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Input-to-State Stability for Impulsive Gilpin-Ayala Competition Model With Reaction Diffusion and Delayed Feedback

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ABSTRACT This study focuses on the input-to-state stability issue for impulsive Gilpin-Ayala competition model with reaction diffusion and delayed feedback. By using a fixed point theorem, variational method and Lyapunov function method, the unique existence of globally asymptotical input-to-state stability of positive stationary solution is established under Dirichlet zero boundary value. Remarkably, it is the first paper to derive the unique existence of the stationary solution of Reaction-Diffusion (RD) Gilpin-Ayala competition model, which is globally asymptotical input-to-state stability. In the end, simulation results are presented to validate the effectiveness and feasibility of the proposed results.

INDEX TERMS Gilpin-Ayala competition model, globally asymptotical stability, Lyapunov function, Markovian jumping.

I. INTRODUCTION

Delayed ecosystem or reaction-diffusion ecosystem has been investigated for a long time (see, e.g. [1]–[4], [10], [12]–[14], [16] and the references therein). But most of the related literature only involved in the Neumann zero boundary value. In real world, Dirichlet zero boundary value can sometimes better simulate the population ecology, for example, the population density of deep-sea fish at the edge of their life circle is zero, and out of the circle may mean that they cannot adapt to the environment. Besides, the delayed feedback model is introduced in this paper, for the larval individuals in the population often have a certain growth period, and only adults can participate in the food competition among populations. Such delayed feedback models are not only suitable for biological population competition model, but also common to other dynamic models ([14]–[16]). In addition, Markov models can always simulate the competition systems of biological population with random factors and other dynamical systems ([9], [17]). Some recent works have been involved in semi-Markov jump systems ([38]–[40]), which greatly enriches the application of Markov process. In addition, multiple-species competition models are always linear ones. For example, even in 2017, Yuanyuan Liu and Youshan Tao investigated the following two-species linear competition model with cross-diffusion for one species under Neumann boundary value ([4]):

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta[(d_1 + a_{12}v)u] + \mu_1 u(1 - u - a_1 v), \\ &\qquad x \in \Omega, t > 0, \\ 0 &= \Delta v + \mu_2 v(1 - v - a_2 u), \quad x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \\ u(x, 0) &= u_0(x), \quad x \in \Omega. \end{aligned} \tag{1}$$

Until 1973, Gilpin and Ayala found that the model did not match a series of experimental data well([5]). Via accurate data analysis, they proposed the following nonlinear

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competition model with two-species:

$$\dot{x}_1(t) = x_1(t)[b_1 - a_{11}x_1^{\theta_1}(t) - a_{12}x_2(t)],$$
$$\dot{x}_2(t) = x_2(t)[b_2 - a_{21}x_1(t) - a_{22}x_2^{\theta_2}(t)], \quad (2)$$

in which θ_1, θ_2 represent the nonlinear density restrictions. As pointed out in [6]–[8] that the nonlinear density restrictions model can match well the experimental data on drosophila melanogasters when θ_i was far less than 1. From then, Gilpin-Ayala ecosystems have been investigated extensively (see, e.g. [3], [12], [13], [17]). Even various reaction-diffusion Gilpin-Ayala competition models were investigated under Neumann boundary value (see, e.g. [2]–[4]). But seldom reaction-diffusion two-species competition models were studied under Dirichlet boundary value. In fact, there are many cases suitable to the Dirichlet boundary problem. For example, deep sea fish live in a certain range of three-dimensional waters, and in their area edge, the population density of deep sea fish is zero. Besides, the living range of some pollens is also affected by their regional environment. They only spread in a certain area, and the population density of the living pollens on the edge of the area is zero. Based on the above-mentioned cases, we shall investigate the global stabilization of reaction-diffusion Gilpin-Ayala competition models. We also assume $\theta_i < 1$ in this paper so that it can match well the experimental data on drosophila melanogasters. Particularly, the two-species ecosystem's globally asymptotical stabilization means that after a long time, population densities of two species will reach to a pair of positive numbers (two positive numbers), respectively. That is, the suitable artificial intervention can make both of species alive. But it is important to prove the existence of a positive stationary solution of the ecosystem. Different from the Neumann boundary problem, our Dirichlet boundary condition of the ecosystem makes us deal with the existence of positive solutions for the corresponding elliptic equations, but ecosystems with Neumann boundary condition only make us deal with the existence of positive solutions for simple algebraic equations. Moreover, in order to achieve global stabilization, we have to solve the uniqueness of the solution of the elliptic equations. To overcome all the above-mentioned difficulties, we have to utilize the fixed point theory and variational methods to derive the unique existence of the positive stationary solution before we use impulse control technique on the ecosystem.

Furthermore, input-to-state stability was studied in many literature involved in various dynamical systems (see [18]- [28]), which is also suitable to ecosystem. In fact, putting a certain amount of food and small fry in the fish pond can be seen as the external input, which can make the dynamic of the ecosystem stabilized at a positive equilibrium point. By employing the methods used in [11], the unique existence of the stable stationary solution of RD Gilpin-Ayala competition model was obtained.

Compared with the existing references, the main contributions of this paper are as follows:

★ It is the first paper to derive the unique existence of the stationary solution of reaction-diffusion Gilpin-Ayala competition model, which is globally asymptotical input-to-state stability.

★ Different from Neumann boundary problem, the nonzero constant equilibrium point is not the solution for the ecosystem with Dirichlet boundary value (see [11]), which brings about more mathematical difficulties. In this paper, we originally propose how to impulsively control the dynamic behavior of a reaction diffusion two species competition model with Dirichlet boundary value.

★ Compared with the existing Lipschitz condition of Theorem 3.1-3.2 of [11], the generalized Lipschitz one is developed.

Besides, dynamical analysis on three-species ecosystem or singer-species ecosystem were investigated in many literature (see,e.g. [32]–[37] and the references therein), and some limit cycles were described and characterized in some previous literature. Actually, the methods of this paper can also applied to stabilize globally such ecosystems.

Throughout of this paper, the author denotes by I the identity matrix. Besides, $\|u_i\| = \sqrt{\int_{\Omega_\sigma} |\nabla u_i|^2 dx}$, and $\|u\|^2 = \sum_{i=1}^{2} \|u_i\|^2$, for $u = (u_1(x), u_2(x))^T$ with $u_i \in H_0^1(\Omega)$. Denote by λ_1 the first positive eigenvalue of Laplace operator $-\Delta$ in $H_0^1(\Omega)$. For vectors $u = (u_1, u_2)^T$, $v = (v_1, v_2)^T$, I denote $|u| = (|u_1|, |u_2|)^T$, and $u \leqslant v$ implies $u_i \leqslant v_i$, $i = 1, 2$. Matrices $A < B$ means that the symmetric matrices A, B satisfies $(B - A)$ is a positive definite matrix. Denote $|C| = (|c_{ij}|)_{2\times 2}$ for matrix $C = (c_{ij})_{2\times 2}$.

II. SYSTEM DESCRIPTIONS AND PREPARATIONS

Let λ_n be the nth positive eigenvalue of the following Dirichlet problem:

$$\Delta\varphi(x) = \lambda\varphi(x), \quad x \in \Omega,$$
$$\varphi(x)) = 0, \quad x \in \partial\Omega,$$

then λ_n satisfies $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$, and $\lambda_n \to +\infty$ if $n \to \infty$. Here, λ_1 is positive, isolated and singled, and its corresponding eigenfunctions space is one dimension space with positive eigenfunctions and negative ones (see,e.g. [11]). Particularly in $H_0^1(\Omega)$, the Poincare inequality $\int_\Omega |\nabla\varphi(x)|^2 dx \geqslant \lambda_1 \int_\Omega \varphi^2(x) dx$ holds.

Denote by $(\Upsilon, \mathcal{F}, \mathbb{P})$ the complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is described similarly as those of [15]. Consider the following delayed feedback system:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= d_1\Delta u_1 + u_1(b_1 - a_{11}u_1^{\theta_1} - a_{12}u_2) \\ &\quad + k_1(r(t))[u_1 - u_1(t - \tau_1(t), x)] + \chi_1, \\ &\quad\quad t \geqslant 0, x \in \Omega, \\ \frac{\partial u_2}{\partial t} &= d_2\Delta u_2 + u_2(b_2 - a_{21}u_1 - a_{22}u_2^{\theta_2})\end{aligned}$$

$$+k_2(r(t))[u_2 - u_2(t-\tau_2(t),x)] + \chi_2,$$
$$t \geqslant 0, x \in \Omega,$$
$$u_1(t,x) = u_2(t,x) = 0, \quad t \geqslant 0, x \in \partial\Omega, \tag{3}$$

where Ω is a domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, χ_i is a bounded continuous disturbance input with $\chi(x) = (\chi_1(x), \chi_2(x))^T$ and $0 < |\chi_i(x)| < J_i (i = 1, 2)$, and $k_1(r(t))$ and $k_2(r(t))$ are feedback benefit coefficients at mode $r(t) = r \in S$. Below, we denote $k_1(r(t)) = k_{1r}$, $k_2(r(t)) = k_{2r}$ for simple.

Remark 2.1: Here, we assume $\Omega \subset \mathbb{R}^3$. And if two species live in two dimensional plane, we can assume $u_i(t,x) = u_i(t,x_1,x_2,x_3) = u_i(t,x_1,x_2,\cdot)$, independent of the third dimension, where $x = (x_1,x_2,x_3)^T \in \Omega$.

Assume that $(u_1^*(x), u_2^*(x))$ is a positive stationary solution of the system (3). Set

$$U_1 = u_1 - u_1^*(x)$$
$$U_2 = u_2 - u_2^*(x), \tag{4}$$

and the stationary solution $(u_1^*(x), u_2^*(x))$ of the system (3) corresponds to the zero solution $(0,0)$ of the following system:

$$\frac{\partial U_1}{\partial t} = d_1 \Delta U_1 + b_1 U_1 - \Phi_1(U_1,U_2)$$
$$+k_{1r}[U_1 - U_1(t-\tau_1(t),x)], \quad t \geqslant 0,\ x \in \Omega,$$
$$\frac{\partial U_2}{\partial t} = d_2 \Delta U_2 + b_2 U_2 - \Phi_2(U_1,U_2)$$
$$+k_{2r}[U_2 - U_2(t-\tau_2(t),x)], \quad t \geqslant 0,\ x \in \Omega,$$
$$U_1(t,x) = U_2(t,x) = 0, \quad t \geqslant 0,\ x \in \partial\Omega, \tag{5}$$

or

$$\frac{\partial U_1}{\partial t} = d_1 \Delta U_1 + (b_1 + k_{1r}) U_1 - \Phi_1(U_1,U_2)$$
$$-k_{1r} U_1(t-\tau_1(t),x), \quad t \geqslant 0,\ x \in \Omega,$$
$$\frac{\partial U_2}{\partial t} = d_2 \Delta U_2 + (b_2 + k_{2r}) U_2 - \Phi_2(U_1,U_2)$$
$$-k_{2r} U_2(t-\tau_2(t),x), \quad t \geqslant 0,\ x \in \Omega,$$
$$U_1(t,x) = U_2(t,x) = 0, \quad t \geqslant 0,\ x \in \partial\Omega, \tag{6}$$

where we denote $U = (U_1, U_2)^T$, and

$$\Phi_1(U) = (U_1 + u_1^*(x))[a_{11}(U_1 + u_1^*(x))^{\theta_1} + a_{12}(U_2 + u_2^*(x))]$$
$$- u_1^*(x)(a_{11} u_1^*(x)^{\theta_1} + a_{12} u_2^*(x)),$$
$$\Phi_2(U) = (U_2 + u_2^*(x))[a_{21}(U_1 + u_1^*(x)) + a_{22}(U_2 + u_2^*(x))^{\theta_2}]$$
$$- u_2^*(x)(a_{21} u_1^*(x) + a_{22} u_2^*(x)^{\theta_2}). \tag{7}$$

The following system is the system (6) in form of vector-matrix:

$$\frac{\partial U}{\partial t} = D\Delta U + (B + K_r)U - \Phi(U)$$
$$- K_r U(t-\tau(t),x), \quad t \geqslant 0,\ x \in \Omega,$$
$$U(t,x) = 0, \quad t \geqslant 0,\ x \in \partial\Omega, \tag{8}$$

where $U = (U_1, U_2)^T$, $U(t-\tau(t),x) = (U(t-\tau_1(t),x), U(t-\tau_2(t),x))^T$, $\Phi(U) = (\Phi_1(U), \Phi_2(U))^T$ and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},\ A_k = \begin{pmatrix} a_1^{(k)} & 0 \\ 0 & a_2^{(k)} \end{pmatrix},$$
$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},\ K_r = \begin{pmatrix} k_{1r} & 0 \\ 0 & k_{2r} \end{pmatrix}. \tag{9}$$

Under impulse control on (8), one can get the following system

$$\frac{\partial U}{\partial t} = D\Delta U + (B + K_r)U - \Phi(U)$$
$$- K_r U(t-\tau(t),x), \quad t \geqslant 0,\ t \neq t_k,\ x \in \Omega,$$
$$U(t_k^+,x) = A_k U(t_k^-,x), \quad k = 1,2\cdots$$
$$U(t,x) = 0, \quad t \geqslant 0,\ x \in \partial\Omega, \tag{10}$$

where $U(t_k^-,x) = U(t_k,x)$ for all $i = 1,2$, $k = 1,2,\cdots$. Besides, the bounded initial value of the system (10) is proposed as follows,

$$U_1(s,x) = \phi_1(s,x) \geqslant 0,$$
$$U_2(s,x) = \phi_2(s,x) \geqslant 0,\ s \in [-\tau,0],\ x \in \Omega, \tag{11}$$

or

$$U(s,x) = \phi(s,x) \geqslant 0, \quad s \in [-\tau,0],\ x \in \Omega, \tag{12}$$

where $\phi(s,x) = (\phi_1(s,x), \phi_2(s,x))^T$.

III. MAIN RESULTS

Firstly assume that $\theta_i \in (0,1)$ for $i = 1,2$, just like [6]- [8].

Next, the following assumption on the population density may be necessary:

(H1) There are positive numbers M_i, N_i such that

$$0 < N_1 \leqslant u_1 \leqslant M_1,\ \ 0 < N_2 \leqslant u_2 \leqslant M_2. \tag{13}$$

Remark 3.1 : Everyone knows the fact that the population density of any species must have the bounded below, or the species will die out. For example, when the population density of whales is lower than a certain degree, it will be difficult for male and female whales to meet each other in the vast sea, leading to the extinction of the species. Besides, due to the limited resource, the population density of any species must have an supper boundedness.

Next, the following existence of positive stationary solution comes mainly from Theorem 3.1 of [11]. Of course, the ecosystem (8) is involved in non-Lipschitz functions, and so the author has to generalize the first conclusion of Theorem 3.1 of [11] from the Lipschitz condition to the generalized Lipschitz condition.

Theorem 3.1: Suppose the condition (H1) holds, $\theta_i \in (0,1)$ for $i = 1,2$ and $0 < |\chi_i| < J_i$ with $J = (J_1, J_2)^T$,

$$0 \leqslant g(u^*(x)) - J \leqslant g(u^*(x)) + J \leqslant cDE, \tag{14}$$

where $g(u) = (g_1(u_1,u_2), g_2(u_1,u_2))^T$, and

$$g_1(u_1,u_2) = u_1(b_1 - a_{11}u_1^{\theta_1} - a_{12}u_2),$$
$$g_2(u_1,u_2) = u_2(b_2 - a_{21}u_1 - a_{22}u_2^{\theta_2}), \tag{15}$$

then the system (3) possesses at least one positive bounded stationary solution (u_1^*, u_2^*).

Proof: Firstly definite the so-called generalized Lipschitz condition: $f(u_1, u_2)$ is said to satisfy the generalized Lipschitz condition if there are constants $\bar{l}_1, \bar{l}_2 > 0$ such that

$$|f(u_1, u_2) - f(v_1, v_2)| \leqslant \bar{l}_1|u_1 - v_1| + \bar{l}_2|u_2 - v_2|, \ u_i, v_i \in \mathbb{R}^1. \quad (16)$$

In fact, the first conclusion of Theorem 3.1 of [11] still holds if the Lipschitz conditions are replaced with the generalized Lipschitz condition. And hence, Theorem 3.1 is the direct corollary of Theorem 3.1 of [11]. However, in view of the integrity of the proof, we are willing to prove it in details.

Indeed, let $(u_1(x), u_2(x))$ is the stationary solution, satisfying

$$\begin{aligned} &d_1\Delta u_1 + g_1(u_1, u_2) + \chi_1 = 0, \quad x \in \Omega,\\ &d_2\Delta u_2 + g_2(u_1, u_2) + \chi_2 = 0, \quad x \in \Omega,\\ &u_1(x) = u_2(x) = 0, \quad x \in \partial\Omega, \end{aligned} \quad (17)$$

The condition (H1) yields that there are four positive constants l_1, l_2, l_3 and l_4 such that

$$|g_1(u_1, u_2) - g_1(v_1, v_2)| \leqslant l_1|u_1 - v_1| + l_2|u_2 - v_2|, \ u_i, v_i \in \mathbb{R}^1 \quad (18)$$

and

$$|g_2(u_1, u_2) - g_2(v_1, v_2)| \leqslant l_3|u_1 - v_1| + l_4|u_2 - v_2|, \ u_i, v_i \in \mathbb{R}^1, \quad (19)$$

where

$$\begin{aligned} &l_1 = b_1 + a_{11}(1+\theta_1)M_1^{\theta_1} + a_{12}M_2, \quad l_2 = a_{12}M_1,\\ &l_3 = a_{21}M_2, \ l_4 = b_2 + a_{22}(1+\theta_2)M_2^{\theta_2} + a_{21}M_1. \end{aligned} \quad (20)$$

In fact, $0 < \theta_i < 1$ and (H1) yield

$$\begin{aligned} &|g_1(u_1, u_2) - g_1(v_1, v_2)|\\ &= \left|[u_1(b_1 - a_{11}u_1^{\theta_1} - a_{12}u_2)] - [v_1(b_1 - a_{11}v_1^{\theta_1} - a_{12}v_2)]\right|\\ &\leqslant [b_1 + a_{11}(1+\theta_1)M_1^{\theta_1} + a_{12}M_2]|u_1 - v_1|\\ &\quad + a_{12}M_1|u_2 - v_2|, \end{aligned} \quad (21)$$

$$\begin{aligned} &|g_2(u_1, u_2) - g_2(v_1, v_2)|\\ &= \left|[u_2(b_2 - a_{21}u_1 - a_{22}u_2^{\theta_2})] - [v_2(b_2 - a_{21}v_1 - a_{22}v_2^{\theta_2})]\right|\\ &\leqslant a_{21}M_2|u_1 - v_1| + [b_2 + a_{22}(1+\theta_2)M_2^{\theta_2}\\ &\quad + a_{21}M_1]|u_2 - v_2|, \end{aligned} \quad (22)$$

which derives (20).

If the stationary solution of the system (3) exists, we may denote it by $u^*(x) = (u_1^*(x), u_2^*(x))^T$.

Define the operator $\mathfrak{M} : [C(\overline{\Omega_\sigma})]^2 \to [C(\overline{\Omega_\sigma})]^2$ as follows,

$$\mathfrak{M} = \begin{pmatrix} -\Delta & 0\\ 0 & -\Delta \end{pmatrix}. \quad (23)$$

The operator $\mathfrak{M}$ has the inverse operator $\mathfrak{M}^{-1}$ as follows,

$$\mathfrak{M}^{-1} = \begin{pmatrix} (-\Delta)^{-1} & 0\\ 0 & (-\Delta)^{-1} \end{pmatrix}, \quad (24)$$

where $\mathfrak{M}^{-1} : [C(\overline{\Omega_\sigma})]^2 \to [C(\overline{\Omega_\sigma})]^2$ is a linear compact positive operator (see, e.g. [11]), and

$$\mathfrak{M}u^*(x) = D^{-1}g(u^*(x)) + D^{-1}\chi, \quad x \in \Omega, \quad (25)$$

$$u^*(x) = 0, \quad x \in \partial\Omega. \quad (26)$$

It is obvious that $\left(D^{-1}g(u^*(x)) + D^{-1}\chi\right)$ is continuous for all the variables x, u_1^*, u_2^*. Define $\mathfrak{K}$ as that of the proof of Theorem 3.1 in [11], then $\mathfrak{K}$ is a positive cone, which must be a closed convex subset of $[C(\overline{\Omega_\sigma})]^2$. Define an operator $\mathfrak{T} : \mathfrak{K} \to \mathfrak{K}$ such that

$$\mathfrak{T}\varphi = \mathfrak{M}^{-1}\left(D^{-1}g(u^*(x)) + D^{-1}\chi\right), \quad \varphi \in \mathfrak{K}. \quad (27)$$

Because $\mathfrak{M}^{-1}$ is the linear positive compact operator, and $\left(D^{-1}g(u^*(x)) + D^{-1}\chi\right)$ is positive continuous, one can conclude that $\mathfrak{T} : \mathfrak{K} \to \mathfrak{K}$ is a positive compact operator.

Next, completely similar as the proof of Theorem 3.1 of [11], one can utilize the fixed point theorem (Lemma 2.1 of [11]) to prove that $\mathfrak{T}$ satisfies all the assumption conditions of Lemma 2.1 of [11], which implies that $\mathfrak{T}$ has at least a fixed point in $\mathfrak{K}$. And u^* is a bounded positive solution of the system (3).

Next, Theorem 3.2 of [11] proposed the methods which may be helpful to conclude the following uniqueness result:

Theorem 3.2: Based on the assumptions of Theorem 3.1, and suppose, in addition, the following condition is satisfied,

(H2) for any mode $r(t) = r$, there exists a scalar $\varepsilon > 0$ such that

$$\begin{pmatrix} l_1 + \varepsilon\dfrac{l_2 + l_3}{2} & 0\\ 0 & l_4 + \varepsilon^{-1}\dfrac{l_2 + l_3}{2} \end{pmatrix} < \lambda_1 D, \quad (28)$$

then the system (3) possesses the unique positive bounded stationary solution $u^*(x)$ for $x \in \Omega_\sigma$ with $u^*|_{\partial\Omega_\sigma} = 0$, where $u^*(x)$ is the positive bounded solution in Theorem 3.1, and $l_i (i = 1, 2, 3, 4)$ is defined in (20).

Proof: Assume both $u(x)$ and $v(x)$ are the stationary solutions of the system (3). Then we claim $u(x) = v(x)$.

In fact,

$$\begin{aligned} &(u(x) - v(x))^T\left(g(u(x)) - g(v(x))\right)\\ &\leqslant |u - v|^T|g(u) - g(v)|\\ &\leqslant l_1|u_1 - v_1|^2 + l_2|u_1 - v_1| \cdot |u_2 - v_2|\\ &\quad + l_3|u_1 - v_1| \cdot |u_2 - v_2| + l_4|u_2 - v_2|^2\\ &\leqslant |u - v|^T \begin{pmatrix} l_1 + \varepsilon\dfrac{l_2 + l_3}{2} & 0\\ 0 & l_4 + \varepsilon^{-1}\dfrac{l_2 + l_3}{2} \end{pmatrix} |u - v|. \end{aligned} \quad (29)$$

Below, I shall employ some methods similar as those of the proof of Theorem 3.2 of [11]. Since both $u(x)$ and $v(x)$ are the stationary solutions of the system (3), one can see it from (29), variational method and the Poincare inequality that

$$\begin{aligned}\lambda_1 \int_\Omega |u(x)-v(x)|^T D|u(x)-v(x)|dx &\leqslant \int_\Omega |\nabla(u(x)-v(x))|^T D|\nabla(u(x)-v(x))|dx \\ &= \int_\Omega (u-v)^T[g(u)-g(v)]dx \\ &\leqslant |u-v|^T \begin{pmatrix} l_1+\varepsilon\frac{l_2+l_3}{2} & 0 \\ 0 & l_4+\varepsilon^{-1}\frac{l_2+l_3}{2}\end{pmatrix}|u-v|. \end{aligned} \tag{30}$$

Now the condition (H2) yields the claim via the proof by contradiction. And so the system (3) possesses a unique positive bounded stationaru solution $u^*(x)$ for $x \in \Omega_\sigma$ with $u^*|_{\partial\Omega_\sigma} = 0$.

Below, I shall prove that the above-mentioned positive bounded vector function $u^*(x)$ is globally exponentially stable, which is the unique stationary solution of the system (3), corresponding to the null solution of the system (10).

Theorem 3.3 : Suppose the conditions (H1),(H2) and (14) hold. In addition, there is a sequence positive definite matrices $P_r(r \in S)$, positive numbers $w_r, \pi_r(r \in S)$, $\varepsilon, \varepsilon_1, \varepsilon_2, \gamma, \varsigma, \lambda$ such that

$$0 < \lambda_{\max} A_k^T A_k < e^{-(\varsigma+\lambda)(t_{k+1}-t_k)},\ k \in \mathbb{Z}^+, \tag{31}$$

$$\begin{aligned}&\frac{1}{w_r}\lambda_{\max}\Big(-2\lambda_1 DP_r + 2(B+K_r)P_r + \sum_{j\in S}\gamma_{rj}P_j \\ &\quad + \varepsilon_1 P_r K_r + \varepsilon_2 P_r + \varepsilon_2^{-1}\pi_r L_\Phi\Big) \\ &\quad + \frac{\gamma e^{\lambda\tau}}{w_r}\lambda_{\max}\left(\varepsilon_1^{-1}P_rK_r\right) \\ &\leqslant \varsigma - \lambda, \end{aligned} \tag{32}$$

$$0 < w_r I \leqslant P_r \leqslant \pi_r I,\ \forall r \in S, \tag{33}$$

where $\gamma \geqslant \frac{1}{\lambda_{\max}A_k^T A_k}$, $k \in \mathbb{Z}^+$, and

$$L_\Phi = 2\begin{pmatrix}\omega_1 & 0 \\ 0 & \omega_2\end{pmatrix}, \tag{34}$$

with $\omega_1 = [a_{11}(1+\theta_1)M_1^{\theta_1} + a_{12}M_2]^2 + a_{21}^2 M_2^2$ and $\omega_2 = [a_{22}(1+\theta_2)M_2^{\theta_2} + a_{21}M_1]^2 + a_{12}^2 M_1^2$,

then the unique positive bounded stationary solution $u^*(x)$ is globally exponential input-to-state stability for $0 < |\chi| < J$. At the same time, the null solution of the impulsive system (10) with initial value (11) is globally exponential input-to-state stability with the convergence rate $\frac{\lambda}{2}$.

Proof: Consider the following Lyapunov function:

$$V(t,r) = \int_\Omega U^T(t,x)P_r U(t,x)dx, \quad \forall r(t) = r \in S. \tag{35}$$

Below, the Poincare inequality is employed to deal with the diffusion item, just like the related literature (see, e.g. [23]). Let $\mathcal{L}$ be the weak infinitesimal operator (see, e.g. [23]) such that for $t \geqslant 0,\ t \neq t_k$,

$$\begin{aligned}\mathcal{L}V(t,r) &\leqslant \int_\Omega \Big(U^T[-2\lambda_1 DP_r + 2(B+K_r)P_r \\ &\quad + \sum_{j\in S}\gamma_{rj}P_j]U + [|U|^T P_r|\Phi(U)| + |\Phi(U)|^T P_r|U|] \\ &\quad + [|U|^T P_r K_r|U(t-\tau(t),x)| \\ &\quad + |U(t-\tau(t),x)|^T K_r P_r|U|]\Big)dx. \end{aligned} \tag{36}$$

On the other hand,

$$\begin{aligned}&|U|^T P_r K_r|U(t-\tau(t),x)| + |U(t-\tau(t),x)|^T K_r P_r|U| \\ &\leqslant \varepsilon_1 U^T P_r K_r U + \varepsilon_1^{-1}U^T(t-\tau(t),x)P_rK_rU(t-\tau(t),x), \end{aligned} \tag{37}$$

and

$$\begin{aligned}&|U|^T P_r|\Phi(U)| + |\Phi(U)|^T P_r|U| \\ &\leqslant \varepsilon_2 U^T P_r U + \varepsilon_2^{-1}\Phi^T(U)P_r\Phi(U) \\ &\leqslant \varepsilon_2 U^T P_r U + \varepsilon_2^{-1}\pi_r\Phi^T(U)\Phi(U). \end{aligned} \tag{38}$$

Besides,

$$\begin{aligned}|\Phi_1(U)| &= u_1(a_{11}u_1^{\theta_1} + a_{12}u_2) \\ &\quad - u_1^*(x)(a_{11}u_1^*(x)^{\theta_1} + a_{12}u_2^*(x)) \\ &\leqslant [a_{11}(1+\theta_1)M_1^{\theta_1} + a_{12}M_2]|U_1| + a_{12}M_1|U_2|. \end{aligned} \tag{39}$$

Similarly,

$$|\Phi_2(U)| \leqslant a_{21}M_2|U_1| + [a_{22}(1+\theta_2)M_2^{\theta_2} + a_{21}M_1]|U_2|. \tag{40}$$

In addition,

$$\begin{aligned}\Phi^T(U)\Phi(U) &\leqslant \Big([a_{11}(1+\theta_1)M_1^{\theta_1} + a_{12}M_2]|U_1| + a_{12}M_1|U_2|\Big)^2 \\ &\quad + \Big(a_{21}M_2|U_1| + [a_{22}(1+\theta_2)M_2^{\theta_2} \\ &\quad + a_{21}M_1]|U_2|\Big)^2 \leqslant U^T L_\Phi U. \end{aligned} \tag{41}$$

It follows by (36)-(41) that for , $t \geqslant 0,\ t \neq t_k$,

$$\begin{aligned}\mathcal{L}V(t,r) &\leqslant \int_\Omega \Big(U^T[-2\lambda_1 DP_r + 2(B+K_r)P_r \\ &\quad + \sum_{j\in S}\gamma_{rj}P_j]U + [|U|^T P_r|\Phi(U)| + |\Phi(U)|^T P_r|U|] \\ &\quad + [|U|^T P_r K_r|U(t-\tau(t),x)| \end{aligned}$$

$$
\begin{aligned}
&\quad + |U(t-\tau(t),x)|^T K_r P_r |U|]\Big)dx \\
&\leqslant \int_\Omega U^T\Big[-2\lambda_1 DP_r + 2(B+K_r)P_r \\
&\quad + \sum_{j\in S}\gamma_{rj}P_j + \varepsilon_1 P_r K_r + \varepsilon_2 P_r + \varepsilon_2^{-1}\pi_r L_\Phi\Big]Udx \\
&\quad + \varepsilon_1^{-1}\int_\Omega U^T(t-\tau(t),x)P_r K_r U(t-\tau(t),x)dx, \qquad (42)
\end{aligned}
$$

which implies that for a small enough positive number ϵ,

$$
\mathbb{E}V(t+\epsilon) - \mathbb{E}V(t) = \int_t^{t+\epsilon}\mathbb{E}\mathcal{L}V(s)ds, \qquad (43)
$$

and letting $\epsilon \to 0$, it leads to

$$
\begin{aligned}
&D^+\mathbb{E}V(t,r) \\
&\leqslant \mathbb{E}\int_\Omega U^T\Big[-2\lambda_1 DP_r + 2(B+K_r)P_r \\
&\quad + \sum_{j\in S}\gamma_{rj}P_j + \varepsilon_1 P_r K_r + \varepsilon_2 P_r + \varepsilon_2^{-1}\pi_r L_\Phi\Big]Udx \\
&\quad + \varepsilon_1^{-1}\mathbb{E}\int_\Omega U^T(t-\tau(t),x)P_r K_r U(t-\tau(t),x)dx \\
&\leqslant \frac{1}{w_r}\lambda_{\max}\Big[-2\lambda_1 DP_r + 2(B+K_r)P_r + \sum_{j\in S}\gamma_{rj}P_j \\
&\quad + \varepsilon_1 P_r K_r + \varepsilon_2 P_r + \varepsilon_2^{-1}\pi_r L_\Phi\Big]\mathbb{E}V(t,r) \\
&\quad + \frac{1}{w_r}\lambda_{\max}\Big(\varepsilon_1^{-1}P_r K_r\Big)\mathbb{E}V(t-\tau(t),r). \qquad (44)
\end{aligned}
$$

Due to the conditions (31)-(33) and the proof of Theorem 3.3 of [15], we can similarly prove and obtain the following inequality:

$$
\mathbb{E}V(t,r) \leqslant M\Big(\mathbb{E}\sup_{s\in[-\tau,0]}V(s,r)\Big)e^{-\lambda(t-t_0)}, \quad t\in[t_{k-1},t_k), \qquad (45)
$$

where $M>1$ is a constant, and $k\in\mathbb{Z}^+$, $r\in S$.

Moreover, it follows by (45) and (33) that for $t\in[t_{k-1},t_k)$, $k\in\mathbb{Z}^+$,

$$
(\min_{r\in S} w_r)\mathbb{E}\|U(t)\|^2_{L^2(\Omega)} \leqslant (\max_{r\in S}\pi_r)M\mathbb{E}\|\phi(s)\|^2_\tau e^{-\lambda(t-t_0)}, \qquad (46)
$$

where $\|\phi(s)\|^2_\tau = \sup_{s\in[-\tau,0]}\int_\Omega|\phi(s,x)|^2dx$. Similarly as the proof of Theorem 2 of [16], one can conclude from (46) that

$$
\mathbb{E}\|U(t)\|^2_{L^2(\Omega)} \leqslant \frac{\max_{r\in S}\pi_r}{\min_{r\in S}w_r}M\mathbb{E}\|\phi(s)\|^2_\tau e^{-\lambda(t-t_0)},\ t\in[t_{k-1},t_k], \qquad (47)
$$

in which the positive constant $\frac{\max_{r\in S}\pi_r}{\min_{r\in S}w_r}M$ is independent of any $r\in S$. Therefore, the unique positive bounded stationary solution $u^*(x)$ is globally exponential input-to-state stability for $0<|\chi|<J$. At the same time, the null solution of the impulsive system (10) with initial value (11) is globally exponential input-to-state stability with the convergence rate $\frac{\lambda}{2}$.

Remark 3.2 : The impulse control condition in Theorem 3.3 is easy to test, which is the advantage of the algorithm. However, we can not find out some methods to improve the Lyapunov function method in the proof of Theorem 3.3, which may make add some free-weighting matrices to the algorithm to reduce its conservatism, which is the important point to be improved in the future.

IV. NUMERICAL EXAMPLE

Example 4.1. Consider the following system:

$$
\begin{cases}
\dfrac{\partial u_1}{\partial t} = d_1\Delta u_1 + u_1(b_1 - a_{11}u_1^{\frac{1}{3}} - a_{12}u_2) \\
\qquad + k_1(r(t))[u_1 - u_1(t-\tau_1(t),x)] + \chi_1, \\
\qquad t\geqslant 0,\ x\in\Omega, \\
\dfrac{\partial u_2}{\partial t} = d_2\Delta u_2 + u_2(b_2 - a_{21}u_1 - a_{22}u_2^{\frac{1}{5}}) \\
\qquad + k_2(r(t))[u_2 - u_2(t-\tau_2(t),x)] + \chi_2, \\
\qquad t\geqslant 0,\ x\in\Omega, \\
u_1(t,x) = u_2(t,x) = 0, \quad t\geqslant 0,\ x\in\partial\Omega,
\end{cases} \qquad (48)
$$

where $\theta_1 = \frac{1}{3}, \theta_2 = \frac{1}{5}$, $\Omega = (-\frac{1}{2},\frac{1}{2})\times(-\frac{1}{2},\frac{1}{2})\times(-\frac{1}{2},\frac{1}{2})$, then $\lambda_1\geqslant 3$ (see Remark 14 of [11]). Let $d_1 = 0.5, b_1 = 1.1$, $d_2 = 0.4, b_2 = 1$, $J_1 = 0.003 = J_2$, $\varepsilon = \varepsilon_1 = \varepsilon_2 = 1$, $a_{11} = 0.002, a_{12} = 0.001, a_{21} = 0.001, a_{22} = 0.002$, $\tau = 0.01$. And $M_1 = 1.5, M_2 = 1.6, N_1 = 0.5, N_2 = 0.6$. and $c = 100000$, then direct computation yields

$$
0\leqslant g(u^*(x)) - J \leqslant g(u^*(x)) + J \leqslant cDE \qquad (49)
$$

and

$$
\begin{pmatrix} l_1 + \varepsilon\dfrac{l_2+l_3}{2} & 0 \\ 0 & l_4 + \varepsilon^{-1}\dfrac{l_2+l_3}{2}\end{pmatrix} < \lambda_1 D, \qquad (50)
$$

which means both the conditions (14) and (28) hold.

Furthermore, let $S = \{1,2\}$,

$$
\begin{aligned}
&\Pi = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -0.1 & 0.1 \\ 0.15 & -0.15\end{pmatrix}, \\
&K_1 = \begin{pmatrix} 0.0013 & 0 \\ 0 & 0.0023\end{pmatrix}, K_2 = \begin{pmatrix} 0.0012 & 0 \\ 0 & 0.0021\end{pmatrix}. \qquad (51)
\end{aligned}
$$

$$
P_1 = \begin{pmatrix} 0.9813 & 0 \\ 0 & 1.0033\end{pmatrix},\ P_2 = \begin{pmatrix} 1.0339 & 0 \\ 0 & 0.9963\end{pmatrix}, \qquad (52)
$$

and $w_1 = 0.98$, $\pi_1 = 1.005$, $w_2 = 0.99$, $\pi_2 = 1.07$, then the condition (33) holds obviously.

Assume the pulse interval $(t_{k+1} - t_k) = 0.25$, for all $k\in\mathbb{Z}^+$, and

$$
A_k \equiv \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2\end{pmatrix}, \quad k\in\mathbb{Z}^+. \qquad (53)
$$

Now, we set $\gamma = 26$, then we get $\gamma = 26 \geqslant 25 = \frac{1}{\lambda_{\max}A_k^TA_k}$, $k\in\mathbb{Z}^+$. Set $\varsigma = 5$ and $\lambda = 1$, then the

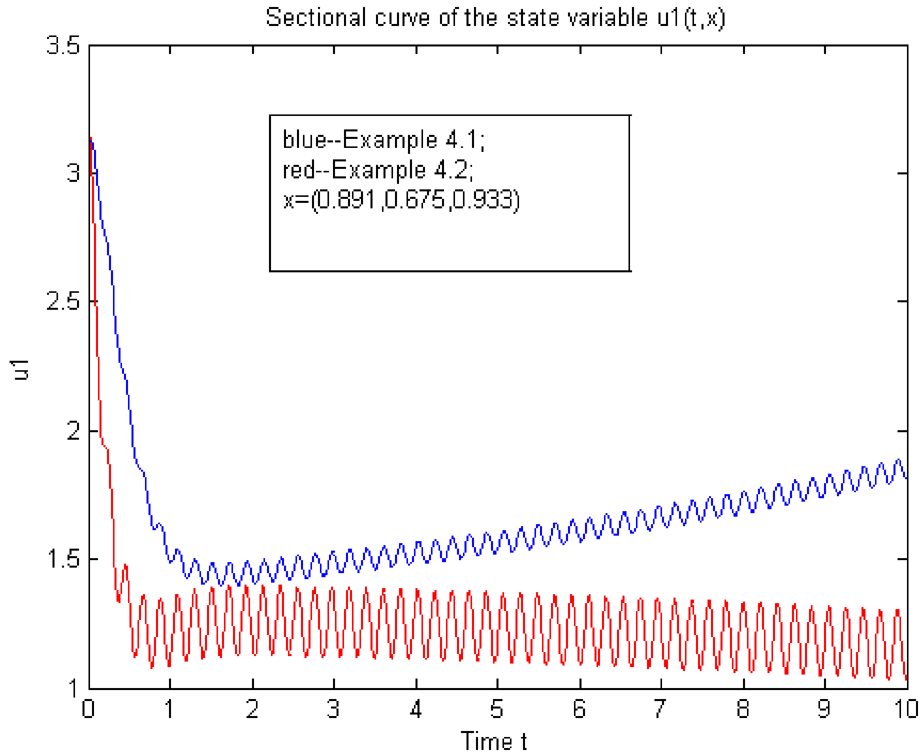

FIGURE 1. Sectional curve of u_1 of Example 4.1-4.2.

direct calculation makes the condition (32) hold. Besides, for $k \in \mathbb{Z}^+$,

$$0 < \lambda_{\max} A_k^T A_k = 0.04 < 0.8607 = e^{-(\varsigma+\lambda)(t_{k+1}-t_k)}, \quad (54)$$

which makes the condition (31) holden.

Now, all the conditions of Theorem 3.3 are satisfied. According to Theorem 3.3, the system (48) possesses the unique positive bounded stationary solution $u^*(x)$, which is globally exponential input-to-state stability with the convergence rate $\frac{\lambda}{2} = 0.5$.

Example 4.2. In Example 4.1, replace the impulse quantity (53) with the following stronger pulse amplitude:

$$A_k \equiv \begin{pmatrix} 0.01 & 0 \\ 0 & 0.02 \end{pmatrix}, \quad k \in \mathbb{Z}^+, \quad (55)$$

and the pulse interval $(t_{k+1} - t_k) \equiv 0.25$ remains unchanged, then we set $\gamma = 630$, and hence $\gamma = 630 \geqslant 625 = \frac{1}{\lambda_{\max} A_k^T A_k}$, $k \in \mathbb{Z}^+$.

Set $\varsigma = 10$ and $\lambda = 2$, then the direct calculation makes the condition (32) hold. Further, for $k \in \mathbb{Z}^+$,

$$0 < \lambda_{\max} A_k^T A_k = 0.0016 < 0.0498 = e^{-(\varsigma+\lambda)(t_{k+1}-t_k)}, \quad (56)$$

which makes the condition (31) hold. Now, all the conditions of Theorem 3.3 are satisfied. According to Theorem 3.3, the system (48) possesses the unique positive stationary solution (u_1^*, u_2^*), which is globally exponentially stabilized under impulse control with the convergence rate $\frac{\lambda}{2} = 1$.

Remark 4.1 : Table 1 illuminates that under the same pulse frequency, the higher the pulse intensity, the faster the convergence speed. And we can see it from FIGURE 1 and FIGURE 2 that the larger pulse can make the system stable at an earlier time (see FIGURE 1 and FIGURE 2).

TABLE 1. Comparisons the influences on the convergence rate $\frac{\lambda}{2}$ under different pulse amplitude with the same other data.

	Example 4.1	Example 4.2
Pulse amplitude	$\lambda_{\max} A_k = 0.2$	$\lambda_{\max} A_k = 0.02$
Pulse intensity	smaller	bigger
Pulse interval	$(t_{k+1} - t_k) \equiv 0.25$	$(t_{k+1} - t_k) \equiv 0.25$
Pulse frequency	same	same
Convergence rate	$\frac{\lambda}{2} = 0.5$	$\frac{\lambda}{2} = 1$

Example 4.3. In Example 4.1, we replace the pulse interval with $(t_{k+1} - t_k) \equiv 0.15$, and pulse amplitude (53) remains unchanged.

Now, we set $\gamma = 26$, then we get $\gamma = 26 \geqslant 25 = \frac{1}{\lambda_{\max} A_k^T A_k}$, $k \in \mathbb{Z}^+$. Set $\varsigma = 5.5$ and $\lambda = 1.5$, then the direct calculation makes the condition (32) hold. Further,

$$0 < \lambda_{\max} A_k^T A_k = 0.04 < 0.3499 = e^{-(\varsigma+\lambda)(t_{k+1}-t_k)}, \quad k \in \mathbb{Z}^+, \quad (57)$$

which makes the condition (31) hold.

Now, all the conditions of Theorem 3.3 are satisfied. According to Theorem 3.3, the system (48) possesses the unique positive stationary solution (u_1^*, u_2^*), which is globally exponentially stabilized under impulse control with the convergence rate $\frac{\lambda}{2} = 0.75$.

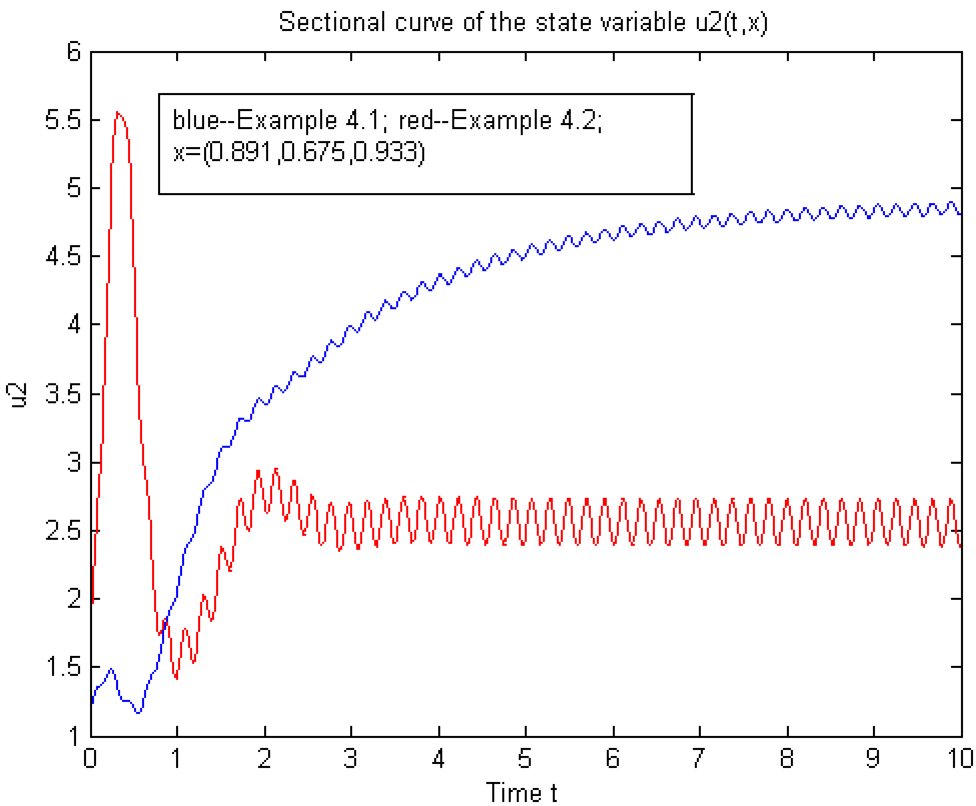

FIGURE 2. **Sectional curve of u_2 of Example 4.1-4.2.**

Remark 4.2: Table 2 reveals that under the same pulse amplitude, the higher the pulse frequency, the faster the convergence speed.

TABLE 2. **Comparisons the influences on the convergence rate $\frac{\lambda}{2}$ under different pulse frequency with the same other data.**

	Example 4.1	Example 4.3
Pulse amplitude	$\lambda_{\max} A_k = 0.2$	$\lambda_{\max} A_k = 0.2$
Pulse intensity	same	same
Pulse interval	$(t_{k+1} - t_k) \equiv 0.25$	$(t_{k+1} - t_k) \equiv 0.15$
Pulse frequency	smaller	bigger
Convergence rate	$\frac{\lambda}{2} = 0.5$	$\frac{\lambda}{2} = 0.75$

V. CONCLUSION AND FURTHER CONSIDERATION

The ecosystem with Dirichelt zero boundary value represents that the nature has limited resources, and population density of the species is zero on the edge of the limited ecological resources, which is entirely in line with some actual situations. Gilpin and Ayala in [5] pointed out that the model did not match a series of experimental data well. Via accurate data analysis, they proposed the nonlinear competition model with two-species, in which θ_1, θ_2 represent the nonlinear density restrictions. As pointed out in [6]- [8] that the nonlinear density restrictions model can match well the experimental data on drosophila melanogasters when θ_i was far less than 1. So, in this paper, the authors considered the nonlinear density restriction models with $\theta_i < 1$. Utilizing the fixed point theorem, variational method and Lyapunov function method resulted in the unique existence of the stationary solution of RD Gilpin-Ayala competition model, which is globally asymptotical input-to-state stability. Numerical examples have illustrated that improving pulse frequency and pulse amplitude is helpful to make the ecosystem stabilized quickly.

Now, the further consideration is, how to study the bi-stabilization of reaction-diffusion two species competition model with Dirichlet boundary value under invasion of infectious diseases. Especially in the novel coronavirus pneumonia epidemic today, it is an interesting problem. In addition, from the point of view of mathematics or mathematical application, the impulsive control of biological mathematical model with switching rules should be further considered, in which some methods of recent literature ([29]–[31]) can be utilized.

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