A Generalization of Lindstedt–Poincaré Perturbation Method to Strongly Mixed-Parity Nonlinear Oscillators

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ABSTRACT In this paper, generalization for the Lindstedt–Poincaré perturbation method for nonlinear oscillators to a class of strongly mixed-parity oscillating system is established. In this extended and enhanced approach, two new odd nonlinear oscillators are introduced in terms of the mixed-parity oscillator. By combining the analytical approximate solutions corresponding to the two new systems, the accurate approximate solutions of the original mixed-parity oscillator are obtained. Comparing with the existing methods such as the perturbation method, the new solution methodology for the singular nonlinear system introduced is not only simple, but the combinatory solution is straight forward and it yields very accurate and physically insightful solutions. Using two typical examples, we demonstrated that this new proposed approach is capable of establishing highly accurate approximate analytical frequency and periodic solutions for small as well as large amplitude of oscillation. The new analytical methodology established will potentially shed new insights to the physical interpretations of strongly nonlinear oscillators including optoelectronic oscillators, pendulums and spring-masses.

INDEX TERMS Analytical approximation, generalization, large amplitude, Lindstedt–Poincaré method, mixed-parity nonlinearity.

I. INTRODUCTION

There exists a number of physical oscillating systems and oscillators in electrical, electronic engineering and mechanical nonlinear systems. Examples of such nonlinear oscillating systems include optoelectronic oscillators [1]–[5], pendulums [6]–[8], spring-masses [9], [10], etc. For such oscillating systems with high nonlinearity and large parameters, there exist very few research works that present analytical or approximate solutions that are sufficiently accurate. Particular challenges are surfaced when the oscillating systems involve a strongly mixed-parity restoring force [11]–[13], and the recently much solicited, tunable optoelectronic meta-oscillators [2]–[4].

As is today, the Lindstedt–Poincaré (L-P) perturbation method [14]–[17] is a well-established approach for constructing approximate analytical solutions of nonlinear system having a small perturbation parameter. But this perturbation theory is generally inapplicable in many

The associate editor coordinating the review of this manuscript and approving it for publication was Hamid Mohammad-Sedighi.

practical engineer problems or a computational failure occurs when the parameters are beyond a certain specified range.

There exists some enhanced L-P perturbation method to the solution of strongly nonlinear systems [18]–[24]. For instance, Cheung *et al.* [18] introduced a new parameter such that the original parameter remains small regardless of the magnitude of the original parameter. Having introduced the new parameter, a strongly nonlinear system with a large parameter is transformed into a small parameter system. However, the impeding issue is this method requires a non-zero linear quantity of the restoring force. By introducing a linear term or a constant term, some researchers, such as Senator and Bapat [19], Amore and Aranda [20], [21], and Wu and his collaborators [22], [23] extended the L-P perturbation method in order to resolve the problem. Based on the homotopy principle, Liao [24] proposed a homotopy analysis method, and He [25] proposed another homotopy perturbation method and both of them effectively extended the application range of the perturbation method. Using the homotopy perturbation method, some further works including the response analysis of fractionally damped

beams subject to external loads [26], nonlinear vibration of cantilever beams with strong nonlinearity [27], nonlinear transversely vibrating beams with an auxiliary term have been investigated [28]. There are some other attempts in this line for improving the existing perturbation methods and for improving the ranges of applicability, including Kattiyapirak and Khovidhungij [29], Fatima [30], Hamed *et al.* [31], etc. However, the effectiveness of these different L-P methods [19], [20], [22] require that the non-linear restoring force to have odd nonlinearity. If this con-dition is not fulfilled, these imporved L-P methods cannot be applied. For mixed-parity nonlinear oscillation system, Mickens [32] constructed approximate analytical solutions by using the usual L-P method [17]. Based on a similar conclu-sion, these solutions are only suitable for week nonlinearity, or for small amplitude, and practicality of the approximate solution is very much restricted. Therefore, a way to extend and generalize the practical application of the classical L-P perturbation method [32], and the various improved L-P methods [19]–[22], for highly nonlinear oscillation system with large oscillation amplitude and with non-odd nonlinear restoring force is critically lacking.

The main purpose of this article is to develop a new approximate analytical method for deriving accurate approxi-mate solutions to strongly mixed-parity nonlinear oscillators. The basic solution methodology of Wu and his collabora-tors [33]–[35] is adopted and further generalized by splitting the original mixed-parity nonlinear system into two new sys-tems with odd nonlinearity. By further adding a linear spring term, we can obtain the approximate analytical solutions to the oscillatory systems with odd nonlinearity. For brevity, the optimal linear spring term is determined according to Wu *et al.* [22]. With analytical matching and prudent combi-natory the two analytical approximate solutions, we establish a new approximate analytical solution to the original mixed-parity nonlinear oscillators over the entire frequency spec-trum. For illustrative purposes and to validate applicability of the new method, two examples are presented to demonstrate the fast-convergent, highly accurate and great simplicity of the new analytical approximation approach over the entire amplitude range.

II. ODD NONLINEAR OSCILLATING SYSTEMS
Before establishing a generalization to the said problem, the existing approach of Wu *et al.* [22] is presented below. This method is only applicable only for odd nonlinearity. Consider an oscillation system governed by

$$\frac{d^2x}{dt^2} + u(x) = 0, \quad x(0) = H, \quad \frac{dx}{dt}(0) = 0 \quad (1)$$

where the nonlinear function $u(x)$ satisfies $u(-x) = u(x)$. Introducing a potential energy function $V(x) = \int u(x)\,dx$, a local center of energy function or a local minimum may exist at $x = x_0$. Without loss of generality, it is well justified to assume $x_0 = 0$. Thus, the system will oscillate in the

interval $[-H, H]$ where H is an appropriate limit to ensure a local minimum at $x_0 = 0$.

Further by taking a coordinate transformation, $\tau = \omega t$, equation (1) becomes

$$\omega^2 x'' + u(x) = 0, \quad x(0) = H, \quad x'(0) = 0 \quad (2)$$

where x' represents the derivative of the variable τ. The new variable is chosen such that the solution $x(\tau)$ to (2) is a periodic function for τ with period 2π, ω is the oscillation frequency, and the frequency and periodic solutions depend on the amplitude H.

A perturbation solution is obtained by introducing an undetermined linear term μx. Hence, equation (2) can be expressed as

$$\omega^2 x'' + \mu x = \varepsilon[\mu x - u(x)] \quad (3)$$

It is noted that (3) can be reduced to the original nonlinear differential equation if $\varepsilon = 1$. Its approximate solution can be expanded as

$$X_m(\tau) = \sum_{k=0}^{m} \varepsilon^k x_k(\tau), \quad \Omega_m^2 = \mu + \sum_{k=1}^{m} \varepsilon^k \mu_k \quad (4)$$

where $X_m(\tau)$ and Ω_m are the mth level approximations to x and ω. The kth terms $x_k(\tau)$ in the corresponding expansions is 2π-periodic with τ and their initial conditions are

$$x_0(0) = H, \quad x_0'(0) = 0,$$
$$x_k(0) = 0, \quad x_k'(0) = 0, \quad (k = 1, 2, \cdots, m) \quad (5)$$

Solving a series of linear equations, we may obtain $X_m(\tau)$ and Ω_m which are dependent on an arbitrary parameter μ, as

$$X_m(\tau) = X_m(\tau; \mu), \quad \Omega_m^2 = \Omega_m^2(\mu) \quad (6)$$

A large number of strategies can be applied to select a good value for μ [19]–[22]. To have an analytical formular as simple as possible, Wu *et al.* [22] suggested

$$\sum_{k=1}^{m} \mu_k = 0. \quad (7)$$

With increasing m, a more accurate μ can be obtained, however, the following approximate solutions involving μ become more complicated. For simplicity, $m = 1$ and $m = 2$ are assigned. The corresponding first and second level approximate solutions are

$$\Omega_1(A) = \sqrt{\mu(H)}, \quad X_1(t) = x_0(\tau) + x_1(\tau),$$
$$\tau = \Omega_1(H)t, \quad \mu(H) = \mu_I(H), \quad (8)$$

and

$$\Omega_2(H) = \sqrt{\mu(H)}, \quad X_2(t) = x_0(\tau) + x_1(\tau) + x_2(\tau),$$
$$\tau = \Omega_2(H)t, \quad \mu(H) = \mu(H) \quad (9)$$

where

$$x_0(\tau) = H\cos\tau,$$
$$\rho_{2n-1}(H) = \frac{4}{\pi}\int_0^{\pi/2} u(x_0(\tau))\cos[(2n-1)\tau]\,d\tau,$$
$$n = 1, 2, \cdots,$$

$$x_1(\tau) = \frac{1}{\mu}\sum_{n=2}^{\infty}\frac{\rho_{2n-1}}{(2n-1)^2-1}\{cos[(2n-1)\tau]-cos\tau\},$$

$$\xi_{2n-1}(H) = \frac{4}{\pi}\int_0^{\pi/2} u_x(x_0(\tau))\,\mu x_1(\tau)cos[(2n-1)\,\tau]\,d\tau,$$

$$n = 1, 2, \cdots,$$

$$x_2(\tau) = \frac{1}{\mu}\sum_{n=2}^{\infty}\left\{\left[\rho_{2n-1}+\frac{\xi_{2n-1}}{\mu}-\left(\frac{\rho_1}{H\mu}\right)\right.\right.$$
$$\left.\left(\frac{(2n-1)^2\,\rho_{2n-1}}{(2n-1)^2-1}\right)\right]\left[\frac{1}{(2n-1)^2-1}\right]\Big\}$$
$$\times\{cos[(2n-1)\,\tau]-cos\tau\},$$

$$\mu_I(A) = \frac{\rho_1}{H}, \mu_{II}(H) = \frac{1}{2A}$$
$$\times\left\{\rho_1+\sqrt{\rho_1^2+4H\xi_1+4\rho_1\left[\sum_{n=2}^{\infty}\frac{\rho_{2n-1}}{(2n-1)^2-1}\right]}\right\}$$

$$(10)$$

It is noted that to obtain an analytical approximation according to the solution procedure above, the coefficients ρ_{2n-1}, $\xi_{2n-1}(n = 1, 2, \cdots)$ for the case of $u(-x) = -u(x)$ should be determined. In fact, these Fourier coefficients can be neglected for $n > 3$ because the series approach to zero with increasing n.

III. A NEW AND GENERALIZED APPROACH TO MIXED-PARITY NONLINEAR OSCILLATING SYSTEMS

Assuming a system governed by (1) to oscillate in an interval $[-B, H]$, where the elastic restoring force $u(x)$ is a mixed-parity function, i.e. $u(-x) \neq u(x)$, we have potential energy at the boundaries as

$$V(-B) = V(H) \tag{11}$$

Following a similar approach [28], [29], the system (1) can be split into two systems, as

$$\frac{d^2x}{dt^2} + lu(x) = 0, \quad x(0) = H, \quad \frac{dx}{dt}(0) = 0 \quad (12a)$$

and

$$\frac{d^2x}{dt^2} + ru(x) = 0, \quad x(0) = B, \quad \frac{dx}{dt}(0) = 0 \quad (12b)$$

where

$$lu(x) = \begin{cases} u(x) & \text{if } x \geq 0, \\ -u(-x) & \text{if } x < 0, \end{cases} \tag{13a}$$

$$ru(x) = \begin{cases} -u(-x) & \text{if } x \geq 0, \\ u(x) & \text{if } x < 0. \end{cases} \tag{13b}$$

Accordingly, two odd nonlinear oscillation systems as governed by (12) and (13) will be further analyzed. The corresponding Fourier coefficients of the two systems are

$$\rho_{(2n-1)lu}(H), \quad \xi_{(2n-1)lu}(H),$$
$$\rho_{(2n-1)ru}(B), \quad \xi_{(2n-1)ru}(B), \quad n = 1, 2, \cdots \quad (14)$$

Based on (14), the analytical approximations to (12a-b) can be denoted by $\Omega_{1lu}(H), X_{1lu}(t), \Omega_{2lu}(H), X_{2lu}(t)$, and

$\Omega_{1ru}(B), X_{1ru}(t), \Omega_{2ru}(B), X_{2ru}(t)$, respectively. Here, it is very important to establish the relation of analytical approximations for (1) and the analytical approximations for (12a-b). Fortunately, the relation of the exact solutions of (12a) and (12b), respectively, to the exact solution of (1) provides a definite guide for establishing the analytical approximation.

Hence, the exact frequency $\omega_e(H)$ and the periodic solution $x_e(t)$ to (1) may be obtained by combinatory piecing of the two solutions above [13], [14]

$$\omega_e(H) = \frac{2\pi}{T_e H}, \quad T_e(H) = \frac{T_{elu}(H)}{2} + \frac{T_{eru}(B)}{2} \quad (15a)$$

and

$$x_e(t)$$
$$= \begin{cases} x_{elu}(t) & \text{for } 0 \leq t \leq \frac{T_{elu}(H)}{4} \\ x_{eru}\left(t-\frac{T_{elu}(H)}{4}+\frac{T_{eru}(B)}{4}\right) & \text{for } \frac{T_{elu}(H)}{4} \leq t \leq T_{lru1} \\ x_{elu}\left(t+\frac{T_{elu}(H)}{2}-\frac{T_{eru}(B)}{2}\right) & \text{for } T_{lru1} \leq t \leq T_{lru2} \end{cases}$$

$$(15b)$$

where $T_{lru1} = \frac{T_{elu}(H)}{4} + \frac{T_{eru}(B)}{2}$, $T_{lru2} = \frac{T_{elu}(H)}{2} + \frac{T_{eru}(B)}{2}$, $T_{elu}(H) = \frac{2\pi}{\omega_{elu}(H)}$, and $T_{eru}(B) = \frac{2\pi}{\omega_{eru}(B)}$.

Applying the approximate analytical solutions to (12a-b) and the formula in (15a) and (15b), the corresponding kth $(k = 1, 2)$ approximation can be obtained as follows

$$\Omega_m(A) = \frac{2\pi}{T_m(H)}, \quad T_m(H) = \frac{T_{mlu}(H)}{2} + \frac{T_{mru}(B)}{2} \quad (16a)$$

and

$$X_m(t)$$
$$= \begin{cases} X_{mru}(t) & \text{for } 0 \leq t \leq \frac{T_{mru}(H)}{4} \\ X_{mlu}\left(t-\frac{T_{mlu}(H)}{4}+\frac{T_{mru}(B)}{4}\right) & \text{for } \frac{T_{mlu}(H)}{4} \leq t \leq T_{mlu1} \\ X_{mlu}\left(t+\frac{T_{mlu}(H)}{2}-\frac{T_{mru}(B)}{2}\right) & \text{for } T_{mlu1} \leq t \leq T_{mlu2} \end{cases}$$

$$(16b)$$

where $T_{mlu1} = \frac{T_{mlu}(H)}{4} + \frac{T_{mru}(B)}{2}$, $T_{mlu2} = \frac{T_{mlu}(H)}{2} + \frac{T_{mru}(B)}{2}$, $T_{mlu}(H) = \frac{2\pi}{\Omega_{mlu}(H)}$, and $T_{mru}(B) = \frac{2\pi}{\Omega_{mru}(B)}$.

In what follows two examples will be presented to verify that (16a-b) with $k = 2$ can provide very accurate and excellent analytical approximations to frequency and the corresponding periodic solution of mixed-parity nonlinear oscillators. We will verify that the approach and solution methodology are valid for the systems with very large amplitude and strong nonlinearities.

IV. ILLUSTRATIVE EXAMPLS

In this section, an example related to the ear membrane vibration of an organism and another a discontinuous vibration system will be used to illustrate the accuracy of the present method. We will show through that the method established in the prior section is able to provide excellent analytical

approximations to the frequency and the corresponding periodic solutions for strong mixed-parity nonlinear oscillators with small, as well as large, oscillation amplitudes.

Example 1. Consider a quadratic nonlinear oscillator governed by [14], [21]

$$\frac{d^2x}{dt^2} + x + x^2 = 0, \quad x(0) = H, \quad \frac{dx}{dt}(0) = 0 \quad (17)$$

For this problem, *t* he corresponding potential energy function is

$$V(x) = \frac{x^2}{2} + \frac{x^3}{3} \quad (18)$$

By solving (11), we have

$$B = \frac{\left[(3+2H) - \sqrt{9 - 12H - 12H^2}\right]}{4} \quad (19)$$

According to (14), the corresponding Fourier coefficients in this example are obtained as follows

$$\rho_{1lu} = H + \frac{8H^2}{3\pi}, \quad \rho_{3lu} = \frac{8H^2}{15\pi}, \quad \rho_{5lu} = -\frac{8H^2}{105\pi},$$

$$\rho_{1ru} = B - \frac{8B^2}{3\pi}, \quad \rho_{3ru} = -\frac{8B^2}{15\pi}, \quad \rho_{5ru} = \frac{8H^2}{105\pi},$$

$$\xi_{1lu} = -\frac{4H^2(736H + 175\pi)}{11025\pi^2},$$

$$\xi_{3lu} = -\frac{H^2(20096H + 6615\pi)}{99225\pi^2},$$

$$\xi_{5lu} = \frac{H^2(17792H - 693\pi)}{218295\pi^2},$$

$$\xi_{1ru} = \frac{4B^2(-736B + 175\pi)}{11025\pi^2},$$

$$\xi_{3ru} = -\frac{B^2(20096B - 6615\pi)}{99225\pi^2},$$

$$\xi_{5ru} = \frac{B^2(17792B + 693\pi)}{218295\pi^2} \quad (20)$$

Two analytical approximations to the exact frequency and periodic solution to (17) will be obtained. For the first oscillator in (12a) with restoring force *lu*(*x*), they are

$$\Omega_{1lu}(H) = \sqrt{1 + 8H/3\pi}$$

$$X_{1lu}(t) = \left(H - \frac{4H^2}{63\pi\varphi_I}\right)\cos\tau + \frac{H^2}{15\pi\varphi_I}\cos3\tau$$

$$- \frac{H^2}{315\pi\varphi_I}\cos5\tau, \quad \tau = \Omega_{1lu}(H)t \quad (21a)$$

and

$$\Omega_{2lu}(H) = \sqrt{\varphi_{II}(H)}$$

$$X_{2lu}(t) = \text{Cos}[\tau](H + \frac{80779H^3}{496125\pi^2\varphi_{II}^2} + \frac{4H^2}{63\pi\varphi_{II}^2} - \frac{8H^2}{63\pi\varphi_{II}})$$

$$+ \text{Cos}[5\tau](\frac{82H^3}{6615\pi^2\varphi_{II}^2} + \frac{H^2}{315\pi\varphi_{II}^2} - \frac{2H^2}{315\pi\varphi_{II}})$$

$$+ \text{Cos}[3\tau](-\frac{1651H^3}{9450\pi^2\varphi_{II}^2} - \frac{H^2}{15\pi\varphi_{II}^2} + \frac{2H^2}{15\pi\varphi_{II}})$$

$$- \frac{121H^3\text{Cos}[7\tau]}{198450\pi^2\varphi_{II}^2} + \frac{H^3\text{Cos}[9\tau]}{9450\pi^2\varphi_{II}^2} - \frac{H^3\text{Cos}[11\tau]}{330750\pi^2\varphi_{II}^2},$$

$$\tau = \Omega_{2lu}(H)t \quad (21b)$$

where

$$\varphi_I = 1 + \frac{8H}{3\pi},$$

$$\varphi_{II} = \frac{840H + 315\pi + \sqrt{3}\sqrt{222272H^2 + 176400H\pi + 33075\pi^2}}{630\pi}$$

For the second oscillator in (12b) with restoring force *ru*(*x*), they are

$$\Omega_{1ru}(B) = \sqrt{1 - 8B/3\pi},$$

$$X_{1ru}(t) = \left(B + \frac{4B^2}{63\pi\varphi_I}\right)\cos\tau - \frac{B^2}{15\pi\varphi_I}\cos3\tau$$

$$+ \frac{B}{315\pi\varphi_I}\cos5\tau, \quad \tau = \Omega_{1ru}(B)t \quad (22a)$$

and

$$\Omega_{2ru}(B) = \sqrt{\varphi_{II}(B)},$$

$$X_{2ru}(t) = \text{Cos}[3\tau]\left(-\frac{1651B^3}{9450\pi^2\varphi_{II}^2} + \frac{B^2}{15\pi\varphi_{II}^2} - \frac{2B^2}{15\pi\varphi_{II}}\right)$$

$$+ \text{Cos}[5\tau]\left(\frac{82B^3}{6615\pi^2\varphi_{II}^2} - \frac{B^2}{315\pi\varphi_{II}^2} + \frac{2B^2}{315\pi\varphi_{II}}\right)$$

$$+ \text{Cos}[\tau]\left(B + \frac{80779B^3}{496125\pi^2\varphi_{II}^2} - \frac{4B^2}{63\pi\varphi_{II}^2} + \frac{8B^2}{63\pi\varphi_{II}}\right)$$

$$- \frac{121B^3\text{Cos}[7\tau]}{198450\pi^2\varphi_{II}^2} + \frac{B^3\text{Cos}[9\tau]}{9450\pi^2\varphi_{II}^2} - \frac{B^3\text{Cos}[11\tau]}{330750\pi^2\varphi_{II}^2},$$

$$\tau = \Omega_{2ru}(B)t \quad (22b)$$

where

$$\varphi_I = 1 - \frac{8B}{3\pi},$$

$$\varphi_{II} = (-840B + 315\pi + \sqrt{3}\sqrt{222272B^2 - 176400B\pi + 33075\pi^2})\frac{1}{630\pi}.$$

Applying the L-P perturbation method, we obtain the second-order analytical approximate frequency $\omega_{LP}(H)$ and the periodic solution $x_{LP}(t)$ as follows

$$\omega_{LP}(H) = 1 - 5H^2/12 \quad (23a)$$

and

$$x_{LP}(t) = H\cos[\omega_{LP2}(H)t]$$

$$+ H^2\left\{-\frac{1}{2} + \frac{1}{3}\cos[\omega_{LP2}(H)t] + \frac{1}{6}\cos[2\omega_{LP2}(H)t]\right\} \quad (23b)$$

The exact frequency $\omega_e(H)$ is

$$\omega_e(H) = \frac{2\pi}{T_e(H)} \quad (24)$$

TABLE 1. Comparison of approximate frequency with exact frequency solution.

H	ω_e	ω_{LP}/ω_e	Ω_1/ω_e	Ω_2/ω_e
0.1	0.995498	1.000337	1.000057	1.000001
0.2	0.980003	1.003398	1.000293	1.000002
0.3	0.947782	1.015529	1.001028	1.000002
0.4	0.881904	1.058316	1.003999	0.999955
0.41	0.871608	1.066945	1.004709	0.999935
0.43	0.847217	1.089400	1.006745	0.999864
0.45	0.815312	1.123036	1.010322	0.999691
0.47	0.769323	1.180204	1.017927	0.999131
0.49	0.682362	1.318888	1.044961	0.994970
0.491	0.674880	1.332903	1.048396	0.994165
0.494	0.647399	1.387581	1.063045	0.989852
0.497	0.604895	1.483033	1.093293	0.974446

where

$$T_e(H) = \int_0^{\frac{\pi}{2}} \frac{2dt}{\sqrt{1 + \frac{[2H(1+\sin t + \sin^2 t)]}{[3(1+\sin t)]}}}$$
$$+ \int_0^{\frac{\pi}{2}} \frac{2dt}{\sqrt{1 - \frac{[2B(1+\sin t + \sin^2 t)]}{[3(1+\sin t)]}}}$$

and B is given, in terms of H, in (19).

For this oscillator, the oscillation amplitude H should satisfy $H < 0.5$, when $H = 0.5$, and (17) has a homoclinic orbit with period $+\infty$. The exact frequency $\omega_e(24)$ and the ratio of approximate ones $\Omega_1(16a)$, $\Omega_2(16a)$, $\omega_{LP}(23a)$ to $\omega_e(24)$ are listed in Table 1. It is observed that (13a) give excellent approximate periods for both small and large oscillation amplitude H except when H approaches to 0.5. This is because at $H = 0.5$, it correspondence to an oscillating system with an infinite oscillation period and hence the assumption of periodic oscillation is invalid and the solution expressions thus derived becomes inapplicable.

Table 1 indicates that (16a) are very accurate. The second approximations provide the most excellent frequencies with respect to the exact one for whole range of oscillation amplitudes.

A comparison of analytical approximate solutions $X_1(16b)$, $X_2(16b)$, and $x_{LP}(t)$ (23b), with respect to the exact periodic solutions $x_e(t)$ obtained by direct numerical integration of (17) is presented in Figs, 1, 2 and 3 for three different amplitudes of oscillation $H = 0.4$, $H = 0.49$ and $H = 0.494$.

These figures demonstrate that the proposed approximate periodic solutions in (16b) are more accurate than the perturbation approximation in (23b) for all permitted oscillation amplitude. Furthermore, the second-order approximation provides better accuracy of approximate analytical periodic solutions for both small and large amplitude of oscillations.
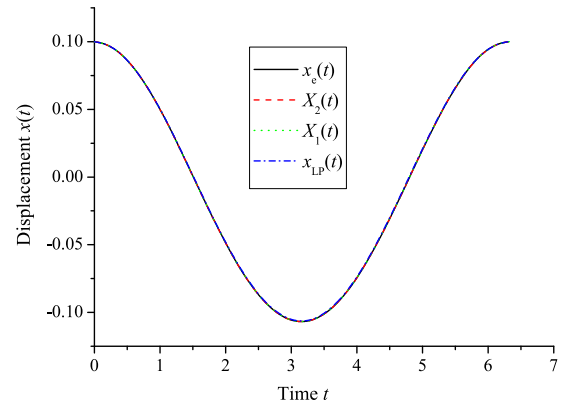

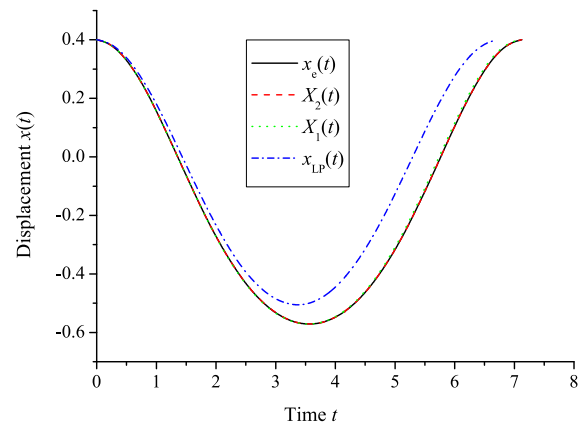**FIGURE 1.** Periodic solutions $H = 0.1$ in example 1.

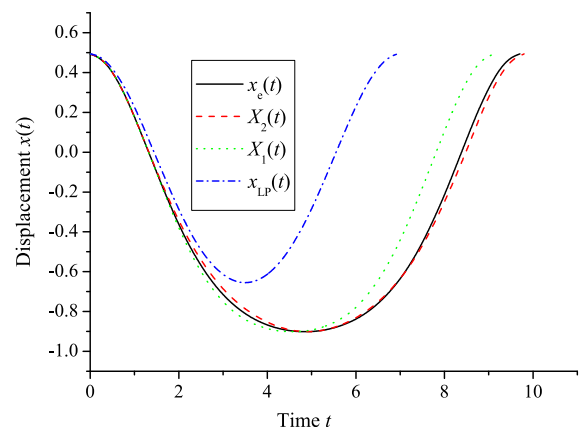**FIGURE 2.** Periodic solutions $H = 0.4$ in example 1.

FIGURE 3. Periodic solutions $H = 0.494$ in example 1.

Example 2. Consider the following nonlinear oscillator [32], [33]

$$\frac{d^2x}{dt^2} + u(x) = 0, \quad x(0) = H, \quad \frac{dx}{dt} = 0. \quad (25)$$

where $u(x) = \begin{cases} x^3 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$

For this problem, we have

$$V(x) = \begin{cases} \dfrac{x^4}{4} & \text{if } x \geq 0, \\ -\dfrac{x^3}{3} & \text{if } x < 0. \end{cases} \quad (26)$$

Furthermore, one can obtain

$$B = \sqrt[3]{\frac{3H^4}{4}}. \qquad (27)$$

The corresponding Fourier coefficients for this example are as follows

$$\rho_{1lu} = \frac{3H^3}{4}, \quad \rho_{3lu} = \frac{H^3}{4}, \quad \rho_{5lu} = 0, \quad \xi_{1lu} = \frac{3H^5}{64},$$

$$\xi_{3lu} = \frac{3H^5}{128}, \quad \xi_{5lu} = \frac{3H^5}{128}, \quad \rho_{1ru} = \frac{8B^2}{3\pi}, \quad \rho_{3ru} = \frac{8B^2}{15\pi},$$

$$\rho_{5ru} = \frac{8B^2}{105\pi}, \quad \xi_{1ru} = -\frac{2944B^3}{11025\pi^2}, \quad \xi_{3ru} = \frac{20096B^3}{99225\pi^2},$$

$$\xi_{5ru} = \frac{17792B^3}{218295\pi^2} \qquad (28)$$

Substituting (28) into (8) and (9) leads to two analytical approximations to the exact period and periodic solution as governed by (25). For the first oscillator with restoring force $lu(x)$, they are

Two analytical approximations to the exact frequency and periodic solution to (25) will be obtained. For the first oscillator in (12a) with restoring force $lu(x)$, they are

$$\Omega_{1lu}(H) = \sqrt{3}H/2, \, X_{1lu}(t) = \left(H - \frac{H^3}{32\varphi_I}\right)\cos\tau$$

$$+ \frac{H^3}{32\varphi_I}\cos 3\tau, \quad \tau = \Omega_{1lu}(H)\,t \qquad (29a)$$

and

$$\Omega_{2lu}(H) = \sqrt{\varphi_{II}(H)}$$

$$X_{2lu}(t) = \text{Cos}[\tau](H + \frac{23H^5}{1024\varphi_{II}^2} - \frac{H^3}{16\varphi_{II}})$$

$$+ \text{Cos}[3\tau](-\frac{3H^5}{128\varphi_{II}^2} + \frac{H^3}{16\varphi_{II}}) + \frac{H^5\text{Cos}[5\tau]}{1024\varphi_{II}^2},$$

$$\tau = \Omega_{2lu}(H)\,t \qquad (29b)$$

where

$$\varphi_I = \frac{3H^2}{4}, \quad \varphi_{II} = \frac{\left(6 + \sqrt{30}\right)H^2}{16}$$

For the second oscillator with restoring force $ru(x)$, they are

$$\Omega_{1ru}(B) = 2\sqrt{2B}/(3\pi)$$

$$X_{1ru}(t) = \left(B - \frac{4B^2}{63\pi\varphi_I}\right)\cos\tau + \frac{B^2}{15\pi\varphi_I}\cos 3\tau$$

$$- \frac{B^2}{315\pi\varphi_I}\cos 5\tau, \quad \tau = \Omega_{1ru}(B)\,t \qquad (30a)$$

and

$$\Omega_{2ru}(B) = \sqrt{\varphi_{II}(B)}$$

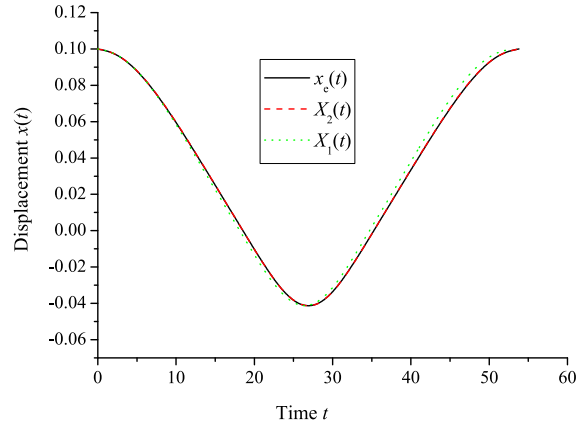$$X_{2ru}(t) = \text{Cos}[\tau](B + \frac{80779B^3}{496125\pi^2\varphi_{II}^2} - \frac{8B^2}{63\pi\varphi_{II}})$$

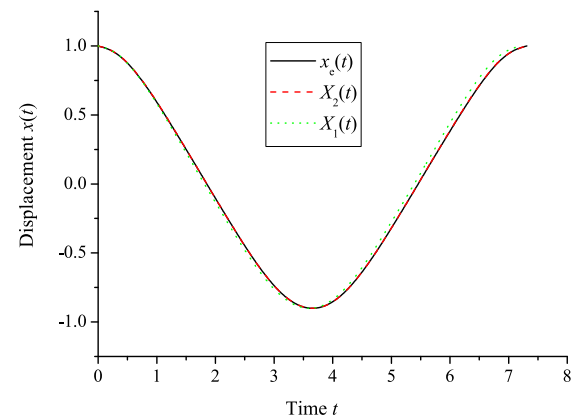**FIGURE 4.** Periodic solutions for $H = 0.1$ in Example 2.

FIGURE 5. Periodic solutions $H = 1$ in example 2.

$$+ \text{Cos}[5\tau](\frac{82B^3}{6615\pi^2\varphi_{II}^2} - \frac{2B^2}{315\pi\varphi_{II}})$$

$$+ \text{Cos}[3\tau](-\frac{1651B^3}{9450\pi^2\varphi_{II}^2} + \frac{2B^2}{15\pi\varphi_{II}})$$

$$- \frac{121B^3\text{Cos}[7\tau]}{198450\pi^2\varphi_{II}^2} + \frac{B^3\text{Cos}[9\tau]}{9450\pi^2\varphi_{II}^2} - \frac{B^3\text{Cos}[11\tau]}{330750\pi^2\varphi_{II}^2},$$

$$\tau = \Omega_{2ru}(B)\,t \qquad (30b)$$

where

$$\varphi_I = \frac{8A^2}{3\pi}, \quad \varphi_{II} = \frac{4B\left(105 + \sqrt{10419}\right)}{315\pi}$$

For this example, the exact frequency $\omega_e(H)$ is

$$\omega_e(H) = \frac{2\pi}{T_e(H)} \qquad (31)$$

where

$$T_e(H) = \frac{2}{H}\int_0^{\frac{\pi}{2}}\sqrt{\frac{2}{1+\sin^2 t}}dt + \frac{2}{\sqrt{B}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{3(1+\sin t)}{2(1+\sin t+\sin^2 t)}}$$

$$dt \approx \frac{3.7081494}{H} + \frac{3.4346307}{\sqrt{B}},$$

and B is given, in terms of H, in (27).

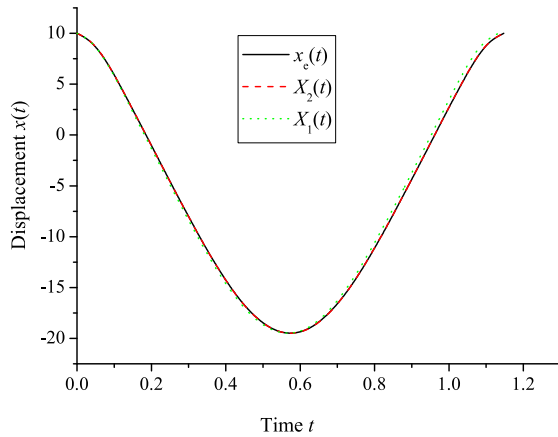**FIGURE 6.** Periodic solutions $H = 10$ in example 2.

TABLE 2. comparison of approximate frequencies with exact frequency.

H	ω_e	Ω_1/ω_e	Ω_2/ω_e
0.1	0.116773	1.017511	0.999853
0.4	0.394976	1.015914	0.999909
0.7	0.636728	1.015226	0.999932
1	0.859360	1.014782	0.999948
4	2.665736	1.013083	1.000007
7	4.148859	1.012435	1.000030
10	5.477322	1.012040	1.000044
40	15.67726	1.010674	1.000092
70	23.69931	1.010207	1.000108
100	30.74975	1.009936	1.000118
1000	158.1019	1.008632	1.000164
10000	772.4653	1.007927	1.000189

The exact frequency $\omega_e(H)$ (31) and the ratio of approximate solutions $\Omega_1(16a)$, $\Omega_2(16a)$ to $\omega_e(H)$ are listed in Table 2.

It is observed from Table 2 that (16a) is valid for the whole range of amplitude H. Furthermore, we have

$$\lim_{H->0^+} \frac{\Omega_1}{\omega_e} = 1.0222049, \qquad \lim_{H->0^+} \frac{\Omega_2}{\omega_e} = 0.9996911,$$

$$\lim_{H->0+\infty} \frac{\Omega_1}{\omega_e} = 1.0072554, \qquad \lim_{H->0+\infty} \frac{\Omega_2}{\omega_e} = 1.0002124.$$
(32)

Hence, we conclude that the proposed method is able to give excellent approximate frequency solutions for the whole range of oscillation amplitude.

A comparison of analytical approximate solutions X_1 (16b) and X_2 (16b), with respect to the exact periodic solutions $x_e(t)$ obtained by direct numerical integration of (25) is presented in Figs, 4, 5 and 6 for three different amplitudes of oscillation $H = 0.1$, $H = 1$ and $H = 10$.

These figures show that both the first-order analytical approximations and the second-order approximations provide excellent solutions with respect to the exact periodic solutions for small as well as large amplitude of oscillation.

V. CONCLUSION

In this study, we proposed a generalized and enhanced L–P perturbation method for solving strong nonlinear oscillation systems with mixed-parity nonlinearity. This new solution methodology has extended the validity range compared to the original method. Comparing with the existing methods, this present solution methodolody by introducing singular nonlinear system has the characteristics of simple form and it yields very accurate and physically insightful solutions. Using two practical examples of nonlinear oscillators, we demonstrated that the proposed approach is capable of establishing highly accurate approximate analytical frequency and periodic solutions for small as well as large amplitude of oscillation, and to a range as wide as the infinity limits at both ends.

Furthermore, the proposed method is capable of establishing solutions with general boundary conditions including an initial ývelocity. With both initial displacement and initial velocity available, the initial velocity can be transformed to a level of zero by system transformation of energy conservation. The proposed method can be further extended to an oscillator with non-polynomial nonlinear restoring force, however, the integral expression will be more symbolically lengthy, yet solvable using symbolic mathematics, for solving the harmonic oscillators.

With reference to the improved perturbation principle [36], solutions to the transient response of highly nonlinear damped vibration systems will be presented in a future work. The modified L-P method will be extended to establish accurate steady-state periodic solutions of nonlinearly damped vibration systems.

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