2)g^2}{2!}=0,\\ \frac{(1-a_1-8a_2)g^3}{3!}=0.\end{cases}$$

As the sampling gap $g > 0$, we have $a_1 = -3$, and $a_2 = 1/2$. Thus, 4-instant general TOA formula is directly given as

$$\dot{x}(t_k)=\frac{x(t_{k+1})}{3}+\frac{x(t_k)}{2}-x(t_{k-1})+\frac{t_{k-2}}{6}. \quad (7)$$

When 4-instant formula (7) is employed for time discretization, according to Result 1 in Appendix [10], [30], its characteristic polynomial is

$$P(\zeta)=\frac{1}{3}\zeta^3+\frac{1}{2}\zeta^2-\zeta+\frac{1}{6}.$$

It is evident that its three roots do not satisfy the 0-stability condition according Result 1 in Appendix [10], [30]. Thus, N-instant general TOA formula does not lead to stable and convergent time discretization when $N = 4$.

Part 3: When the instant number $N = 5$, five instants, i.e., $x(t_{k+1})$ and $x(t_{k-i})$, $i = 0, 1, 2, 3$, to discretize $\dot{x}(t_k)$. To eliminate the second and third order derivatives by the three expansion equations, the following equation have to be satisfied:

$$\frac{(1+a_1+4a_2+9a_3)g^2}{2!}=0,$$
$$\frac{(1-a_1-8a_2-27a_3)g^3}{3!}=0,$$

Thus, a_1 and a_2 are substituted by a_3 as below so that the second and third order derivatives could be eliminated:

$$\begin{cases} a_1 = -3 + 9a_3, \\ a_2 = \frac{1}{2} - \frac{9}{2}a_3. \end{cases} \tag{8}$$

Then, based on the derivative process of N-instant general TOA formula, 5-instant general TOA formula (6) is obtained.

Based on the stable condition in Appendix [10], [30], when 5-instant general TOA formula (6) is employed for time discretization, its characteristic polynomial is

$$P(\zeta) = \zeta^4 + (\frac{3}{2} - \frac{11a_3}{2})\zeta^3 + (9a_3 - 3)\zeta^2 + (\frac{1}{2} - \frac{9a_3}{2})\zeta + a_3. \tag{9}$$

In order to investigate its roots, the bilinear transformation is employed, i.e., define $\zeta = (1+\omega)/(1-\omega)$ in equation (11), and then the following equation is obtained

$$\left(\frac{1+\omega}{1-\omega}\right)^4 + \left(\frac{3}{2} - \frac{11a_3}{2}\right)\left(\frac{1+\omega}{1-\omega}\right)^3 + (9a_3 - 3)\left(\frac{1+\omega}{1-\omega}\right)^2 + (\frac{1}{2} - \frac{9a_3}{2})\left(\frac{1+\omega}{1-\omega}\right) + a_3 = 0,$$

which is simplified as

$$(6 - 6a_3)\omega^3 + (12 - 12a_3)\omega^2 + (2 - 2a_3)\omega + 20a_3 - 4 = 0.$$

According to Routh's stability criterion, the following inequalities should be satisfied:

$$\begin{cases} 6 - 6a_3 > 0 \\ 12 - 12a_3 > 0 \\ 2 - 2a_3 > 0 \\ 20a_3 - 4 > 0 \\ \dfrac{(2a_3 - 2)(12a_3 - 12) + (6a_3 - 6)(20a_3 - 4)}{12 - 12a_3} > 0. \end{cases}$$

Solving the above inequalities, we have $1/5 < a_3 < 1/3$. As the Routh's stability criterion is the necessary and sufficient condition for the stability, 5-instant general TOA formula (6) leads to stable time discretization with $1/5 < a_3 < 1/3$. Besides, based on the results in Theorem 1, the consistency is guaranteed, and thus, the convergence is proved based on Result 3 in Appendix [10], [30].

Part 4: When the instant number $N > 5$, we have more than five instants to discretize the first-order derivative, and more Taylor expansion equations are utilized. It means that more parameters such as a_4 and a_5 can be adjusted to make N-instant general TOA formula (2) satisfy the 0-stability condition. For more convenience, N-instant general TOA formulas with $N = 5, 6, 7$ are presented in Table 2. As shown in Table 2, 6-instant general TOA formula is a special case of 7-instant general TOA formula when parameter $a_5 = 0$, and 5-instant general TOA formula is a special case of 6-instant general TOA formula when parameter $a_4 = 0$. As proved in Part 3, the effective domain is $1/5 < a_3 < 1/3$ for 5-instant general TOA formula, and thus, it must exist an effective domain for 6-instant general TOA formula, which is a plane composed of a_3 and a_4, and the plane includes the range $1/5 < a_3 < 1/3, a_4 = 0$. Furthermore, it must exist an effective domain for 7-instant general TOA formula, which is a three-dimensional space composed of a_3, a_4 and a_5, and the space includes the effective domain for 6-instant general TOA formula. Similarly, if more instants are utilized, then N-instant general TOA formula has lager effective domain, which includes the effective domain for the general TOA formula with less instants. The proof is thus completed. □

To further investigate the connection of effective domain for N-instant general TOA formula with different values of N, the effective domain composed of a_3 and a_4 for 6-instant general TOA formula is studied. 6-instant general TOA formula is expressed as

$$\begin{aligned} \dot{x}(t_k) &= \frac{x(t_{k+1})}{(3 - 3a_3 - 12a_4)g} + \frac{(3 - 11a_3 - 42a_4)x(t_k)}{(6 - 6a_3 - 24a_4)g} \\ &\quad + \frac{(9a_3 + 32a_4 - 3)x(t_{k-1})}{(3 - 3a_3 - 12a_4)g} + \frac{(1 - 9a_3 - 24a_4)x(t_{k-2})}{(6 - 6a_3 - 24a_4)g} \\ &\quad + \frac{a_3 x(t_{k-3})}{(3 - 3a_3 - 12a_4)g} + \frac{a_4 x(t_{k-4})}{(3 - 3a_3 - 12a_4)g} + O(g^3). \end{aligned} \tag{10}$$

Based on the stable condition in Appendix [10], [30], when 6-instant general TOA formula (10) is employed for time discretization, its characteristic polynomial is

$$P(\zeta) = \zeta^5 + (\frac{3}{2} - \frac{11a_3}{2} - 21a_4)\zeta^4 + (9a_3 + 32a_4 - 3)\zeta^3 + (\frac{1}{2} - \frac{9a_3}{2} - 12a_4)\zeta^2 + a_3\zeta + a_4. \tag{11}$$

We define $\zeta = (1+\omega)/(1-\omega)$ in equation (11) by the bilinear transformation and have

$$\left(\frac{1+\omega}{1-\omega}\right)^5 + \left(\frac{3}{2} - \frac{11a_3}{2} - 21a_4\right)\left(\frac{1+\omega}{1-\omega}\right)^4 + (9a_3 + 32a_4 - 3)\left(\frac{1+\omega}{1-\omega}\right)^3 + (\frac{1}{2} - \frac{9a_3}{2} - 12a_4)\left(\frac{1+\omega}{1-\omega}\right)^2 + a_3\left(\frac{1+\omega}{1-\omega}\right) + a_4 = 0,$$

which is simplified as

$$\begin{aligned} &(6 - 24a_4 - 6a_3)\omega^4 + (18 - 72a_4 - 18a_3)\omega^3 \\ &\quad + (14 - 56a_4 - 14a_3)\omega^2 + (18a_3 + 88a_4 - 2)\omega \\ &\quad + 20a_3 + 64a_4 - 4 = 0. \end{aligned}$$

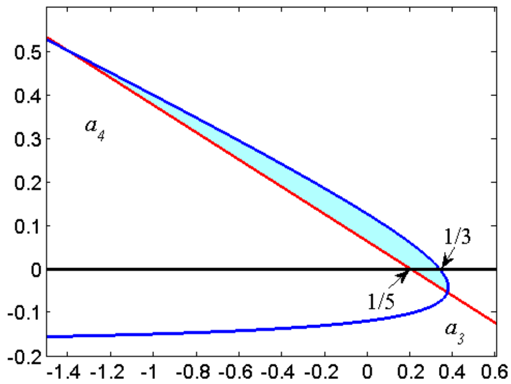


FIGURE 1. Effective domain for 6-instant general TOA formula (10).

According to Routh's stability criterion, the following inequalities should be satisfied:

$$\begin{cases} 6-24a_4-6a_3>0 \\ 18-72a_4-18a_3>0 \\ 14-56a_4-14a_3>0 \\ 18a_3+88a_4-2>0 \\ 20a_3+64a_4-4>0 \\ -\frac{4}{3}(15a_3+64a_4-11)>0 \\ \frac{64}{3}a_4-128a_3-704a_3a_4-\frac{8704}{3}a_4^2+\frac{128}{3}>0. \end{cases}$$

Feasible domain of the inequalities is shown in Fig. 1, which is exactly the effective domain for 6-instant general TOA formula (10). It is observed from Fig. 1 that the effective domain of a_3 is $1/5 < a_3 < 1/3$ when $a_4 = 0$ for formula (10), which is consistent with the results in Part 4 of the proof process of Theorem 2.

III. TIME-VARYING OPTIMIZATION

In this section, a series of general solution models are proposed based on the zeroing neural dynamics and the N-instant general TOA formula (2).

The problem of time-varying optimization is formulated as below with $\mathbf{x}_{k+1}$ to be obtained at each computational time interval $[t_k, t_{k+1}) = [kg, (k+1)g) \subseteq [0, t_f]$:

$$\begin{aligned} &\min.\, f(\mathbf{x}(t_{k+1}), t_{k+1}) \\ &\text{s. t. } A(t_{k+1})\mathbf{x}(t_{k+1}) = \mathbf{b}(t_{k+1}). \end{aligned} \tag{12}$$

The object function $f(\cdot,\cdot): \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ is time varying, nonlinear, and convex with respect to $\mathbf{x}$, and the rank of $A_{k+1} \in \mathbb{R}^{m\times n}$ is constantly equal to m with $m < n$. Thus, problem (12) is a time-varying convex problem, and any local optimum of this convex problem is constantly global.

Based on previous work [4], [31], the continuous-time solution model based on the zeroing neural dynamics method is obtained as

$$\dot{\mathbf{y}}(t) = -H^{-1}(\mathbf{y}(t),t)\left(\lambda\mathbf{h}(\mathbf{y}(t),t) + \dot{\mathbf{h}}_t(\mathbf{y}(t),t)\right), \tag{13}$$

with $\mathbf{y}(t) = \left[\mathbf{x}^{\mathrm{T}}(t), \mathbf{l}^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{n+m}$. Moreover, $\mathbf{l}^{\mathrm{T}}(t) \in \mathbb{R}^m$ is the Lagrange-multiplier vector, and $\mathbf{h}(\mathbf{y}(t),t)$ is defined as

$$\mathbf{h}(\mathbf{y}(t),t) = \begin{bmatrix} \frac{\partial f(\mathbf{x}(t),t)}{\partial \mathbf{x}(t)} + A^{\mathrm{T}}(t)\mathbf{l}(t) \\ A(t)\mathbf{x}(t) - \mathbf{b}(t) \end{bmatrix} \in \mathbb{R}^{n+m}.$$

Matrix $H(\mathbf{y}(t),t)$ is defined as

$$H(\mathbf{y}(t),t) = \begin{bmatrix} \frac{\partial f^2(\mathbf{x}(t),t)}{\partial \mathbf{x}(t)\partial^{\mathrm{T}}\mathbf{x}(t)} & A^{\mathrm{T}}(t) \\ A(t) & \mathbf{0} \end{bmatrix}.$$

$H^{-1}(\mathbf{y}(t),t)$ denotes the inverse of $H(\mathbf{y}(t),t)$. $\dot{\mathbf{h}}_t(\mathbf{y}(t),t)$ is the partial derivative of $\mathbf{h}(\mathbf{y}(t),t)$ with respect to t, which is defined as

$$\dot{\mathbf{h}}_t(\mathbf{y}(t),t) = \frac{\partial \mathbf{h}(\mathbf{y}(t),t)}{\partial t}.$$

The design parameter $\lambda > 0$ should be set as large as the hardware permits or set appropriately for simulative/experimental purposes.

To solve optimization problem (12) in real time, i.e., calculate $\mathbf{x}_{k+1}$ at each computational time interval $[t_k, t_{k+1}) = [kg, (k+1)g) \subseteq [0, t_f]$, N-instant general TOA formula (2) is employed to discretize continuous-time solution model (13). Then, N-instant general solution model is obtained as

$$\begin{aligned} \mathbf{y}(t_{k+1}) = &-\left(\frac{3}{2} - \sum_{i=3}^{N-2}\frac{3i^3-7i^2+4}{4}a_i\right)\mathbf{y}(t_k) \\ &+\left(3 - \sum_{i=3}^{N-2}\left(i^3-2i^2\right)a_i\right)\mathbf{y}(t_{k-1}) \\ &-\left(\frac{1}{2} - \sum_{i=3}^{N-2}\frac{i^3-i^2}{4}a_i\right)\mathbf{y}(t_{k-2}) - \sum_{i=3}^{N-2}a_i\mathbf{y}(t_{k-i}) \\ &-\left(3 + \sum_{i=3}^{N-2}\frac{3i^2-i^3-2i}{2}a_i\right)H^{-1}(\mathbf{y}(t_k),t_k) \\ &\cdot\left(\kappa\mathbf{h}(\mathbf{y}(t_k),t_k) + g\dot{\mathbf{h}}_t(\mathbf{y}(t_k),t_k)\right), \end{aligned} \tag{14}$$

where $\kappa = \lambda g$. Different values of instant number N lead to different general solution models.

The solution precision of N-instant general solution model (14) is analyzed and guaranteed by the following theorem.

Theorem 3: If parameters $a_i, i = 1,2,\ldots,N-2$ lie in effective domain, N-instant general solution model (14) to solve time-varying optimization problem (12) is 0-stable and convergent, which converges with the truncation error order of $\mathbf{O}(g^4)$, which denotes a vector and every element being $O(g^4)$.

Proof: It is known from Theorem 1 that the truncation error of N-instant general TOA formula (2) is $O(g^3)$. When it is employed to discretize solution model (13) without omitting the truncation error, the following equation is obtained:

$$\mathbf{y}(t_{k+1}) = -\left(\frac{3}{2} - \sum_{i=3}^{N-2}\frac{3i^3-7i^2+4}{4}a_i\right)\mathbf{y}(t_k)$$

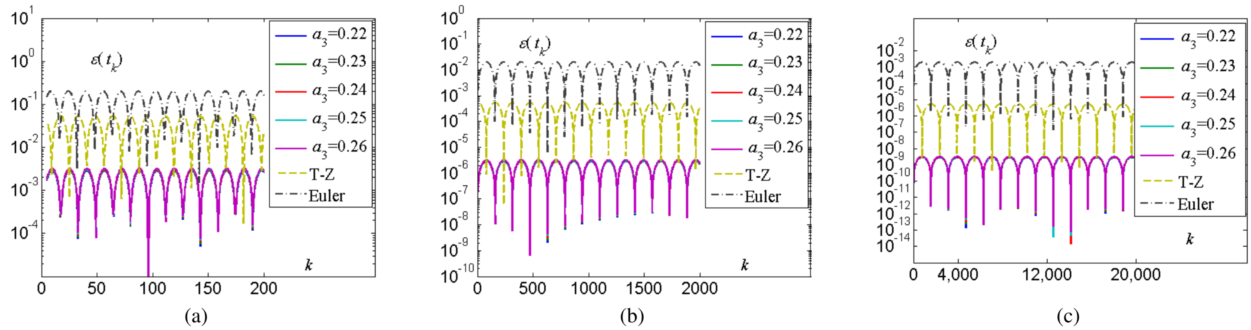


FIGURE 2. Approximation errors defined as $\varepsilon(t_k) = |\dot{x}(t_k) - \dot{x}^*(t_k)|$ when using 5-instant general TOA formula as well as conventional Euler formula and Taylor-Zhang formula to approximate the first-order derivative of $x(t_k) = \sin(2t_k)$ with different sampling gap g. (a) With $g = 0.1$ s. (b) With $g = 0.01$ s. (c) With $g = 0.001$ s.

$$+\left(3-\sum_{i=3}^{N-2}\left(i^3-2i^2\right)a_i\right)\mathbf{y}(t_{k-1})$$
$$-\left(\frac{1}{2}-\sum_{i=3}^{N-2}\frac{i^3-i^2}{4}a_i\right)\mathbf{y}(t_{k-2})-\sum_{i=3}^{N-2}a_i\mathbf{y}(t_{k-i})$$
$$-\left(3+\sum_{i=3}^{N-2}\frac{3i^2-i^3-2i}{2}a_i\right)\dot{\mathbf{y}}(t_k)+\mathbf{O}(g^4),$$

It is concluded that the truncation error of N-instant general solution model (14) is $\mathbf{O}(g^4)$ by the comparison of this equation and solution model (14).

Furthermore, based on Theorem 2, N-instant general solution model (14) to solve time-varying optimization problem (12) is 0-stable and convergent if parameters $a_i, i = 1, 2, \ldots, N-2$ lie in effective domain. Finally, based on Result 4 in Appendix [10], [30], it is proved that N-instant general solution model (14) to solve time-varying optimization problem (12) converges with the truncation error order of $\mathbf{O}(g^4)$. The proof is thus completed. □

Based on Theorems 2 and 3, the following two corollaries are obtained.

Corollary 1: If parameter a_3 satisfy $1/5 < a_3 < 1/3$, then 5-instant general solution model expressed as

$$\begin{aligned}\mathbf{y}(t_{k+1}) &= (3a_3-3)\,H^{-1}(\mathbf{y}(t_k),t_k)\left(\kappa\mathbf{h}(\mathbf{y}(t_k),t_k)+g\dot{\mathbf{h}}_t(\mathbf{y}(t_k),t_k)\right)\\ &\quad+\left(\frac{11a_3}{2}-\frac{3}{2}\right)\mathbf{y}(t_k)+(3-9a_3)\,\mathbf{y}(t_{k-1})\\ &\quad+\left(\frac{9a_3}{2}-\frac{1}{2}\right)\mathbf{y}(t_{k-2})-a_3\mathbf{y}(t_{k-3})\end{aligned}\tag{15}$$

to solve time-varying optimization problem (12) is 0-stable and convergent, which converges with the truncation error order of $\mathbf{O}(g^4)$.

Corollary 2: If parameters a_3 and a_4 lie in the effective domain in Fig. 1, then 6-instant general solution model expressed as

$$\begin{aligned}\mathbf{y}(t_{k+1}) &= (3a_3+12a_4-3)\,H^{-1}(\mathbf{y}(t_k),t_k)\\ &\quad\times\left(\kappa\mathbf{h}(\mathbf{y}(t_k),t_k)+g\dot{\mathbf{h}}_t(\mathbf{y}(t_k),t_k)\right)\\ &\quad+\left(\frac{11a_3}{2}+21a_4-\frac{3}{2}\right)\mathbf{y}(t_k)+(3-9a_3-32a_4)\,\mathbf{y}(t_{k-1})\\ &\quad+\left(\frac{9a_3}{2}+12a_4-\frac{1}{2}\right)\mathbf{y}(t_{k-2})-a_3\mathbf{y}(t_{k-3})-a_4\mathbf{y}(t_{k-4})\end{aligned}\tag{16}$$

to solve time-varying optimization problem (12) is 0-stable and convergent, which converges with the truncation error order of $\mathbf{O}(g^4)$.

IV. NUMERICAL EXPERIMENTS AND VERIFICATION

In this section, some numerical experiments are conducted to verify the good performances of proposed N-instant general TOA formula (2) and N-instant general solution model (14). Numerical results are separated into two parts: first-order approximation by N-instant general TOA formula (2) and time-varying optimization by N-instant general solution model (14).

A. FIRST-ORDER APPROXIMATION

In this subsection, 5-instant general TOA formula and 6-instant general TOA formula are used to approximate the first-order derivative [31] of

$$x_k = x(t_k) = \sin(2t_k). \tag{17}$$

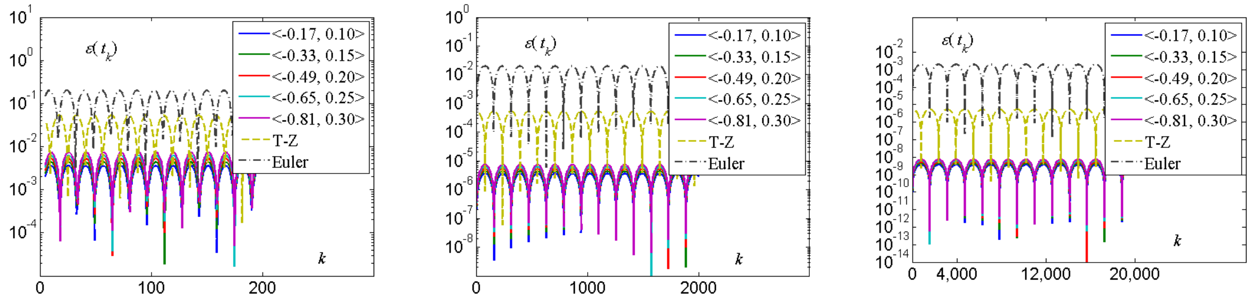
For comparisons, conventional Euler formula [10] expressed as

$$\dot{x}(t_k) = \frac{x(t_{k+1})-x(t_k)}{g} + O(g), \tag{18}$$

and Taylor-Zhang formula [32] expressed as

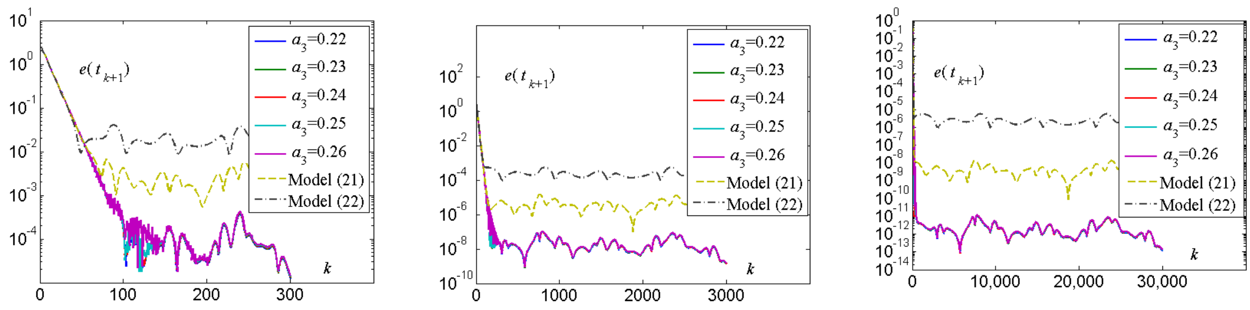
$$\dot{x}(t_k) = \frac{2x(t_{k+1})-3x(t_k)+2x(t_{k-1})-x(t_{k-2})}{2\tau} + O(g^2) \tag{19}$$

are presented. The numerical results are presented in Figs. 2 and 3. Fig. 2 shows approximation errors defined as $\varepsilon(t_k) = |\dot{x}(t_k) - \dot{x}^*(t_k)|$ when using 5-instant general TOA formula as well as conventional Euler formula (18) and Taylor-Zhang formula (19) to approximate the first-order derivative of $x(t_k) = \sin(2t_k)$ with different sampling gap g. Different values of parameter a_3 are employed, and corresponding results



(a) (b) (c)

FIGURE 3. Approximation errors defined as $\varepsilon(t_k) = |\dot{x}(t_k) - \dot{x}^*(t_k)|$ when using 6-instant general TOA formula as well as conventional Euler formula and Taylor-Zhang formula to approximate the first-order derivative of $x(t_k) = \sin(2t_k)$ with different sampling gap g. (a) With $g = 0.1$ s. (b) With $g = 0.01$ s. (c) With $g = 0.001$ s.



(a) (b) (c)

FIGURE 4. Residual errors defined as $e(t_{k+1}) = \|\mathbf{h}(\mathbf{y}(t_{k+1}), t_{k+1})\|$ when using 5-instant general solution model as well as conventional models (21) and (22) to solve time-varying optimization (20) with effective parameter a_3 and different sampling gap g. (a) With $g = 0.1$ s. (b) With $g = 0.01$ s. (c) With $g = 0.001$ s.

are consistent with theoretical analyses. Specifically, approximation errors are of order 10^{-3}, 10^{-6} and 10^{-9} when sampling gap $g = 0.1$ s, 0.01 s and 0.001 s, respectively, which substantiate that 5-instant general TOA formula has third order accuracy. In contrast, conventional Euler formula (18) and Taylor-Zhang formula (19) have $O(g)$ and $O(g^2)$ accuracy. Fig. 3 shows approximation errors when using 6-instant general TOA formula as well as conventional formulas to approximate the same signal with different sampling gap g, which shows the good performances of 6-instant general TOA formula.

B. TIME-VARYING OPTIMIZATION

In this subsection, the following time-varying optimization [31] is considered and solved at each computational time interval $[t_k, t_{k+1}) = [kg, (k+1)g) \subseteq [0, 30]$ s:

$$\begin{aligned}\text{min. } & \frac{1}{4}(\sin(0.1t_{k+1})+1)x_{1,k+1}^4 + \frac{1}{4}(\cos(0.1t_{k+1}) \\ & +1)x_{2,k+1}^4 + \frac{1}{2}x_{1,k+1}^2 + \frac{1}{2}x_{2,k+1}^2 \\ \text{s. t. } & \sin(0.2t_{k+1})x_{1,k+1} + \cos(0.2t_{k+1})x_{2,k+1} \\ & = \cos(0.5t_{k+1}). \end{aligned} \tag{20}$$

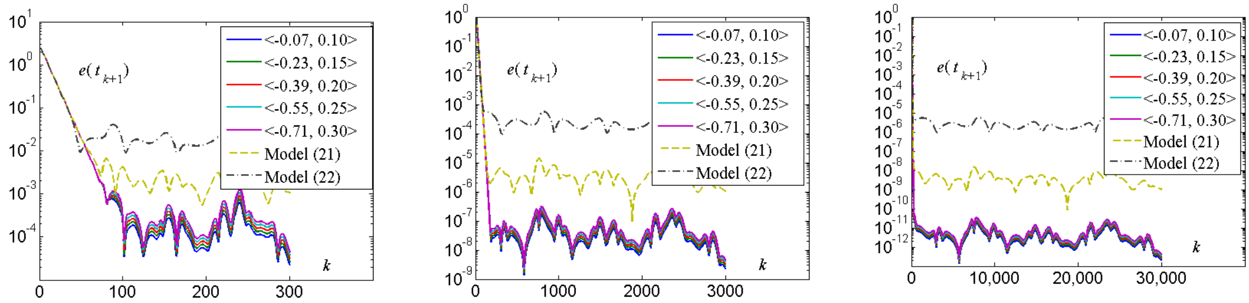
N-instant general solution model (14) is employed to solve this problem. For the purpose of comparison, traditional Taylor-type solution model and Euler-type solution model are also employed to solve this problem [31], and they are presented as

$$\begin{aligned}\mathbf{y}(t_{k+1}) & \\ = & -H^{-1}(\mathbf{y}(t_k), t_k)\left(\kappa \mathbf{h}(\mathbf{y}(t_k), t_k) + g\dot{\mathbf{h}}_t(\mathbf{y}(t_k), t_k)\right) \\ & + \frac{3}{2}\mathbf{y}(t_k) - \mathbf{y}(t_{k-1}) + \frac{1}{2}\mathbf{y}(t_{k-2}) \end{aligned} \tag{21}$$

and

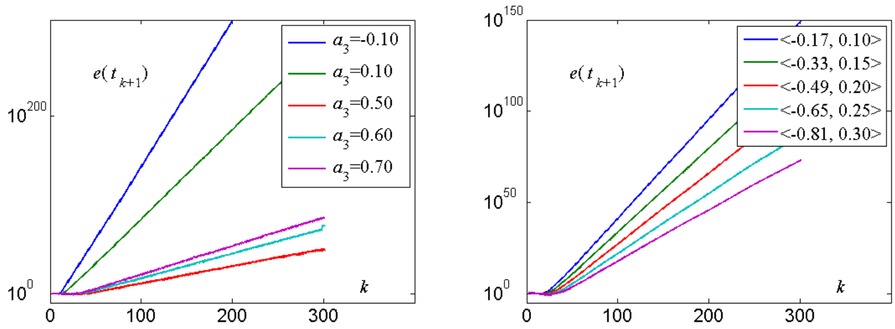
$$\begin{aligned}\mathbf{y}(t_{k+1}) = & \ \mathbf{y}(t_k) \\ & -H^{-1}(\mathbf{y}(t_k), t_k)\left(\kappa \mathbf{h}(\mathbf{y}(t_k), t_k) + g\dot{\mathbf{h}}_t(\mathbf{y}(t_k), t_k)\right). \end{aligned} \tag{22}$$

Numerical results are presented in Figs. 4-6. Specifically, Fig. 4 shows residual errors defined as $e(t_{k+1}) = \|\mathbf{h}(\mathbf{y}(t_{k+1}), t_{k+1})\|$ when using 5-instant general solution model as well as conventional models (21) and (22) to solve time-varying optimization (20) with effective parameter a_3 and different sampling gap g. When sampling gap $g = 0.1$ s, 0.01 s and 0.001 s, steady-state residual errors of 5-instant models are of order 10^{-4}, 10^{-8} and 10^{-12}, which substantiates that 5-instant general solution model has $O(g^4)$ error. However, it is observed that conventional models (21) and (22) only have $O(g^3)$ and $O(g^2)$ errors based on the variation of their residual errors. Note that the parameter a_3 is seted



(a) (b) (c)

FIGURE 5. **Residual errors defined as $e(t_{k+1}) = \|h(y(t_{k+1}), t_{k+1})\|$ when using 6-instant general solution model as well as conventional models (21) and (22) to solve time-varying optimization (20) with effective parameters a_3, a_4 and different sampling gap g. (a) With $g = 0.1$ s. (b) With $g = 0.01$ s. (c) With $g = 0.001$ s.**



(a) (b)

FIGURE 6. **Residual errors defined as $e(t_{k+1}) = \|h(y(t_{k+1}), t_{k+1})\|$ when using (a) 5-instant general solution model and (b) 6-instant general solution model to solve time-varying optimization (20) with ineffective parameters and sampling gap $g = 0.1$ s.**

in the effective domain $1/5 < a_3 < 1/3$, which is consistent with the theoretical results. Fig. 4 shows residual errors when using 6-instant general solution model as well as conventional models (21) and (22) to solve time-varying optimization (20) with effective parameters a_3, a_4 and different sampling gap g. Parameters $< a_3, a_4 >$ are seted as $< -0.07, 0.10 >$, $< -0.23, 0.15 >$, $< -0.39, 0.20 >$, $< -0.55, 0.25 >$ and $< -0.71, 0.30 >$. They are all in the effective domain shown in Fig. 1 and lead to great performances to solve time-varying optimization. Fig. 6 shows residual errors when using 5-instant general solution model and 6-instant general solution model with ineffective parameters, which are beyond the effective domains presented in theoretical parts. It is evident that they fail to solve time-varying optimization (20).

V. CONCLUSION

N-instant general third-order-accuracy formula has been proposed to deeply study the time discretization in this paper. Different N values lead to different general formulas, and different general formulas have different effective domain for their parameters. The connections of infinite general formulas have been studied, which show that general formulas using less instants are special cases of these using more instants. Besides, the connection of effective domains for different general formulas have been investigated. Furthermore, N-instant general third-order-accuracy formula have been employed to solve time-varying optimization, and N-instant general solution model has been proposed. Numerical results have verified the effectiveness and superiority of proposed general formulas and solution models.

APPENDIX

We have the following four results for an M-step method [10], [30].

Result 1: An M-step method $\sum_{i=0}^{M} \alpha_i x_{k+i} = g \sum_{i=0}^{M} \beta_i \psi_{k+i}$ can be checked for 0-stability by determining the roots of its characteristic polynomial $P_M(\zeta) = \sum_{i=0}^{M} \alpha_i \zeta^i$. If all roots denoted by ζ of the polynomial $P_N(\zeta)$ satisfy $|\zeta| \leq 1$ with $|\zeta| = 1$ being simple, then the corresponding M-step method is 0-stable (i.e., has 0-stability).

Result 2: An M-step method is said to be consistent (i.e., has consistency) of order p if the truncation error for the exact solution is of order $O(g^{p+1})$ where $p > 0$.

Result 3: An M-step method is convergent, i.e., $x_{[t/g]} \to x^*(t)$, for all $t \in [0, t_f]$, as $g \to 0$, if and only if the method is 0-stable and consistent. That is, 0-stability plus consistency means convergence, which is also known as Dahlquist equivalence theorem.

Result 4: A 0-stable consistent method converges with the order of its truncation error.

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