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Reliability Analysis for Multi-Phase Wiener Processes Considering Phase-Varying Nonlinearity

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ABSTRACT Degradation reliability analysis incorporating two-phase or even multi-phase features has become a research focus in recent years. Motivated by the fact that a practical multi-phase degradation procedure may probably involve both linear phases and nonlinear phases, a multi-phase degradation model is constructed that can consider the phase-varying nonlinear property. The unit-to-unit variability is incorporated. Meanwhile, the reliability function based on the first passage time (FPT) conception is derived, where the solution for the two-phase situation is first obtained, and the one for the multi-phase circumstance is constructed via a reclusive scheme. To verify the reasonability of the proposed reliability function, a simulation study has been complemented. Then two real applications are given to demonstrate the validity and the effectiveness of the proposed method.

INDEX TERMS Multi-phase degradation, phase-varying nonlinearity, reliability evaluation, unit specific property.

I. INTRODUCTION

For highly reliable complex systems, degradation data-based methods have been recognized as one of the most feasible schemes to assess the reliability [1]. There is considerable interest on the part of scientists and engineers in constructing a reasonable model to properly describe the degradation mechanism. Most literatures have focused on single-phase deterioration problems, where the whole degradation procedure can be governed by a single mechanism or regulation, and then can be depicted by a single mathematical model. Extensive studies have been implemented and two main models have been proposed and widely adopted, including general path models [2] and stochastic process models [3]. Because of the uncertainties or dynamics of the deterioration progression over time, stochastic process-based methods have been well recognized and commonly applied.

Various stochastic processes have been incorporated in degradation modeling according to the literature. Due to the good mathematical properties and clear physical explanations, Wiener processes (WPs) [4], [5] have become the most commonly adopted one for modeling non-monotonous degradations. In addition, Markov chain [6], Gamma processes [7] and inverse Gaussian processes [8] have also been investigated for various circumstances. It is worth noticing that extensive researches have been implemented to settle the nonlinear degradation problems which are commonly encountered in real applications [9]–[13].

Recent literatures have demonstrated that two-phase or even multi-phase properties have been shown by many practical degradation procedures where every two adjacent phases are divided by a so-called change point. To the best of our knowledge, the phase transformation can be caused by incomplete burn-in of devices, the change of inner physical mechanism and external working conditions. Thereafter, multiple degradation phases may probably illustrate different regulations and dynamics inherent properties. Consequently, two-phase degradation modeling has become a great concern to analyze the failure procedures. Two-phase WP models have also been commonly adopted, where most current

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studies suppose a same linear WP model form for all stages and different parameters are involved as distinctions. A series of researches can be found in Refs. [14]–[21].

In two-phase circumstances, one of the biggest challenges lies in the reasonable modeling of the nonlinear deterioration characteristics which may be probably caused by the integrity and complexity of the system itself or the multiple missions for more and more modern products and systems. Particularly, one should note that the nonlinearity property of a multi-phase deterioration procedure may demonstrate two situations. First, all degradation stages illustrate a similar nonlinear regulation and different parameters are introduced for the multiple phases. Meanwhile, based on extensive practical investigations, we found that a more common circumstance lies in the phase-varying nonlinearity which means that both linear stages and nonlinear stages are involved in a whole deterioration procedure. For example, some electronic devices initially experience a rapid and nonlinear decrease stage in the case of incomplete burn-in [22]–[24], and the degradation rate in this initial phase gradually decreases and then a nonlinear characteristic is illustrated [14], [17], [19], [25]. Second, there is also another class of devices with a stable first phase and a rapid second phase, because a defect may have been initiated [26], [27]. It may keep in a normal operation stage in the early time and then become defective rapidly [8], [26]–[32].

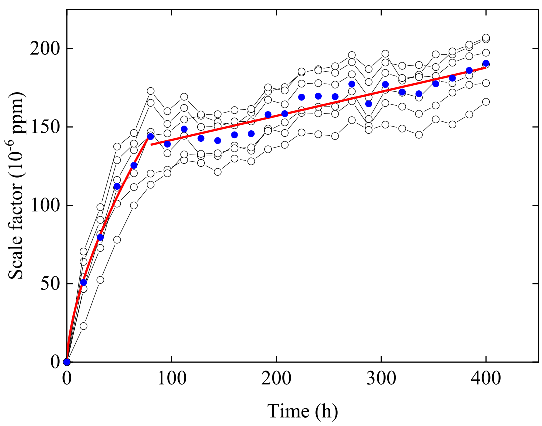


FIGURE 1. Degradation measurements of accelerometers [25], where the sample averages illustrate a nonlinear (power law) regulation in the first phase and a linear one for the second stage.

To illustrate a better understanding, two representative test datasets are given. Fig. 1 shows the two-phase degradation measurements of accelerometers tested by Wei and Chen [25], where a fast and nonlinear first phase and a linear second phase are demonstrated. Fig. 2 illustrates the gyros' drift test result in Ref. [9], where the first degradation phase is linear, and the second one is nonlinear. It is worth mentioning that the red lines are obtained through the piecewise curve fitting of the sample average values (blue dots). It can be also seen from Fig. 1 and Fig. 2 that heterogeneities (including change point location and degradation rate) do exist among a batch of units.

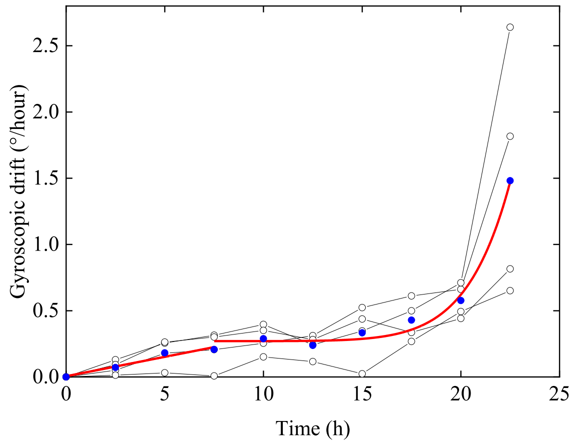


FIGURE 2. Degradation investigations of gyros [9], where the sample averages illustrate a linear regulation in the first phase and a nonlinear (power law) one in the second stage.

It is well known that the proper description of phase-varying nonlinearity regulation is very important to guarantee an effective two-phase degradation reliability analysis. This is because the precision of reliability assessment over working time and the reliability prediction depends on the proper description of the degradation characteristics in all the multiple phases. However, few researches have been reported in the literature regarding this issue. Therefore, the first objective of the current study is to construct a generalized two-phase WP degradation model with random effects (where the common degradation characteristics can be depicted by the fixed model parameters and the unit specific properties can be captured by the random ones), which can be easily extended to multi-phase model if necessary. It can reasonably consider the phase-varying nonlinearity and heterogeneity.

To complement an effective degradation reliability analysis, it is very necessary to derive the reliability function based on the FPT conception for a WP deterioration model. However, the two-phase degradation property brings enormous challenges for the derivation, that is, the reliability of the second phase has to be accessed on the precondition that the product does not fail in the first phase. It is very hard to derive the conditional reliability function considering FPT concept, because the distribution property of the degradation state at the change point (i.e., the distribution property of the performance index at the change point) is unknown. Furthermore, the random effects existing in the devices of the same batch should also be considered in reliability evaluation.

According to the literature, only a few studies regarding two-phase linear WP model have been reported. A feasible scheme to simplify the derivation was constructed by assuming the change point as the FPT when the degradation value exceeds a change-point value [34], i.e., the degradation value at the change point is prefixed. To reasonably consider the randomness property of the degradation value at change point, studies were conducted for the two-phase linear WP situation and a closed form reliability function was proposed [35]. Furthermore, Ref. [36] gave a reclusive reliability function for the more complex circumstance involving three or more linear WP degradation phases. To this end, the second objective of the current study is to construct a reliability function under the concept of FPT for the proposed two-phase nonlinear WP model and extend it to multi-phase situation.

The remainder of paper is organized as follows. In Section 2, a generalized multi-phase WP degradation model that can properly consider the phase-varying nonlinearity property is proposed. Then, analytical reliability analysis methods for the proposed model are derived in Section 3. For practical applications, the approach of model identification and initial guesses for optimization algorithm are given in Section 4. Simulation studies considering two-phase and multi-phase degradation cases are given to illustrate the rationality in Section 5, and two real applications are given to demonstrate the effectiveness in Section 6. Finally, a conclusion and summary are given in Section 7.

II. DEGRADATION MODEL FORMULATION

This section focuses on constructing a generalized multi-phase WP degradation model to reasonably consider the phase-varying nonlinearity and unit specific properties. To this end, the following effective multi-phase degradation model can be constructed by developing the mean degradation path as a more generalized form.

$$X(t) = \begin{cases} x_0 + \mu_1\Lambda_1(t) + \sigma_1 B(t), \\ \quad 0 < t \le \tau_1 \\ x_{\tau_1} + \mu_2\Lambda_2(t-\tau_1) + \sigma_2 B(t-\tau_1), \\ \quad \tau_1 < t \le \tau_2 \\ \cdots \\ x_{\tau_{k-1}} + \mu_k\Lambda_k(t-\tau_{k-1}) + \sigma_k B(t-\tau_{k-1}), \\ \quad t > \tau_{k-1} \end{cases} \tag{1}$$

where $X(t)$ denotes the performance value at time t; $B(t)$ is standard Brownian motion with drift coefficients $\mu_1, \mu_2, \ldots, \mu_k$ and diffusion coefficients $\sigma_1, \sigma_2, \ldots, \sigma_k$ for the k phases; $\tau_1, \tau_2, \ldots, \tau_{k-1}$ are change points where τ_i denotes the one dividing the i-th phase and $(i+1)$-th phase, $i = 1, 2, \ldots, k-1$; x_0 is the initial degradation state (degradation value at the initial test point), and x_{τ_i} is the degradation state at the change point τ_i, $i = 1, 2, \ldots, k-1$; $\Lambda_i(t)$ denotes the transformed time scale for the i-th phase, and all of them can be linear or nonlinear functions, $i = 1, 2, \ldots, k$.

Based on extensive studies in the literature, one can see that many practical degradation procedures demonstrate a two-phase deterioration mechanism involving nonlinear phases. From this view point, without loss of generality, a two-phase degradation model considering phase-varying nonlinearity is focused as a typical representative, denoted as

$$X(t) = \begin{cases} x_0 + \mu_1\Lambda_1(t) + \sigma_1 B(t), & 0 < t \le \tau_1 \\ x_\tau + \mu_2\Lambda_2(t-\tau) + \sigma_2 B(t-\tau_1), & t > \tau_1 \end{cases} \tag{2}$$

for the commonly encountered two-phase situation.

Remark 1: The two-phase model of Eq. (2) can be considered as a generalized one: 1) it can become the conventional two-phase linear WP model when $\Lambda_1(t) = \Lambda_2(t) = t$; 2) it can reasonably depict a degradation process involving a linear phase and a nonlinear phase by setting a linear transformed time scale and a nonlinear one; 3) it can also model a deterioration procedure involving two nonlinear stages when both $\Lambda_1(t)$ and $\Lambda_2(t)$ are nonlinear. It is worth noticing that the transformation functions can be obtained by engineering experience (see the log-transformation depicting fatigue crack growth data), mechanistic knowledge (see the wearing theory in Wang and Dan [37]), or data plotting (see the complex logarithmic transformation in Zuo *et al.* [38]).

To further capture the heterogeneity within a population, a common scheme is to let the fixed model parameters capture the common characteristics while the random ones describe the unit specific properties. From this viewpoint, the degradation rates μ_1, μ_2 are assumed to be normally distributed as $\mu_1 \sim N\left(\mu_{1,0}, \sigma_{1,0}^2\right)$ and $\mu_2 \sim N\left(\mu_{2,0}, \sigma_{2,0}^2\right)$, and the change point τ_1 is supposed as a Gamma distributed variable with shape parameter α_1 and scale parameter β_1. The schemes of random effects and Gaussian assumptions on degradation rates have been widely recognized in degradation modeling. Meanwhile, the assumption that the change point τ_1 follows a Gamma distribution is because the changing time has to be positive. In addition, Gamma distribution encompasses a number of important distributions, and when the shape parameter is large enough (approximately $\alpha_1 > 10$), a Gamma distribution can approximate a normal distribution [35].

As we know, the lifetime is usually defined as the FPT when the performance index reaches a pre-specified threshold D; i.e., failure time (lifetime) T can be defined as

$$T = \inf\{t | X(t) \ge D\} \tag{3}$$

Then, the reliability can be defined as

$$R(t) = \Pr\{T > t\} \tag{4}$$

Since p-percentile lifetime of the failure time distribution (FTD) is a great concern in practical engineering, the calculation of t_p is further focused. Suppose t_p as 100pth percentile of FTD, which means an average of $100(1-p)\%$ for the population of the products will not fail before t_p. Then t_p can be obtained by solving $R(t) = 1 - p$ with respect to t.

III. RELIABILITY FUNCTION DERIVATION

From the model introduced in Section 2, it can be concluded that the derivation of reliability function is very important. For a multi-phase degradation, two aspects of randomness have to be considered: first is the randomness of the change-point degradation state which comes from the change of degradation phase; second is the unit-to-unit variability where the degradation drifts and change points are random variables. To this end, necessary lemmas and inference are introduced in Part A. Then the solution of Eq. (4) for the two-phase circumstance is derived in Part B. Finally, the multi-phase situation (3 or more phases are involved) is further focused in Part C.

A. BASIC LEMMAS AND STATISTICAL INFERENCE

For a single-phase linear WP situation, it has been recognized that the failure time follows an Inverse Gaussian distribution [39] under the concept of FPT. Regarding the nonlinear circumstance, Ref. [40] obtained the distribution of FPT from the extreme value distribution of nonlinear drift Brownian motion, which was applied in Refs. [41] and [42] for single-phase nonlinear WP. Then the following Lemma 1 gives the expression without considering the unit specific properties.

Lemma 1 [40]: For the degradation process given by $X(t) = x_0 + \mu\Lambda(t) + \sigma B(t)$, if $\Lambda(t)$ is a one-dimensional, continuous and derivable function of time t in $[0, +\infty)$, then, under the concept of FPT, the reliability function can be obtained with an explicit form as

$$R(t) = H(t|x_0, \mu, \Lambda(\cdot), \sigma, D) = \Phi\left(\frac{D - x_0 - \mu\Lambda(t)}{\sigma\sqrt{t}}\right) - \exp\left[\frac{2\mu\Lambda(t)(D-x_0)}{\sigma^2 t}\right] \times \Phi\left(\frac{-D + x_0 - \mu\Lambda(t)}{\sigma\sqrt{t}}\right) \tag{5}$$

As discussed above, the reliability of a later phase has to be derived considering the precondition the device does not fail in all the former stages. For a two-phase situation, therefore, the probability distribution function (PDF) of degradation state x_τ conditional on $X(t) < D (0 \le t \le \tau)$ has to be obtained. Then Lemma 2 based on measure theory is introduced to derive the analytical form of the PDF of x_τ.

Lemma 2 [36]: For a Brownian process $X(t)$ whose start point x_0, end point x_τ and diffusion coefficient σ are constants, suppose $M^X_{[0,\tau]}$ is the maximum value of $X(t)$ in $[0, \tau]$, then we have

$$\Pr\left\{M^X_{[0,\tau]} \le D|x_0, x_\tau\right\} = 1 - \exp\left[-\frac{2(x_\tau - D)(x_0 - D)}{\sigma^2\tau}\right] \tag{6}$$

Furthermore, the unit-to-unit variability has to be reasonably considered in real applications. Therefore, the following Lemma 3 focusing on integration problem is given.

Lemma 3 [43]: If $Y \sim N(\mu_Y, \sigma_Y^2)$, $A, B, C \in R$, then the following holds

$$E_Y\left[\exp(AY)\left(\Phi(B + CY)\right)\right] = \exp\left[A\mu_Y + \frac{A^2}{2}\sigma_Y^2\right]\Phi\left(\frac{B + C\mu_Y + AC\sigma_Y^2}{\sqrt{1 + C^2\sigma_Y^2}}\right) \tag{7}$$

Based on Lemma 3, considering the unit-to-unit variability in nonlinear degradation model $X(t) = x_0 + \mu\Lambda(t) + \sigma B(t)$, i.e., $\mu \sim N(\mu_0, \sigma_0^2)$, the reliability function given by Lemma 1 can be rewritten as the following explicit form.

$$\begin{aligned} R(t) &= G(t|x_0, \mu_0, \sigma_0^2, \Lambda(\cdot), \sigma, D) \\ &= \Phi\left(\frac{D - x_0 - \mu_0\Lambda(t)}{\sqrt{\sigma^2 t + \sigma_0^2\Lambda^2(t)}}\right) \\ &= \exp\left[\frac{2\mu_0\Lambda(t)(D-x_0)}{\sigma^2 t} + \frac{2\sigma_0^2\Lambda^2(t)(D-x_0)^2}{\sigma^4 t^2}\right] \\ &\quad \times \Phi\left(-\frac{(D - x_0 + \mu_0\Lambda(t))\,\sigma^2 t + 2(D-x_0)\sigma_0^2\Lambda^2(t)}{\sigma^2 t\sqrt{\sigma^2 t + \sigma_0^2\Lambda^2(t)}}\right) \end{aligned} \tag{8}$$

Eq. (8) can be achieved by setting $A = 2\Lambda(t)(D - x_0)/\sigma^2 t$, $B = (-D + x_0)/\sigma\sqrt{t}$, $C = -\Lambda(t)/\sigma\sqrt{t}$, and taking the expectation of Eq.(5) with respect to μ.

Based on above lemmas and inference, the reliability functions for the proposed two-phase and multi-phase models can be derived respectively.

B. RELIABILITY ANALYSIS FOR THE TWO-PHASE MODEL

In this subsection, we begin without considering random effects of the parameters for the two-phase model, which means that all parameters of the proposed model are fixed. Based on Lemma 1, the reliability function of the first phase can be easily expressed as

$$R(t) = H(t|x_0, \mu_1, \Lambda_1(\cdot), \sigma_1, D), \quad 0 < t \le \tau_1 \tag{9}$$

For the second phase, the reliability function can be defined by

$$\begin{aligned} R(t) &= \Pr\{T > t\} = \Pr\left\{M^X_{[0,\tau_1]} < D, M^X_{[\tau_1,t]} < D\right\} \\ &= \Pr\left\{M^X_{[0,\tau_1]} < D\right\}\Pr\left\{M^X_{[\tau_1,t]} < D|M^X_{[0,\tau_1]} < D\right\} \\ &= H(\tau_1|x_0, \mu_1, \Lambda_1(\cdot), \sigma_1, D) \\ &\quad \times \int_{-\infty}^{D} H(t - \tau_1|u, \mu_2, \Lambda_2(\cdot), \sigma_2, D) \\ &\quad \times f_{x_{\tau_1}}(u|x_0, M^X_{[0,\tau_1]} < D)du, \; t > \tau_1 \end{aligned} \tag{10}$$

where $M^X_{[0,\tau_1]}$ and $M^X_{[\tau_1,t]}$ denotes the maximum value of $X(t)$ among time period $[0, \tau_1]$ and $[\tau_1, t]$, separately. It is worth noticing that $R(\tau_1)$ is the reliability at change point τ_1.

The conditional PDF $f_{x_{\tau_1}}(u|x_0, M^X_{[0,\tau_1]} < D)$ of the change point state x_{τ_1} can be rewritten as [36]

$$\begin{aligned} & f_{x_{\tau_1}}(u|x_0, M^X_{[0,\tau_1]} < D) \\ &= \int_{-\infty}^{D} f_{x_{\tau_1}}(u|x_0, M^X_{[0,\tau_1]} = v) f_{M^X_{[0,\tau_1]}}(v|x_0, M^X_{[0,\tau_1]} < D)dv \\ &= \int_{-\infty}^{D} \frac{f_{x_{\tau_1}}(u|x_0) f_{M^X_{[0,\tau_1]}}(v|x_0, x_{\tau_1} = u)}{f_{M^X_{[0,\tau_1]}}(v|x_0)} \frac{f_{M^X_{[0,\tau_1]}}(v|x_0)}{\Pr\left\{M^X_{[0,\tau_1]} < D|x_0\right\}} dv \end{aligned}$$

$$= \int_{-\infty}^{D} \frac{f_{x_{\tau_1}}(u|x_0) f_{M^X_{[0,\tau_1]}}(v|x_0, x_{\tau_1}=u)}{\Pr\left\{M^X_{[0,\tau_1]} < D|x_0\right\}} dv$$
$$= \frac{f_{x_{\tau_1}}(u|x_0) \Pr\left\{M^X_{[0,\tau_1]} < D|x_0, x_{\tau_1}=u\right\}}{\Pr\left\{M^X_{[0,\tau_1]} < D|x_0\right\}} \quad (11)$$

Substitute Eq. (11) into Eq. (10), the reliability function of the second phase can be simplified as

$$\begin{aligned} R(t) &= H(\tau_1|x_0, \mu_1, \Lambda_1(\cdot), \sigma_1, D) \\ &\quad \times \int_{-\infty}^{D} H(t-\tau_1|u, \mu_2, \Lambda_2(\cdot), \sigma_2, D) \\ &\quad \times \frac{f_{x_{\tau_1}}(u|x_0) \Pr\left\{M^X_{[0,\tau_1]} < D|x_0, x_{\tau_1}=u\right\}}{H(\tau_1|x_0, \mu_1, \Lambda_1(\cdot), \sigma_1, D)} du \\ &= \int_{-\infty}^{D} H(t-\tau_1|u, \mu_2, \Lambda_2(\cdot), \sigma_2, D) \\ &\quad \times f_{x_{\tau_1}}(u|x_0) \Pr\left\{M^X_{[0,\tau_1]} < D|x_0, x_{\tau_1}=u\right\} du, \ t > \tau_1 \end{aligned} \quad (12)$$

where, based on the property of WP, the form of $f_{x_\tau}(u|x_0)$ can be obtained as

$$f_{x_{\tau_1}}(u|x_0) = \frac{1}{\sqrt{2\pi\sigma_1^2\tau_1}} \exp\left[-\frac{(u-\mu_1\Lambda_1(\tau_1)-x_0)^2}{2\sigma_1^2\tau_1}\right] \quad (13)$$

$\Pr\left\{M^X_{[0,\tau_1]} \le D|x_0, x_{\tau_1}=u\right\}$ can be given by Lemma 2. Therefore, by substituting Eq. (6) and Eq. (13) into Eq. (12), the reliability function of the two-phase model with fixed parameters can be presented as the following Proposition 1.

Proposition 1: Given a generalized two-phase WP degradation model as Eq. (2) with fixed parameters, the reliability function under the concept of FPT can be written as

$$R(t) = \begin{cases} H(t|x_0, \mu_1, \Lambda_1(\cdot), \sigma_1, D), \ 0 < t \le \tau_1 \\ \int_{-\infty}^{D} H(t-\tau_1|u, \mu_2, \Lambda_2(\cdot), \sigma_2, D) \\ \times \left(\frac{1}{\sqrt{2\pi\sigma_1^2\tau_1}} \exp\left[-\frac{(u-\mu_1\Lambda_1(\tau_1)-x_0)^2}{2\sigma_1^2\tau_1}\right]\right) \\ \times \left(1-\exp\left[-\frac{2(u-D)(x_0-D)}{\sigma_1^2\tau_1}\right]\right) du, \ t > \tau_1 \end{cases} \quad (14)$$

In the above derivation, all model parameters are supposed as constants. In the next step, the variability of degradation rates is first considered, and then the distribution function $f_{x_{\tau_1}}(u|x_0)$ can be obtained as

$$f_{x_{\tau_1}}(u|x_0) = \frac{1}{\sqrt{2\pi(\sigma_1^2\tau_1 + \sigma_{1,0}^2\tau_1^2)}} \times \exp\left[-\frac{(u-\mu_{1,0}\Lambda_1(\tau_1)-x_0)^2}{2(\sigma_1^2\tau + \sigma_{1,0}^2\tau_1^2)}\right] \quad (15)$$

Then the reliability function can be further presented by the following Proposition 2.

Proposition 2: Given a generalized two-phase WP degradation model as Eq. (2) with random degradation rates and fixed change-point τ_1, the reliability function under the concept of FPT can be expressed as

$$R(t) = \begin{cases} G(t|x_0, \mu_{1,0}, \sigma_{1,0}^2, \Lambda_1(\cdot), \sigma_1, D), \ 0 < t \le \tau_1 \\ \int_{-\infty}^{D} G(t-\tau_1|u, \mu_{2,0}, \sigma_{2,0}^2, \Lambda_2(\cdot), \sigma_2, D) \\ \times \left(\frac{1}{\sqrt{2\pi(\sigma_1^2\tau_1+\sigma_{1,0}^2\tau_1^2)}} \times \exp\left[-\frac{(u-\mu_{1,0}\Lambda_1(\tau_1)-x_0)^2}{2(\sigma_1^2\tau_1+\sigma_{1,0}^2\tau_1^2)}\right]\right) \\ \times \left(1-\exp\left[-\frac{2(u-D)(x_0-D)}{\sigma_1^2\tau_1}\right]\right) du, \ t > \tau_1 \end{cases} \quad (16)$$

In general, the expressions of reliability functions given by Eqs. (14) and (16) still contain integrals. Fortunately, only univariate integrals need to be calculated, which can be easy to calculate via numerical integration methods.

The above presented results can deal with the reliability evaluation well for the two-phase degradation processes with deterministic change points, when the two-phase features may cause by operation state switches or when we evaluate the reliability for a certain product. In a further step, the randomness of change point is considered for a group of the same products with different change points. Based on the total probability law, the reliability function can be represented by

$$\begin{aligned} R(t) &= \int_{t}^{+\infty} G(t|x_0, \mu_{1,0}, \sigma_{1,0}^2, \Lambda_1(\cdot), \sigma_1, D) f_{\tau_1}(v) dv \\ &\quad + \int_0^t \int_{-\infty}^{D} G(t-v|u, \mu_{2,0}, \sigma_{2,0}^2, \Lambda_2(\cdot), \sigma_2, D) \\ &\quad \times \left(\frac{1}{\sqrt{2\pi(\sigma_1^2 v + \sigma_{1,0}^2 v^2)}} \times \exp\left[-\frac{(u-\mu_{1,0}\Lambda_1(v)-x_0)^2}{2(\sigma_1^2 v+\sigma_{1,0}^2 v^2)}\right]\right) \\ &\quad \times \left(1-\exp\left[-\frac{2(u-D)(x_0-D)}{\sigma_1^2 v}\right]\right) f_{\tau_1}(v) du dv \end{aligned} \quad (17)$$

where $f_{\tau_1}(v)$ denotes the PDF of change point τ_1.

In practical, the integrals in Eq. (17) is intractable due to the randomness of change point, thus a natural way is to use the Monte Carlo method by getting enough change point samples from its distribution. And so on, the reliability for each change point sample can be calculated by Eq. (16), then the reliability can be approximated by the mean value of reliability for all samples.

C. RELIABILITY ANALYSIS FOR THE MULTI-PHASE MODEL

Considering the multi-phase degradation situation, the analytical results in Section 3.2 can be utilized for the first two phase. Similarly, the multi-phase degradation model is considered with no random effects first for simplicity. Then a recursive scheme is adopted; i.e., the reliability of the k-th phase can be derived based on the reliability of the (k-1)-th phase by

$$\begin{aligned} R(t) &= \Pr\{T > t\} = \Pr\left\{M^X_{[0,\tau_{k-1}]} < D, M^X_{[\tau_{k-1},t]} < D\right\} \\ &= \Pr\left\{M^X_{[0,\tau_{k-1}]} < D\right\} \Pr\left\{M^X_{[\tau_{k-1},t]} < D | M^X_{[0,\tau_{k-1}]} < D\right\} \\ &= R(\tau_{k-1}) \int_{-\infty}^{D} H(t-\tau_{k-1}|u, \mu_k, \Lambda_k(\cdot), \sigma_k, D) \\ &\quad \times f_{x_{\tau_{k-1}}}(u|x_0, \bigcap_{i=1}^{k-1} M^X_{[\tau_{i-1},\tau_i]} < D)du, \; t > \tau_{k-1} \end{aligned} \tag{18}$$

where $M^X_{[\tau_{i-1},\tau_i]}$ is the maximum value of $X(t)$ in $[\tau_{i-1}, \tau_i]$ and $R(\tau_{k-1})$ is the reliability at change point τ_{k-1}.

Then the following recursive formulation can also be constructed to represent the change-point state $x_{\tau_{k-1}}$ distribution (19), as shown at the bottom of the page.

By substituting Eq. (19) into Eq. (18), the recursive relationship of reliability function for multi-phase model can be presented by the following Proposition 3.

Proposition 3: Given a generalized multi-phase WP degradation model as Eq. (1) with change-points $\tau_1, \tau_2, \ldots, \tau_{k-1}$, the recursive formulation of reliability function under the concept of FPT can be expressed as (20), shown at the bottom of the page, where, based on the property of WP, the form of $f_{x_{k-1}}(u|x_{\tau_{k-2}} = v)$ can be obtained as

$$\begin{aligned} & f_{x_{\tau_{k-1}}}(u|x_{\tau_{k-2}} = v) \\ &= \frac{1}{\sqrt{2\pi\sigma^2_{k-1}(\tau_{k-1}-\tau_{k-2})}} \\ &\quad \times \exp\left[-\frac{(u-\mu_{k-1}\Lambda_{k-1}(\tau_{k-1}-\tau_{k-2})-v)^2}{2\sigma^2_{k-1}(\tau_{k-1}-\tau_{k-2})}\right] \end{aligned} \tag{21}$$

$\Pr\left\{M^X_{[\tau_{k-2},\tau_{k-1}]} < D | x_{\tau_{k-2}} = v, x_{\tau_{k-1}} = u\right\}$ can be given by Lemma 2. Besides, when $k = 3$,

$$f_{x_{\tau_{k-2}}}(v|x_0, \bigcap_{i=1}^{k-2} M^X_{[\tau_{i-1},\tau_i]} < D) = f_{x_{\tau_1}}(v|x_0, M^X_{[0,\tau_1]} < D),$$

which can be given by Eq. (11). Then, Eq. (20) contains bivariate integrals which can be solved by numerical integration. However, when the number of phases increases, the above PDF becomes a form of high-dimension integrals. We can simplify the PDF via acceptable approximations. Particularly in practical engineering, most of failures occurred in the last one or two phases. Hence, when the number of degradation phases $k > 3$, assuming that the failure probability before change point τ_{k-2} is small, i.e.,

$$\Pr\left\{\bigcap_{i=1}^{k-2} M^X_{[\tau_{i-1},\tau_i]} < D\right\} \approx 1,$$

we have

$$f_{x_{\tau_{k-2}}}(v|x_0, \bigcap_{i=1}^{k-2} M^X_{[\tau_{i-1},\tau_i]} < D) \approx f_{x_{\tau_{k-2}}}(v),$$

which is the unconditional PDF of $x_{\tau_{k-2}}$. Then Eq. (20) can be calculated by numerical integration. When unit-to-unit variability is considered, we can utilize the law of the total probability similar to Eq. (17). This section provides a feasible solution to evaluate the reliability of proposed multi-phase model. In current study, the two-phase model is focused as a typical representative for frequently faced two-phase degradation. Next, the identification of model parameters is introduced.

IV. STATISTICAL INFERENCE

A. PARAMETER ESTIMATION

To evaluate the reliability, model parameters have to be estimated based on measurements. A maximum likelihood method is proposed. Without loss of generality, take the two-phase model for example to demonstrate the estimation procedure. All products are assumed to be inspected at a same time sequence $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)'$ (n is the total observation number). It is worthwhile noticing that this situation can be

$$\begin{aligned} & f_{x_{\tau_{k-1}}}(u|x_0, \bigcap_{i=1}^{k-1} M^X_{[\tau_{i-1},\tau_i]} < D) \\ &= \int_{-\infty}^{D} f_{x_{\tau_{k-1}}}(u|x_{\tau_{k-2}} = v, M^X_{[\tau_{k-2},\tau_{k-1}]} < D) \times f_{x_{\tau_{k-2}}}(v|x_0, \bigcap_{i=1}^{k-2} M^X_{[\tau_{i-1},\tau_i]} < D)dv \\ &= \int_{-\infty}^{D} \frac{f_{x_{\tau_{k-1}}}(u|x_{\tau_{k-2}} = v)\Pr\left\{M^X_{[\tau_{k-2},\tau_{k-1}]} < D|x_{\tau_{k-2}} = v, x_{\tau_{k-1}} = u\right\}}{H(\tau_{k-1}-\tau_{k-2}|v, \mu_{k-1}, \Lambda_{k-1}(\cdot), \sigma_{k-1}, D)} \times f_{x_{\tau_{k-2}}}(v|x_0, \bigcap_{i=1}^{k-2} M^X_{[\tau_{i-1},\tau_i]} < D)dv \end{aligned} \tag{19}$$

$$\begin{aligned} R(t) = R(\tau_{k-1}) & \int_{-\infty}^{D} H(t-\tau_{k-1}|u, \mu_{k-1}, \Lambda_{k-1}(\cdot), \sigma_{k-1}, D) \\ & \times \int_{-\infty}^{D} \frac{f_{x_{\tau_{k-1}}}(u|x_{\tau_{k-2}} = v)\Pr\left\{M^X_{[\tau_{k-2},\tau_{k-1}]} < D|x_{\tau_{k-2}} = v, x_{\tau_{k-1}} = u\right\}}{H(\tau_{k-1}-\tau_{k-2}|v, \mu_{k-1}, \Lambda_{k-1}(\cdot), \sigma_{k-1}, D)} \\ & \times f_{x_{\tau_{k-2}}}(v|x_0, \bigcap_{i=1}^{k-2} M^X_{[\tau_{i-1},\tau_i]} < D)dvdu, \; t > \tau_{k-1} \end{aligned} \tag{20}$$

easily extended to the circumstance involving different test time sequences among units. Test data are denoted as $x_{ij} = x_i(t_j)$ where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and m, n are total number of test units and time points, respectively.

To construct the log-likelihood function (log-LF) of unknown parameters, the two-phase test datasets have to be discussed separately based on the change point position. For degradation sequence $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})'$ of unit i, a most general circumstance lies in that the change point τ_{1i} locates between two sequential test time points (denoted as t_{c_i} and t_{c_i+1}) which can vary from unit to unit, $i = 1, 2, \ldots, m$. Consequently, three segments have to be considered for degradation data $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})'$. The first segment is comprised of the observations before test time point t_{c_i}, denoted as $\boldsymbol{x}_{1i} = (x_{i1}, x_{i2}, \ldots, x_{ic_i})'$. Then the measurement after test time point t_{c_i+1} are considered in the second segment. Since different degradation regulations are demonstrated because of the two-phase property, the measurement sequence of the second segment incorporated in the log-LF construction is denoted as $\boldsymbol{x}_{2i} = \big(x_{i(c_i+2)} - x_{i(c_i+1)}, x_{i(c_i+3)} - x_{i(c_i+1)}, \ldots, x_{in} - x_{i(c_i+1)}\big)'$. Finally, the third segment is the performance increment from t_{c_i} to t_{c_i+1}, denoted as $x_{3i} = x_{i(c_i+1)} - x_{ic_i}$.

For the first segment, define $\boldsymbol{L}_{1i} = \big(\Lambda_1(t_1), \Lambda_1(t_2), \ldots, \Lambda_1(t_{c_i})\big)'$. According to the properties of WP, $\boldsymbol{x}_{1i}$ follows a multivariate normal distribution with mean $\boldsymbol{\mu}_{1i} = \mu_{1,0}\boldsymbol{L}_{1i}$ and covariance $\boldsymbol{\Sigma}_{1i} = \sigma_1^2\boldsymbol{P}_{1i} + \sigma_{1,0}^2\boldsymbol{L}_{1i}\boldsymbol{L}_{1i}'$, where

$$\boldsymbol{P}_{1i} = \begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_{c_i} \end{bmatrix}.$$

For the second segment, define $\boldsymbol{L}_{2i} = \big(\Lambda_2(t_{c_i+1} - \tau_{1i}) - \Lambda_2(t_{c_i} - \tau_{1i}), \Lambda_2(t_{c_i+2} - \tau_{1i}) - \Lambda_2(t_{c_i} - \tau_{1i}), \ldots, \Lambda_2(t_n - \tau_{1i}) - \Lambda_2(t_{c_i} - \tau_{1i})\big)'$. Also, $\boldsymbol{x}_{2i}$ follows a multivariate normal distribution with mean $\boldsymbol{\mu}_{2i} = \mu_{2,0}\boldsymbol{L}_{2i}$ and covariance $\boldsymbol{\Sigma}_{2i} = \sigma_2^2\boldsymbol{P}_{2i} + \sigma_{2,0}^2\boldsymbol{L}_{2i}\boldsymbol{L}_{2i}'$, where

$$\boldsymbol{P}_{2i} = \begin{bmatrix} t_{c_i+2}-t_{c_i+1} & t_{c_i+2}-t_{c_i+1} & \cdots & t_{c_i+2}-t_{c_i+1} \\ t_{c_i+2}-t_{c_i+1} & t_{c_i+3}-t_{c_i+1} & \cdots & t_{c_i+3}-t_{c_i+1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{c_i+2}-t_{c_i+1} & t_{c_i+3}-t_{c_i+1} & \cdots & t_n-t_{c_i+1} \end{bmatrix}.$$

For the third segment, because change point τ_i locates between t_{c_i} and t_{c_i+1}, increment x_{3i} is normally distributed with mean $\mu_{3i} = \mu_{1,0}\left(\Lambda_1(\tau_{1i}) - \Lambda_1(t_{c_i})\right) + \mu_{2,0}\Lambda_2(t_{c_i+1} - \tau_{1i})$ and variance $\sigma_{3i}^2 = \sigma_1^2(\tau_{1i} - t_{c_i}) + \sigma_2^2(t_{c_i+1} - \tau_{1i}) + \sigma_{1,0}^2\left(\Lambda_1(t_{c_i+1}) - \Lambda_1(\tau_{1i})\right)^2 + \sigma_{2,0}^2\Lambda_2^2(t_{c_i+1} - \tau_{1i})$.

To facilitate the estimation, we re-parameterize the parameters by $\tilde{\sigma}_1^2 = \sigma_1^2/\sigma_{1,0}^2$, $\tilde{\sigma}_2^2 = \sigma_2^2/\sigma_{2,0}^2$, $\tilde{\boldsymbol{\Sigma}}_{1i} = \boldsymbol{\Sigma}_{1i}/\sigma_{1,0}^2 = \tilde{\sigma}_1^2\boldsymbol{P}_{1i} + \boldsymbol{L}_{1i}\boldsymbol{L}_{1i}'$ and $\tilde{\boldsymbol{\Sigma}}_{2i} = \boldsymbol{\Sigma}_{2i}/\sigma_{2,0}^2 = \tilde{\sigma}_2^2\boldsymbol{P}_{2i} + \boldsymbol{L}_{2i}\boldsymbol{L}_{2i}'$. Let $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ denote the unknown parameter vector in $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$, respectively. Then we define $\boldsymbol{\Theta}$ as the unknown parameters vector to be estimated. The log-LF of i-th unit can be constructed as

$$\begin{aligned}
&\ln L(\mu_{1,0}, \sigma_{1,0}^2, \sigma_1^2, \boldsymbol{\theta}_1, \mu_{2,0}, \sigma_{2,0}^2, \sigma_2^2, \boldsymbol{\theta}_2, \tau_{1i}|\boldsymbol{x}_i) \\
&\quad = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\left(\ln\left|\tilde{\boldsymbol{\Sigma}}_{1i}\right| + c_i\ln(\sigma_{1,0}^2)\right) \\
&\quad\quad - \frac{1}{2\sigma_{1,0}^2}(\boldsymbol{x}_{1i} - \boldsymbol{\mu}_{1i})'(\tilde{\boldsymbol{\Sigma}}_{1i})^{-1}(\boldsymbol{x}_{1i} - \boldsymbol{\mu}_{1i}) \\
&\quad\quad - \frac{1}{2}\left(\ln\left|\tilde{\boldsymbol{\Sigma}}_{2i}\right| + (n - c_i - 1)\ln(\sigma_{2,0}^2)\right) \\
&\quad\quad - \frac{1}{2\sigma_{2,0}^2}(\boldsymbol{x}_{2i} - \boldsymbol{\mu}_{2i})'(\tilde{\boldsymbol{\Sigma}}_{2i})^{-1}(\boldsymbol{x}_{2i} - \boldsymbol{\mu}_{2i}) \\
&\quad\quad - \frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\sigma_{3i}^2 - \frac{(x_{3i} - \mu_{3i})^2}{2\sigma_{3i}^2} \qquad (22)
\end{aligned}$$

Therefore, the maximum likelihood estimation (MLE) of unknown parameters $\boldsymbol{\Theta}$ can be derived by

$$\hat{\boldsymbol{\Theta}} = \arg\max_{\boldsymbol{\Theta}} \sum_{i=1}^{m} \ln L(\mu_{1,0}, \sigma_{1,0}^2, \sigma_1^2, \boldsymbol{\theta}_1, \mu_{2,0}, \sigma_{2,0}^2, \sigma_2^2, \boldsymbol{\theta}_2, \tau_{1i}|\boldsymbol{x}_i) \qquad (23)$$

By taking the first order partial derivatives of $\Sigma_{i=1}^{m}\ln L(\mu_{1,0}, \sigma_{1,0}^2, \sigma_1^2, \boldsymbol{\theta}_1, \mu_{2,0}, \sigma_{2,0}^2, \sigma_2^2, , \boldsymbol{\theta}_2, \tau_{1i}|\boldsymbol{x}_i)$ with respect to $\mu_{1,0}$, $\mu_{2,0}$, $\sigma_{1,0}^2$, $\sigma_{2,0}^2$ and let them equal to zero, we have

$$\begin{aligned}
\mu_{1,0} &= \frac{\sum_{i=1}^{m}\boldsymbol{L}_{1i}'(\tilde{\sigma}_1^2\boldsymbol{P}_{1i} + \boldsymbol{L}_{1i}\boldsymbol{L}_{1i}')^{-1}\boldsymbol{x}_{1i}}{\sum_{i=1}^{m}\boldsymbol{L}_{1i}'(\tilde{\sigma}_1^2\boldsymbol{P}_{1i} + \boldsymbol{L}_{1i}\boldsymbol{L}_{1i}')^{-1}\boldsymbol{L}_{1i}} \\
\mu_{2,0} &= \frac{\sum_{i=1}^{m}\boldsymbol{L}_{2i}'(\tilde{\sigma}_2^2\boldsymbol{P}_{2i} + \boldsymbol{L}_{2i}\boldsymbol{L}_{2i}')^{-1}\boldsymbol{x}_{2i}}{\sum_{i=1}^{m}\boldsymbol{L}_{2i}'(\tilde{\sigma}_2^2\boldsymbol{P}_{2i} + \boldsymbol{L}_{2i}\boldsymbol{L}_{2i}')^{-1}\boldsymbol{L}_{2i}} \qquad (24)
\end{aligned}$$

and

$$\begin{aligned}
&\sigma_{1,0}^2 \\
&= \frac{\sum_{i=1}^{m}(\boldsymbol{x}_{1i} - \mu_{1,0}\boldsymbol{L}_{1i})'(\tilde{\sigma}_1^2\boldsymbol{P}_{1i} + \boldsymbol{L}_{1i}\boldsymbol{L}_{1i}')^{-1}(\boldsymbol{x}_{1i} - \mu_{1,0}\boldsymbol{L}_{1i})}{\sum_{i=1}^{m}c_i} \\
&\sigma_{2,0}^2 \\
&= \frac{\sum_{i=1}^{m}(\boldsymbol{x}_{2i} - \mu_{2,0}\boldsymbol{L}_{2i})'(\tilde{\sigma}_2^2\boldsymbol{P}_{2i} + \boldsymbol{L}_{2i}\boldsymbol{L}_{2i}')^{-1}(\boldsymbol{x}_{2i} - \mu_{2,0}\boldsymbol{L}_{2i})}{\sum_{i=1}^{m}(n - c_i - 1)} \qquad (25)
\end{aligned}$$

By substituting Eq. (24) and Eq. (25) into Eq. (23), we can obtain the profile log-LF of $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\tilde{\sigma}_1^2$, $\tilde{\sigma}_2^2$, τ_{1i}, and then the corresponding estimates can be obtained by maximizing this profile log-LF. After that the estimates of $\mu_{1,0}$, $\mu_{2,0}$, $\sigma_{1,0}^2$, $\sigma_{2,0}^2$ can be obtained by substituting estimation results $\hat{\boldsymbol{\theta}}_1$, $\hat{\boldsymbol{\theta}}_2$, $\hat{\tilde{\sigma}}_1^2$, $\hat{\tilde{\sigma}}_2^2$, $\hat{\tau}_{1i}$ in Eq. (24) and Eq. (25). It is reasonable to consider estimates $\hat{\tau}_{11}, \hat{\tau}_{12}, \cdots, \hat{\tau}_{1m}$ as the observations of the random point τ, and hence the corresponding distributions can be obtained by statistical analysis. Finally, with the identified parameters, the reliability of a batch of devices can be assessed.

In real applications, however, it is still very tough for direct constrained optimization of the profile log-LF function. The genetic algorithm (GA) provides a possible solution for this problem, and it is adopted in the current study. A more comprehensive overview of GA can be found in Ref. [44].

B. INITIAL GUESSES

When GA is applied to numerically maximize the profile log-LF, reasonable initial guesses for decision variables are necessary. First, the rough estimates of change point are obtained separately for each test unit. Then, motivated by the three-step procedure given in Refs. [12] and [45], a similar method is developed to obtain an educated guess for the range of each decision variable. The detailed procedure is as follows.

1) For i-th test unit, suppose a series of rough change point locations as $\left\{\tau_{1i}^* = t_3, \tau_{1i}^* = t_3, \cdots, \tau_{1i}^* = t_{n-3}\right\}$ and construct a corresponding series of two-phase deterioration models by maximizing a corresponding series of log-LF $\left\{\text{log-LF}_{i|\tau_{1i}^*=t_3}, \text{log-LF}_{i|\tau_{1i}^*=t_4}, \cdots, \text{log-LF}_{i|\tau_{1i}^*=t_{n-3}}\right\}$. Then the rough estimate of the change point is τ_{1i}^* corresponding to maximum value among the log-LF sequence. Then the monitoring data sequence $\boldsymbol{x}_i$ can be divided into two subsequences by this rough result τ_{1i}^*, denoted as $\boldsymbol{x}_{1i}^* = (x_{i1}, x_{i2}, \ldots, x_{ic_i^*})'$ and $\boldsymbol{x}_{2i}^* = \left(x_{i(c_i^*+1)} - x_{ic_i^*}, x_{i(c_i^*+2)} - x_{ic_i^*}, \ldots, x_{in} - x_{ic_i^*}\right)'$, where c_i^* is the serial number of the rough result τ_{1i}^*.

2) Based on the least square method, the rough estimates ($\mu_{11}, \mu_{12}, \ldots, \mu_{1m}$ and $\boldsymbol{\theta}_1$ for phase 1, $\mu_{21}, \mu_{22}, \ldots, \mu_{2m}$ and $\boldsymbol{\theta}_2$ for phase 2), can be obtained by minimizing the mean squared error (MSE), respectively.

$$\text{MSE}_1 = \sum_{i=1}^{m} \left(\boldsymbol{x}_{1i}^* - \mu_{1i}\boldsymbol{L}_{1i}^*\right)' \left(\boldsymbol{x}_{1i}^* - \mu_{1i}\boldsymbol{L}_{1i}^*\right) \tag{26}$$

$$\text{MSE}_2 = \sum_{i=c_i^*+1}^{m} \left(\boldsymbol{x}_{2i}^* - \mu_{2i}\boldsymbol{L}_{2i}^*\right)' \left(\boldsymbol{x}_{2i}^* - \mu_{2i}\boldsymbol{L}_{2i}^*\right) \tag{27}$$

where $\boldsymbol{L}_{1i}^* = \left(\Lambda_1(t_1), \Lambda_1(t_2), \ldots, \Lambda_1(\tau_{1i}^*)\right)'$ and $\boldsymbol{L}_{2i}^* = \left(\Lambda_2(t_{c_i^*+1} - \tau_{1i}^*), \Lambda_2(t_{c_i^*+2} - \tau_{1i}^*), \ldots, \Lambda_2(t_n - \tau_{1i}^*)\right)'$.

3) The rough estimates of $\mu_{1,0}$, $\sigma_{1,0}^2$ and $\mu_{2,0}$, $\sigma_{2,0}^2$, can be calculated by fitting the estimations $\mu_{11}, \mu_{12}, \ldots, \mu_{1m}$ and $\mu_{21}, \mu_{22}, \ldots, \mu_{2m}$, respectively.

4) Based on the estimates of $\mu_{1,0}$, $\sigma_{1,0}^2$, $\boldsymbol{\theta}_1$ for phase 1, and $\mu_{2,0}$, $\sigma_{2,0}^2$, $\boldsymbol{\theta}_2$ for phase 2, the rough estimates of σ_1^2 and σ_2^2 can be obtained by maximizing the profile log-LF of σ_1^2 and σ_2^2.

Therefore, the range of each decision variable for maximizing the log-LF via GA is determined.

V. SIMULATION STUDY

In this section, a comprehensive simulation study is conducted to verify the reasonability of the reliability function proposed in Section 3. Without loss of generality, two cases for two-phase situation (defined as Case 1) and three-phase circumstance (as a representative of three-phase situation and defined as Case 2) are presented. According to empirical studies, power law can be considered as a commonly encountered degradation regulation [7]. To this end, the transformed time scale for the k-th phase is supposed as $\Lambda_k(t) = t^{b_k}$, which becomes a linear one when $b_k = 1$.

Case 1 incorporates a nonlinear first phase and a linear second phase with both random drift and random changing point. The parameters are set as $\mu_{1,0} = 10$, $\sigma_{1,0} = 0.1$, $b_1 = 0.5$, $\sigma_1 = 2$, $\mu_{2,0} = 1$, $\sigma_{2,0} = 0.1$, $b_2 = 1$, $\sigma_2 = 1$, $\alpha_1 = 400$, $\beta_1 = 0.01$ and $D = 20$.

Case 2 involves a nonlinear first phase, a linear second phase and a nonlinear third phase considering random effects. The phase duration of the second phase is also assumed to follow gamma distribution. The parameters are predefined as $\mu_{1,0} = 10$, $\sigma_{1,0} = 0.1$, $b_1 = 0.5$, $\sigma_1 = 1$, $\mu_{2,0} = 0.5$, $\sigma_{2,0} = 0.05$, $b_2 = 1$, $\sigma_2 = 1$, $\mu_{3,0} = 1.5$, $\sigma_{3,0} = 0.1$, $b_3 = 1.5$, $\sigma_3 = 2$, $\alpha_1 = 400$, $\beta_1 = 0.01$, $\alpha_2 = 400$, $\beta_2 = 0.01$ and $D = 20$.

Based on the datasets generated via above simulation models, reliability curves can be obtained by the Propositions proposed in Section 3. To illustrate the reasonability of the analytical results constructed in the current study, a numerical method is adopted as a reference. It is an approximation approach based on the Euler-Maruyama discretization policy, which has been commonly applied in complex reliability analysis [46]. The following four-step simulation procedure is involved to generate a number of failure times based on the above preset model parameters (including the initial degradation value x_0, change point distribution parameters α_1 and β_1, drift coefficient distribution parameters $\mu_{1,0}$, $\sigma_{1,0}$, $\mu_{2,0}$, $\sigma_{2,0}$, parameters b_1, b_2 for two phases, drift coefficient σ_1^2, σ_2^2 and failure thresholds D), and then the reliability can be approximated.

1) Initialize the total number of sampling paths 10000 and discretization step Δt.
2) For the i-th sampling path, generate parameters τ_1, μ_1, μ_2 based on the distributions, and let $j = 0$ indicating that it degrades from the initial time point zero.
3) At time point $j\Delta t$, calculate $X_{(j+1)\Delta t}^{(i)}$ from $X_{j\Delta t}^{(i)}$ using a Euler approximation for sampling path i. If $X_{(j+1)\Delta t}^{(i)} \geq D$ holds for the first time, the FPT of the i-th sampling path can be obtained as $T^{(i)} = (j+1)\Delta t$, and sampling path i terminated. Otherwise, set $j = j + 1$ and continue the i-th sampling path until $X_{(j+1)\Delta t}^{(i)}$ exceeds D. As such, set $i = i + 1$ and go back to Step 2 until $i = 10000$.
4) Repeat Steps 2-3 until 10000 FPTs are simulated; i.e., $\boldsymbol{T} = \left\{T^{(1)}, T^{(2)}, \cdots, T^{(10000)}\right\}$.

5) The reliability at time point $j\Delta t$ can be calculated by

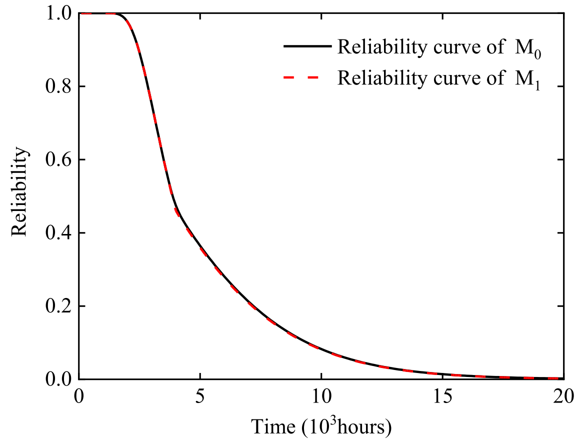
$$R(j\Delta t) \approx \frac{N_{j\Delta t}}{10000} \quad (28)$$

where $N_{j\Delta t}$ is the number of $T^{(i)}$ satisfying $T^{(i)} > j\Delta t$ at time point $j\Delta t$.

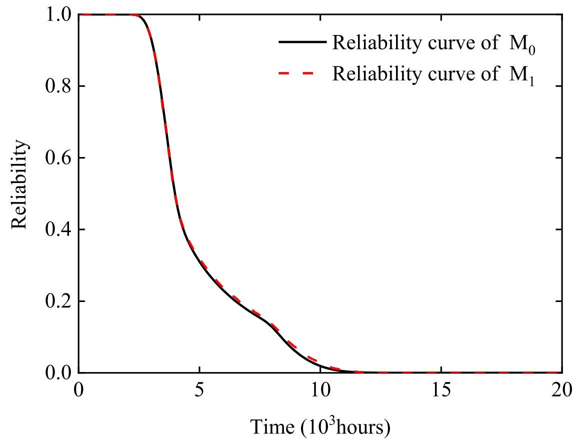
It can be seen from the above numerical simulation procedure, that the reliability curve is approximated by plotting the reliability at a series of discrete time points with a constant interval Δt. Consequently, a considerabye small Δt is necessary to guarantee the reasonability and accuracy of the numerical result. From this viewpoint, the discretization step size is defined as Δt=0.01 in 10^3hours. It can be seen that the FPTs $\boldsymbol{T} = \{T^{(1)}, T^{(2)}, \cdots, T^{(10000)}\}$ are realizations of the true FPT, and the histogram can reasonably approach the PDF of the true FPT because the sample size 10000 is large enough. To this end, the above numerical approach can demonstrate the reasonability of the proposed method.

To illustrate a better understanding, the reliability curves obtained by the proposed method constructed in Section 3 (defined as M_0) and the reference numerical approach described above (defined as M_1) are shown in Fig. 3, where Fig. 3(a) illustrates the two-phase situation and Fig. 3(b) demonstrates the three-phase circumstance.

From Fig. 3, one can see that the reliability curves derived from the two methods fit quite well for both the two cases. According to the literature, Pearson Correlation [47] is adopted to further quantify the agreement. Based on correlation analysis, the two-phase case produces a correlation coefficient of 0.99998, and the three-phase case gives a correlation coefficient of 0.99994. Meanwhile, the p values of both cases are almost zero. These results demonstrate that the curves obtained from the two methods fit quite well; i.e., the constructed reliability function can be numerically verified.



(a)



(b)

FIGURE 3. Comparative reliability curves for simulation study (a) Two-phase situation (b) Three-phase situation.

VI. EMPIRICAL RESULTS

In this section, the two motivating examples shown in Fig. 1 and Fig. 2 are reinvestigated to verify the validity and effectiveness of the proposed method (denoted as M_0) in real applications. The failure threshold of accelerometers can be predefined as $D = 300$ [25] and the failure threshold of gyros can be pre-set as $D = 0.6$ [9].

A. APPLICATION TO ACCELEROMETERS

Due to the influences of material degradation and residual stress changes in machining and manufacturing process, scale factor of accelerometer changes with time, which influence storage life and reliability of accelerometers [24]. To investigate the behaviour of the accelerometers, a degradation test was implemented [25]. 6 units were tested for 384 hours at 60 °C, and the measurements are shown in Fig. 1.

From Fig. 1, it can be found that the degradation trajectories show an obvious two-phase property. Thus, Ref. [25] utilized a two-phase linear WP model (denoted as reference model M_2) to fit the degradation data, which has also been studied by Wen *et al.* [48], Yan *et al.* [20] and Feng *et al.* [49]. The main difference between M_0 and M_2 is the time scale function, where M_0 supposes $\Lambda_1(t) = t^{b_1}$, $\Lambda_2(t - \tau_1) = t - \tau_1$, and M_2 assumes $\Lambda_1(t) = t$, $\Lambda_2(t - \tau_1) = t - \tau_1$. To further emphasize the necessity of considering the two-phase property, the single-phase nonlinear model (denoted as reference approach M_3 which was utilized by Si *et al.* [9], Wang *et al.* [10], [11], and Li *et al.* [12]) is also adopted as a reference. It can be represented as $X = \mu\Lambda(t) + \sigma B(t)$, $\mu \sim N(\mu_0, \sigma_0^2)$, $\Lambda(t) = t^b$.

It is worth noticing that, the proposed method M_0 and the reference approach M_2 both consider the two-phase property and the unit specific characteristics, and the difference lies in that M_0 can concern the nonlinearity while M_2 cannot. Meanwhile, models M_0 and M_3 both can consider the nonlinearity of the degradation procedure, and the key difference lies in that M_0 can concern the phase variability while M_3 cannot.

The model fitting goodness and accuracy of lifetime indexes are two main considerations for comparisons. To demonstrate the fitting goodness, log-LF and the corresponding Akaike information criterion (AIC) values are

TABLE 1. Comparative results of model parameters and fitting goodness for accelerometers.

Model	Phase	Unknown parameters						log-LF	AIC
		$\Lambda_k(\cdot)/\Lambda(\cdot)$	$\mu_{k,0}/\mu_0$	$\sigma_{k,0}^2/\sigma_0^2$	σ_k^2/σ^2	α_1	β_2		
M_0	1st	$t^{0.6963}$	7.180	2.627×10^{-3}	6.951	49.09	1.470	-522.6	1084.2
	2nd	$(t-\tau_1)$	0.1575	4.399×10^{-4}	3.446				
M_2	1st	t	2.054	$1{,}846\times10^{-3}$	15.74	41.41	1.605	-532.1	1100.2
	2nd	$(t-\tau_1)$	0.1531	4.579×10^{-4}	3.534				
M_3	/	$t^{0.4283}$	16.45	2.438×10^{-2}	5.832	/	/	-553.1	1114.2

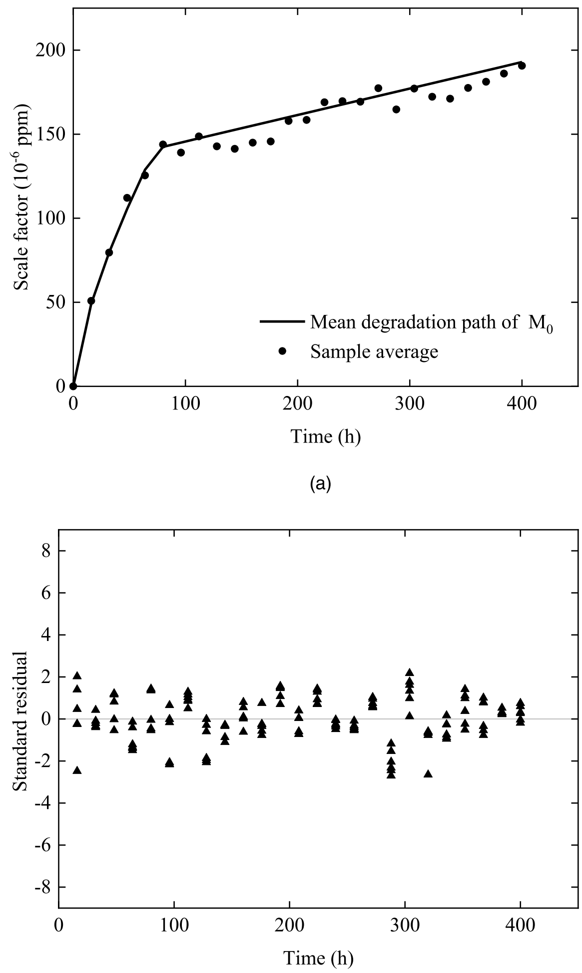
derived, where AIC can be defined as

$$\text{AIC} = -2 \times \{\max(\log - \text{LF})\} + 2q \tag{29}$$

where q is the number of unknow parameters in the adopted model. A larger log-LF and a lower AIC indicate a better fitting goodness. Comparative results including parameter estimations and fitting goodness indexes are summarized in Table 1.

From Table 1, one can see that the proposed method M_0 illustrates a best fitting goodness in both terms of log-LF and AIC. Based on the unknown parameter results derived by M_0 and M_2, each of μ_{k0}, σ_{k0}^2, σ_k^2 $(k = 1, 2)$ shows significant difference between the two values corresponding to the two phases. This demonstrates the necessity to construct a two-phase degradation model. It can then be understood why reference model M_3 derives a poorest fitting. Furthermore, the similar estimations for α and β from methods M_0 and M_2 illustrate the existence of the change point. Meanwhile, results of parameter b_k (for M_0) and b (for M_3) illustrate a clear understanding of the nonlinearity property. Then it can also be understood why M_0 can guarantee a better fitting than reference approach M_2. To this end, the constructed methodology M_0 demonstrates a best modeling reasonability because it can rationally depict the two-phase property and phase-varying nonlinearity characteristic.

Furthermore, the mean degradation curve derived by M_0 is shown in Fig. 4(a). One can observe that the mean degradation curve estimated by M_0 matches the sample average quite well. From this viewpoint, M_0 can give a better description of the first phase, and this indicates the reasonability of the nonlinear assumption. Meanwhile, the standard residuals over time is further given in Fig. 4(b). Consequently, the constructed model M_0 can be considered as reasonable.



(a)

(b)

FIGURE 4. (a) Mean degradation path of accelerometers derived by M_0. (b) Standard residual plot of accelerometers by M_0.

TABLE 2. Lifetime estimations of accelerometers by different models / hours.

Model	M_0		M_2		M_3	
Lifetime indexes	$t_{0.5}$	$t_{0.1}$	$t_{0.5}$	$t_{0.1}$	$t_{0.5}$	$t_{0.1}$
	1015.0	623.7	1063.3	615.6	825.6	463.3

As previous discussed, median life $t_{0.5}$ and FTD percentile $t_{0.1}$ are important indexes of great concern to support maintenance schedules. The results from three methods M_0, M_2 and M_3 are obtained and listed in Table 2. From the rationality and validity of the proposed method M_0 demonstrated above, the estimations from M_0 can be considered as more reasonable and reliable. Reference approach M_3 gives a lowest life prediction where both $t_{0.5}$ and $t_{0.1}$ are far lower than those from M_0 and M_2. This is because the initial rapid degradation (nonlinear fast deterioration in phase 1) gives a significant impact on the single-stage modeling of M_3 and then leads to an excessively conservative life prediction. Meanwhile, the difference between the results of M_0 and M_2 is caused by the phase-varying nonlinearity. To demonstrate an intuitive

TABLE 3. Comparative results of model parameters and fitting goodness for gyros.

Model	Phase	Unknown parameters						log-LF	AIC
		$\Lambda_k(\cdot)/\Lambda(\cdot)$	$\mu_{k,0}/\mu_0$	$\sigma_{k,0}^2/\sigma_0^2$	σ_k^2/σ^2	α_1	β_2		
M_0	1st	t	2.860×10^{-2}	1.345×10^{-14}	1.196×10^{-3}	147.2	7.354×10^{-2}	37.93	-53.86
	2nd	$(t-\tau_1)^{11.5}$	3.854×10^{-13}	8.690×10^{-54}	6.382×10^{-3}				
M_3	/	$t^{18.09}$	2.939×10^{-25}	7.469×10^{-50}	6.571×10^{-2}	/	/	28.38	-48.75

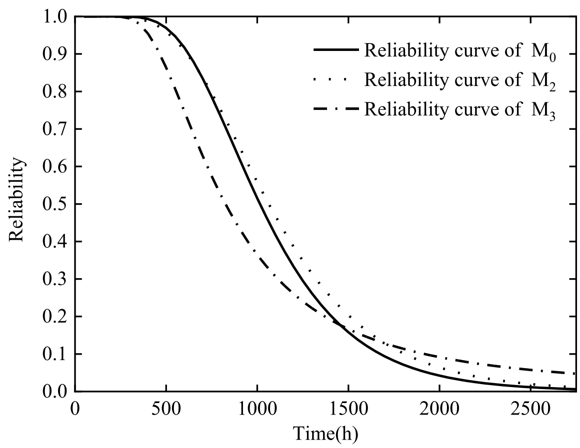


FIGURE 5. Comparative reliability curves of accelerometers by three models M_0, M_2 and M_3.

understanding, Fig. 5 shows the reliability curves derived by the adopted three methods M_0, M_2 and M_3.

B. DEGRADATION ANALYSIS FOR GYROS

To give a better understanding that phase-varying nonlinearity commonly exist in practical engineering and also to further demonstrate the efficiency of the proposed method, the drift coefficient in an inertial navigation platforms given by Si [9] is utilized for verification. The inertial platform is a key component in the inertial navigation systems of weapon systems and space equipment. Its operating state has a direct influence on navigation precision. The degradation data, including 5 tested items (9 measurements for each item) were obtained from inertial platforms' precision tests, where the conditions were similar to a field setting. The investigations are shown in Fig. 2.

Because the two-phase linear model M_2 does not match the degradation paths apparently, only the nonlinear WP model M_3 is adopted to fit the degradation dataset for comparison. The proposed parameter estimation method is applied, and the MLE estimations of model parameters and fitting goodness are obtained and summarized in Table 3.

From Table 3, it is obvious that model M_0 derives a better fitting in both terms of Log-LF and AIC. Although the reference model M_3 adopts a nonlinear function to describe the degradation process, it cannot perform well without considering the phase-varying nonlinearity. The fitting goodness can be also shown in Fig. 6, which shows that the mean degradation path of M_0 matches the sample average better than M_3. And the residual plots over time for model M_0 (shown in Fig. 6) demonstrates that it can appropriately describe the drift of gyros.

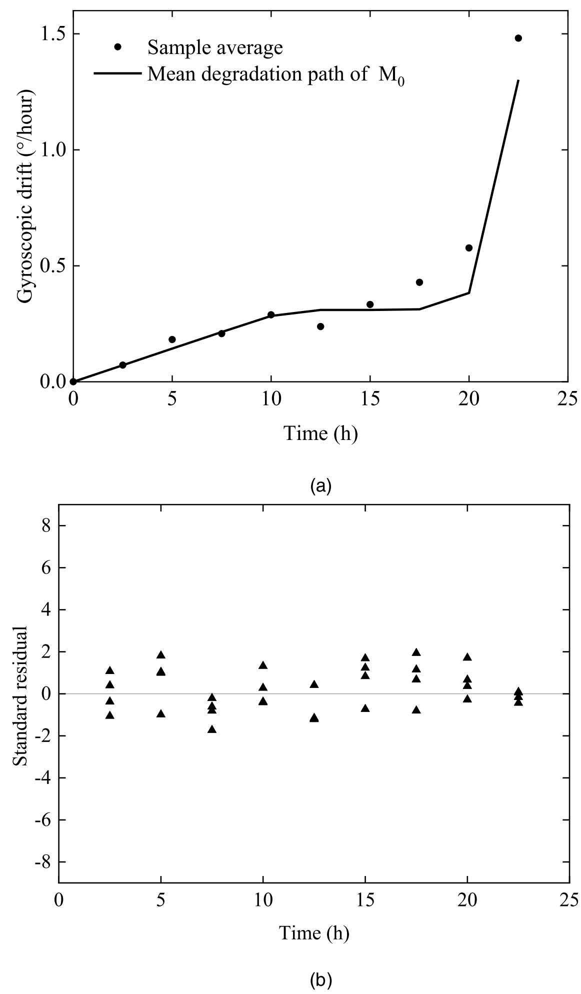


FIGURE 6. (a) Mean degradation path of gyros derived by M_0. (b) Standard residual plot of gyros by M_0.

The results of median life $t_{0.5}$ and FTD percentile $t_{0.1}$ from methods M_0 and M_3 are obtained and listed in Table 4, and the corresponding reliability curves are shown in Fig. 7. From the rationality and validity of the proposed method M_0 demonstrated above, the estimations from M_0 can be considered as more reasonable and reliable. Based on the experiment, the real $t_{0.5}$ is about 21.5 hours. Then the relative errors for

TABLE 4. Lifetime estimations of gyros by different models / hours.

Model	M_0		M_3	
Lifetime indexes	$t_{0.5}$	$t_{0.1}$	$t_{0.5}$	$t_{0.1}$
	21.15	13.75	23.74	20.35
$t_{0.5}$ prediction relative error	1.628%		10.41%	

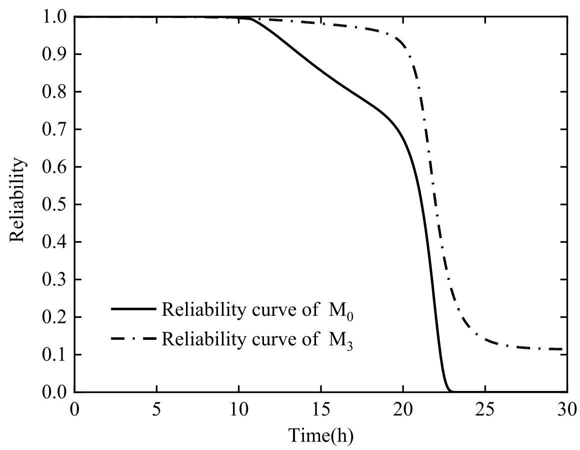


FIGURE 7. Comparative reliability curves of gyros by two models M_0 and M_3.

lifetime prediction can be obtained. The proposed method M_0 can give a more accurate lifetime estimation because the multi-phase variety can be considered.

VII. CONCLUSION

Motivated by the degradation property of accelerometers and gyros, a generalized two-phase degradation model which can properly describe the phase-varying nonlinearity characteristic is proposed, which can be extended to multi-phase one if necessary. Unit-to-unit heterogeneity is depicted by randomly distributed model parameters to demonstrate the unit specific properties. Then an MLE method is constructed to effectively solve the parameter estimation problem. Thereafter, a novel analytical reliability analysis method is proposed, which can give theoretical reliability solutions for multi-phase degradations.

The modeling reasonability and reliability assessment rationality of the proposed method are demonstrated via a comprehensive simulation study and two real applications. The widely applied two-phase linear and single-phase nonlinear model are adopted as references to demonstrate a better understanding. Based on the results, the proposed method can guarantee a reasonable modeling and an effective reliability inference.

Regarding future research, one issue is worth noticing. In real application, sometimes measurement errors are inevitable because of imperfect inspection, thus the multi-phase degradation with measurement errors needs to be studied.

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