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Further Results on Large-Scale Complex Logical Networks

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ABSTRACT In the past few decades, how to reduce the computational complexity of dealing with k-valued logical networks (LNs) has become a heated research topic. This paper firstly presents a brief survey on the recent efforts of dealing with large-scale LNs, including approximation of LNs, network aggregation approach, and logical matrix factorization technique. Then, by using the network aggregation approach, the stability of large-scale network pairing problem is studied. The network aggregation approach is also used to study probabilistic logical networks (PLNs). Finally, an illustrative example is given to demonstrate the effectiveness of the obtained results.

INDEX TERMS Large-scale system, complex logical network, algebraic state space representation, network aggregation, approximation, logical matrix factorization.

I. INTRODUCTION

As one of significant complex logical dynamic systems, genetic regulatory networks (GRNs), which are effective to forecast genetic diseases and design treatment schemes, have attracted increasing attention due to the broad applications [1]. There are many mathematical tools to study GRNs, such as differential equations [2]–[6] and Boolean networks [7]. GRNs are in fact a kind of complex networks [8]–[13]. Multi-valued logical networks (LNs) are a natural extension of Boolean networks [14], [15], which were firstly proposed by Jan Lukasiewicz in order to solve the problems in computer and engineering. The values of nodes in LNs are taken from $\mathcal{D}_k := \{0, 1, \cdots, k-1\}$, and the update rules are determined by logical mappings. LNs with control inputs and outputs are called k-valued logical control networks (LCNs). Furthermore, LNs can be generalized to probabilistic logical networks (PLNs), which are powerful to solve the stochasticity in the process of model establishment.

Recently, based on the semi-tensor product of matrices (STP), a new framework called algebraic state space representation (ASSR) method has been proposed for the analysis and control of LCNs [16]–[20]. Under this framework, one can convert a logical expression into an equivalent algebraic form. Many landmark results about LCNs have been presented, such as controllability [21]–[24], observability [25]–[28], stability and stabilization [29]–[34], optimal control [35]–[37], output tracking control [38]–[41], disturbance decoupling problem [42], [43], pinning control design [44]–[47], and so on [48]–[53]. In addition, the STP method is also applied to PLNs. Correspondingly, the stability [54]–[60], stabilization [61]–[65], controllability [66]–[68], and other issues of PLNs [69]–[73] are investigated. There also exist some results in complex networked evolutionary games theory [74]–[79], multi-agent systems [80], and other fields [81]–[86], which show the extensive applications of STP. Lu *et al.* [87] pointed out that computational complexity is one of control challenges faced by control theory. When the number of nodes is large, the results mentioned above are hardly to use in large-scale LCNs, which shows that how to reduce the computational load is urgently needed.

Based on the ASSR framework, many efficient techniques have been introduced to solve the control problems of large-scale LCNs, including approximation method [88], network aggregation approach [89]–[91], logical matrix factorization technique [92], and pinning control design method [44]. The approximation of LNs was firstly proposed by Cheng and Zhao [88] to obtain a simplified network of large-scale LNs. Zhao *et al.* [90] firstly introduced the network aggregation approach for the attractors analysis of large-scale LNs. Another effective way of dealing with large-scale LNs is logical matrix factorization [92].

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Li and Wang [92] firstly proposed the logical matrix factorization technique for the topological structure analysis of LNs, and a size-reduced structure-equivalent LNs is constructed for a given LNs. Zhong *et al.* [44] proposed a network-structure-based distributed pinning control framework to design pinning control with lower dimensional controllers and less computational load. In addition, Yu *et al.* [93] firstly proposed decompositions of non-negative integer vectors. By resorting to logical matrix equations, original networks can be decomposed into many independent subsystems.

Network pairing problem was proposed in [94] to generalize conventional pairing problem, which can be modeled as multi-valued LNs. How to find a stable arrangement of applicants and employers is an interesting issue, but it is hard to obtain an arrangement when the number of players is large. In this paper, we explore the properties of large-scale network pairing problem. Topological structure is a fundamental issue for LNs. In this paper, we consider a version of this problem in which the network is PLNs.

In this paper, we give a review of the research progress for large-scale LCNs and propose some new results about large-scale LCNs. The remainder of this paper is presented as follows. In Section II, some preliminaries about the STP of matrices and ASSR of LCNs are introduced. In Section III, the methods of dealing with large-scale LCNs are reviewed, including approximation of LNs, network aggregation approach and logical matrix factorization technique. In Section IV, we explore the stability of large-scale network pairing problem. In Section V, we investigate the topological structure of PLNs. A numerical example is given in Section VI, which is followed by the conclusion in Section VII.

II. PRELIMINARIES

A. NOTATIONS

In this subsection, we give some notations. Please refer to [16] for more details.

- $\mathbf{1}_k^\top = [\underbrace{1 \cdots 1}_{k}]$.
- $Col_j(A)\big(Row_j(A)\big)$: the j-th column (row) of A. $Col(A)$ denotes the set of columns of A.
- $\Delta_k := \{\delta_k^i = Col_i(I_k) : i = 1, \cdots, k\}$, where I_k denotes the k-dimensional identity matrix.
- An $n \times t$ real matrix L is called a logical matrix, if $Col(L) \subseteq \Delta_n$, and the set of $n \times t$ logical matrices is denoted by $\mathcal{L}_{n\times t}$.
- If $L \in \mathcal{L}_{n\times t}$, then it can be expressed as $L = [\delta_n^{i_1} \ \cdots \ \delta_n^{i_t}]$; for brevity, $L = \delta_n[i_1 \ \cdots \ i_t]$.
- The Khatri-Rao product of two matrices $A \in \mathbb{R}^{p\times n}$ and $B \in \mathbb{R}^{q\times n}$ is

$$A * B = [Col_1(A) \otimes Col_1(B) \ \cdots \ Col_n(A) \otimes Col_n(B)],$$

where "$\otimes$" denotes the Kronecker product.

B. ALGEBRAIC EXPRESSION OF LOGICAL FUNCTION

Given a logical function $f : \mathcal{D}_k^n \to \mathcal{D}_k$. By using the STP method introduced in [16], one can obtain a unique logical matrix M_f such that

$$f(x_1, x_2, \cdots, x_n) = M_f \ltimes_{i=1}^n x_i, \tag{1}$$

where $M_f \in \mathcal{L}_{k\times k^n}$ is the structural matrix of f, $x_i \in \Delta_k$, $i = 1, \cdots, n$ and $\ltimes_{i=1}^n x_i = x_1 \ltimes \cdots \ltimes x_n$.

For example, given a Boolean function

$$f(x_1, x_2, x_3) = x_1 \vee x_2 \vee \neg x_3,$$

where $x_1, x_2, x_3 \in \mathcal{D}_2$ are Boolean variables. The corresponding algebraic form is

$$f(x_1, x_2, x_3) = M_f \ltimes_{i=1}^3 x_i,$$

where $x_i \in \Delta_2$, $i = 1, 2, 3$, $M_f = \delta_2[1\ 1\ 1\ 1\ 1\ 1\ 2\ 1] \in \mathcal{L}_{2\times 8}$ is the structural matrix.

For a given a logical mapping $F : \mathcal{D}_k^n \to \mathcal{D}_k^m$, which contains m logical functions. Then, by using the Khatri-Rao product of matrices [16], it holds that

$$F(x_1, x_2, \cdots, x_n) = M_F \ltimes_{i=1}^n x_i, \tag{2}$$

where $M_F \in \mathcal{L}_{k^m\times k^n}$ is the so-called structural matrix of F.

For example, given a Boolean mapping $F : \mathcal{D}_2^3 \to \mathcal{D}_2^2$, which is described as follows:

$$\begin{cases} f_1(x_1, x_2, x_3) = x_1 \vee x_2, \\ f_2(x_1, x_2, x_3) = \neg x_2 \wedge x_3, \end{cases}$$

where $x_1, x_2, x_3 \in \mathcal{D}_2$ are Boolean variables. Then, the corresponding algebraic form is

$$F(x_1, x_2, x_3) = M_F \ltimes_{i=1}^3 x_i,$$

where $M_F = M_{f_1} * M_{f_2} = \delta_4[2\ 2\ 1\ 2\ 2\ 2\ 3\ 4] \in \mathcal{L}_{4\times 8}$ is the structural matrix.

C. ALGEBRAIC STATE SPACE REPRESENTATION OF LOGICAL NETWORKS

Consider the following LN:

$$\begin{cases} x_1(t+1) = \tilde{f}_1(x_1(t), \cdots, x_n(t)), \\ \cdots \\ x_n(t+1) = \tilde{f}_n(x_1(t), \cdots, x_n(t)), \end{cases} \tag{3}$$

where $x_i(t) \in \mathcal{D}_k$, $i = 1, \cdots, n$ are state variables, $\tilde{f}_i : \mathcal{D}_k^n \to \mathcal{D}_k$, $i = 1, \cdots, n$ are k-valued logical functions.

By using the algebraic form of logical variables, Cheng *et al.* [16] established the ASSR of system (3) as follows:

$$x(t+1) = \tilde{L}x(t), \tag{4}$$

where $x(t) = \ltimes_{i=1}^n x_i(t)$, $\tilde{L} \in \mathcal{L}_{k^n\times k^n}$ is the state transition matrix.

If system (3) has input and output variables, then system (3) becomes an LCN which is expressed as

$$\begin{cases} x_1(t+1) = f_1(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)), \\ \cdots \\ x_n(t+1) = f_n(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)), \\ y_j(t) = h_j(x_1(t), \cdots, x_n(t)), j = 1, \cdots, \rho, \end{cases} \tag{5}$$

where $x_i(t) \in \mathcal{D}_k$, $i = 1, \cdots, n$ are state variables, $u_j(t) \in \mathcal{D}_k, j = 1, \cdots, m$ are control inputs, $y_j(t) \in \mathcal{D}_k, j = 1, \cdots, \rho$

are outputs, $f_i:\mathcal{D}_k^{m+n}\to\mathcal{D}_k$, $i=1,\cdots,n$ and $h_j:\mathcal{D}_k^n\to\mathcal{D}_k$, $j=1,\cdots,\rho$ are k-valued logical functions.

Similarly, the ASSR of system (5) is obtained as

$$\begin{cases} x(t+1)=Lu(t)x(t),\\ y(t)=Hx(t), \end{cases} \tag{6}$$

where $x(t)=\ltimes_{i=1}^n x_i(t)$, $u(t)=\ltimes_{j=1}^m u_j(t)$, $y(t)=\ltimes_{j=1}^{\rho} y_j(t)$, $L\in\mathcal{L}_{k^n\times k^{m+n}}$ is the state transition matrix, and $H\in\mathcal{L}_{k^\rho\times k^n}$ is the output matrix.

Example 1: Consider the following Boolean network:

$$\begin{cases} x_1(t+1)=x_2(t)\wedge x_3(t),\\ x_2(t+1)=x_1(t)\vee x_3(t),\\ x_3(t+1)=\neg x_2(t),\\ x_4(t+1)=x_1(t)\vee\neg x_4(t), \end{cases} \tag{7}$$

where $x_i\in\mathcal{D}$, $i=1,\cdots,4$. By using the STP method, the ASSR of system (7) can be obtained as

$$\begin{cases} x_1(t+1)=\delta_2[1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2]x(t),\\ x_2(t+1)=\delta_2[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2]x(t),\\ x_3(t+1)=\delta_2[2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1]x(t),\\ x_4(t+1)=\delta_2[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1]x(t), \end{cases} \tag{8}$$

where $x(t)=\ltimes_{i=1}^4 x_i(t)$. Then, the state transition matrix of system (7) is

$$M_F=\delta_{16}[3\ 3\ 11\ 11\ 9\ 9\ 9\ 9\ 4\ 3\ 16\ 15\ 10\ 9\ 14\ 13].$$

III. EXISTING METHODS OF STUDYING LARGE-SCALE COMPLEX LOGICAL NETWORKS

A. APPROXIMATION METHOD

The approximation of large-scale LNs (3) was firstly investigated in [88] by aggregating the whole networks into several subsystems. According to the observed data, the best approximation of original networks is given in [88].

Consider system (3). Setting node set $\mathcal{X}=\{x_1,\cdots,x_n\}$, then partition $\mathcal{X}$ into s blocks as

$$\mathcal{X}=\mathcal{X}_1\cup\cdots\cup\mathcal{X}_s, \tag{9}$$

where $\mathcal{X}_i=\{x_{i,1},\cdots,x_{i,n_i}\}$, $i=1,\cdots,s$ is the subset of $\mathcal{X}$. We emphasize that $\mathcal{X}_i\cap\mathcal{X}_j=\emptyset$, $i,j=1,\cdots,s$, $i\neq j$.

In addition, for the i-th block $\mathcal{X}_i$, if there exist incoming (outgoing) edges from (to) other blocks, then the starting nodes of these incoming (outgoing) edges are called input (output) nodes of block $\mathcal{X}_i$. Denote the set of input nodes and output nodes of the i-th block by $\mathcal{Z}_i=\{z_{i,1},\cdots,z_{i,p_i}\}$ and $\mathcal{Y}_i=\{y_{i,1},\cdots,y_{i,q_i}\}$, respectively. In the sequel, without loss of generality, we assume that the elements in $\mathcal{X}_i$, $\mathcal{Z}_i$ and $\mathcal{Y}_i$ keep the order in $\mathcal{X}$ if they are nonempty sets.

From the above partition, we can obtain that $\sum_{i=1}^{s} n_i=n$, $\sum_{i=1}^{s} p_i=\sum_{j=1}^{s} q_j:=\vartheta$.

Each block forms a subnetwork. Denote the i-th block by Σ_i, $i=1,\cdots,s$. The dynamics of Σ_i is described as:

$$\begin{cases} x_{i,j}(t+1)=\tilde{f}_{i,j}(z_{i,1}(t),\cdots,z_{i,p_i}(t),x_{i,1}(t),\\ \qquad\qquad\cdots,x_{i,n_i}(t)),\ j=1,\cdots,n_i\\ y_{i,j}(t)=g_{i,j}(x_{i,1}(t),\cdots,x_{i,n_i}(t)),\ j=1,\cdots,q_i. \end{cases} \tag{10}$$

The ASSR of system (10) is

$$\begin{cases} X_i(t+1)=\tilde{L}_iZ_i(t)X_i(t),\\ Y_i(t)=G_iX_i(t), \end{cases} \tag{11}$$

where $X_i(t)=\ltimes_{j=1}^{n_i}x_{i,j}(t)$, $Z_i(t)=\ltimes_{j=1}^{p_i}z_{i,j}(t)$, $Y_i(t)=\ltimes_{j=1}^{q_i}y_{i,j}(t)$, $\tilde{L}_i\in\mathcal{L}_{k^{n_i}\times k^{n_i+p_i}}$, $G_i\in\mathcal{L}_{k^{q_i}\times k^{n_i}}$.

Combining the equation in (11), it holds that

$$\begin{aligned} Y_i(t+1)&=G_iX_i(t+1)\\ &=G_i\tilde{L}_iZ_i(t)X_i(t)\\ &=G_i\tilde{L}W_{[k^{n_i},k^{p_i}]}X_i(t)Z_i(t)\\ &:=\tilde{L}_i(t)Z_i(t), \end{aligned}$$

where $\tilde{L}_i(t)$ is called the input-output transition matrix of Σ_i, which is a time-varying matrix.

Setting $Y(t)=\ltimes_{i=1}^{s}Y_i(t)$, then there exists a dummy operator Ψ_i such that

$$Z_i(t)=\Psi_iY(t),$$

where Ψ_i is in the form of $(\mathbf{1}_{a_i}^\top\otimes I_{b_i}^\top\otimes\mathbf{1}_{c_i}^\top)$, and a_i,b_i,c_i are dependent on the partition. Therefore, system (11) can be written as

$$Y_i(t+1)=\hat{L}_i(t)Y(t), \tag{12}$$

where $\hat{L}_i(t)=\tilde{L}_i(t)\Psi_i$, $i=1,\cdots,s$.

By using the observed data, system (12) is approximately estimated in [88], and a simplified logical network for system (3) can be obtained.

More importantly, $q_i,p_i\ll n$, which shows that the size of the structural matrix in (12) is quite small. Therefore, one can use the new simplified networks to explore the properties of original large-scale LNs, which will further reduce the computational load.

B. NETWORK AGGREGATION METHOD

1) ATTRACTORS ANALYSIS OF LARGE-SCALE LNs

Consider system (3) with aggregation (9). The input-state transition graph of Σ_i, $i=1,\cdots,s$ can be defined as

$$\{(\alpha_1,\beta_1)\to(\alpha_2,\beta_2):\alpha_j\in\Delta_{k^{n_i}},\beta_j\in\Delta_{k^{p_i}},j=1,2,\\ X_i(\alpha_1,\beta_1)=\alpha_2\}.$$

According to the input-state transition graph, the control fixed point and control limit cycle of Σ_i can be derived, $i=1,\cdots,s$.

Firstly, a natural mapping from the state in set $\mathcal{P}=\{\ltimes_{i=1}^n\alpha_i:\alpha_i\in\Delta_k,i=1,\cdots,n\}\subseteq\Delta_{k^n}$ to the state in set $\mathcal{Q}=\{\ltimes_{j=1}^m\alpha_{i_j}:\alpha_{i_j}\in\Delta_k,i_j\in\{1,\cdots,n\},j=1,\cdots,m\}\subseteq\Delta_{k^m}$ is defined as:

$$\pi(\mathcal{P},\mathcal{Q}):\Delta_{k^n}\to\Delta_{k^m}.$$

Split state transition matrix $\tilde{L}_i$ defined in (11) into k^{p_i} blocks:

$$\tilde{L}_i=[\tilde{L}_i^1\ \cdots\ \tilde{L}_i^{k^{p_i}}]. \tag{13}$$

For the control fixed points of Σ_i, Zhao *et al.* [90] pointed out that $(\delta_{k^{n_i}}^\alpha,\delta_{k^{p_i}}^\beta)$ is a control fixed point of Σ_i, iff $X_i(\delta_{k^{n_i}}^\alpha,\delta_{k^{p_i}}^\beta)=\delta_{k^{n_i}}^\alpha$, that is, $(\tilde{L}_i^\beta)_{\alpha,\alpha}=1$.

In [90], Zhao *et al.* proposed that a chain of distinct components $\{(\delta_{k^{n_i}}^{\alpha_1}, \delta_{k^{p_i}}^{\beta_1}) \rightarrow \cdots \rightarrow (\delta_{k^{n_i}}^{\alpha_l}, \delta_{k^{p_i}}^{\beta_l})\}$ is a control limit cycle with length l of subnetwork Σ_i, iff $X_i(\delta_{k^{n_i}}^{\alpha_j}, \delta_{k^{p_i}}^{\beta_j}) = \delta_{k^{n_i}}^{\alpha_{j+1}}$, $j = 1, \cdots, l-1$, $X_i(\delta_{k^{n_i}}^{\alpha_l}, \delta_{k^{p_i}}^{\beta_l}) = \delta_{k^{n_i}}^{\alpha_1}$, that is, $(\tilde{L}_i^{\beta_j})_{\alpha_{j+1},\alpha_j} = 1$, $(\tilde{L}_i^{\beta_l})_{\alpha_1,\alpha_l} = 1, j = 1, \cdots, l-1$.

In [90], the fixed points and limit cycles of large-scale LNs are obtained by combining the control fixed points and limit cycles of each subnetwork which satisfy the matching criteria.

Therefore, the complete solution of attractors for large-scale LNs is obtained via network aggregation approach.

2) CONTROLLABILITY ANALYSIS OF LARGE-SCALE LCNs

Consider system (5). The node set of (5) is denoted by $\mathcal{N} = \{x_1, \cdots, x_n, u_1, \cdots, u_m\}$, where state node set is $\mathcal{X} = \{x_1, \cdots, x_n\}$, and control input node set is $\mathcal{U} = \{u_1, \cdots, u_m\}$. Partition $\mathcal{N}$ into s blocks as

$$\mathcal{N} = \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_s, \tag{14}$$

where $\mathcal{N}_i = \{x_{i,1}, \cdots, x_{i,n_i}, u_{i,1}, \cdots, u_{i,m_i}\}$ is the proper subset of $\mathcal{N}$, and $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$, $i, j = 1, \cdots, s$, $i \neq j$.

Similarly, the input nodes and output nodes of the i-th block can be described by $\mathcal{Z}_i = \{z_{i,1}, \cdots, z_{i,p_i}\}$ and $\mathcal{Y}_i = \{y_{i,1}, \cdots, y_{i,q_i}\}$, respectively. Σ_i is also used to express the i-th block, $i = 1, \cdots, s$. The dynamics of Σ_i can be described as:

$$\begin{cases} x_{i,j}(t+1) = f_{i,j}(z_{i,1}(t), \cdots, z_{i,p_i}(t), u_{i,1}(t), \\ \cdots, u_{i,m_i}(t), x_{i,1}(t), \cdots, x_{i,n_i}(t)), j = 1, \cdots, n_i, \\ y_{i,j}(t) = h_{i,j}(x_{i,1}(t), \cdots, x_{i,n_i}(t)), j = 1, \cdots, q_i. \end{cases} \tag{15}$$

Then, the ASSR of system (15) is

$$\begin{cases} X_i(t+1) = L_i Z_i(t) U_i(t) X_i(t), \\ Y_i(t) = H_i X_i(t), \end{cases} \tag{16}$$

where $X_i(t) = \ltimes_{j=1}^{n_i} x_{i,j}(t)$, $Z_i(t) = \ltimes_{j=1}^{p_i} z_{i,j}(t)$, $U_i(t) = \ltimes_{j=1}^{m_i} u_{i,j}(t)$, $Y_i(t) = \ltimes_{j=1}^{q_i} y_{i,j}(t)$, $L_i \in \mathcal{L}_{k^{n_i} \times k^{n_i+p_i+m_i}}$, $H_i \in \mathcal{L}_{k^{q_i} \times k^{n_i}}$.

The weak connectivity of network graph for large-scale LCN (5) and each subnetwork is needed. In addition, there exists at least one state node in each block.

From the controllability of the whole system (5) with aggregation (14), the controllability of each subnetworks is derived. Zhao *et al.* [91] proved that if system (5) with aggregation (14) is controllable, then all the subnetworks Σ_i are controllable, $i = 1, \cdots, s$.

Remark 1: It is worth pointing out that, the result for controllability of large-scale LCNs via network aggregation is just a necessary condition. The sufficient conditions need to be explored further.

3) OBSERVABILITY ANALYSIS OF LARGE-SCALE LCNs

The definition of observability considered in this part is similar to Definition 1.1 (D4) in [26].

Noticing that the observability of system (5) needs to consider the output sequence, in the following, we give a new node partition which contains the output node set $\Gamma = \{\gamma_1, \cdots, \gamma_\rho\}$.

Consider system (5). Denote the node set of (5) as $\mathcal{N} = \mathcal{X} \cup \mathcal{U} \cup \Gamma = \{x_1, \cdots, x_n, u_1, \cdots, u_m, \gamma_1, \cdots, \gamma_\rho\}$.

Partition $\mathcal{N}$ into s blocks as

$$\mathcal{N} = \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_s, \tag{17}$$

where $\mathcal{N}_i$ is a proper subset of $\mathcal{N}$, and $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$, $i, j = 1, \cdots, s$, $i \neq j$.

In like manner, the set of input nodes and output nodes of Σ_i can be denoted by $\mathcal{Z}_i$ and $\mathcal{Y}_i$, $i = 1, \cdots, s$, respectively.

The observability of large-scale LCNs was investigated in [89], which shows that if system (5) has an acyclic aggregation and all the subnetworks Σ_i, $i = 1, \cdots, s$ are observable, then system (5) is observable.

We emphasize that the observability of large-scale LCNs in general network aggregation structure needs further exploration.

4) STABILIZATION OF LARGE-SCALE LCNs

Consider system (5) with network aggregation (14). Given equilibrium point X_e. System (5) is called globally stabilizable, if for any initial state, there exists a control sequence $\{U(t) : t \geq 0\}$ such that system (5) can converge to X_e.

For the stabilization of system (5), a necessary condition is given in [91]. If system (5) is stabilized to X_e, then subnetwork Σ_i is stabilizable to $\pi(X_e, \mathcal{X}_i)$, $i = 1, \cdots, s$.

In the following, the sufficient condition for stabilization of large-scale LCNs is explored via acyclic aggregation.

Some terms need to be defined before the sufficient condition is stated:

- Level-0 blocks: the blocks with in-degree 0;
- Level-r blocks: the block which the longest path from it to the level-0 block is of length r;
- $\mathcal{N}^r$:={level-0 block}$\cup \cdots \cup${level-r block};
- The stabilized fixed output of $\mathcal{N}^r$: if subnetworks corresponding to the level-0 block to the level-r block are stabilized after time t, then the output of the subnetworks corresponding to $\mathcal{N}^r$ are fixed after time t, which is called the stabilized fixed output of $\mathcal{N}^r$.

Given an acyclic aggregation of system (5). Suppose that all the subnetworks corresponding to level-0 block are globally stabilizable. For a certain $r > 0$, there is a stablilized fixed output of $\mathcal{N}^r$ such that, when the input of the subnetworks corresponding to the level-r block are fixed as the mapping of the stabilized fixed output, all the subnetworks corresponding to the level-r block are stabilizable. Then, the whole system (5) is globally stabilizable.

Remark 2: The sufficient condition of stabilization for large-scale LCNs is dependent on the special network structure. The corresponding results in general network structure deserve further discussion.

C. LOGICAL MATRIX FACTORIZATION METHOD

Consider system (3). Suppose that the state transition matrix is $\tilde{L} = \delta_{k^n}[i_1 \ \cdots \ i_{k^n}]$. There exists a permutation matrix

$Q \in \mathcal{L}_{k^n \times k^n}$ such that

$$\tilde{\tilde{L}} = \tilde{L}Q = \delta_{k^n}\Big[\underbrace{\hat{i}_1 \cdots \hat{i}_1}_{s_1} \cdots \underbrace{\hat{i}_r \cdots \hat{i}_r}_{s_r}\Big] \in \mathcal{L}_{k^n \times k^n},$$

where $\hat{i}_j$ denotes the different components in the vector $(i_1, \cdots, i_{k^n})$, r is the number of distinct components, and s_j is the number of components in the vector $(i_1, \cdots, i_{k^n})$ coinciding with $\hat{i}_j, j = 1, \cdots, r$.

In [92], Li and Wang firstly proposed a matrix factorization form for state transition matrix $\tilde{L}$. In details, the state transition matrix $\tilde{L}$ of system (3) can be factorized as $\tilde{L} = \tilde{L}^1\tilde{L}^2$, where $\tilde{L}^1 = \delta_{k^n}[\hat{i}_1 \cdots \hat{i}_r] \in \mathcal{L}_{k^n \times r}$, $\tilde{L}^2 = \delta_r\Big[\underbrace{1 \cdots 1}_{s_1} \cdots \underbrace{r \cdots r}_{s_r}\Big]Q^{-1} \in \mathcal{L}_{r \times k^n}$.

In order to obtain the equivalent size-reduced logical network of system (3), a bijection from $\{\delta_{k^n}^{\hat{i}_\mu} : \mu = 1, \cdots, r\}$ to Δ_r is defined as $\Phi(\delta_{k^n}^{\hat{i}_\mu}) = \delta_r^\mu$. Then, setting

$$\tilde{\tilde{L}} = \tilde{L}^2\tilde{L}^1 \in \mathcal{L}_{r \times r}, \tag{18}$$

the size-reduced logical network with state transition matrix $\tilde{\tilde{L}} \in \mathcal{L}_{r \times r}$ is obtained as

$$w(t+1) = \tilde{\tilde{L}}w(t), \tag{19}$$

where $w(t) \in \Delta_r$.

By using logical matrix factorization technique, the dimension of state transition matrix of system (3) is reduced from k^n to r. In [92], the topological structure of system (3) is proved to be equivalent to that of size-reduced system (19).

Similarly, logical matrix factorization technique can also be applied to large-scale LCNs in order to reduce the dimension of state transition matrix which can be further studied in the future.

IV. NETWORK PAIRING PROBLEM

A network pairing problem is denoted by a tuple

$$\Theta = \{A, M, G, \mathcal{X}, R\}, \tag{20}$$

where A is the set of n applicants and M is the set of n employers. $G := \{A, M, E\}$ is called a graph, where $E \subseteq A \times M$ is the set of edges. In this section, the graph is undirected. $\mathcal{X} := \{x_1, \cdots, x_n\}$, $x_i \in N_i$ is the proposal of applicant i, $i = 1, \cdots, n$, where $N_i \subseteq M$ is the set of neighbors for applicant i. $R : A \times M \to \mathbb{N}^2$ is the ranking of pairs, which comes from the preferences of applicants and employers.

Considering the ranking of pairs, the dynamics of a network pairing problem can be modeled by

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)), \\ \cdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), \end{cases} \tag{21}$$

where $x_i \in \mathcal{D}_{k_i}$, $i = 1, \cdots, n$ are applicants, $f_i : \prod_{j=1}^{n} \mathcal{D}_{k_j} \to \mathcal{D}_{k_i}$ are mix-valued logical functions, and $k_i := |N_i|$. These mix-valued logical functions are determined by the overall ranking of pairs.

The ASSR of system (21) is

$$x(t+1) = Lx(t), \tag{22}$$

where $x(t) = \ltimes_{i=1}^n x_i(t)$, $L \in \mathcal{L}_{k \times k}$, $k = \prod_{i=1}^{n} k_i$ is the structural matrix.

One problem worth discussing is whether or not an arrangement is stable. Zhang and Cheng [94] pointed out that the arrangements of network pairing problem are the fixed points and limit cycles of system (22). However, if the number of players is large, it is hard to verify whether or not one arrangement is stable. In the following, based on the network aggregation method, we consider the stability of network pairing problem.

Denoting $\mathcal{X} = \{x_1, \cdots, x_n\}$, partition $\mathcal{X}$ into s blocks:

$$\mathcal{X} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_s, \tag{23}$$

where $\mathcal{X}_i$ is a proper subset of $\mathcal{X}$ consisting of state nodes $\mathcal{X}_i = \{x_{i,1}, \cdots, x_{i,n_i}\}$ and $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$, $i \neq j$. $x_{i,j} \in N_{i,j}$ is the proposal of applicant (i,j), where (i,j) denotes the j-th applicant in the i-th block, and $N_{i,j} \subseteq M$ is the set of neighbors for applicant (i,j), $i = 1, \cdots, s$, $j = 1, \cdots, n_i$. Set $k_{i,j} := |N_{i,j}|, j = 1, \cdots, n_i, i = 1, \cdots, s$.

The node partition is similar to aggregation (9), then the set of input nodes and output nodes of the i-th block is denoted by $\mathcal{Z}_i = \{z_{i,1}, \cdots, z_{i,p_i}\}$ and $\mathcal{Y}_i = \{y_{i,1}, \cdots, y_{i,q_i}\}$, respectively.

The dynamics of subnetwork $\Sigma_i, i = 1, \cdots, s$ is described as:

$$x_{i,j}(t+1) = f_{i,j}(x_{i,1}(t), \cdots, x_{i,n_i}(t), z_{i,1}(t), \cdots, z_{i,q_i}(t)). \tag{24}$$

The ASSR of system (24) is

$$X_i(t+1) = F_i Z_i(t) X_i(t), \tag{25}$$

where $X_i(t) = \ltimes_{j=1}^{n_i} x_{i,j}(t)$, $Z_i(t) = \ltimes_{j=1}^{q_i} z_{i,j}(t)$, $F_i \in \mathcal{L}_{d_i \times r_i d_i}$, $d_i = \prod_{j=1}^{n_i} k_{i,j}$, $r_i = \prod_{j=1}^{q_i} k_{i,j}$, $i = 1, \cdots, s$.

By using the method introduced in Section III-B1, the fixed points and limit cycles of system (22) can be obtained, that is, all the arrangements of network pairing problem are obtained. The next question is to check which arrangement is stable.

By extending the Algorithm 1 in [94], the unstable degree of each subnetwork can be obtained. If the unstable degree is 0, then the corresponding arrangement is stable.

Remark 3: The algorithm provided in [94] can be applied to the case of square matrix, while our method can be used in the situation where the rank matrix is not a square matrix.

In the following, based on the acyclic network aggregation, the global stability of network pairing problem is investigated.

Denote the set of all arrangements of network pairing problem by $\mathcal{M}$. Set $\mathcal{M}_i = \pi(\mathcal{X}_i, \mathcal{M})$.

Definition 1: Subnetwork Σ_i is said to be globally set stabilizable to $\mathcal{M}_i$, if for any initial state, there exists a control sequence $\{Z_i(t) : t \geq 0\}$ such that Σ_i converges to set $\mathcal{M}_i$.

Algorithm 1 Unstable Degree Computation for Subnetwork Σ_i

1: Extract the rank matrix of applicants and employers for Σ_i, denoted by M_i and N_i, respectively.
2: Mark the elements of M_i and N_i, which are corresponding to the arrangement to be checked.
3: For each row of M_i, if there exists an element which is less than the marked one in the same row, use 1 to replace it; otherwise, all elements of this row are 0. Denote the new matrix by $\tilde{M}_i$.
4: For each column of N_i, if there exists an element which is less than the marked one in the same column, use 1 to replace it, and use 0 to replace the rest; otherwise, all elements of this column are 0. Denote the new matrix by $\tilde{N}_i$.
5: Set $Q_i = \tilde{M}_i \cap \tilde{N}_i$. If $\sum_{j=1}^{n_i}\sum_{l=1}^{n} Q_i = \varrho_i$, then the unstable degree of Σ_i corresponding to this checked arrangement is ϱ_i, $i = 1, \cdots, s$

Theorem 1: Subnetwork Σ_i is globally set stabilizable to the set $\mathcal{M}_i$, iff the following hold:

(i) The elements in $\mathcal{M}_i$ are fixed points of Σ_i;
(ii) In Σ_i, $\mathcal{M}_i$ is globally reachable.

Theorem 2: Consider system (22) whose network graph is acyclic. It is globally set stable at $\mathcal{M}$ if the following hold:

(i) All the subnetworks corresponding to level-0 blocks are globally set stable;
(ii) For any positive integer r, there exists a stabilized fixed output from blocks $\mathcal{L}^{r-1}$, such that subnetworks corresponding to level-r block are set stabilizable to their own mapping set from $\mathcal{M}$, while the input states of level-r blocks are the projections of the stabilized fixed output from $\mathcal{L}^{r-1}$.

V. TOPOLOGICAL STRUCTURE OF LARGE-SCALE PROBABILISTIC LOGICAL NETWORKS

A. ALGEBRAIC FORM OF LARGE-SCALE PLNs

Consider the following PLN:

$$\begin{cases} x_1(t+1) = f^{(1)}\left(x(t), \cdots, x_n(t)\right), \\ \cdots\cdots \\ x_n(t+1) = f^{(n)}\left(x_1(t), \cdots, x_n(t)\right), \end{cases} \tag{26}$$

where $x_i \in \mathcal{D}_k$ is state variable, $i = 1, \cdots, n$. Assume that there exist $c(i)$ candidates of logical function $f^{(i)}$ as

$$f_l^{(i)}(x_1(t), \cdots, x_n(t)), l = 1, \cdots, c(i), i = 1, \cdots, n. \tag{27}$$

The probability of choosing logical function $f_l^{(i)}$ as the update function is $\rho_l^{(i)}$, that is,

$$\rho_l^{(i)} = \mathbb{P}\{f^{(i)} = f_l^{(i)}\}, l = 1, \cdots, c(i), i = 1, \cdots, n. \tag{28}$$

It is obvious that

$$\sum_{l=1}^{c(i)} \rho_l^{(i)} = 1, i = 1, \cdots, n. \tag{29}$$

Due to the randomness in genetic networks, the state of gene i at the next moment is determined by $c(i)$ logical functions $f_1^{(i)}, \cdots, f_{c(i)}^{(i)}$ with probabilities $\rho_1^{(i)}, \cdots, \rho_{c(i)}^{(i)}$, respectively. Moreover, notice that a PLN can be regarded as an extension of LNs with a probabilistic setting. Then, in the j-th LN, assume that the network update function is given by $f_j = (f_{j_1}^{(1)}, \cdots, f_{j_n}^{(n)})$, $j_i \in \{1, \cdots, c(i)\}$, $j = 1, \cdots, \tau$, $i = 1, \cdots, n$, $\tau = \prod_{i=1}^{n} c(i)$. Denote the probability that the j-th LN is selected by ρ_j. The ASSR of PLN (26) is

$$x(t+1) = Lx(t), \tag{30}$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{k^n}$, $L \in \{L_1 \cdots, L_\tau\}$, and $\mathbb{P}\{L = L_j\} = \rho_j$, $j = 1, \cdots, \tau$.

B. TOPOLOGICAL STRUCTURE OF PLNs

The topological structure of PLN (30) includes positive probability fixed point and positive probability basic cycle, which were analyzed in [95].

Firstly, we give the one-step transition probability matrix of PLN (30) as

$$M = \sum_{j=1}^{\tau} \rho_j L_j. \tag{31}$$

It is easy to see that $(M)_{\alpha,\beta} = \mathbb{P}\{x(t+1) = \delta_{k^n}^{\alpha} \mid x(t) = \delta_{k^n}^{\beta}\}$. By induction, the θ-step transition probability matrix of PLN (30) is M^θ, that is, $(M^\theta)_{\alpha,\beta} = \mathbb{P}\{x(t+\theta) = \delta_{k^n}^{\alpha} \mid x(t) = \delta_{k^n}^{\beta}\}$.

Definition 2 [96]: $x_e = \delta_{k^n}^{\zeta}$ is said to be a positive probability fixed point of system (30), if $\mathbb{P}\{x(t+1) = \delta_{k^n}^{\zeta} \mid x(t) = \delta_{k^n}^{\zeta}\} > 0$, that is, $(M)_{\zeta,\zeta} > 0$.

Definition 3 [96]: A chain of distinct states $\left\{\delta_{k^n}^{\zeta_1} \to \cdots \to \delta_{k^n}^{\zeta_l}\right\}$ is said to be a positive probability basic cycle with length l of system (30), if $\mathbb{P}\left\{x(t+1) = \delta_{k^n}^{\zeta_{s+1}} \mid x(t) = \delta_{k^n}^{\zeta_s}\right\} > 0$, $s = 1, \cdots, l-1$, and $\mathbb{P}\left\{x(t+1) = \delta_{k^n}^{\zeta_1} \mid x(t) = \delta_{k^n}^{\zeta_l}\right\} > 0$, that is, $(M)_{\zeta_{s+1},\zeta_s} > 0$, $s = 1, \cdots, l-1$, and $(M)_{\zeta_1,\zeta_l} > 0$.

C. AGGREGATION OF PLNs

Considering the network graph composed by all modes of PLN (30), then partition the new network graph using the similar partition method introduced in Section III. The dynamics of subnetwork Σ_i is described as

$$x_{i,j}(t+1) = f^{(i,j)}(x_{i,1}(t), \cdots, x_{i,n_i}(t), z_{i,1}(t), \cdots, z_{i,q_i}(t)), \tag{32}$$

where $\rho_l^{(i,j)} = \mathbb{P}\{f^{(i,j)} = f_l^{(i,j)}\}$, $l = 1, \cdots, c(i,j)$, $c(i,j)$ denotes the number of candidates for logical function $f^{(i,j)}$, $j = 1, \cdots, n_i$, $i = 1, \cdots, s$.

The ASSR of subnetwork (32) is

$$X_i(t+1) = F_i Z_i(t) X_i(t), \tag{33}$$

where $X_i(t) = \ltimes_{j=1}^{n_i} x_{i,j}(t)$, $Z_i(t) = \ltimes_{j=1}^{q_i} z_{i,j}(t)$, $F_i \in \{F_{i,1}, \cdots, F_{i,\tau_i}\}$, $\mathbb{P}\{F_i = F_{i,j}\} = \rho_j^{(i)}$, $j = 1, \cdots, \tau_i$, $\tau_i = \prod_{k=1}^{n_i} c(i,k)$, $i = 1, \cdots, s$.

D. TOPOLOGICAL STRUCTURE ANALYSIS

In this part, we consider the topological structure of large-scale PLN (30) based on the topological structure of subnetworks.

Consider subnetwork Σ_i. Split structural matrix F_i into k^{q_i} blocks. According to $F_i \in \{F_{i,1}, \cdots, F_{i,\tau_i}\}$, we have

$$F_{i,j} = [F_{i,j}^1 \cdots F_{i,j}^{k^{q_i}}],\ j = 1, \cdots, \tau_i. \tag{34}$$

Then, under the control $Z_i(t) = \delta_{k^{q_i}}^{\xi}$, the transition probability matrix of subnetwork Σ_i is

$$M_{i,\xi} = \sum_{j=1}^{\tau_i} \rho_j^{(i)} F_{i,j}^{\xi}. \tag{35}$$

Definition 4: Consider subnetwork Σ_i.

(i) The input-state positive probability transition graph of Σ_i is defined as a directed graph $\Big\{\Delta_{k^{n_i}} \times \Delta_{k^{q_i}}, \Lambda_i, \{M_{i,\xi}, \xi = 1, \cdots, k^{q_i}\}\Big\}$, where

$$\Lambda_i = \Big\{(\theta, \beta) \to (\theta', \beta') \mid \theta, \theta' \in \Delta_{k^{n_i}}, \beta, \beta' \in \Delta_{k^{q_i}}, (M_{i,\beta})_{\theta',\theta} > 0\Big\}; \tag{36}$$

(ii) A period path of the input-state positive probability transition graph with period 1 is called a positive probability control fixed point of Σ_i;

iii) A period path with distinct components of the input-state positive probability transition graph with period l, $l > 1$ is called a positive probability control basic cycle with length l.

Lemma 1: $(\delta_{k^{n_i}}^{\alpha}, \delta_{k^{q_i}}^{\beta})$ is an positive probability control fixed point of subnetwork Σ_i, iff $\mathbb{P}\{X_i(t+1) = \delta_{k^{n_i}}^{\alpha} \mid X_i(t) = \delta_{k^{n_i}}^{\alpha}, Z_i(t) = \delta_{k^{q_i}}^{\beta}\} > 0$, that is, $(M_{i,\beta})_{\alpha,\alpha} > 0$.

Lemma 2: A chain of distinct components $\big\{(\delta_{k^{n_i}}^{\alpha_1}, \delta_{k^{q_i}}^{\beta_1}) \to \cdots \to (\delta_{k^{n_i}}^{\alpha_l}, \delta_{k^{q_i}}^{\beta_l})\big\}$ is an positive probability control basic cycle with length l of subnetwork Σ_i, iff $\mathbb{P}\big\{X_i(t+1) = \delta_{k^{n_i}}^{\alpha_{s+1}} \mid X_i(t) = \delta_{k^{n_i}}^{\alpha_s}, Z_i(t) = \delta_{k^{q_i}}^{\beta_s}\big\} > 0$, $s = 1, \cdots, l-1$, and $\mathbb{P}\big\{X_i(t+1) = \delta_{k^{n_i}}^{\alpha_1} \mid X_i(t) = \delta_{k^{n_i}}^{\alpha_l}, Z_i(t) = \delta_{k^{q_i}}^{\beta_l}\big\} > 0$, that is, $(M_{i,\beta_s})_{\alpha_{s+1},\alpha_s} > 0$, $s = 1, \cdots, l-1$, and $(M_{i,\beta_l})_{\alpha_1,\alpha_l} > 0$.

Firstly, we analyze the positive probability fixed points of large-scale PLN (30) via network aggregation.

According to Lemma 1, one can obtain all positive probability control fixed points of subnetwork Σ_i as

$$A_i^j = (\delta_{k^{n_i}}^{\alpha_i^j}, \delta_{k^{q_i}}^{\beta_i^j}), \tag{37}$$

where $j = 1, \cdots, e_i$, e_i is the number of positive probability control fixed points in Σ_i, $i = 1, \cdots, s$. Define $\mathcal{Z}_i^{(h)} = \mathcal{Y}_h^{(i)} := \mathcal{Z}_i \cap \mathcal{Y}_h$, $i, h = 1, \cdots, s, i \neq h$. Given a set of positive probability control fixed points as

$$\{A_i^{j_i} : i = 1, \cdots, s\}, \tag{38}$$

where $j_i \in \{1, \cdots, e_i\}$. If

$$\pi\big(A_i^{j_i}, \mathcal{Z}_i^{(h)}\big) = \pi\big(A_h^{j_h}, \mathcal{Y}_h^{(i)}\big) \tag{39}$$

holds for any $i, h = 1, \cdots, s$, $i \neq h$, then the positive probability control fixed points in (38) form a positive probability fixed point of system (30). Suppose that there exist ε sets of positive probability control fixed points satisfying condition (39), denoted by $\mathcal{A}_i$, $i = 1, \cdots, \varepsilon$. Assuming that $x_\alpha \in \mathcal{X}_\beta$, let $x_\alpha^e := \pi(A_\beta^{j_\beta}, \{x_\alpha\})$, $\alpha \in \{1, \cdots, n\}$, $\beta \in \{1, \cdots, s\}$. Then, the corresponding positive probability fixed point of system (30) is $X_e = \ltimes_{\alpha=1}^n x_\alpha^e$.

Based on the above analysis, we give the following algorithm to obtain all positive probability fixed points of large-scale PLN (30).

Algorithm 2 The Following Steps Are Used to Calculate All Positive Probability Fixed Points of Large-Scale PLN (30)

1: Calculate all positive probability control fixed points of Σ_i in the form of (37).
2: Find out all sets of positive probability control fixed points satisfying condition (39). If there exists any, denote them by $\mathcal{A}_i$, $i = 1, \cdots, \varepsilon$. Otherwise, stop.
3: According to $\mathcal{A}_i$, $i = 1, \cdots, \varepsilon$, calculate all positive probability fixed points of large-scale PLN (30), denoted by $X_{e,i} = \ltimes_{j=1}^n x_j^{e,i}$, $i = 1, \cdots, \varepsilon$.

Similarly, in the following, we analyze the positive probability basic cycles of large-scale PLN (30) via network aggregation.

According to Lemma 2, it is easy to obtain all positive probability control basic cycles of subnetwork Σ_i, denoted by

$$B_i^j = \Big\{\Big(\delta_{k^{n_i}}^{\gamma_{j,1}^i}, \delta_{k^{q_i}}^{\lambda_{j,1}^i}\Big) \to \cdots \to \Big(\delta_{k^{n_i}}^{\gamma_{j,l_j}^i}, \delta_{k^{q_i}}^{\lambda_{j,l_j}^i}\Big)\Big\}, \tag{40}$$

where the length of basic cycle B_i^j is l_j, $j = 1, \cdots, \varpi_i$, and ϖ_i is the number of positive probability control basic cycles in Σ_i, $i = 1, \cdots, s$.

Given a set of positive probability control basic cycles as

$$\{B_i^{j_i} : i = 1, \cdots, s\}, \tag{41}$$

where $j_i \in \{1, \cdots, \varpi_i\}$.

Define $\Pi\big(B_i^j, D\big) := \big\{\pi(B_i^{j_1}, D) \to \cdots \to \pi(B_i^{j_{l_j}}, D)\big\}$, $D \subseteq \big(\mathcal{X}_i \cup \mathcal{Z}_i\big) \subseteq \mathcal{X}$. If

$$\Pi\Big(B_i^{j_i}, \mathcal{Z}_i^{(h)}\Big) = \Pi\Big(B_h^{j_h}, \mathcal{Y}_h^{(i)}\Big), \tag{42}$$

holds for any $i, h = 1, \cdots, s$, $i \neq h$, then the positive probability control basic cycles in (41) form a positive probability basic cycle of system (30). Suppose that there exist ϵ sets of positive probability control basic cycles satisfying condition (42), denoted by $\mathcal{B}_r$, $r = 1, \cdots, \epsilon$. For

$$\mathcal{B}_r := \{B_i^{j_i^r} : i = 1, \cdots, s\}, \tag{43}$$

denote the least common multiple of the length of $\mathcal{B}_i^{j_i^r}$, $i = 1, \cdots, s$ by l_r. By repeating the elements in $\mathcal{B}_i^{j_i^r}$ (with possible adjusting the order), one can extend the length of $\mathcal{B}_i^{j_i^r}$ to l_r, denoted the obtained positive probability control basic cycle by $\tilde{\mathcal{B}}_i^{j_i^r}$. Then, according to $\tilde{\mathcal{B}}_i^{j_i^r}$, $i = 1, \cdots, s$, one can obtain $x_j^{r,t}$, $j = 1, \cdots, n$, $t = 1, \cdots, l_r$. Thus, the positive probability basic cycles of system (30) corresponding to $\mathcal{B}_r$ can be obtained as $\{X_{r,1} \to \cdots \to X_{r,l_r}\}$, where $X_{r,t} = \ltimes_{j=1}^{n} x_j^{r,t}$, $t = 1, \cdots, l_r$.

Based on the following algorithm, all positive probability basic cycles of large-scale PLN (30) can be obtained.

Algorithm 3 The Following Steps Are Used to Obtain All Positive Probability Basic Cycles of Large-Scale PLN (30)

1: Calculate all positive probability control basic cycles of Σ_i in the form of (40).
2: Find out all sets of positive probability control basic cycles satisfying condition (42). If there exists any, denote them by $\mathcal{B}_r$, $r = 1, \cdots, \epsilon$. Otherwise, stop.
3: For each $\mathcal{B}_r$, $r = 1, \cdots, \epsilon$, construct the corresponding $\tilde{\mathcal{B}}_r$.
4: According to $\tilde{\mathcal{B}}_r$, $r = 1, \cdots, \epsilon$, calculate the positive probability basic cycles of large-scale PLN (30) as $\{X_{r,1} \to \cdots \to X_{r,l_r}\}$.

Remark 4: There exists τ modes in PLN (30). When partitioning, we consider the network graph composed of all the modes rather than the network graph of one mode.

Remark 5: When combining the positive probability control fixed points and basic cycles of subnetworks to obtain the positive probability fixed points and basic cycles of system (30), we use the information of different modes instead of just one mode.

VI. EXAMPLE

Example 2: Considering the following probabilistic Boolean network, and the update rules are given as follows:

$$\begin{cases} x_1(t+1) = x_2(t),\ \rho_1^{(1)} = 1 \\ x_2(t+1) = \begin{cases} x_3(t) \wedge x_7(t),\ \rho_1^{(2)} = 0.7 \\ x_3(t),\ \rho_2^{(2)} = 0.3 \end{cases} \\ x_3(t+1) = x_1(t) \vee x_2(t),\ \rho_1^{(3)} = 1 \\ x_4(t+1) = \begin{cases} x_1(t) \vee x_5(t) \vee x_7(t),\ \rho_1^{(4)} = 0.5 \\ x_1(t),\ \rho_2^{(4)} = 0.5 \end{cases} \\ x_5(t+1) = \neg x_4(t),\ \rho_1^{(5)} = 1 \\ x_6(t+1) = x_6(t),\ \rho_1^{(6)} = 1 \\ x_7(t+1) = x_6(t),\ \rho_1^{(7)} = 1 \\ x_8(t+1) = \begin{cases} x_7(t) \vee x_9(t),\ \rho_1^{(8)} = 0.6 \\ x_7(t) \wedge x_9(t),\ \rho_2^{(8)} = 0.4 \end{cases} \\ x_9(t+1) = x_6(t),\ \rho_1^{(1)} = 1. \end{cases} \quad (44)$$

Partition the node set $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ into 3 blocks:

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3,$$

where $\mathcal{X}_1 = \{x_1, x_2, x_3\}$, $\mathcal{X}_2 = \{x_4, x_5\}$, $\mathcal{X}_3 = \{x_6, x_7, x_8, x_9\}$. One can find that $\mathcal{Z}_1 = \{x_7\}$, $\mathcal{Z}_2 = \{x_1, x_7\}$, $\mathcal{Z}_3 = \emptyset$. Figure. 1 is the network graph of system (44), where the weight on edge represents the probability of interaction.

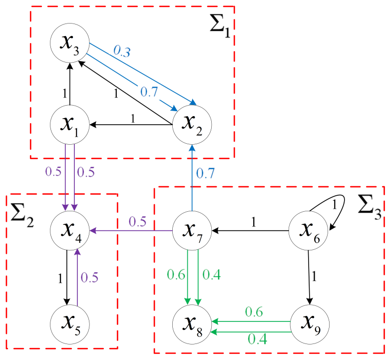

FIGURE 1. The network graph of system (44) in Example 2.

In this figure, the black line means that the probability of interaction is 1, while lines in other colors represent different modes of a node.

The ASSR of subnetwork Σ_i is:

$$X_i(t+1) = F_i Z_i(t) X_i(t), \quad (45)$$

where $X_i(t) = \ltimes_{j=1}^{n_i} x_{i,j}(t)$, $Z_i(t) = \ltimes_{j=1}^{q_i} z_{i,j}(t)$, $\mathbb{P}\{F_1 = F_{1,1}\} = 0.7$, $\mathbb{P}\{F_1 = F_{1,2}\} = 0.3$, $\mathbb{P}\{F_2 = F_{2,1}\} = 0.5$, $\mathbb{P}\{F_2 = F_{2,2}\} = 0.5$, $\mathbb{P}\{F_3 = F_{3,1}\} = 0.6$, $\mathbb{P}\{F_3 = F_{3,2}\} = 0.4$, and

$$F_{1,1} = \delta_8[1\ 3\ 5\ 7\ 1\ 3\ 6\ 8\ 3\ 3\ 7\ 7\ 3\ 3\ 8\ 8],$$
$$F_{1,2} = \delta_8[1\ 3\ 5\ 7\ 1\ 3\ 6\ 8\ 1\ 3\ 5\ 7\ 1\ 3\ 6\ 8],$$
$$F_{2,1} = \delta_4[2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 4\ 1\ 3],$$
$$F_{2,2} = \delta_4[2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 4\ 4\ 3\ 3\ 4\ 4\ 3\ 3],$$
$$F_{3,1} = \delta_{16}[1\ 1\ 1\ 1\ 1\ 3\ 1\ 3\ 14\ 14\ 14\ 14\ 14\ 16\ 14\ 16],$$
$$F_{3,2} = \delta_{16}[1\ 3\ 1\ 3\ 3\ 3\ 3\ 3\ 14\ 16\ 14\ 16\ 16\ 16\ 16\ 16].$$

According to Lemma 1, the positive probability control fixed points of Σ_1 are

$$A_1^1 = (\delta_8^1, \delta_2^1), A_1^2 = (\delta_8^1, \delta_2^2), A_1^3 = (\delta_8^8, \delta_2^1), A_1^4 = (\delta_8^8, \delta_2^2).$$

The positive probability control fixed points of Σ_2 are

$$A_2^1 = (\delta_4^2, \delta_4^1),\quad A_2^2 = (\delta_4^2, \delta_4^2),\quad A_2^3 = (\delta_4^2, \delta_4^3),$$
$$A_2^4 = (\delta_4^3, \delta_4^3),\quad A_2^5 = (\delta_4^3, \delta_4^4).$$

The positive probability control fixed points of Σ_3 are

$$A_3^1 = \delta_{16}^1,\ A_3^2 = \delta_{16}^{16}.$$

The sets of positive probability control fixed points satisfying condition (39) are

$$\mathcal{A}_1 = \{A_1^1, A_2^1, A_3^1\},\quad \mathcal{A}_2 = \{A_1^2, A_2^2, A_3^2\},$$
$$\mathcal{A}_3 = \{A_1^3, A_2^3, A_3^1\},\quad \mathcal{A}_4 = \{A_1^3, A_2^4, A_3^1\},$$
$$\mathcal{A}_5 = \{A_1^4, A_2^5, A_3^2\}. \quad (46)$$

Therefore, from Algorithm 1, the positive probability fixed points of system (44) are obtained as

$$X_{e,1} = \delta_{512}^{17},\quad X_{e,2} = \delta_{512}^{32},\quad X_{e,3} = \delta_{512}^{465},$$
$$X_{e,4} = \delta_{512}^{481},\quad X_{e,5} = \delta_{512}^{496}. \quad (47)$$

According to Lemma 2, one can obtain all positive probability control basic cycles of Σ_i, $i = 1, 2, 3$. It is worth pointing out that there exists no basic cycle in subnetwork Σ_3. Therefore, there exists no positive probability basic cycle in system (44).

One can see from this example that network aggregation method makes it possible to investigate the topological structure of large-scale PLNs.

VII. CONCLUSION

In this paper, we have reviewed some efforts made for the analysis and control of large-scale LNs, including approximation of LNs, controllability, stabilization, and observability of LCNs via network aggregation method, and the topological structure of LNs via logical matrix factorization technique. By using the method introduced above, we have explored the stability of large-scale version of network pairing problem and the topological structure of large-scale PLNs. Future works will devote to investigating other issues of large-scale LCNs and PLNs by virtue of approximation method, network aggregation approach and logical matrix factorization technique. In addition, the methods reviewed in this paper can be used to investigate the impulsive control [97]–[99] of large-scale LCNs.

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