Impulsive Control Via Variable Impulsive Perturbations on a Generalized Robust Stability for Cohen–Grossberg Neural Networks With Mixed Delays

JINDE CAO [1,2], (Fellow, IEEE), TRAYAN STAMOV[3], SOTIR SOTIROV[4],
EVDOKIA SOTIROVA[5], AND IVANKA STAMOVA [6]
[1]School of Mathematics, Southeast University, Nanjing 211189, China
[2]Yonsei Frontier Laboratory, Yonsei University, Seoul 03722, South Korea
[3]Department of Machine Elements and Non-metallic Constructions, Technical University of Sofia, 1000 Sofia, Bulgaria
[4]Department of Computer Systems and Technologies, Burgas ''Prof. Dr. Assen Zlatarov'' University, 8010 Burgas, Bulgaria
[5]Faculty of Public Health and Health Care, Burgas ''Prof. Dr. Assen Zlatarov'' University, 8010 Burgas, Bulgaria
[6]Department of Mathematics, The University of Texas at San Antonio, San Antonio, TX 78249, USA

Corresponding author: Ivanka Stamova (ivanka.stamovar@utsa.edu)

ABSTRACT Cohen–Grossberg neural networks with delays provide a very powerful tool in the study of information processing, parallel computation, pattern recognition and solving of optimization problems. The robust stability behavior of such neural network models is essential in their numerous applications. Also, since the effect of various types of impulsive perturbations has been found to be remarkably important in the implementation of complex networks, the hybrid impulsive networks paradigm has gained increasing popularity during the last few decades. In this paper, an impulsive control strategy is proposed via variable impulsive perturbations for the robust stability with respect to manifolds for a class of Cohen–Grossberg neural networks with mixed delays and uncertain parameters. To this end, first new stability criteria are established for the nominal system under impulsive control. Then, the robust stability results are proposed. Finally, examples are considered to illustrate our impulsive control strategy. We generalize and extend some known robust stability results considering stability with respect to manifolds instead of isolated states stability.

INDEX TERMS Cohen–Grossberg neural networks, h-manifolds, impulsive control, variable impulsive perturbations, robust stability, mixed delays, uncertain parameters.

I. INTRODUCTION

Classification, information processing, parallel computing, associative memory and nonlinear optimization problems are crucial tasks in the investigation of many real-life phenomena and engineering systems. The role of the specific class of Cohen–Grossberg neural networks, initially proposed in [1] becomes progressively prominent in the above artificial intelligence technologies. The significance of their applications stimulated a great deal of research interest to the analysis of their dynamic properties. Since 1983 Cohen–Grossberg

neural networks are ones of the most investigated neural network models, and numerous qualitative results have been reported in the literature [2]–[6]. The emergence of new results on Cohen–Grossberg type of neural networks demonstrates the importance of research on their dynamic analysis for mathematical, computational, neurophysiological and engineering theories, as well as of their real applications [7], [8].

Also, many authors introduced delays in Cohen–Grossberg type of neural networks and studied the dynamics of delayed classes of Cohen–Grossberg neural network models. In fact, the effects of time delays, that are mainly due to signal transmissions and switching, cannot be ignored in practical

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neural network systems. It is well known that time delays may corrupt the neural network system performance or destabilize the system [9], [10]. The importance of delayed Cohen–Grossberg neural networks for the theory and applications is evidenced by the presence of numerous results on their analysis [11]–[14].

One type of time delays which effects are extremely studied is the type of distributed delays that more adequately reproduce the realistic actions [15]. Also, neural network systems with distributed delays extend the differential models to integro-differential ones. Hence, the behavior of Cohen–Grossberg neural networks with distributed and mixed delays have been intensively investigated. See, for example, [16]–[18] and some of the references therein.

In addition, uncertain parameters and robustness in neural network models are principal research challenges for applied fields such as biology, neurosciences and computing. For Cohen–Grossberg types of neural networks, several researches have developed robust stability criteria in order to advance the analysis in the fields of applications. For example, in the article [19] the global robust exponential stability behavior of the equilibrium point and the periodic solution of a type of interval Cohen–Grossberg neural networks with time-varying delays and infinite distributed delays is investigated. The research in [20] is devoted to sufficient conditions for the global exponential robust stability of the equilibrium point of Cohen–Grossberg neural networks with both time-varying and distributed delays. The authors in [21] studied the global robust exponential stability of the equilibrium point of interval Cohen–Grossberg neural networks with time-varying delays. In [22] important robust stability criteria have been obtained for a class of switched Cohen–Grossberg neural networks with mixed time-varying delays. The global robust exponential stability for second-order Cohen–Grossberg neural networks with multiple delays has been studied in [23]. In a very resent paper [24] the global robust stability behavior of the equilibrium point for Cohen–Grossberg neural networks with time-varying delays is investigated.

Nonetheless, the cited above robust stability results treated only single states of the considered Cohen–Grossberg neural network models, such as equilibrium points and periodic solutions. Indeed, it has been proven that the stability analysis of an equilibrium point or a periodic solution is important in the design and applications of neural network models. Then, a question arises: which approach is appropriate for the cases where it is difficult or even impossible to design a controller that guarantees the robust stability properties of single states? For such cases and in order to extend the research in the field of robustness, in this paper we propose to study a more general robust stability behavior, namely robust stability with respect to manifolds. The results in [19]–[24] will be particular cases of the proposed here findings.

The $h-$stability concept is related to a stability defined by a real-valued function, not necessarily, the distance. The motivation to consider such a generalized concept of stability with respect to manifolds comes from its properties.

From one side, it generalizes the stability of a single state notion, when the used function is the distance from the selected state. From the other side, it relaxes this stability concept allowing the state trajectories to converge to a neighborhood of the equilibrium or even to a set of solutions without disrupting its stability properties. Among the proved applications of stability with respect to manifolds notion are observer designs [25], celestial mechanics [26], maneuvering systems [27]. Because of the great possibilities for applications, the topic of $h-$stability whether or not related to equilibrium states has been studied for different types of systems [28]–[30], including some very recent results [31]–[34].

However, due to the complexity of the problem, the reported results on $h-$manifolds stability for Cohen–Grossberg neural network systems are still quite limited. To the best of our knowledge, there is only one paper [35] devoted to the study of this extended stability concept for a class of bidirectional associative memory Cohen–Grossberg neural networks with time-varying delays. However, distributed delays, uncertain parameters and robust stability are not considered in this unique investigation. Hence, the $h-$manifold robust stability notion is not developed for Cohen–Grossberg neural network models and is still a challenging problem.

In order to study the robust stability behavior of a Cohen–Grossberg neural network with uncertain parameters and distributed delays with respect to $h-$manifolds, an effective approach is proposed in this research using discrete control via variable impulsive perturbations. To this end, we will apply the theory of impulsive differential equations, which are useful tools in the design of impulsive control strategies [30], [36]–[38]. Impulsive controllers and impulsive control methods have been intensively applied to different mathematical models [39]–[45] including neural network systems [46]–[50]. The fact that such discontinuous controllers are preferred by researchers is mainly due to their very simple structure. Also, ''when the synchronization impulses are sent to the receiving systems at discrete time instants, it can greatly decrease the information redundancy in the transmitted signal and increase robustness against the disturbances'' [48]. Hence, some progress in the investigations of impulsive Cohen–Grossberg neural network systems has been made [51]–[56].

Robust stability analysis has been conducted for impulsive Cohen–Grossberg neural network models in just few research papers. In [57] the global robust exponential stability of equilibrium point for impulsive Cohen–Grossberg neural networks with distributed delays and reaction–diffusion terms has been investigated. The authors in [58] studied the contribution of impulsive perturbations at fixed times to the robust stability of a class of the impulsive bidirectional associative memories neural networks that possesses a Cohen–Grossberg dynamics and incorporates time delays and variable coefficients. The paper [59] is devoted to the robust exponential stability of the zero solution for

Markovian jump impulsive stochastic Cohen-Grossberg neural networks with mixed time delays and known or unknown parameters.

However, all existing robust stability results for impulsive Cohen–Grossberg neural network models again treated only a solo state of the model under consideration and stability with respect to the norm. Also, only impulsive perturbations at fixed instances are investigated. To extend and improve the existing results we will consider robust stability with respect to $h-$manifolds in our analysis. Also, different from all proposed techniques, we will first propose stability results via an impulsive control to the "nominal" system without uncertain parameters, and then we will add boundedness criteria for these system parameters. In addition, the great possibilities for applications motivated our control strategy to robustness which includes a discrete control via variable impulsive perturbations rather than at fixed instances. In fact, impulses at variable instances are more realistic and include as a special case fixed moments of impulsive jumps [60]–[62]. To the best of the authors' knowledge, variable impulsive perturbations have been discussed for Cohen–Grossberg neural networks in the recent papers [35] and [63]. However, the research in [35] does not include robust stability analysis, and [63] is devoted to the stability of a single almost periodic solution with respect to the distance. Both papers do not consider distributed delays.

Motivated by the above analysis, in this paper we will present results on impulsive control via impulses at variable times on the $h-$manifolds robust stability for Cohen–Grossberg neural networks with time-varying and distributed delays.

The novel contribution of our research is in the following aspects:

(1) we consider $h-$robust stability of the model, which generalizes and extends numerous robust stability concepts;

(2) the developed approach consists of two steps, including stability analysis of the "nominal" system without uncertain parameters, which allows the flexibility in the conditions about their effect;

(3) the proposed impulsive control strategy is in variable times, which improves some existing results on impulsive inputs at fixed times.

The structure of the manuscript is the following. Section II will propose our Cohen–Grossberg-type neural network model with mixed delays. The impulsive control strategy via variable impulsive perturbations will be also introduced. In addition, some main definitions will be given. Section III is devoted to the stability analysis of the "nominal" system without uncertain parameters. Section IV offers robust stability criteria. In section V we illustrate our results via examples and simulations. The developed approaches and results are summarized in section VI.

II. MODEL DESCRIPTION AND PRELIMINARIES
In this paper we will use the following standard notations: $\mathbb{R}^n$ is the $n-$dimensional Euclidean space with elements $x \in \mathbb{R}^n$,

$x = (x_1, x_2, \ldots, x_n)^T$, $||x|| = \sum_{i=1}^{n} |x_i|$ is the norm of $x \in \mathbb{R}^n$, $\mathbb{R}_+ = [0, \infty)$.

A. MODEL DESCRIPTION
Let $u = u(t)$, $u \in \mathbb{R}^n$, $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$, and $t \in \mathbb{R}_+$. We will study the robust stability behavior of the following Cohen–Grossberg neural network model with time-varying and distributed delays and uncertain parameters

$$
\begin{cases}
\dot{u}_i(t) = -d_i(u_i(t)) \Big[\gamma_i(u_i(t)) \\
\qquad - \sum_{j=1}^{n} \left(a_{ij} + \tilde{a}_{ij} \right) f_j(u_j(t)) \\
\qquad - \sum_{j=1}^{n} \left(b_{ij} + \tilde{b}_{ij} \right) g_j(u_j(t - s_{ij}(t))) \\
\qquad - \sum_{j=1}^{n} \left(c_{ij} + \tilde{c}_{ij} \right) \int_{-\infty}^{t} m_{ij}(t - \tau) v_j(u_j(\tau)) d\tau \\
\qquad - (I_i + \tilde{I}_i) \Big], \quad i = 1, 2, \ldots, n,
\end{cases}
\tag{1}
$$

where n denotes the number of the nodes in the model, $t > 0$, $u_i(t)$ correspond to the state of the $i-$th node at time t, f_j, g_j and v_j are the activation functions of the $j-$th node at the corresponding times, $d_i(u_i(t))$ is the amplification function for the state u_i at time t, $\gamma_i(u_i(t)$ is an appropriately behaved function at time t that guarantees the boundedness of the states of the model (1), a_{ij}, b_{ij} and c_{ij} are the connection coefficients, I_i is an input on the $i-$th node from outside of the network, the transmission time-varying delay $s_{ij}(t)$ satisfies $0 \leq s_{ij}(t) \leq s_{ij} = const$, the functions $m_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ are the delay kernels for the distributed delays, $\tilde{a}_{ij}$, $\tilde{b}_{ij}$ and $\tilde{c}_{ij}$ are the constant uncertain parameters in the connection coefficients, $\tilde{I}_i$ represents the uncertainty in the external input of the $i-$th unit.

It is clear that the considered uncertain parameters may affect the stability behavior of the models of type (1). Hence, finding efficient criteria for their robust stability is an important and challenging issue.

Let $\phi : (-\infty, 0] \to R^n$ be a bounded and continuous function. Consider an initial condition for the model (1) in the form

$$
u(\tau) = \phi(\tau), \quad \tau \leq 0. \tag{2}
$$

B. IMPULSIVE CONTROL VIA VARIABLE IMPULSIVE PERTURBATIONS INTRODUCTION
We will now introduce a discrete control strategy on the qualitative behavior of the units in the model (1) via impulsive perturbations at not fixed times.

Let $\tau_0(u) = 0$ for $u \in \mathbb{R}^n$. Consider the continuous functions $\tau_k : \mathbb{R}^n \to \mathbb{R}_+$, $k = 1, 2, \ldots$ such that

$$
0 < \tau_1(u) < \tau_2(u) < \ldots, \quad \tau_k(u) \to \infty \text{ as } k \to \infty
$$

uniformly on $u \in \mathbb{R}^n$. Define the hypersurfaces

$$
\sigma_k : t = \tau_k(u), \quad k = 1, 2, \ldots.
$$

The impulsive control is applied at some moments t_{l_k} at which the integral curve $(t, u(t))$ of the model (1) meets the hypersurfaces σ_k, so that the states trajectories are instantly changed from the position $(t, u(t)) = (t^-, u(t^-))$ into the position $(t^+, u(t^+))$, where $u(t^+) = u(t^-) + P_k(u(t^-)) + \tilde{P}_k(u(t^-))$, P_k and $\tilde{P}_k$ are matrices of continuous functions. The matrices $P_k = diag(P_{1k}, P_{2k}, \ldots, P_{nk})$, P_{ik} : $\mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots$. The matrices $\tilde{P}_k = diag(\tilde{P}_{1k}, \tilde{P}_{2k}, \ldots, \tilde{P}_{nk})$, $\tilde{P}_{ik}$: $\mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots$ are uncertainties in the impulsive controls.

On the basic of the above assumptions, we formulate the following impulsive control model with uncertain parameters

$$
\begin{cases}
\dot{u}_i(t) = -d_i(u_i(t)) \Big[\gamma_i(u_i(t)) \\
\qquad - \sum_{j=1}^{n} \left(a_{ij} + \tilde{a}_{ij} \right) f_j(u_j(t)) \\
\qquad - \sum_{j=1}^{n} \left(b_{ij} + \tilde{b}_{ij} \right) g_j(u_j(t - s_{ij}(t))) \\
\qquad - \sum_{j=1}^{n} \left(c_{ij} + \tilde{c}_{ij} \right) \int_{-\infty}^{t} m_{ij}(t - \tau) v_j(u_j(\tau)) d\tau \\
\qquad - (I_i + \tilde{I}_i) \Big], \ t \neq \tau_k(u), \\
u_i(t^+) = u_i(t) + P_{ik}(u_i(t)) \\
\qquad\qquad + \tilde{P}_{ik}(u_i(t)), \ t = \tau_k(u),
\end{cases} \tag{3}
$$

where $i = 1, 2, \ldots, n, t > 0$. The impulsive control instances t_k $(0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \ldots)$ in the model (3) are defined such that:

- t_1 is the first moment when the integral curve of the problem without impulses (1) meets some of the hypersurfaces σ_k. The number of this hypersurface is denoted by l_1.
- t_2 is defined as $t_2 = \tau_{l_2}(u(t_2)) = \min\{t : t = \tau_k(u(t))\}$, $t > t_1$, i.e., t_2 is the first bigger than t_1 moment when the integral curve of (3) meets some of the hypersurfaces σ_k. The number of this hypersurface is denoted by l_2,

etc.

Note that, it is possible for the integral curve $(t, u(t))$ of the model (3) to not meet the hypersurface σ_k at the moment t_k, i.e., $k \neq l_k, k = 1, 2, \ldots$, in general.

The proposed control technique generalizes the considered in [57]–[59] impulsive control at fixed times. It is clear that, in the case of variable impulsive perturbations the discrete control instances depend on the solution $u(t)$. Hence, different solutions may have different control inputs. This leads to complexities in the investigation of systems of type (3).

C. NOMINAL SYSTEM

When all uncertain parameters are zeros, we will receive the so-called "nominal" [64], [65] system corresponding to (3)

in the form

$$
\begin{cases}
\dot{u}_i(t) = -d_i(u_i(t)) \Big[\gamma_i(u_i(t)) \\
\qquad - \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) \\
\qquad - \sum_{j=1}^{n} b_{ij} g_j(u_j(t - s_{ij}(t))) \\
\qquad - \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} m_{ij}(t - \tau) v_j(u_j(\tau)) d\tau \\
\qquad - I_i \Big], \ t \neq \tau_k(u), \\
\Delta u_i(t) = P_{ik}(u_i(t)), \ t = \tau_k(u),
\end{cases} \tag{4}
$$

where $\Delta u_i(t) = u_i(t^+) - u_i(t)$, $i = 1, 2, \ldots, n$, $t > 0$.

We will analyse the system (4) with initial conditions of the form

$$
\begin{cases}
u(\tau; 0, \varphi_0) = \varphi_0(\tau), \ \tau \leq 0, \\
u(0^+) = \varphi_0(0),
\end{cases} \tag{5}
$$

where the initial function $\varphi_0 : (-\infty, 0] \to R^n$ is bounded and piecewise continuous with points of discontinuity of the first kind at which it is continuous from the left. The set of all such functions will be denoted by $\mathcal{PC}$.

We will first study the Cohen–Grossberg neural network model (4) under the following basic assumptions on the system parameters and impulsive functions:

(A1) The amplification functions d_i, $i = 1, 2, \ldots, n$ are continuous and there exist positive constants $\underline{d}_i$ and $\overline{d}_i$ such that $1 < \underline{d}_i \leq d_i(\chi) \leq \overline{d}_i$ for $\chi \in \mathbb{R}$.

(A2) For the functions γ_i there exist positive constants Γ_i such that

$$
\frac{\gamma_i(\chi_1) - \gamma_i(\chi_2)}{\chi_1 - \chi_2} \geq \Gamma_i,
$$

for any $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$ and $i = 1, 2, \ldots, n$.

(A3) For the activation functions f_i, g_i, v_i there exist positive constants F_i, G_i, V_i such that

$$
\begin{aligned}
F_i &= \sup \left| \frac{f_i(\chi_1) - f_i(\chi_2)}{\chi_1 - \chi_2} \right|, \\
G_i &= \sup \left| \frac{g_i(\chi_1) - g_i(\chi_2)}{\chi_1 - \chi_2} \right|, \\
V_i &= \sup \left| \frac{v_i(\chi_1) - v_i(\chi_2)}{\chi_1 - \chi_2} \right|,
\end{aligned}
$$

for any $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$ and $i = 1, 2, \ldots, n$.

(A4) The delay kernels m_{ij} are continuous, and there exist positive numbers μ_{ij} such that

$$
\int_{-\infty}^{t} m_{ij}(t - \tau) d\tau \leq \mu_{ij} < \infty
$$

for all $t \geq 0$, and $i, j = 1, 2, \ldots, n$.

(A5) The jump functions $P_{ik}(\chi)$ are continuous for $\chi \in \mathbb{R}$, $i = 1, 2, \ldots, n, k = 1, 2, \ldots$.

(A6) The functions $\tau_k(u)$ are continuous for $u \in \mathbb{R}^n$ and $\tau_k(u + P_k(u)) \leq \tau_k(u)$ for any $u \in \mathbb{R}^n$ and $k = 1, 2, \ldots$.

In addition, we introduce the following assumption

(A7) For the model (4) there exists a unique equilibrium $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ such that

$$
\begin{cases}
\gamma_i(u_i^*) = \sum_{j=1}^{n} a_{ij} f_j(u_j^*) \\
\qquad + \sum_{j=1}^{n} b_{ij} g_j(u_j^*)) \\
\qquad + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} m_{ij}(t - \tau) v_j(u_j^*) d\tau \\
\qquad + I_i], \ t \neq \tau_k(u), \\
\Delta u_i^*(t) = 0, \ t = \tau_k(u).
\end{cases}
\tag{6}
$$

Note that, the existence of the equilibrium state u^* can be easily proven using conditions (A1)-(A4) for the continuous part of the model and conditions (A5) and (A6). In fact, since all conditions of Theorem 1 in [16] are satisfied, then the equilibrium u^* exists for $t \neq \tau_k(u)$ for any $k = 1, 2, \ldots$. Conditions (A5) and (A6) [30], [60], [63], [65] guarantee that it exists for any $t \geq 0$.

D. H-STABILITY

Consider a function $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^l$, $l \leq n$, such that the set

$$
M = \{u \in \mathbb{R}^n : h(t, u) = 0\}
$$

is an $(n - l)$−dimensional manifold in $\mathbb{R}^n$, and any solution of the $u(t)$ model (3) that satisfies

$$
||h(t, u(t))|| \leq H < \infty
$$

is defined for $t \geq 0$ for any values of the uncertain parameters.

We will adopt the notion of stability of solutions with respect to h−manifolds [28]–[34] to the equilibrium state of the Cohen–Grossberg model u^* of (4) as follows:

Definition 1: The equilibrium state u^* of (4) is said to be globally exponentially stable with respect to the function h, if for any initial function $\varphi_0 \in \mathcal{PC}$, we have

$$
||h(t, u(t) - u^*)|| \leq \mathcal{M}(\varphi_0) e^{-\mu t}
$$

for $t \geq 0$, where $u(t) = u(t; 0, \varphi_0)$, $\mathcal{M}(0) = 0$, and $\mathcal{M}$ is Lipschitz continuous with respect to its variable $\varphi \in \mathcal{PC}$.

Remark 1: Note that, the function h is of a very general structure. For example, if $l = 1$, and the function $h(t, u - u^*) = u - u^*$, then Definition 1 reduces to the Lyapunov-type global exponential stability definition of the equilibrium state u^*.

Definition 2: The equilibrium state u^* of (4) is said to be globally robustly exponentially stable with respect to the function h, if for any initial function $\varphi_0 \in \mathcal{PC}$, and any values of the uncertain parameters $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{c}_{ij}$ and $\tilde{I}_i$, the equilibrium of (3) is globally exponentially stable with respect to the function h.

E. LYAPUNOV METHOD ESSENTIALS

In order to investigate the robust stability of the equilibrium state of (4) we will apply the proven Lyapunov function approach.

We will consider a class V_0 of piecewise continuous Lyapunov-type functions defined as: $V_0 = \{V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+, \ V$ is continuous for $(t, u) \in \mathbb{R}_+ \times \mathbb{R}^n$, $t \neq \tau_k(u)$, $k = 1, 2, \ldots$, locally Lipschitz continuous with respect to its second argument u, $V(t, 0) = 0$, for any $k = 1, 2, \ldots$ and $(\tilde{t}, u(\tilde{t})) \in \sigma_k$, $V(\tilde{t}^-, u(\tilde{t}^-))$ and $V(\tilde{t}^+, u(\tilde{t}^+))$ are well defined and finite, and $V(\tilde{t}^-, u(\tilde{t}^-)) = V(\tilde{t}, u(\tilde{t}))\}$.

Following [30], for a function $V \in V_0$, we will consider the following *upper right-hand derivative of V* given by

$$
D^+ V(t, \varphi(0))
$$
$$
= \lim_{\chi \to 0^+} \sup \frac{1}{\chi} [V(t + \chi, u(t + \chi)) - V(t, \varphi(0))],
$$

where $(t, \varphi) \in \mathbb{R}_+ \times \mathcal{PC}$.

Here is a basic assumption for a function $V \in V_0$.

(A8) For the function $h(t, u)$ there exists a Lyapunov-type function $V \in V_0$ such that

$$
||h(t, u)|| \leq V(t, u) \leq \Omega(H)||h(t, u)||, \quad t \in \mathbb{R}_+,
$$

where $\Omega(H) \geq 1$ exists for any $0 < H < \infty$.

In the next section we will use the following lemma from [30].

Lemma 1: If $V \in V_0$ is such that for $t \in [0, \infty)$, $\varphi \in \mathcal{PC}$ satisfies the following assumptions:

(i) $V(t^+, \varphi(0) + \Delta\varphi) \leq V(t, \varphi(0))$, $t = \tau_k(\varphi)$, $k = 1, 2, \ldots$,

(ii) $D^+ V(t, \varphi(0)) \leq -\mu V(t, \varphi(0))$, $t \neq \tau_k(\varphi)$, $k = 1, 2, \ldots$ for $V(t + \xi, \varphi(\xi)) \leq V(t, \varphi(0))$, $-\infty < \xi \leq 0$.

then

$$
V(t, u(t; 0, \varphi_0)) \leq \sup_{-\infty < \xi \leq 0} V(0^+, \varphi_0(\xi)) \exp(-\mu t)
$$

for $t \in [0, \infty)$.

III. IMPULSIVE CONTROL VIA VARIABLE IMPULSIVE PERTURBATIONS ON THE NOMINAL SYSTEM

In this section, we will use a suitable Lyapunov-type function to derive criteria for global exponential stability with respect to a function h of the steady state u^* of the nominal system (4) under variable impulsive perturbations.

The impulsive control is designed as follows.

(A9) For $t = \tau_k(u(t))$, the impulsive functions P_{ik} are such that

$$
P_{ik}(u_i(t)) = -v_{ik}(u_i(t) - u_i^*), \ 0 < v_{ik} < 2
$$

and

$$
\underline{d}|1 - v_{ik}| \leq |1 - \hat{v}_k| \leq \frac{\underline{d}}{\bar{d}},
$$

where $\hat{v}_k = \min_{1 \leq i \leq n} v_{ik}$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots$, $\underline{d}_i = \min_{1 \leq i \leq n} d_i$, $\bar{d} = \max_{1 \leq i \leq n} d_i$.

Theorem 1: Under the assumptions (A1)–(A8), if there is a positive number $\mu = \mu_1 - \mu_2$ such that

$$\mu_1 = \underline{d} \min_{1 \le i \le n} \left(\Gamma_i - a_{ii}^+ F_i - \sum_{j=1, \, j \ne i}^n |a_{ji}|F_i \right)$$

$$> \mu_2 = \overline{d} \max_{1 \le i \le n} \sum_{j=1}^n \left(|b_{ji}|G_i + |c_{ji}|\mu_{ji}V_i \right), \qquad (7)$$

where $a_{ii}^+ = \max\{|a_{ii}|, 0\}, i = 1, 2, \ldots, n$, then the equilibrium u^* of the model (4) is globally exponentially stable with respect to the function h under the impulsive control (A9).

Proof: Denote $y_i(t) = u_i(t) - u^*$, $i = 1, 2, \ldots, n$. Consider the Lyapunov-type function

$$V(t, y(t)) = \sum_{i=1}^n \int_0^{y_i(t)} \frac{sgn(\xi)}{\theta_i(\xi)} d\xi,$$

where $\theta_i(y_i(t)) = d_i(y_i(t) + u_i^*)$.

Since

$$\frac{1}{\overline{d}}||y(t)|| \le V(t, y(t)) \le \frac{1}{\underline{d}}||y(t)||, \qquad (8)$$

then, for $t = \tau_k(u)$, $k = 1, 2, \ldots$ and $\varphi \in \mathcal{PC}$, from (A9) we derive the following estimate

$$V(t^+, \varphi(0) + \Delta\varphi) \le \frac{1}{\underline{d}}||\varphi(0) + \Delta\varphi||$$

$$= \frac{1}{\underline{d}} \sum_{i=1}^n |\varphi_i(0) + \Delta\varphi_i| = \frac{1}{\underline{d}} \sum_{i=1}^n |1 - v_{ik}||\varphi_i(0)|$$

$$< \sum_{i=1}^n |1 - v_{ik}||\varphi_i(0)| \le \frac{|1 - \hat{v}_k|}{\underline{d}} \sum_{i=1}^n |\varphi_i(0)|$$

$$\le \frac{1}{\underline{d}} \sum_{i=1}^n |\varphi_i(0)| = \frac{1}{\underline{d}}||\varphi(0)|| \le V(t, \varphi(0)). \qquad (9)$$

For the derivative of the Lyapunov-type function V for $\tau_{k-1}(u) < t < \tau_k(u)$, we calculate

$$\frac{d}{dt} V(t, y(t)) = - \sum_{i=1}^n sgn(y_i(t)) \left(\tilde{\gamma}_i(y_i(t)) - \sum_{j=1}^n a_{ij}\tilde{f}_j(y_j(t)) \right.$$

$$- \sum_{j=1}^n b_{ij}\tilde{g}_j(y_j(t - s_{ij}(t)))$$

$$\left. - \sum_{j=1}^n c_{ij} \int_{-\infty}^t m_{ij}(t - \tau)\tilde{v}_j(y_j(\tau))d\tau \right),$$

where $\tilde{\gamma}_i(y_i(t)) = \gamma_i(y_i(t) + u_i^*) - \gamma_i(u_i^*)$, $\tilde{f}_j(y_j(t)) = f_j(y_j(t) + u_j^*) - f_j(u_j^*)$, $\tilde{g}_j(y_j(t)) = g_j(y_j(t) + u_j^*) - g_j(u_j^*)$, $\tilde{v}_j(y_j(t)) = v_j(y_j(t) + u_j^*) - v_j(u_j^*)$.

From the above estimate, using the conditions of Theorem 2, we get for $t \ne \tau_k(\varphi)$, $k = 1, 2, \ldots$

$$D^+V(t, \varphi(0))$$

$$\le \sum_{i=1}^n \left(-\Gamma_i|\varphi_i(0)| + a_{ii}^+ F_i|\varphi_i(0)| \right.$$

$$+ \sum_{j=1, \, j \ne i}^n |a_{ij}|F_j|\varphi_j(0)| + \sum_{j=1}^n |b_{ij}|G_j|\varphi_j(-s_{ij}(0))|$$

$$+ \sum_{j=1}^n |c_{ij}|\mu_{ij}V_j \sup_{-\infty < \tau \le 0} |\varphi_j(\tau)| \Bigg)$$

$$\le - \sum_{i=1}^n \left(\Gamma_i - a_{ii}^+ F_i - \sum_{j=1, \, j \ne i}^n |a_{ji}|F_i \right) |\varphi_i(0)|$$

$$+ \sum_{i=1}^n \sum_{j=1}^n \left(|b_{ji}|G_i + |c_{ji}|\mu_{ji}V_i \right) \sup_{-\infty < \tau \le 0} |\varphi_i(\tau)|.$$

From the last inequality and (8), for $t \ne \tau_k(\varphi)$, $k = 1, 2, \ldots$, we get

$$D^+V(t, \varphi(0)) \le -\mu_1 V(t, \varphi(0)) + \mu_2 V(t, \varphi(0)),$$

and by (7) it follows that

$$D^+V(t, \varphi(0)) \le -\mu V(t, \varphi(0)), \qquad (10)$$

for $V(t + \tau, \varphi(\tau)) \le V(t, \varphi(0))$, $-\infty < \tau \le 0$.

It follows from (9), (10) and Lemma 1, that

$$V(t, u(t; 0, \varphi_0) - u^*) \le \sup_{-\infty < \tau \le 0} V(0^+, \varphi_0(\tau) - u^*) \exp(-\mu t)$$

for $t \in [0, \infty)$.

Hence, from assumption (A8), we obtain

$$||h(t, u(t; , 0, \varphi_0) - u^*|| \le V(t, u(t) - u^*)$$

$$\le \Omega(H) \sup_{-\infty < \tau \le 0} ||h(0^+, \varphi_0(\tau) - u^*)|| \exp(-\mu t), \quad t \ge 0.$$

Set $\mathcal{M} = \mathcal{M}(\varphi_0) = \Omega(H) \sup_{-\infty < \tau \le 0} ||h(0^+, \varphi_0(\tau) - u^*)||$.

Then,

$$||h(t, u(t) - u^*)|| \le \mathcal{M}(\varphi_0)e^{-\mu t}, \quad t \in \mathbb{R}_+$$

which shows that the equilibrium state u^* of the nominal system (4) is globally exponentially stable with respect to the function h under the impulsive control (A9), and the proof is complete. ∎

Remark 2: Theorem 2 offers global exponential stability criteria with respect to the function h of the equilibrium u^* of the nominal system (4). This result generalizes the existing global stability results for equilibria considering a manifold defined by the function h. In addition, the result can be used alone in the absence of uncertain terms, and is a first step in the robust stability analysis of the model (4).

IV. ROBUST STABILITY CRITERIA

In this section we will present global robust stability criteria for the equilibrium u^* of the model (4) with respect to the function h.

We consider the following assumptions for the uncertain parameters and the uncertainties in the impulsive controls.

(A10) The constants $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{c}_{ij}$ and $\tilde{I}_i$ are bounded and

$$\underline{d} \min_{1 \le i \le n} \left(\Gamma_i - \left(a_{ii}^+ + \tilde{a}_{ii}^+ \right) F_i - \sum_{j=1, \, j \ne i}^n \left(|a_{ji}| + |\tilde{a}_{ji}| \right) F_i \right)$$

$$> \overline{d} \max_{1 \le i \le n} \sum_{j=1}^n \left(\left(|b_{ji}| + |\tilde{b}_{ji}| \right) G_i + \left(|c_{ji}| + |\tilde{c}_{ji}| \right) \mu_{ji} V_i \right),$$

where $\tilde{a}_{ii}^+ = \max\{|\tilde{a}_{ii}|, 0\}$, $i, j = 1, 2, \ldots, n$.

(A11) The uncertainties in the impulsive controls $\tilde{P}_{ik}(u_i(t)) = -\tilde{v}_{ik}(u_i(t) - u_i^*)$ and the unknown constants $\tilde{v}_{ik}$ are such that $0 < \tilde{v}_{ik} < 2 - v_{ik}$, $i = 1, 2, \ldots, n$ $k = 1, 2, \ldots$.

The next result is a direct corollary of Theorem 2.

Theorem 2: Under the assumptions (A1)–(A8), (A10), the equilibrium u^* of the model (4) is globally robustly exponentially stable with respect to the function h under the impulsive control (A9), (A11).

Remark 3: The researchers in [19]–[24] proposed robust stability results for some classes of Cohen–Grossberg-type neural networks with delays. However, the impulsive control is not considered in these papers. For impulsive models, robust stability results have been proposed in [57]–[59]. The considered impulses in these papers are at fixed points. It is well known, that systems with fixed moments of impulsive perturbations are particular cases of systems with variable impulsive perturbations. In fact, if we denote the fixed instances of discrete impulsive jumps as $t = t_k$, $k = 1, 2, \ldots$, then for σ_k : $t = t_k$} the results in [57]–[59] can be considered as corollaries of Theorem 3.

Remark 4: Different from the proposed in [19]–[24] and in [57]–[59] robust stability approaches, the developed in this paper approach consists of two steps. The first one is the global stability of the nominal system. This allows a flexibility in the criteria for the uncertain parameters in the system.

In addition, we apply the generalized h−stability concept in our investigation. Thus, our results extend and generalize all existing results on robust stability of delayed Cohen–Grossberg neural network models.

V. ILLUSTRATIONS

Example 1: As an example, consider the model (3) for $n = 2$, $u(t) = (u_1(t), u_2(t))^T$, $t > 0$, $f_i(u_i) = g_i(u_i) = v_i(u_i) = \frac{1}{2}(|u_i + 1| - |u_i - 1|)$, $s_{ij}(t) = e^t/(1 + e^t)$, $0 \le s_{ij}(t) \le 1$, $d_i(u_i) = 1$, $\gamma_i(u_i) = 4$, $\mu_{ij}(s) = e^{-s}$, $i, j = 1, 2$, $I_1 = 2.85$, $I_2 = 4.3$,

$$(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.7 \\ -0.8 & 0.5 \end{pmatrix},$$

$$(b_{ij}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} -0.4 & 0.3 \\ 0.6 & -0.5 \end{pmatrix},$$

$$(c_{ij}) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{pmatrix},$$

$\tau_k(u_i) = |u_i| + k$, $k = 1, 2, \ldots$.

Let the impulsive functions $P_{ik}(u_i)$ be such that

$$P_{1k}(u_1) = \left(\frac{1}{k} - 2\right)(u_1 - 0.5),$$

$$P_{2k}(u_2) = \left(\frac{1}{2k} - 2\right)(u_2 - 1.5), \tag{11}$$

for any $k = 1, 2, \ldots$.

Then, all assumptions (A1)–(A3) are satisfied for $\underline{d}_i = \overline{d}_i = 1$, $\Gamma_i = 4$, $F_i = G_i = V_i = 1$. The assumption (A4) is also satisfied, since $\int_0^\infty e^{-s} ds = 1$.
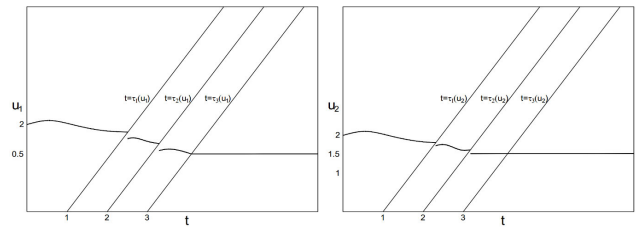

FIGURE 1. The global robustly exponentially stable behavior of u_1 and u_2 with respect to the function h under the impulsive control (11), (12).

We can also check that the nominal model has an equilibrium state at $u^* = (u_1^*, u_2^*)^T = (0.5, 1.5)^T$ and the condition (7) is satisfied for $\mu_1 = 2.6 > \mu_2 = 1.5$.

In addition, the impulsive control on the nominal system via the impulsive functions $P_{ik}(u_i)$ is designed according to (A9) and is performed via variable impulsive perturbations that satisfy (A5) and (A6).

Consider the function h : $\mathbb{R}^2 \to \mathbb{R}^2$, $h = (h_1, h_2)^T$, $h_1 = u_1 - u_1^*$, $h_2 = u_2 - u_2^*$, for which assumption (A8) is achieved.

Since all conditions of Theorem 2 are satisfied, it guarantees that the equilibrium u^* of the nominal model is globally exponentially stable with respect to the function h under the impulsive control (11).

We can also realize that, if the uncertain parameters satisfy the boundedness condition

$$1.1 + \min_{1 \le i \le n} \left(\tilde{a}_{ii}^+ + \sum_{j=1, j \ne i}^n |\tilde{a}_{ji}| \right)$$
$$> \max_{1 \le i \le n} \sum_{j=1}^n \left(|b_{ji}| + |\tilde{c}_{ji}| \right),$$

and the uncertainties in the impulsive controls $\tilde{P}_{ik}(u_i(t)) = -\tilde{v}_{ik}(u_i(t) - u_i^*)$, $i = 1, 2$, are such that

$$0 < \tilde{v}_{1k}(u_1) < \frac{1}{k},$$
$$0 < \tilde{v}_{2k}(u_2) < \frac{1}{2k} \tag{12}$$

for any $k = 1, 2, \ldots$, then according to Theorem 3 the equilibrium u^* is globally robustly exponentially stable with respect to the function h under the impulsive control (11), (12). The stable behavior for initial data $u_1(\tau) = u_2(\tau) = 2$, $-\infty < \tau < 0$, $u_1(0^+) = u_2(0^+) = 2$, is shown in Figure 1.

Remark 5: Criteria for robust stability of impulsive Cohen–Grossberg-type neural networks with delays are proposed in [57]–[59]. The proposed conditions are specific for each uncertain parameter in the corresponding model. Example 1 shows again that our boundedness criteria are more general, and therefore our approach allows more flexibility in the work with uncertain terms, and that is how it outperforms the proposed in the existing literature approaches.

Example 2: Consider again the model (3) with particular parameters determined in Example 1. Let, instead of the boundedness conditions for the uncertainties in the impulsive
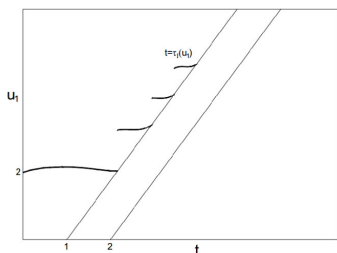
FIGURE 2. The unstable behavior of u_1^* with respect to the function h. Under the impulsive perturbations (11), (13) u_1 is "beating" on $\tau_1(u_1)$.

controls (12), the constants $\tilde{v}_{ik}$, $i = 1, 2$ be such that

$$\tilde{v}_{1k} = \frac{2}{k},$$
$$0 < \tilde{v}_{2k}(u_2) < \frac{1}{2k} \quad (13)$$

$k = 1, 2, \ldots$.

Since the condition (A11) is not satisfied, we can not make a conclusion about the global robust exponential stable behavior of the equilibrium u^* with respect to the function h by means of Theorem 3. However, the numerical simulations show that in this case, even the uncertain parameters satisfy (A10), the impulses destroy the exponential stability of the equilibrium which is shown in Figure 2. In fact, we observe the "beating" phenomenon, when $u_1(t)$ beets infinitely many times $\tau_1(u_1)$, and is not continuable after it.

Remark 6: Example 2 demonstrates that the impulsive control strategy is essential for the robust stability of the equilibrium states. Particularly, when the control is conducted via variable impulsive perturbations, the control instances for distinct states are not, in general, the same, and difficulties such as "beating" of solutions exist. However, since such perturbations are more relevant to reality, it is important to study their effect on the behavior and control of numerous systems studied in widespread areas of the mathematical, physical, chemical, engineering, and statistical sciences.

VI. CONCLUSION

In this research we investigate a Cohen–Grossberg neural network model with mixed delays. The main research question evaluated in the paper is about the robust stability of the model with respect to a general type function not necessarily a distance. The proposed notion extends various robust stability notions. Thus, our robust stability criteria generalize and complement some existing robust stability results. We also propose an impulsive control strategy via variable impulsive perturbations. Since the proposed robust stability concept and the impulsive control technique have a great potential in applications, it is expected that our research will inspire the researchers to apply the proposed approach to different neural networks models of diverse interest.

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JINDE CAO (Fellow, IEEE) received the B.S. degree in mathematics/applied mathematics from Anhui Normal University, Wuhu, China, in 1986, the M.S. degree in mathematics/applied mathematics from Yunnan University, Kunming, China, in 1989, and the Ph.D. degree in mathematics/applied mathematics from Sichuan University, Chengdu, China, in 1998. He is an Endowed Chair Professor, the Dean of the School of Mathematics, the Director of the Jiangsu Provincial Key Laboratory of Networked Collective Intelligence of China, and the Director of the Research Center for Complex Systems and Network Sciences, Southeast University. He is a member of the Academy of Europe and the European Academy of Sciences and Arts, a fellow of the Pakistan Academy of Sciences, and an IASCYS Academician. He was a recipient of the National Innovation Award of China, the Obada Prize and the Highly Cited Researcher Award in Engineering, Computer Science, and the Mathematics by Thomson Reuters/Clarivate Analytics.

TRAYAN STAMOV received the B.S. and M.S. degrees in engineering design and the Ph.D. degree in industrial design from the Technical University of Sofia, Bulgaria, in 2008, 2010, and 2014, respectively. He is currently an Associate Professor with the Department of Machine Elements and Non-metallic Construction, Technical University of Sofia. His research interests include neural networks applications in industrial design practice, emotional design, and experiments. He is also awarded for his design achievements in numerous product design competitions, including the Silver A' Design Award Winner for Lighting Products and Lighting Projects Design Category, in 2015, from the A' Design Award and Competition, Italy. He is serving at the Editorial Board of VE Vehicle Engineering (VE).

SOTIR SOTIROV received the M.S. degree in electronics and automation from the Technical University of Sofia, Bulgaria, in 1994, and the Ph.D. degree in computer sciences from the University "Prof. Dr. Assen Zlatarov", Burgas, Bulgaria, in 2005.

He is currently a Professor of electronics with the Department of Computer Systems and Technologies, Burgas "Prof. Dr. Assen Zlatarov" University. He is also the Head of the Laboratory of Intelligent Systems and a Vice-Rector for international relations with Burgas "Prof. Dr. Assen Zlatarov" University. He is the Coordinator of the Group Intuitionistic Fuzzy Sets: Theory, Applications and Related Topics in European Society for Fuzzy Logic and Technology (EUSFLAT). His current research interests include intuitionistic fuzzy sets and neural network modeling.

EVDOKIA SOTIROVA received the M.S. degree in computer systems and technologies from the Technical University of Sofia, Sofia, Bulgaria, in 1994, and the Ph.D. degree in computer sciences from Burgas "Prof. Dr. Assen Zlatarov" University, Burgas, Bulgaria, in 2004.

She is a member of the Laboratory of Intelligent Systems and the Vice-Dean of education process of the Faculty of Public Health and Health Care, Burgas "Prof. Dr. Assen Zlatarov" University. She is also the Head of the Department of Computer and Information Technologies and a Research Scientist with the Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences, Sofia. Her research interests include fuzzy logic, generalized nets (extensions of Petri nets), data mining, and algorithms.

IVANKA STAMOVA received the M.S. degree in mathematics from Plovdiv University, Bulgaria, in 1987, and the Ph.D. degree in stability analysis of nonlinear systems and the D.Sc. degree in applied mathematics from the Higher Accreditation Commission of Bulgaria, in 1996 and 2009, respectively.

She is currently an Associate Professor with the Department of Mathematics, The University of Texas of San Antonio. She is the author of numerous books and articles on stability and control of nonlinear systems, including the books *Stability Analysis of Impulsive Functional Differential Equations* (Walter de Gruyter, Berlin, 2009), *Applied Impulsive Mathematical Models* (Springer, Berlin, 2016), and *Functional and Impulsive Differential Equations of Fractional Order: Qualitative Analysis and Applications* (CRC Press, Boca Raton, 2017). Her current research interests include qualitative analysis of nonlinear dynamical systems, nonlinear observer design, and impulsive control and applications. She is an Associate Editor of the journal *Advances in Difference Equations*.

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