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# Design of LQR Tracking Controller Combined With Orthogonal Collocation State Planning for Process Optimal Control

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**ABSTRACT** As one of the most important optimization methods for process optimal control, orthogonal collocation method has been widely used. However, this kind of optimization method is generally an open-loop optimization frame, which makes the model disturbances greatly affect the control performance and quality. In order to improve the optimal control performance of dynamic systems with disturbances, this work proposed an orthogonal collocation optimization-based linear quadratic regulator (OC-LQR) control method for process optimal control problems. Firstly, the orthogonal collocation method is derived in detail to transform the optimal control problem so that optimal state curves can be accordingly calculated. Then, an improved LQR controller design is proposed by using the obtained optimal state curves so as to construct the feedback control frame. On this basis, a state planning-based LQR tracking control is established with closed-loop control characteristics to tackle model disturbance problem. Meanwhile, the detailed Simulink model of the proposed control method is constructed for simulation execution. Finally, the proposed control structure is tested on a classical process optimal control problem with model disturbance (white noise) test and Gaussian mixture noise test to verify the performance of the proposed method. Simulation studies show that the control performance of the proposed method is excellent, where the mean absolute error of state curve tracking averagely reduces by 34.73% and the minimal performance index improves by 64.44% when compared with the classical orthogonal optimal control method. Simulation results demonstrate the effectiveness of the proposed planning-based tracking control framework.

**INDEX TERMS** Optimal control, orthogonal collocation method, state planning, LQR tracking control, process control.

## I. INTRODUCTION

With the rapid development of optimal control theory [1], optimal control methods have been widely used in industry process and aerospace fields [2]–[4]. Optimal control theory can be traced back to the 1950s, then Siebenthal [5] solved the stirred reactor optimization control problem based on the Pontryagin maximum principle. Generally, the objective of an optimal control problem is to establish a performance index function by selecting different independent variables to find a control strategy that maximizes or minimizes the performance index under certain constraints [6]. According to previous classification, optimal control methods are usually divided into indirect methods [7], direct methods [8] and intelligent optimization algorithms [9]. Among them, indirect methods mainly use the Pontryagin maximum principle [10] to convert the optimal control problem into a two-point boundary problem; alternatively, direct methods always employ discrete strategies to transform the optimal control problem into a nonlinear programming (NLP) problem [11]; the characteristic of intelligent optimization algorithms is to solve discrete optimization problems by intelligent ways. Compared with the intelligent optimization methods, direct methods have the advantages of less

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calculation amount and high calculation efficiency. Meanwhile, convergence range of direct methods is wider and does not require a high-precision initial value when compared to indirect methods. Therefore, direct methods become popular numerical methods for solving optimal control problems.

In recent years, as one of the most efficient direct methods, orthogonal collocation (OC) method has been favored by many scholars because of its advantages such as high accuracy, low calculation, simple structure and high efficiency [12]. The idea of this method is to discretize the continuous optimal control problem at orthogonal points, then approximate the state variables and control variables through global interpolation polynomials [13] so as to transform optimal control problem into a NLP problem. This method has been demonstrated efficient in military and civilian fields. For instance, Li *et al.* [14] used orthogonal collocation (also named pseudospectral) method for trajectory optimization of variable trust missile to intercept short-range targets; Huang *et al.* [15] proposed a *p*k-adaptive mesh refinement for orthogonal collocation method to improve the computation efficiency; Zhang *et al.* [16] employed the orthogonal collocation method to solve unmanned helicopter optimal control problems so as to obtain the optimal trajectories.

Based on these discussions in literatures [14]–[16], it can be found that orthogonal collocation method has good spectral accuracy in differential approximation theory and owns larger convergence region and faster convergence speed when compared with other indirect method and direct method (such as control vector parameterization method [17] and iterative dynamic programming algorithm [18]. Therefore, orthogonal collocation method becomes a terrific candidate for high precision trajectory optimization. Theoretically, orthogonal collocation method can obtain optimal control and state planning, where the optimal state is guaranteed by using the optimal control strategy under ideal optimization model. However, it is difficult to obtain the exact models of dynamic processes and disturbances will also make the model mismatch. When the dynamic system is mismatch, the control performance may be affected and the control effect is greatly reduced as the orthogonal collocation method does not have the characteristics of closed-loop control [19]. Thus, how to guarantee the optimal control quality in the presence of model parameter disturbance and unknown external disturbances is a key problem for orthogonal collocation method application.

To enhance the control quality of orthogonal collocation method under disturbances, an online re-optimization strategy in two layers is proposed for handling disturbance and uncertainty within semi batch process by Rohman *et al.* [20]. Furthermore, adding estimator in optimal control problem is also a meaningful way [21]. While, these methods are still open-loop methods, closed-loop characteristics cannot be guaranteed under disturbances. In essence, the obtained optimal state trajectory of OC method can be regarded as ideal input for dynamic system. If the state trajectory can be tracked, the performance index then will also be guaranteed. Correspondingly, the design of high-efficiency tracking controller for OC optimization is another efficient solution for tackling optimal control problem with disturbances.

In recent years, tracking control has been widely studied, for instance, model predictive control [22], robust tracking control [23], linear quadratic regulator (LQR) control [24], etc. As a classical control strategy, LQR control has very high importance and representativeness in modern control theory [24] and has good control ability for dynamic systems with uncertainties [25]. Meanwhile, LQR controller is a fundamental control strategy for plenty of improved control methods and has closed-loop characteristics. LQR controller thus becomes a potential candidate for combining the orthogonal collocation optimization to improve the optimization and control performances of dynamic systems with disturbances.

Therefore, a Gauss distribution point reconstruction OC method combined with LQR control is proposed to form a closed-loop control frame so as to weaken the influence of model disturbances for optimal control methods and further ensure the control quality. Firstly, the formula of Legendre Gauss distribution point is given in the framework of orthogonal collocation method to discrete the process optimal control problems so as to discretize the continuous dynamic processes; then, the optimal state vector is obtained by solving the transformed nonlinear programming problem. In order to improve the control quality of the optimal control method for dynamic systems with disturbances, the state-planning-based LQR controller design method is proposed to establish a closed-loop control. Meanwhile, the detailed Simulink model of the proposed LQR tracking is also constructed for simulation testing. Numerical experimental tests of a well-known process optimal control problem with considering white noise and Gaussian mixture noise are carried out to verify the efficiency of the proposed method, where the classical OC method and some literature results are employed to make comparison.

The organizational structure of this paper is stated as follows: Section II describes the orthogonal collocation optimization method to obtain the optimal state curves; Section III introduces the optimal state planning-based LQR tracking control method; Section IV gives the specific implementation method of the proposed OC-LQR control algorithm; numerical tests are carried out in Section V; between Section V and VII, Section VI discusses the tests results in detail; Section VII summarizes the work of this paper.

## II. OPTIMAL STATE PLANNING METHOD

### A. PROCESS OPTIMAL CONTROL PROBLEM FORMULATION

In process optimal control, the objective of control is usually to find the feasible control/state curves to minimize/maximize the performance index. Generally, this control process can be attributed to an optimal control problem. In this work, the following Bolza form process optimal control problem is considered,

$$\begin{aligned} \text{Min} \quad & J = \Phi\left(\boldsymbol{x}(t_0), t_0, \boldsymbol{x}(t_f), t_f\right) + \int_{t_0}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\mathrm{d}t \\ \text{s.t.} \quad & \dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \\ & E\left(\boldsymbol{x}(t_0), t_0, \boldsymbol{x}(t_f), t_f\right) = 0 \\ & C(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \le 0, \ t \in \left[t_0 \quad t_f\right] \end{aligned} \tag{1}$$

where $t_0$ and $t_f$ are the initial and the terminal time, $\boldsymbol{x}(t) \in \mathbb{R}$ is the state vector, $\boldsymbol{u}(t) \in \mathbb{R}$ is the control vector. The nonlinear dynamic process $\dot{\boldsymbol{x}}(t)$ is described by ordinary differential function $f(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$ with the initial and terminal conditions $E\left(\boldsymbol{x}(t_0), t_0, x(t_f), t_f\right) = 0$. $C(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$ describes the inequality path constraints. $J$ is the objective function (also named performance index), where $\Phi\left(\boldsymbol{x}(t_0), t_0, \boldsymbol{x}(t_f), t_f\right)$ and $\int_{t_0}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\mathrm{d}t$ are the terminal cost and integrand cost, respectively.

In problem (1), if an admissible control $\boldsymbol{u}(t)$ satisfies inequality path constraints, then it can be regarded as a feasible control. Let $\boldsymbol{F}$ denote the class of all such feasible controls. Therefore, the goal of problem (1) is to find a feasible control vector $\boldsymbol{u}(t) \in \boldsymbol{F}$ over $t \in [t_0, t_f]$ to minimize an objective function $J$ and then to obtain the optimal state planning curves.

### B. ORTHOGONAL COLLOCATION OPTIMIZATION METHOD

In this work, orthogonal collocation optimization method is applied for solving the process optimal control problem and then to obtain the optimal state planning curves. Orthogonal collocation method, also known as pseudo-spectral method [26], is a kind of approximate analytic method for differential equations which has been widely used in recent years [27]–[29]. The idea of orthogonal collocation method is to approximate continuous state and control vector with discrete points, then the optimal control problem can be transformed into a NLP problem. The specific process of orthogonal collocation method is stated as follows.

Firstly, transform the time interval $[t_0, t_f]$ of optimal control problem (1). In orthogonal collocation method, the new time vector $\tau$ is introduced for scale transformation by using the following formula:

$$t = \frac{t_f - t_0}{2}\tau + \frac{t_f + t_0}{2} \tag{2}$$

Then, it is easy to verify that time interval $[t_0, t_f]$ is transformed into a new unit time interval $[\tau_0, \tau_f] = [-1, 1]$. By using the time interval transformation, problem (1) is converted into the following Bolza problem:

$$\begin{aligned} \text{Min} \quad & J = \Phi\left(x(-1), t_0, x(1), t_f\right) \\ & \quad + \frac{t_f - t_0}{2}\int_{-1}^{1} g\big(x(\tau), u(\tau), \tau, t_0, t_f\big)\mathrm{d}\tau \\ \text{s.t.} \quad & \frac{dx}{d\tau} = \frac{t_f - t_0}{2} f\left(x(\tau), u(\tau), \tau, t_0, t_f\right) \\ & E\left(x(-1), t_0, x(1), t_f\right) = 0 \\ & C(x(\tau), u(\tau), \tau) \le 0, \tau \in [-1 \quad 1] \end{aligned} \tag{3}$$

On this basis, the control vector and state vector are further simultaneously approximated by Lagrange polynomials. Suppose the $N+1$ group of discrete points $\{\tau_1, \tau_2, \cdots, \tau_{N+1}\}$ and the corresponding values $\{g_1, g_2, \cdots, g_{N+1}\}$ are known, the interpolation can be performed using the $N$-th degree polynomial to obtain:

$$g(\tau) \approx G(\tau) = \sum_{i=1}^{N+1} \hat{L}_i(\tau) g_i \tag{4}$$

where $G(\tau)$ is the approximation of function $g(\tau)$ at time point $\tau$; $\hat{L}_i(\tau)$ is the $N$-th order Lagrange interpolation polynomial, which is calculated by the follow equation:

$$\hat{L}_i(\tau) = \prod_{j=1, j\neq i}^{N+1} \frac{\tau - \tau_j}{\tau_i - \tau_j} \tag{5}$$

Assume $\tau_s$ $(s = 1, 2, \ldots, N+1)$ is the $s$-th discrete time point. It is easy to know that Eq. (5) has the following characteristic:

$$\hat{L}_i(\tau_s) = \begin{cases} 1, & i = s \\ 0, & i \neq s \end{cases} \tag{6}$$

Therefore, $G(\tau_s) = g_s$ can be obtained at discrete point $\tau_s$.

However, how to choose these discrete points is a key. In this work, Legendre polynomial [30] (defined by the equation stated below) is employed for choosing these points in time interval $[-1, 1]$:

$$\begin{aligned} & P_n(\tau) = \frac{1}{(2^n)n!}\frac{\mathrm{d}^n}{\mathrm{d}\tau^n}\left[(\tau^2 - 1)^n\right], \ (n = 1, 2, \ldots) \\ & P_0(\tau) = 1 \end{aligned} \tag{7}$$

By solving the $N$-th order Legendre polynomial of Eq. (7), $N$ discrete points can then be obtained in the interval $(-1, 1)$. These points are called Legendre Gauss (LG) collocation points in this work and the calculation is based on Theorem 1.

*Theorem 1:* ***Assume the N-th order Legendre polynomial is:***

$$P_N(\tau) = \frac{1}{(2^N)N!}\frac{\mathrm{d}^N}{\mathrm{d}\tau^N}\left[(\tau^2 - 1)^N\right] \tag{8}$$

***It can be recursively obtained by using the equation below:***

$$\begin{aligned} & P_{n+1}(\tau) = \frac{(2n+1)}{(n+1)}\tau P_n(\tau) - \frac{n}{(n+1)}P_{n-1}(\tau), \\ & \qquad n = 1, 2, \ldots, N-1 \\ & P_0(\tau) = 1, \quad P_1(\tau) = \tau \end{aligned} \tag{9}$$

The proof of Theorem 1 please see Ref. [31].

Therefore, by solving $P_N(\tau) = 0$, these $N$ LG collocation points will be obtained. Since $\tau_0 = -1$, the approximation of state vector on these $N+1$ LG collocation points can be obtained as follows:

$$\begin{aligned} \boldsymbol{x}(\tau) &\approx \boldsymbol{X}(\tau)=\sum_{i=0}^{N} L_i(\tau)\boldsymbol{X}(\tau_i)=\sum_{i=0}^{N} L_i(\tau)\boldsymbol{X}_i \\ L_i(\tau) &= \prod_{j=0,j\neq i}^{N} \frac{\tau-\tau_j}{\tau_i-\tau_j}=\frac{b(\tau)}{(\tau-\tau_i)\dot{b}(\tau_i)} \\ b(\tau) &= \prod_{i=0}^{N}(\tau-\tau_i) \end{aligned} \tag{10}$$

where $\boldsymbol{X}_i$ is the value of state vector at discrete point $\tau_i$.

Similarly, control vector can also be approximately expressed by these LG collocation points as follows:

$$\boldsymbol{u}(\tau)\approx \boldsymbol{U}(\tau)=\sum_{i=0}^{N} L_i(\tau)\boldsymbol{U}(\tau_i) \tag{11}$$

Consequently, it is easy to calculate the derivation of state vector as follows,

$$\dot{\boldsymbol{x}}(\tau)\approx \dot{\boldsymbol{X}}(\tau)=\sum_{i=0}^{N} \dot{L}_i(\tau)\boldsymbol{X}_i \tag{12}$$

By considering the derivative of the state vector on LG collocation points, the derivative values of Eq. (12) then can be expressed as follows,

$$\dot{\boldsymbol{x}}(\tau_k)\approx \dot{\boldsymbol{X}}(\tau_k)=\sum_{i=0}^{N} \dot{L}_i(\tau_k)\boldsymbol{X}_i=\sum_{i=0}^{N} D_{k,i}\boldsymbol{X}_i \tag{13}$$

where,

$$D_{k,i}=\dot{L}_i(\tau_k)=\begin{cases} \dfrac{\dot{b}(\tau_k)}{(\tau_k-\tau_i)\dot{b}(\tau_i)}, & k\neq i \\ \dfrac{\ddot{b}(\tau_k)}{2\dot{b}(\tau_k)}, & k=i \end{cases} \quad 0=1,2,\ldots,N \tag{14}$$

Thus, the equation of state can be replaced by the following constraints:

$$\sum_{i=0}^{N} D_{k,i}\mathbf{X}_i=\frac{t_f-t_0}{2}f(\boldsymbol{X}_k,\boldsymbol{U}_k,\tau_k;t_0,t_f),\ k=1,2,\ldots,N \tag{15}$$

Based on this, the integration of the Lagrange item can be expressed as:

$$\int_{-1}^{1} g(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau),\tau;t_0,t_f)\mathrm{d}\tau \approx \sum_{k=1}^{N} w_k g(\boldsymbol{x}(\tau_k),\boldsymbol{u}(\tau_k),\tau_k;t_0,t_f) \tag{16}$$

where $w_k$ is the integral weight in Gauss integral formula and is stated as follows:

$$w_k=\frac{2}{(1-\tau_k^2)(\dot{P}_N(\tau_k))^2},\quad k=1,2,\ldots,N \tag{17}$$

Finally, by using the LG collocation discretization, Problem (3) is translated into the following NLP problem:

$$\begin{aligned} \text{Min}\quad & J=\phi(\boldsymbol{X}_0,t_0,\boldsymbol{X}_f,t_f) + \frac{t_f-t_0}{2}\sum_{k=1}^{N} w_k g(\boldsymbol{X}_k,\boldsymbol{U}_k,\tau_k;t_0,t_f) \\ \text{s.t.}\quad & \sum_{i=0}^{N} D_{k,i}\boldsymbol{X}_i-\frac{t_f-t_0}{2}f(\boldsymbol{X}_k,\boldsymbol{U}_k,\tau_k;t_0,t_f)=0 \\ & \boldsymbol{E}(\boldsymbol{X}_0,t_0,\boldsymbol{X}_f,t_f)=0 \\ & \boldsymbol{C}(\boldsymbol{X}_k,\boldsymbol{U}_k,\tau_k;t_0,t_f)\leq 0 \\ & \boldsymbol{X}_f-\boldsymbol{X}_0-\sum_{i=0}^{N}\boldsymbol{X}_i\sum_{k=1}^{N} w_i D_{k,i}=0,\ k=1,2,\ldots,N \end{aligned} \tag{18}$$

*Remark 1:* The convergence analysis of orthogonal collocation optimization is omitted as which has been proved in Ref. [32], thus this method is stable. Furthermore, it is obvious that the goal of Problem (18) is to find a set of variables $(\boldsymbol{X}_k, \boldsymbol{U}_k)$, $k = 1, 2, \ldots, N$ to minimize the performance index $J$ and the solution is the approximation of Problem (3). Typically, gradient-based NLP methods, such as sequential quadratic method or interior point method can be used to solve the reformulated Problem (3) with high precision [33]. Eventually, the optimal control curves and corresponding state curves can be obtained.

## III. STATE PLANNING-BASED LQR TRACKING CONTROL

As it is discussed in Section II, the optimal control strategy and state curves are obtained after solving optimization problem (18). Theoretically, control strategy should be used as the control input. However, this is a kind of open-loop control, when the dynamic system is mismatch or disturbance occurs, the control performance may be affected. Since linear quadratic regulator (LQR) control has good ability to tackle disturbances and is efficient for tracking control, the optimal state planning-based LQR tracking control method is proposed in this work to tackle these disturbances. Generally, the functional performance of LQR is to minimize the tracking error and control consumption [34]. Therefore, the optimal state vector obtained from problem (18) is regarded as the ideal output and LQR controller is designed to track the route. The detailed process of the proposed method is stated as follows.

Firstly, suppose the dynamic system of problem (1) can be converted into linear system in state space.

$$\begin{cases} \dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+B\boldsymbol{u}(t) \\ \boldsymbol{y}(t)=C\boldsymbol{x}(t) \end{cases} \tag{19}$$

Denote the ideal output as $\boldsymbol{y}_d(t)$, which is the optimal state vector of problem (18) in time interval $[t_0, t_f]$. Then, the state planning-based LQR tracking control problem can be described below.

Giving the ideal output $\boldsymbol{y}_d(t)$ and initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, find a feasible control strategy to minimize the functional performance $J_{\text{LQR}}$ listed below in finite time interval $t \in [t_0, t_f]$.

$$J_{\text{LQR}} = \frac{1}{2}\int_{t_0}^{t_f} \{\boldsymbol{e}^T(t)Q\boldsymbol{e}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)\}\mathrm{d}t \tag{20}$$

where $\boldsymbol{e}(t) = \boldsymbol{y}_d(t) - \boldsymbol{y}(t)$, $Q$ is the dynamic output error weight matrix and $R$ is the dynamic control weight matrix.

To achieve the minimal $J_{\text{LQR}}$, Hamilton function is constructed by introducing Lagrange multiplier $\boldsymbol{\lambda}(t)$:

$$\begin{aligned} &H(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t), t) \\ &= \frac{1}{2}\left[\boldsymbol{e}^T(t)Q\boldsymbol{e}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)\right] + \boldsymbol{\lambda}^T(t)\left[A\boldsymbol{x}(t) + B\boldsymbol{u}(t)\right] \end{aligned} \tag{21}$$

Accordingly, by using the minimum principle, the following equations can be correspondingly calculated:

1) Governing equation of the system:

$$\begin{aligned} \frac{\partial H}{\partial \boldsymbol{u}(t)} &= R\boldsymbol{u}(t) + B^T\boldsymbol{\lambda}(t) = 0 \\ \boldsymbol{u}(t) &= -R^{-1}B^T\boldsymbol{\lambda}(t) \end{aligned} \tag{22}$$

2) State equation of dynamic system:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) = A\boldsymbol{x}(t) - BR^{-1}B^T\boldsymbol{\lambda}(t) \tag{23}$$

3) Concomitant equation of the system:

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial H}{\partial \boldsymbol{x}(t)} = -C^TQC\boldsymbol{x}(t) + C^TQ\boldsymbol{y}_d(t) - A^T\boldsymbol{\lambda}(t) \tag{24}$$

4) Since the functional performance function $J_{\text{LQR}}$ only contains comprehensive performance indicators, the cross-section condition $\boldsymbol{\lambda}(t_f) = 0$.

To sum up, based on Eq. (22) and Eq. (24), the optimal control input of system can be described as follows:

$$\begin{aligned} \boldsymbol{u}^*(t) &= -R^{-1}B^T\boldsymbol{\lambda}^*(t) \\ \boldsymbol{\lambda}^*(t) &= \bar{P}\boldsymbol{x}(t) - \bar{g} \end{aligned} \tag{25}$$

where $\bar{P}$ is the solution of Riccati algebraic equation and $\bar{g}$ is defined as a feedback gain vector.

Based on above definitions and derivation, the optimal control $\boldsymbol{u}^*(t)$ can be obtained based on Inference 1 defined below.

**Inference 1. Suppose $[A, B]$ is controllable, $[A, C]$ is observable and $R$ is positive definite, the optimal control of controlled system (19) under performance indicator (20) is:**

$$\boldsymbol{u}^*(t) = -R^{-1}B^T\bar{P}\boldsymbol{x}(t) + R^{-1}B^T\bar{g} \tag{26}$$

**where matrix $\bar{P}$ is the positive definite solution of the following Riccati algebraic equation:**

$$\bar{P}A + A^T\bar{P} - PBR^{-1}B^T\bar{P} + C^TQC = 0 \tag{27}$$

Generally, feedback gain vector is approximately obtained by using the follow equation:

$$\bar{g} \approx (\bar{P}BR^{-1}B^T - A^T)^{-1}C^TQ\boldsymbol{y}_d(t) \tag{28}$$

Correspondingly, it can be seen that the optimal state trajectory should satisfy the following equation:

$$\dot{\boldsymbol{x}}^*(t) = \left[A - BR^{-1}B^T\bar{P}\right]\boldsymbol{x}(t) + BR^{-1}B^T\bar{g} \tag{29}$$

By defining $K = -R^{-1}B^T\bar{P}$, the structure of state planning-based linear quadratic regulator (LQR) tracking control is established in Fig. 1.

**Remark 2.** It should be noted that the reference input of tracking system is ideal output $\boldsymbol{y}_d(t)$, which comes from the optimal solution of problem (18). Thus, constant vector $\bar{g}$ is constituted by a sub-feedback module and the input is $\boldsymbol{y}_d(t)$. After solving the Riccati matrix differential equation, $\bar{g}$ then is uniquely calculated. Meanwhile, by adjusting $Q$ and $R$, the system output can track the expected output. In this work, the trial and error method is employed to tune $Q$ and $R$.

The computation complexity of the proposed method can be analyzed as follows. Suppose the dimensional of state vector $\boldsymbol{x}(t)$ is $x_n$ and $\boldsymbol{u}(t)$ is a $u_n$ dimensional vector. By using the orthogonal collocation discretization, it is obvious that $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ are then discretized into a set of variables $(\boldsymbol{X}_k, \boldsymbol{U}_k)$, $k = 1, 2, \ldots, N$. Furthermore, since the optimal control problem (1) is finally transformed into a NLP problem (18). Assume that the optimization iteration of the NLP solver is $N_{opt}$, the computation complexity of OC optimization is,

$$\text{T(OC)} = \text{O}((x_n + u_n) \times N \times N_{opt}) \tag{30}$$

Next, since OC-LQR method introduces the LQR design, the computation complexity of LQR should be considered. From above discussion, it can be found that the core of LQR design is to calculate Eq. (28) and Eq. (29). Using linear transformation strategy to obtain the ideal input $\boldsymbol{y}_d(t)$ and then to calculate feedback $\bar{g}$ and matrix $\bar{P}$ is the main computation. Assume the dimensional of $\boldsymbol{y}_d(t)$ is $y_d$, the computation complexity of LQR is

$$\text{T(LQR)} = \text{O}(x_n \times y_d \times u_n) \tag{31}$$

Finally, the computation complexity of OC-LQR method is,

$$\text{T(OC-LQR)} = \text{O}(\max\{(x_n + u_n) \times N \times N_{opt}, x_n \times y_d \times u_n\}) \tag{32}$$

## IV. ALGORITHM IMPLEMENTATION

Based on the specific derivation of state planning-based linear quadratic regulator (LQR) tracking control method, the implementation of LQR tracking control design combined with orthogonal collocation optimization is given. The flow chart of the algorithm is shown in Fig 2. The detailed algorithm steps are stated below:

*Step 1: Input parameters of the Bolza optimal control problem and assign initial values of state vector and control vector.*

*Step 2:Employ the proposed orthogonal collocation optimization method to solve optimal control problem so as to obtain the optimal control vector and corresponding optimal state vector.*

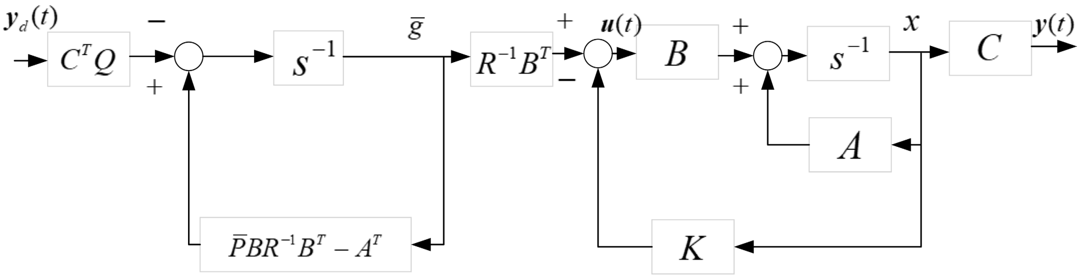



**FIGURE 1.** Structure of state planning-based LQR tracking control system.

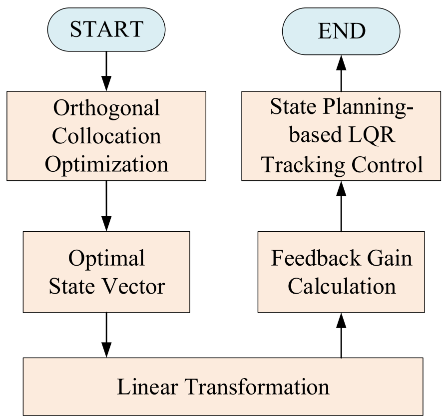



**FIGURE 2.** Flow chart of the proposed algorithm.

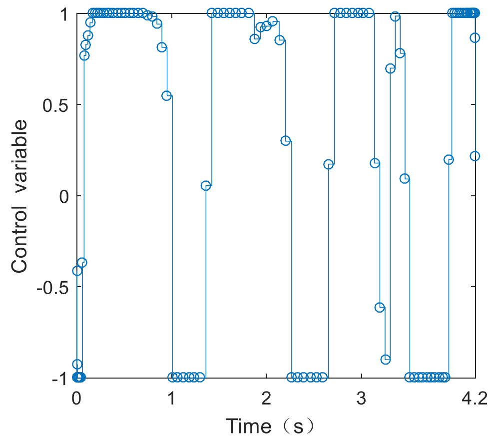



**FIGURE 3.** Optimal control vector curve.

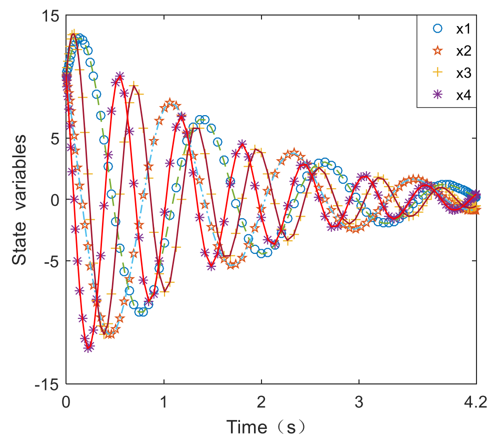



**FIGURE 4.** Optimal state vector curves.

*Step 3: Rearrange the obtained optimal state vector and get the ideal input* $\mathbf{y}_d(t)$. *Select appropriate weight matrices Q and R. Using the linear transformation strategy to calculate feedback gain vector* $\bar{g}$ *and then calculate matrix* $\bar{P}$ *based on Riccati algebraic Eq.* (27).

*Step 4: Obtain the feedback gain matrix K. Then, calculate the state planning-based optimal control rate* $\mathbf{u}^*(t)$ *by using Eq.* (26).

*Step 5: Using the obtained feedback gain matrix K to establish the feedback control structure.*

*Step 6: Input the optimal control vector* $\mathbf{u}^*(t)$ *into the dynamic model and monitor the output.*

## V. SIMULATION TESTING

In order to verify the effectiveness of the proposed algorithm for process optimization with disturbances, the classical Nishida optimal control problem [35] is tested. In addition, results of the proposed method are also compared with the optimal strategy obtained by orthogonal collocation optimization control. All simulation studies are carried out on a personal computer (Intel CORE i5/2.3GHz CPU and DDR4/2400MHz 4GB memory) in MATLAB platform.

### A. ORTHOGONAL COLLOCATION OPTIMIZATION-BASED STATE PLANNING

Briefly, the mathematical model of Nishida problem is arranged as follows:

$$\min\ J = x_1^2(t_f) + x_2^2(t_f) + x_3^2(t_f) + x_4^2(t_f)$$

$$\begin{aligned} \text{s.t. } \dot{x}_1(t) &= -0.5x_1(t) + 5x_2(t) \\ \dot{x}_2(t) &= -5x_1(t) - 0.5x_2(t) + u(t) \\ \dot{x}_3(t) &= -0.6x_3(t) + 10x_4(t) \\ \dot{x}_4(t) &= -10x_3(t) - 0.6x_4(t) + u(t) \\ 1 &\leq u(t) \leq 1 \\ t_f &= 4.2\ \ (s) \end{aligned} \tag{33}$$

By calculating the roots of this system, it can be found that all poles are located in the left half region of the origin, thus the system is stable. During the orthogonal

**TABLE 1.** Optimization results of Nishida problem.

| Method | Control vector discretization | State vector discretization | $J_{(\min)}$ | Computation time (s) |
|---|---|---|---|---|
| CVP[36] | Piecewise constant parameterization | Runge-Kutta | 1.2342 | 2.73 |
| DOT-CVP[37] | Piecewise constant parameterization | —— | 1.2347 | —— |
| OC | LG collocation | LG collocation | 1.2214 | 0.65 |

Note: —— means unknown.

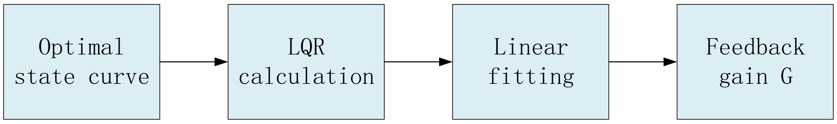



**FIGURE 5.** Calculation process of feedback gain.

collocation optimization, the initial state variables are set as $x(0) = [10; 10; 10; 10]$ and the initial decision variable is set to $u(0) = 0$. The time interval of the dynamic process is $[0, 4.2]$. To verify the validity of optimization results obtained by orthogonal collocation method, the classical piecewise-constant control vector parameterization (CVP) method presented in Ref. [36] and an improved DOT-CVP method proposed in Ref. [37] are also employed to make comparisons. The corresponding test results are shown in Table 1. It can be seen that OC method obtains the similar result as CVP and DOT-CVP methods. Correspondingly, Fig. 3 the gives optimal control curve of the orthogonal collocation method and Fig. 4 shows the optimal state curves. Next, by using linear transformation, the optimal state vector is employed to calculate the feedback gain $G$. Furthermore, the OC control, which inputs the optimal control curve obtained by OC method directly into dynamic model, is also executed to compare the control performance with the proposed state planning-based LQR tracking control.

### B. OC-LQR AND OC CONTROL SIMULATION

Based on the optimization results of OC method, the obtained optimal state vector is employed as input signal for LQR control, the calculation process of feedback gain $G$ is shown in Fig. 5. The Simulink structures of OC-LQR control and OC control in MATALB are given in Fig. 6 and Fig. 7, respectively. OC control is regarded as a conventional optimal controller to make comparison with OC-LQR controller. In Fig. 6, feedback increment $G$ is calculated according to the optimal state vector and then is used as the desired output. Fig. 7 is an open-loop Simulink model of OC control, where the optimal control vector of OC optimization is directly used as the control signal.

In order to make the comparison of OC and OC-LQR control methods fair, all simulation studies of the two control methods are carried out on a same platform. Meanwhile, the corresponding parameters, including the running time, model parameters and simulation parameters are all set the same.

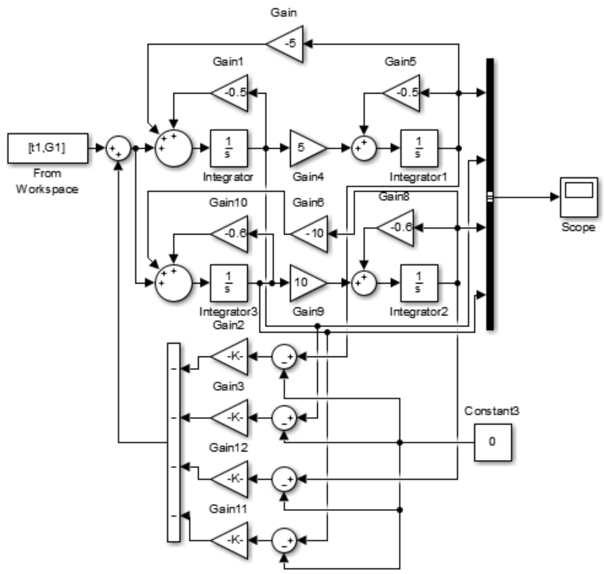



**FIGURE 6.** Simulink structure of OC-LQR control.

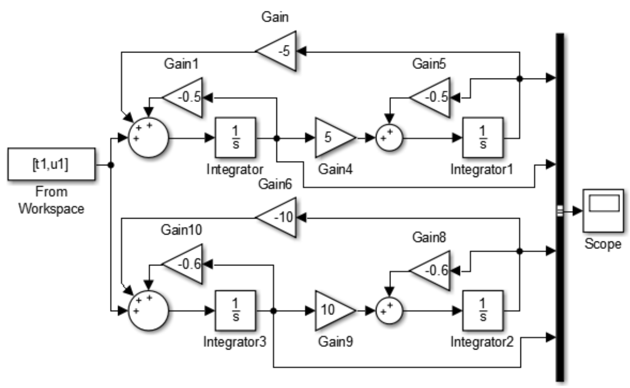



**FIGURE 7.** Simulink structure of OC control.

By setting the simulation time as 4.2 seconds, test results of the above two structures are then obtained. The numerical results are shown in Fig. 8-10, where Fig. 8 shows the control

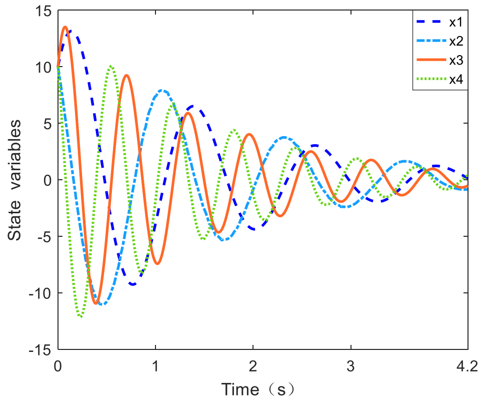



**FIGURE 8.** OC-LQR tracking control performance.

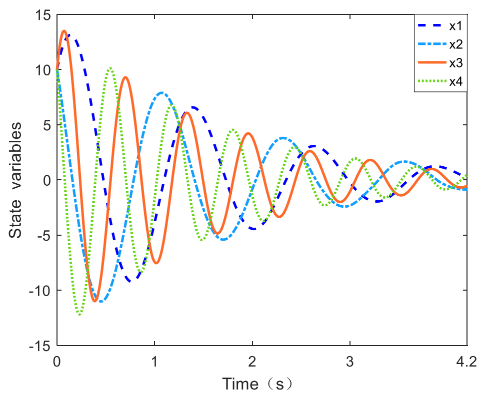



**FIGURE 9.** OC control performance.

performance of state planning-based LQR tracking control and Fig. 9 reveals the state tracking curve under OC control. The state tracking curve comparison of the two methods is given in Fig. 10. It is obvious that the two methods obtain the similar and excellent tracking performance. Results show that the vertical error of state tracking is less than 0.01, revealing the effectiveness of the OC-LQR tracking control algorithm.

### C. MODEL DISTURBANCE TEST

In Section V.B, OC-LQR and OC control methods both can efficiently track the optimal state vector when the dynamic model is accurate. However, since various disturbances exist in engineering processes, it is difficult to obtain the exact models of dynamic processes and the models will also mismatch. In order to test the adaptability of OC-LQR tracking control algorithm for dynamic model disturbances. The model disturbance test is carried out. Similarly, the OC control method is executed as a comparison. During the simulation test, the same disturbances are directly added to the model parameter module in the Simulink structures of OC control and OC-LQR control.

Without loss of generality, 10% and 30% white noise are inputted into each feedback gain of the Nishida dynamic model as model disturbance in Fig. 6 and Fig. 7. The corresponding test results are given in Fig. 11 and Fig. 12, respectively. It can be clearly obtained that the state tracking curves obtained by OC-LQR tracking control algorithm have less fluctuation and higher stability than that of OC control method under 10% and 30% white noise. Especially between the 2 and 3 process time, it can be clearly observed in Fig. 11-12 that the state tracking curves obtained by OC control method fluctuate greatly, and the anti-interference ability is inferior to the method proposed in this paper. These results show that the proposed algorithm has good control stability and can reduce tracking errors caused by the dynamic model disturbance.

Furthermore, the mean absolute error (MAE) is employed to analyze the tracking errors (compare with the optimal state curves) of the two methods under 10 and 30% white noise. Table 2 shows these results. It can be found that the MAEs of OC-LQR and OC methods are 0.8974 and 1.4039 under 10% white noise input. Compared with OC control method, statistical analysis shows that the proposed method can reduce 36.08% MAE tracking error (0.5065). Meanwhile, when the white noise is 30%, tracking errors of both two methods increase, where the OC-LQR method adds 0.6461 and the increment of OC method is 1.0171. MAE analysis results show that the proposed method reduces 36.17% tracking error averagely. Correspondingly, by analyzing the minimal performance indexes (MPIs) in Table 3, it is found that disturbances will affect the MPI, but the MPI increasing amounts of OC-LQR method are obviously smaller than those of OC method under 10% and 30% white noise disturbances, which revealing the effectiveness of the proposed OC-LQR method in this paper.

### D. GAUSSIAN MIXTURE NOISE TEST

To further verify the efficacy of the proposed approach in practical scenario, Gaussian mixture noise [38], which is commonly used to simulate the real-world disturbance, is introduced for testing. Together with model disturbances (10% and 30% white noise), Gaussian noise (mean: 1, variance: 1) is added to the system input channel of dynamic model (see Fig. 6 and Fig. 7) as external disturbance. For convenience, these disturbances are named as 10% and 30% Gaussian mixture noise. The corresponding test results are shown in Fig. 13 and Fig. 14, respectively. Based on the simulation results, it is obvious that OC-LQR tracking control algorithm can still track the state curves well and the stability is better than OC control method. By comparing the state tracking curves of two methods under 30% Gaussian mixture noise in Fig. 14, it can be also found that oscillation amplitude of OC-LQR tracking control method is smaller than that of OC control method, further revealing the effectiveness of OC-LQR method.

In addition, MAE analysis and MPI analysis are given in Table 4 and Table 5, respectively. Results in Table 4 show that the MAEs of OC-LQR control method are 0.9501 and 1.5812 under 10% and 30% Gaussian mixture noise. While, the corresponding results of OC control method are 1.3894 and 2.4035, which reveals that the increase of Gaussian mixture noise causes smaller impact to

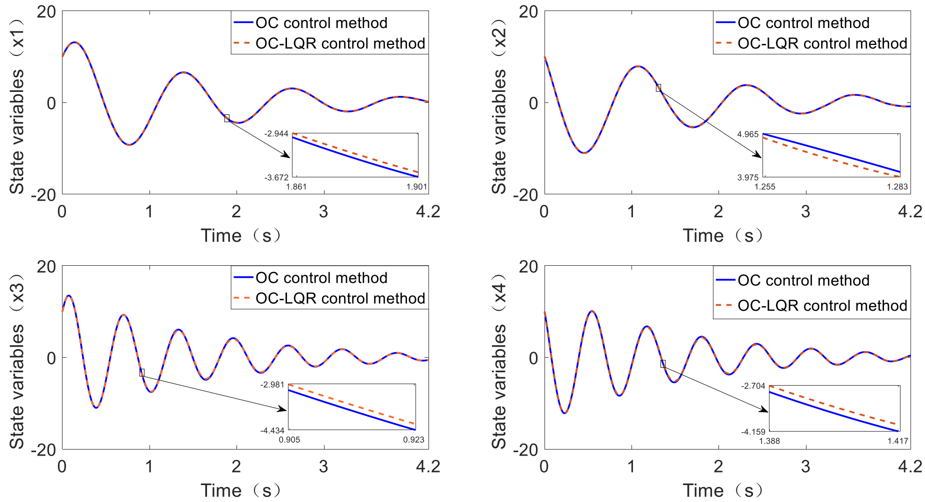
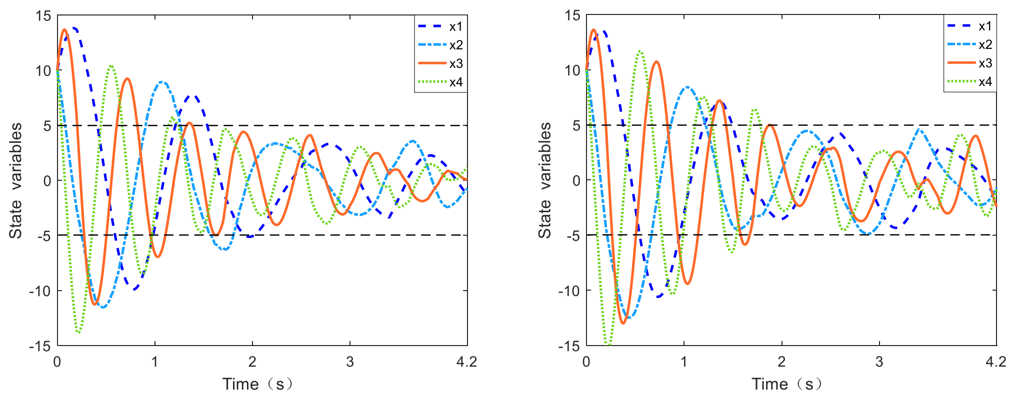


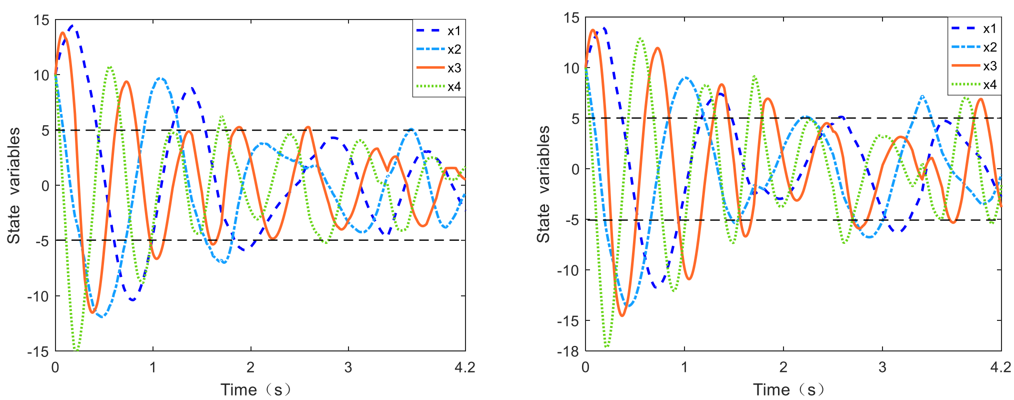

**FIGURE 10.** State tracking curves of OC and OC-LQR methods.




(a) OC-LQR tracking control method

(b) OC control method

**FIGURE 11.** State tracking curves of two methods with 10% white noise model disturbance.




(a) OC-LQR tracking control method

(b) OC control method

**FIGURE 12.** State tracking curves of two methods with 30% white noise model disturbance.

OC-LQR control than that of OC control. Compared with OC method, it is summed up that the MAE of OC-LQR averagely decreases by 32.92%. Therefore, the proposed LQR tracking controller design is efficient to track the state curves with internal model disturbance and external disturbance. Meanwhile, MPI results in Table 5 also indicate that LQR-OC

**TABLE 2.** Mean absolute error analysis results of model disturbance test.

| Method | 10% white noise | 30% white noise | MAE average value | Running time (s) |
|---|---|---|---|---|
| OC-LQR control | 0.8974 | 1.5435 | 1.2205 | 4.2 |
| OC control | 1.4039 | 2.4210 | 1.9124 | 4.2 |
| Tracking error decrease percentage | 36.08% | 36.25% | 36.17% | — |

**TABLE 3.** Minimal performance index analysis of model disturbance test.

| Method | No disturbance | 10% white noise | 30% white noise | Average |
|---|---|---|---|---|
| OC-LQR control | 1.2239 | 3.4945 | 8.8165 | 6.1555 |
| OC control | 1.2214 | 9.3294 | 25.2590 | 17.2942 |
| MPI improvement | — | 62.54% | 65.10% | 64.41% |

**TABLE 4.** Mean absolute error analysis results of Gaussian mixture noise test.

| Method | 10% white noise + Gaussian noise | 30% white noise + Gaussian noise | Average | Running time (s) |
|---|---|---|---|---|
| OC-LQR control | 0.9501 | 1.5812 | 1.2657 | 4.2 |
| OC control | 1.3894 | 2.4035 | 1.8965 | 4.2 |
| Error decrease percentage | 31.62% | 34.21% | 32.92% | — |

**TABLE 5.** Minimal performance index analysis of Gaussian mixture noise test.

| Method | 10% white noise + Gaussian noise | 30% white noise + Gaussian noise | Average |
|---|---|---|---|
| OC-LQR control | 3.3912 | 8.6289 | 6.0100 |
| OC control | 9.0097 | 24.8404 | 16.9250 |
| MPI Improvement | 62.36% | 65.26% | 64.49% |

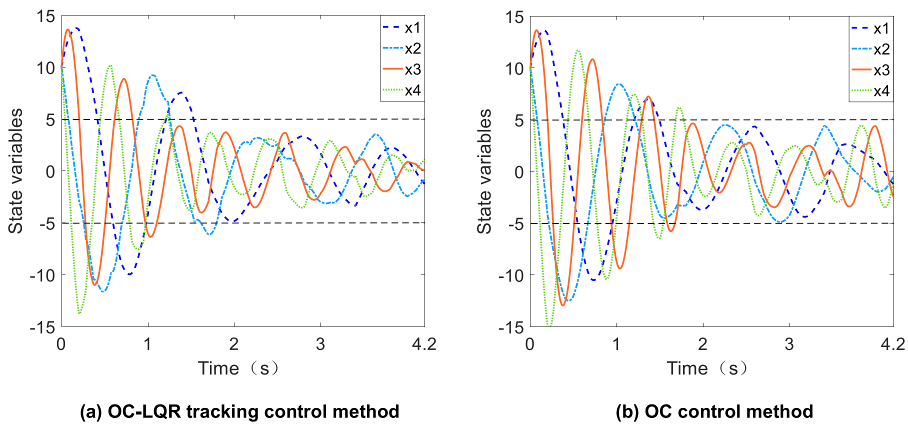



**FIGURE 13.** State tracking curves of two methods with 10% Gaussian mixture noise disturbance.

control can efficiently reduce the impact of Gaussian mixture noise to MPI when compared with OC control, showing the effectiveness of the proposed OC-LQR control frame for handling internal and external disturbances.

## VI. RESULT ANALYSIS

By analyzing the simulation testing results, it can be summarized that orthogonal collocation optimization is efficient to obtain the optimal control strategy under certain objective function, where the computation time is less than that of CVP method (see Table 1), which shows that OC method has good potential for online control application. Meanwhile, by using the obtained optimal control curve, corresponding optimal state curves can be transformed (see Fig. 5). Thus, the LQR control can easily employ these curves to construct feedback control so that the OC-LQR control can be established. Simulation results of Fig. 8-10 reveal that OC control and OC-LQR control both can guarantee the state curve performance when there is no disturbance in dynamic process. However, model disturbance test show that state performances of OC control and OC-LQR control are both influenced by white noise disturbance. States curves in Fig. 11 and 12 indicate that the fluctuation of OC control is bigger than that of OC-LQR control. Meanwhile, by using MAE analysis to compare the

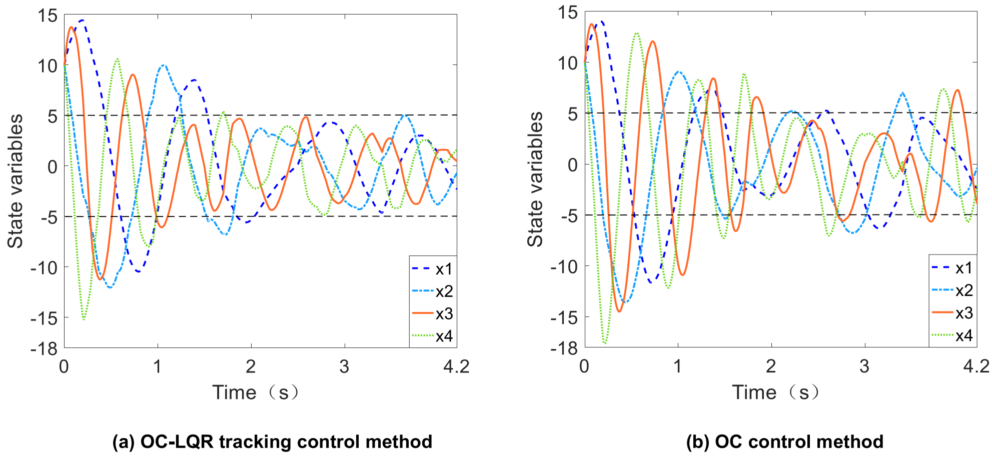



**FIGURE 14.** State tracking curves of two methods with 30% Gaussian mixture noise disturbance.

**TABLE 6.** Statistical analysis of test results.

| Method | Performance Type | Model disturbance test | Gaussian mixture noise test | Average improvement |
|---|---|---|---|---|
| OC-LQR control<br>OC control | MAE | 1.2205<br>1.9124 | 1.2657<br>1.8965 | 34.73% |
| OC-LQR control<br>OC control | MPI | 6.1555<br>17.2942 | 6.0100<br>16.9250 | 64.44% |

state curves of OC and OC-QR methods with optimal state curves, it is found that the MAE of OC control changes from 1.4039 to 2.4210 under 10% and 30% white noise disturbance (see Table 2 ). While, by introducing the LQR feedback strategy, OC-LQR averagely decreases the MAE by 36.17% when compared with OC control. Therefore, although model disturbance will influence the control quality of OC and OC-LQR control, the closed-loop OC-LQR can efficiently reduce the impact.

Furthermore, MPI analysis in Table 3 indicates that the influence of model disturbance to performance index is huge. MPIs of OC control increase from 1.2214 to 9.3294 and 25.2590 under 10% and 30% white noise disturbance. Correspondingly, MPIs of OC-LQR control change from 1.2239 to 3.4945 and 8.8165, respectively. It can be concluded that OC-LQR control averagely reduces the MPI influence by 64.49% when compared with OC control, revealing the effectiveness of the proposed OC-LQR design.

Gaussian mixture test, which inputs the internal white noise and external Gaussian noise to the dynamic process simultaneously, is carried out to verify the efficacy of the proposed approach. Compared with model disturbance test, results in Fig. 13 and 14 indicate that external Gaussian noise disturbance has a limited influence on MAE and MPI. The proposed OC-LQR control is efficient to handle the internal and external disturbances (see Table 4 and Table 5). Statistical analysis of model disturbance test and Gaussian mixture noise test is shown in Table 6. Statistical results show that OC-LQR control averagely decreases the MAE index by 34.73% and improves the MPI performance by 64.44%, demonstrating the efficacy of the proposed approach.

To sum up, by introducing LQR control design, the feedback control of OC-LQR can be established so that the proposed method can own the closed-loop characteristics. Thus, the control quality and performance index can be improved under internal and external disturbances.

## VII. CONCLUSION

This paper aims at combining the OC optimization with LQR tracking control to tackle the process optimal control problem with internal and external disturbances. The main concept of this work is to adopt integral design in state vector planning and tracking control for process dynamic systems. By using OC dynamic optimization method, the optimal state curves can be accordingly obtained and transformed into input signal for LQR controller, so that LQR feedback tracking control can be easily established to strengthen the control performance of dynamic system with internal or external disturbances. The performance studies of model disturbance test and Gaussian mixture noise test show that the control quality of proposed approach is steady, the mean absolute error of state curve tracking averagely reduces by 34.73% and the minimal performance index improves by 64.44% when compared with the classical orthogonal collocation optimal control method. Thus, the proposed OC-LQR control design is efficient to obtain the state planning and track the optimal state with model disturbances. Simulation results indicate that the proposed method has good application potential for model disturbance optimal control, which provides a way to design the state planning-based LQR controller. Currently, the proposed method is limited to linear dynamic systems and nonlinear systems should be linear approximated firstly, which is a shortcoming of the proposed method. Meanwhile, the proposed method is mainly applicable to systems with observability. Extensions of the proposed method can be establishing adaptive state-planning-based control methods for complex constrained nonlinear industrial dynamic processes and multiple control variables optimal control.

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