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# The Interval Efficiency Evaluation Model Based on Incentive Compatibility

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**ABSTRACT** Data envelopment analysis (DEA) is a method of measuring the efficiency of peer decision-making units (DMUs). Conventional DEA evaluates the performance of each DMU only from the optimistic point of view. In this paper, the efficiency of a DMU is measured by using an interval efficiency evaluation model based on incentive compatibility, which considers both the optimistic and pessimistic attitude of the DMU during the evaluation process. The efficiency of a DMU, which is computed by combining the two attitudes, can be expressed by interval and provide a more reasonable assessment of the DMU. The lower bound of the interval efficiency are computed by measuring the worst relative efficiency of the ADMU, which is between the results of bounded DEA models from the optimistic or pessimistic points of view. Two numerical examples were examined using the proposed interval DEA model to show its potential application and validity.

**INDEX TERMS** Interval linear programming, data envelopment analysis, interval ranking.

## I. INTRODUCTION

Data envelopment analysis (DEA), first proposed by Charnes *et al.* [1], is an extensively used nonparametric method to measure the performance of decision-making units (DMUs) by comparing them in order to obtain the best evaluation. The corresponding Charnes–Cooper–Rhodes efficiencies (CCR efficiencies) are referred to as the best relative efficiencies, which are restricted to be no greater than one. If the CCR efficiency of a DMU is equal to one, then it is considered to be DEA efficient; otherwise, it is DEA non-efficient.

The traditional CCR model only considers the most efficient that the DMU can achieve, but does not consider the lower efficiency limit of DMU. Searching for the optimal efficiencies of DMUs is an optimistic evaluation, while searching for the worst efficiency of DMUs is a pessimistic evaluation. Unilateral evaluation is only considering optimistic evaluation or pessimistic evaluation. The result of unilateral evaluation is crisp efficiency, and interval efficiency can be obtained by considering optimistic and pessimistic perspectives.

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To consider only unilateral evaluation is one-sided and unrealistic. Some researchers considered that interval efficiency, which can give a perspective assessment of each DMU, can be used to evaluate a DMU better than a crisp efficiency [2]–[5]. Therefore, they proposed many different models to assess DMUs by using interval efficiency. Theoretically, the upper and lower limits of the interval efficiency constitute the maximization and minimization of the objective function under the same constraint conditions, respectively. For example, the structures of the interval DEA models that Doyle *et al.* and Entani *et al.* proposed to compare multiattribute objects were similar, but they had the same drawback that only one input and one output data item of the DMU were utilized effectively and all the remaining input and output data were not included in the evaluation process. Therefore, several scholars developed bounded models to overcome these defects [6]–[12].

For the existing models, the upper limit of the intervals of a DMU can be computed easily, which even equals CCR efficiency in most cases [6], [7], [9], [10], [13], [14]. However, the lower limits of the intervals cannot be reasonably measured. In most cases, it is easy to define the best production standards, whereas it is frequently difficult to provide the worst production standards. Wang *et al.* [13] defined a virtual anti-ideal DMU (ADMU), which consumed the most inputs only to produce the least outputs. They formulated the bounded DEA by using the CCR efficiency of the ADMU as the lower bound of all DMUs, and the upper bound of all DMUs was set to an efficiency of one. In their view, the relative efficiency of every DMU must be restricted by the standard absolutely. However, certain researchers considered it unreasonable to make the CCR efficiency of ADMU the lower bound of all DMUs, and developed many models to modify it. Azizi and Adjirlu [8] presented a virtual ideal DMU (IDMU), which consumed the least inputs to produce the most outputs, and they constructed the bounded model by using the ratio of the CCR efficiency of an ADMU to the pessimistic efficiency of the IDMU as the lower bound, and retained the efficiency of one as the upper bound. Chen [15] retained the worst relative efficiency of the ADMU as the lower bound efficiency and modified the traditional bounded DEA models without distortion of frontiers. Nevertheless, the improved bounded DEA models have a common defect: the lower limit of interval efficiency of each DMU, which is computed by the improved bounded models, is too small and very close to zero [8], [10], [15].

In the market economy, a DMU has both positive and negative attitudes in the production process and the two attitudes exist in the self-evaluation. In this paper, compared to the bounded DEA models, we does not restrict the efficiency of a DMU by making the standard compulsorily, but introduce an incentive mechanism to explain the relationship between the two attitudes and make a DMU as efficient as possible while imposing the standard. We compute two main sets of weights for the two attitudes of a DMU and set up the participation constraints and incentive compatibility constraints to conceive an interval DEA model. Then, we discuss whether the upper bound of each DMU is equal to its CCR efficiency in most cases. Furthermore, we explain the reasonableness of the bounded DEA models and redefine the widest interval efficiency of all DMUs.

The main contents of the remainder of this paper are as follows. The basic DEA models for evaluating DMUs from both the optimistic and pessimistic viewpoint are presented in Section II. In Section III, Wang *et al.*'s bounded model is introduced, and the analysis of the irrationalities and the development of the interval DEA model based on incentive compatibility are described. In Section IV, the Hurwicz criterion approach for comparing and ranking interval efficiencies of DMUs is briefly introduced. This is followed by two example implementations using real data to show the practical application of the interval DEA model based on incentive compatibility. The summary of the paper is presented in Section VI.

## II. PRELIMINARY

In DEA, the maximum ratio of outputs to inputs is called the efficiency, which is obtained from the optimistic viewpoints for each DMU.S.ppose there are $n$ DMUs with $m$ inputs and $s$ outputs. Let $x_{ij}(i = 1, 2, \cdots, m)$ and $y_{rj}(r = 1, 2, \cdots, s)$ be the amount of inputs and outputs for the $j^{\text{th}}$ DMU ($j = 1, 2, \cdots, n$); we can assume $x_{ij}(i = 1, 2, \cdots, m)$ and $y_{rj}(r = 1, 2, \cdots, s)$ are all positive. The efficiency of the $j^{\text{th}}$ DMU is defined as

$$\theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}$$

where $u_r$ is the weight of the $r^{\text{th}}$ output and $v_i$ is the weight of the $i^{\text{th}}$ input. The classic CCR model proposed by Charnes *et al.* to measure the efficiency of $\text{DMU}_0$ (the subscript zero represents the DMU under evaluation) relative to the other DMUs from the optimistic viewpoint is

$$\begin{aligned} \text{Max } \theta_0 &= \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\ \text{s.t. } \theta_j &= \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1, 2, \cdots, n, \\ & u_r, v_i \ge \varepsilon, \quad r = 1, 2, \cdots, s; \ i = 1, 2, \cdots, m. \end{aligned} \tag{1}$$

The variables $u_r$ and $v_i$ are decision variables and $\varepsilon$ is the non-Archimedean infinitesimal. Eq. (1) can be solved by an equivalent linear programming (LP) model:

$$\begin{aligned} \text{Max } \theta_0 &= \sum_{r=1}^{s} u_r y_{r0} \\ \text{s.t. } & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1, 2, \cdots, n, \\ & \sum_{i=1}^{m} v_i x_{i0} = 1, \\ & u_r, v_i \ge \varepsilon, \quad r = 1, 2, \cdots, s; i = 1, 2, \cdots, m. \end{aligned} \tag{2}$$

Denote by $u_r^*(r = 1, \cdots, s)$ and $v_i^*(i = 1, \cdots, m)$ the optimal solution to Eq. (2). Then, let $\theta_0^{\text{CCR}} = \sum_{r=1}^{s} u_r^* y_{r0}$ be the CCR-efficiency or the best relative efficiency. If $\theta_0^{\text{CCR}} = 1$, we consider $\text{DMU}_0$ to be DEA efficient (CCR efficient); otherwise, the DMU is considered DEA non-efficient.

## III. INTERVAL DATA ENVELOPMENT ANALYSIS MODELS BASED ON INCENTIVE COMPATIBILITY

### A. BOUNDED DATA ENVELOPMENT ANALYSIS MODEL

In actual production, in order to determine the worst production standard, we usually need to consider the waste of resource. Therefore, we can define the ADMU as follows.

*Definition 1 [13]:* An anti-ideal DMU (ADMU) is a virtual DMU that consumes the most inputs only to produce the least outputs.

By this definition, $x_i^{\max}(i = 1, 2, \cdots, m)$ and $y_r^{\min}(r = 1, 2, \cdots, s)$ are the inputs and outputs of the ADMU, respectively. They are determined by

$$\begin{aligned} x_i^{\max} &= \max_j\{x_{ij}\}, \quad i = 1, \cdots, m \\ y_r^{\min} &= \min_j\{y_{rj}\}, \quad r = 1, \cdots, s \end{aligned}$$

Denote by $\theta_{\text{ADMU}}^{\text{CCR}}$ the CCR efficiency of the ADMU, which is determined by the model

$$\begin{aligned}
&\text{Max } \theta_{\text{ADMU}}^{\text{CCR}} = \frac{\sum_{r=1}^{s} u_r y_r^{\min}}{\sum_{i=1}^{m} v_i x_i^{\max}} \\
&\text{s.t. } \theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \quad j = 1, 2, \cdots, n, \\
&\quad u_r, v_i \geq \varepsilon, \quad r = 1, 2, \cdots, s; i = 1, 2, \cdots, m. \qquad (3)
\end{aligned}$$

This equation can be solved by an equivalent LP model:

$$\begin{aligned}
&\text{Max } \theta_{\text{ADMU}}^{\text{CCR}} = \sum_{r=1}^{s} u_r y_r^{\min} \\
&\text{s.t. } \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, 2, \cdots, n, \\
&\quad \sum_{i=1}^{m} v_i x_i^{\max} = 1, \\
&\quad u_r, \quad v_i \geq \varepsilon, \ r = 1, 2, \cdots, s; \ i = 1, 2, \cdots, m. \qquad (4)
\end{aligned}$$

Since the ADMU is the worst virtual DMU that consumes the most inputs to produce the least outputs, the CCR-efficiency of all the DMUs cannot be less than $\theta_{\text{ADMU}}^{\text{CCR}}$. The following theorem shows this.

*Theorem 1:* $\theta_{\text{ADMU}}^{\text{CCR}} \leq \theta_0^{\text{CCR}}$.

*Proof:* Let $\theta_{\text{ADMU}}^{\text{CCR}}$ be the optimum solution of Eq (1), and $u_{r\text{ADMU}}^*, v_{i\text{ADMU}}^*$ be the optimal weights of the ADMU corresponding to $\theta_{\text{ADMU}}^{\text{CCR}}$. By Definition 1.

$$\frac{\sum_{r=1}^{s} u_{r\text{ADMU}}^* y_{rj}}{\sum_{i=1}^{m} v_{i\text{ADMU}}^* x_{ij}} \geq \frac{\sum_{r=1}^{s} u_{r\text{ADMU}}^* y_r^{\min}}{\sum_{i=1}^{m} v_{i\text{ADMU}}^* x_i^{\max}} = \theta_{\text{ADMU}}^*$$

and clearly $u_{r\text{ADMU}}^*, v_{i\text{ADMU}}^*$ is the solution of the model represented by Eq. (1). Let $\theta_0^{\text{CCR}}$ be the optimum solution of Eq. (1); then, we have $\theta_0^{\text{CCR}} \geq \frac{\sum_{r=1}^{s} u_{r\text{ADMU}}^* y_{rj}}{\sum_{i=1}^{m} v_{i\text{ADMU}}^* x_{ij}} \geq \frac{\sum_{r=1}^{s} u_{r\text{ADMU}}^* y_r^{\min}}{\sum_{i=1}^{m} v_{i\text{ADMU}}^* x_i^{\max}} = \theta_{\text{ADMU}}^{\text{CCR}}$. This completes the proof. □

Wang *et al.* (2007) [13] considered the ADMU the worst virtual DMU, and then it could be considered the worst standard of all DMUs. This means that the lower and upper bounds of every DMU should be within the range of interval $[\theta_{\text{ADUM}}^{\text{CCR}}, 1]$ absolutely. Therefore, the bounded DEA model was developed to measure the best and worst relative efficiency:

$$\begin{aligned}
&\text{Max/Min } \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\
&\text{s.t. } \theta_{\text{ADUM}}^{\text{CCR}} \leq \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \quad j = 1, 2, \cdots, n \\
&\quad u_r, v_i \geq \varepsilon, \quad r = 1, 2, \cdots, s; \ i = 1, 2, \cdots, m. \qquad (5)
\end{aligned}$$

Eq. (5) can be solved by an equivalent LP model

$$\begin{aligned}
&\text{Max/Min } \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\
&\text{s.t. } \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, 2, \cdots, n, \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \theta_{\text{ADUM}}^{\text{CCR}} * \sum_{i=1}^{m} v_i x_{ij} \geq 0, \quad j = 1, 2, \cdots, n, \\
&\quad u_r, v_i \geq \varepsilon, \quad r = 1, 2, \cdots, s; \ i = 1, 2, \cdots, m. \qquad (6)
\end{aligned}$$

Denote by $\theta_0^{U*}$ and $\theta_0^{L*}$ the optimal solution of Eqs. (5) and (6), respectively. They are the best and worst relative efficiencies of $\text{DMU}_0$, and therefore, the interval relative efficiency of $\text{DMU}_0$ is $[\theta_0^{L*}, \theta_0^{U*}]$. After the study, we can find $\theta_0^{U*} \leq \theta_0^{\text{CCR}}$. We prove this conclusion as follows.

*Theorem 2:* Let $\theta_0^{U*}$ be the upper bound of $\text{DMU}_0$ based on the bounded DEA model. Then, we have $\theta_0^{U*} \leq \theta_0^{\text{CCR}}$.

*Proof:* Let $\theta_0^{U*}$ be the maximum solution of Eq. (5) and $u_r^{U*}, v_i^{U*}$ be the optimal weights of $\text{DMU}_0$ corresponding to $\theta_0^{U*}$; let $\theta_0^{\text{CCR}}$ be the optimum solution of Eq. (2) and $u_r^*, v_i^*$ the optimal weights of $\text{DMU}_0$ corresponding to $\theta_0^{\text{CCR}}$. Clearly, $u_r^{U*}, v_i^{U*}$ is also the solution of Eq. (2), and therefore, we have $\theta_0^{U*} \leq \theta_0^{\text{CCR}}$. This completes the proof. □

In most cases, if $\theta_{\text{ADMU}}^{\text{CCR}}$ is sufficiently small, $\theta_0^{U*}$ is equal to $\theta_0^{\text{CCR}}$ [6], [7], [9], [10], [13], [14].

In the bounded DEA model, Wang *et al.* (2007) [13] attempted to explore the lower limit of DMU efficiency by assuming that all DMU efficiencies are higher than $\theta_{\text{ADMU}}^{\text{CCR}}$. In fact, $\theta_{ADMU}^{\text{CCR}}$ is the best relative efficiency of an ADMU. If the inputs and outputs of the DMU are close to those of the ADMU, it is unreasonable that their worst relative efficiencies must be higher than $\theta_{\text{ADMU}}^{\text{CCR}}$. Recently, Azizi and Adjirlu [8], Azizi [10], and Chen [15] put forward a smaller lower bound of all DMUs instead of $\theta_{\text{ADMU}}^{\text{CCR}}$. However, the lower bound of each DMU is too small and even close to the zero in these models, because the bounded models required that all the relative efficiencies be not less than the lower bound. Moreover, most researchers considered that if the lower bound were sufficiently small, the model would be more reasonable. In our opinion, an incentive mechanism is required to ensure that the DMUs can be as efficient as possible according to the restriction of the lower bound. Therefore, we introduce a new model that considers the incentive mechanism.

### B. INTERVAL DATA ENVELOPMENT ANALYSIS MODEL BASED ON INCENTIVE COMPATIBILITY

There exists some kind of incentive mechanism to make a human pursue personal interests, coinciding with the goal of enterprise, this institutional arrangement constitutes "incentive compatibility." Incentive mechanisms have been widely used in many fields [17]–[21].

In fact, the production process is complex. The DMU may have positive and negative attitudes. When a DMU pursues the best efficiency, the positive attitude makes the process increasingly better. The negative attitude is also caused by the restriction of the production conditions and in contrast it causes the process to be increasingly worse. However, the positive attitude can emerge by means of the incentive mechanism, so that the worst relative efficiency of DMUs cannot be close to zero. In other words, the two attitudes exist at the same time and they influence each other. Therefore, it is necessary to consider the two attitudes when a DMU measures its relative efficiency. In this paper, we denote two sets of inputs/outputs weights for the two attitudes, respectively. Let $\{\underline{u_r}, \underline{v_i}, | r = 1, 2, \cdots, s, i = 1, 2, \cdots, m\}$ be a set of weights corresponding to the positive attitude, which must satisfy the conditions

$$\sum_{r=1}^{s} \underline{u_r} y_{rj} - \sum_{i=1}^{m} \underline{v_i} x_{ij} \leq 0, \quad j = 1, 2, \cdots, n \tag{7}$$

The above equation means the efficiency of the DMU computed by using the weights of the positive attitude cannot be greater than one. Then, let $\{\overline{u_r}, \overline{v_i} | r = 1, 2, \cdots, s, i = 1, 2, \cdots, m\}$ be a set of weights corresponding to a negative attitude, which must satisfy

$$\sum_{r=1}^{s} \overline{u_r} y_{rj} - \theta_{\mathrm{ADMU}}^{\mathrm{CCR}} * \sum_{i=1}^{m} \overline{v_i} x_{ij} \geq 0, \quad j = 1, 2, \cdots, n \tag{8}$$

As Wang *et al.* [13] assumed, we need to provide an estimate of the lower bound to ensure that the worst relative efficiencies of the DMU are not infinitely close to zero. Since the ADMU is the most wasteful of all DMUs, we can hope that most of the efficiencies of the DMUs will be higher than $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$. Then, we consider that the efficiency of each DMU when the weights of the negative attitude are applied cannot be less than $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$, as shown in Eq. (8).

As noted above, whether a DMU is pursuing a good or a bad evaluation, it has two attitudes at the same time. Obviously, the DMU is more positive when pursuing a better evaluation and more negative when pursuing a worse evaluation. Under the different circumstances, the two attitudes show different priorities. The relationship can be explained as follows.

*Definition 2:* The two pursuits are incentive compatible. The positive attitude is weakly preferred to the negative attitude for the pursuit of a higher efficiency evaluation, while the negative attitude is weakly preferred to the positive attitude for the pursuit of a lower efficiency evaluation.

This definition refers to *The Theory of Incentives – The Principal-Agent Model* (2001).

As is known, when DMUs seek to obtain a better evaluation, they compare their efficiency with the best efficiency. In contrast, when they have to seek the lower limit of their own efficiency, they determine whether they can achieve the relative minimum standard. Mathematically, Definition 2 can be expressed as the incentive compatibility constraints

$$\sum_{r=1}^{s} \underline{u_r} y_{rj} - \sum_{i=1}^{m} \underline{v_i} x_{ij} \geq \sum_{r=1}^{s} \overline{u_r} y_{rj} - \sum_{i=1}^{m} \overline{v_i} x_{ij}, \tag{9}$$

$$\sum_{r=1}^{s} \overline{u_r} y_{rj} - \theta_{\mathrm{ADMU}}^{\mathrm{CCR}} * \sum_{i=1}^{m} \overline{v_i} x_{ij} \geq \sum_{r=1}^{s} \underline{u_r} y_{rj} - \theta_{\mathrm{ADMU}}^{\mathrm{CCR}} * \sum_{i=1}^{m} \underline{v_i} x_{ij}. \tag{10}$$

Eq. (9) illustrates that when DMUs compare their efficiency with the best efficiency, they are more positive and the positive attitude is weakly preferred to the negative attitude, and therefore, the difference between the outputs and inputs caused by the set of weights of the positive attitude is no less than that caused by the set of weights of the negative attitude. Eq. (10) illustrates that when DMUs compare their efficiency with $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$, this means they want to explore the worst relative efficiency, and therefore, they are more negative and the negative is weakly preferred to the positive attitude. Therefore, the difference between the outputs and inputs caused by the set of weights of the negative attitude is no less than that caused by the set of weights of the positive attitude. The incentive compatibility constraints ensure that the two sets of weights for the two attitudes can be configured more reasonably under the different situations.

From Eqs. (7) and (8), the efficiency of one and $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$ can be considered as relative standards, such as the efficiency of one is only for the positive attitude and the efficiency $\theta_{ADMU}^{CCR}$ is only for the negative attitude. This means that the efficiency of a DMU may be lower than $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$ or higher than one. Therefore, we need to ensure that the DMU is as efficient as possible. It is noteworthy that some researchers disagreed with $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$ as the lower bound of DMUs [8]–[10], [15]. In their opinion, $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$ is considered the absolute standard, because it is unrealistic that the lower limit of interval efficiency of every DMU should be strictly not less than $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$. However, $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$ was regarded as the relative minimum standard in this study. Furthermore, we hope DMUs can be better evaluated by using an incentive mechanism. It may be reasonable to take $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$ as the relative standard for the negative attitudes because of the incentive mechanism.

The incentive and participation constraints together define a set of incentive feasible allocations achievable through a menu of contracts. This leads to the following definition.

*Definition 3:* A menu of collocation is incentive feasible if it satisfies both the incentive and participation constraints represented by Eqs. (7) to (10).

Since we should discuss the interval efficiency of a DMU based on the incentive and participation constraints of the two attitudes, we adopt the mean value of the two sets of the weights to set up the objective function. The principal problem is written as

$$\text{Max/Min } \theta_0 = \frac{\sum_{r=1}^{s} \underline{u_r} y_{r0} + \sum_{r=1}^{s} \overline{u_r} y_{r0}}{\sum_{i=1}^{m} \underline{v_i} x_{i0} + \sum_{i=1}^{m} \overline{v_i} x_{i0}}$$

$$\text{s.t.} \sum_{r=1}^{s} \underline{u_r} y_{rj} - \sum_{i=1}^{m} \underline{v_i} x_{ij} \leq 0, \quad j = 1, 2, \cdots, n;$$

$$\begin{aligned}
&\sum_{r=1}^{s}\overline{u_r}y_{rj}-\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}*\sum_{i=1}^{m}\overline{v_i}x_{ij}\geq 0,\quad j=1,2,\cdots,n;\\
&\sum_{r=1}^{s}\underline{u_r}y_{rj}-\sum_{i=1}^{m}\underline{v_i}x_{ij}\geq\sum_{r=1}^{s}\overline{u_r}y_{rj}-\sum_{i=1}^{m}\overline{v_i}x_{ij},\\
&j=1,2,\cdots,n;\\
&\sum_{r=1}^{s}\overline{u_r}y_{rj}-\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}*\sum_{i=1}^{m}\overline{v_i}x_{ij}\geq\sum_{r=1}^{s}\underline{u_r}y_{rj}\\
&-\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}*\sum_{i=1}^{m}\underline{v_i}x_{ij},j=1,2,\cdots,n;\\
&\overline{u_r},\overline{v_i},\underline{u_r},\underline{v_i}\geq\varepsilon,r=1,2,\cdots,s,i=1,2,\cdots,m.
\end{aligned}\tag{11}$$

Eq. (11) can be solved by the equivalent LP model:

$$\begin{aligned}
&\text{Max/Min }\theta_0=\sum_{r=1}^{s}\underline{u_r}y_{r0}+\sum_{r=1}^{s}\overline{u_r}y_{r0}\\
&\text{s.t. }\sum_{i=1}^{m}\underline{v_i}x_{i0}+\sum_{i=1}^{m}\overline{v_i}x_{i0}=1;\\
&\sum_{r=1}^{s}\underline{u_r}y_{rj}-\sum_{i=1}^{m}\underline{v_i}x_{ij}\leq 0,\quad j=1,2,\cdots,n;\\
&\sum_{r=1}^{s}\overline{u_r}y_{rj}-\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}*\sum_{i=1}^{m}\overline{v_i}x_{ij}\geq 0,\quad j=1,2,\cdots,n;\\
&\sum_{r=1}^{s}\underline{u_r}y_{rj}-\sum_{i=1}^{m}\underline{v_i}x_{ij}\geq\sum_{r=1}^{s}\overline{u_r}y_{rj}-\sum_{i=1}^{m}\overline{v_i}x_{ij},\\
&j=1,2,\cdots,n;\\
&\sum_{r=1}^{s}\overline{u_r}y_{rj}-\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}*\sum_{i=1}^{m}\overline{v_i}x_{ij}\geq\\
&\sum_{r=1}^{s}\underline{u_r}y_{rj}-\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}*\sum_{i=1}^{m}\underline{v_i}x_{ij},\quad j=1,2,\cdots,n;\\
&\overline{u_r},\overline{v_i},\underline{u_r},\underline{v_i}\geq\varepsilon,\quad r=1,2,\cdots,s,i=1,2,\cdots,m.
\end{aligned}\tag{12}$$

Let $\theta_0^{L*}$ and $\theta_0^{U*}$ be the minimum and maximum respectively of the objective function in Eq. (11). By repeating the above solution process for each DMU, we can obtain the interval efficiencies of all the DMUs $[\theta_j^{L*}, \theta_j^{U*}](j=1,2,\cdots,n)$.

We now discuss a special case of incentive feasible allocation of contracts obtained when the contracts targeted for each situation coincide, i.e., $\underline{u_r}=\overline{u_r}=u_r$, $r=1,\cdots,m$, $\underline{v_i}=\overline{v_i}=v_i, i=1,\cdots,s$. The model represented by Eq. (12) is changed to

$$\begin{aligned}
&\text{Max/Min }\theta_0=\frac{\sum_{r=1}^{s}u_r y_{r0}}{\sum_{i=1}^{m}v_i x_{i0}}\\
&\text{s.t. }\sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij}\leq 0,\quad j=1,2,\cdots,n,\\
&\sum_{r=1}^{s}u_r y_{rj}-\theta_{\mathrm{ADUM}}^{\mathrm{CCR}}*\sum_{i=1}^{m}v_i x_{ij}\geq 0,\\
&j=1,2,\cdots,n,\\
&u_r,\quad v_i\geq\varepsilon, r=1,2,\cdots,s;\ i=1,2,\cdots,m.
\end{aligned}\tag{13}$$

The model represented by Eq. (13) is the same as the bounded DEA model in [13] Wang's paper (2007), and we can call it bunching allocations. In Eq. (13), the incentive constraints are all trivially satisfied by these contracts. Incentive compatibility is thus easy to satisfy, but at the cost of a clear loss of flexibility in allocations that are no longer dependent on the state of nature. Only the participation constraints are now important.

As compared with the bounded DEA models, the interval DEA model based on incentive compatibility does not simply set the absolute standard of the lower bound for all the DMUs, but also provides the relative standard. Moreover, the individual differences of DMUs were fully taken into account in this study. As a result, the worst relative efficiencies of DMUs from Eq. (12) are lower than the results in Eq. (6), and some may be lower than $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$. This is in line with the facts. $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$ is the best relative efficiency of the ADMU, but the lower limit of interval efficiency means only the worst relative efficiency of a DMU. The evaluation perspectives of these two kinds of efficiency are different, and therefore, it is unreasonable to require the worst relative efficiency of every DMU to be higher than $\theta_{ADMU}^{\mathrm{CCR}}$. Therefore, we take $\theta_{\mathrm{ADMU}}^{\mathrm{CCR}}$ only as the relative standard to restrict the negative attitude in Eq. (12). As the negative attitude is weakly preferred to the positive attitude when the DMU pursues a worse evaluation, the relative efficiency obtained by positive attitudes is no higher than the result obtained by negative attitudes, as shown in Eq. (10), and therefore, the final relative efficiency of a DMU is lower than when only the negative attitudes are considered, as shown in Eqs. (6) and (13).

### C. LOWER AND UPPER BOUND OF THE INTERVAL EFFICIENCY BASED ON THE INCENTIVE COMPATIBILITY

In the models presented in [7], the best relative efficiency of a DMU was equivalent to the CCR efficiency for every DMU, when it was further transformed. Similarly, we have proved that the best relative efficiency of a DMU is no larger than its CCR efficiency in the bounded DEA model in [13]. In this paper, we should discuss whether the results also hold. We prove the conclusion as follows.

*Theorem 3:* Let $\theta_0^{U*}$ be the best relative efficiency of the DMU $\mathrm{DMU}_0$ resulting from the interval efficiency model based on incentive compatibility. Then, we have $\theta_0^{U*}\leq\theta_0^{\mathrm{CCR}}$.

*Proof:* Let $\{\underline{u_r},\underline{v_i},|r=1,2,\cdots,s,i=1,2,\cdots,m\}$, $\{\overline{u_r},\overline{v_i}|r=1,2,\cdots,s,i=1,2,\cdots,m\}$ satisfy the constraints in Eqs. (7) and (9), and therefore, we have that $\{\overline{u_r},\overline{v_i}|r=1,2,\cdots,s,i=1,2,\cdots,m\}$ satisfy $\sum_{r=1}^{s}\overline{u_r}y_{rj}-\sum_{i=1}^{m}\overline{v_i}x_{ij}\leq 0$, which means $\sum_{r=1}^{s}(\underline{u_r}+\overline{u_r})y_{rj}-$

$\sum_{i=1}^{m}(\underline{v_i}+\overline{v_i})x_{ij}\leq 0$. Consider the model

$$\begin{aligned}
&\text{Max } \theta_0=\sum_{r=1}^{s}(\underline{u_r}+\overline{u}_r)y_{r0}\\
&\text{s.t. } \sum_{r=1}^{s}(\underline{u_r}+\overline{u_r})y_{rj}-\sum_{i=1}^{m}(\underline{v_i}+\overline{v_i})x_{ij}\leq 0,\\
&\quad j=1,2,\cdots,n,\\
&\quad \sum_{i=1}^{m}(\underline{v_i}+\overline{v_i})x_{i0}=1,\\
&\quad \overline{u_r},\overline{v_i},\underline{u_r},\underline{v_i}\geq\varepsilon,\quad r=1,2,\cdots,s,i=1,2,\cdots,m
\end{aligned}\tag{14}$$

If we let $u_r'=\underline{u_r}+\overline{u_r}$, $v_i'=\underline{v_i}+\overline{v_i}$, the model above represented by Eq. (14) is the same as the CCR model. Then, let $\theta_0^{U*}$ be the maximum solution of Eq. (11) and $\underline{u_r^{U*}}$, $\underline{v_i^{U*}}$, $\overline{u_r^{U*}}$, $\overline{v_i^{U*}}$ be the optimal weights of DMU$_0$ corresponding to $\theta_0^{U*}$. Clearly, $\underline{u_r^{U*}}$, $\underline{v_i^{U*}}$, $\overline{u_r^{U*}}$, $\overline{v_i^{U*}}$ satisfy the constraints of the model represented by Eq. (14), and therefore, $\theta_0^{U*}\leq\theta_0^{CCR}$. This completes the proof. □

In order to compute the lower bound of all DMUs, we can consider the worst relative efficiency of the ADMU to be obtained by the model

$$\begin{aligned}
&\text{Min } \theta_{\text{ADMU}}=\frac{\sum_{r=1}^{s}\underline{u_r}y_r^{\min}+\sum_{r=1}^{s}\overline{u_r}y_r^{\min}}{\sum_{i=1}^{m}\underline{v_i}x_i^{\max}+\sum_{i=1}^{m}\overline{v_i}x_i^{\max}}\\
&\text{s.t. } \sum_{r=1}^{s}\underline{u_r}y_{rj}-\sum_{i=1}^{m}\underline{v_i}x_{ij}\leq 0,\quad j=1,2,\cdots,n\\
&\quad \sum_{r=1}^{s}\overline{u_r}y_{rj}-\theta_{\text{ADMU}}^{\text{CCR}}*\sum_{i=1}^{m}\overline{v_i}x_{ij}\geq 0,\quad j=1,2,\cdots,n\\
&\quad \sum_{r=1}^{s}\underline{u_r}y_{rj}-\sum_{i=1}^{m}\underline{v_i}x_{ij}\geq\sum_{r=1}^{s}\overline{u_r}y_{rj}-\sum_{i=1}^{m}\overline{v_i}x_{ij},\\
&\quad j=1,2,\cdots,n\\
&\quad \sum_{r=1}^{s}\overline{u_r}y_{rj}-\theta_{\text{ADMU}}^{\text{CCR}}*\sum_{i=1}^{m}\overline{v_i}x_{ij}\geq\\
&\quad \sum_{r=1}^{s}\underline{u_r}y_{rj}-\theta_{\text{ADMU}}^{\text{CCR}}*\sum_{i=1}^{m}\underline{v_i}x_{ij},\quad j=1,2,\cdots,n\\
&\quad \overline{u_r},\overline{v_i},\underline{u_r},\underline{v_i}\geq\varepsilon,\quad r=1,2,\cdots,s,i=1,2,\cdots,m
\end{aligned}\tag{15}$$

Eq. (15) can be solved by an equivalent LP model:

$$\begin{aligned}
&\text{Min } \theta_{\text{ADMU}}=\sum_{r=1}^{s}\underline{u_r}y_r^{\min}+\sum_{r=1}^{s}\overline{u_r}y_r^{\min}\\
&\text{s.t. } \sum_{i=1}^{m}\underline{v_i}x_i^{\max}+\sum_{i=1}^{m}\overline{v_i}x_i^{\max}=1\\
&\quad \sum_{r=1}^{s}\underline{u_r}y_{rj}-\sum_{i=1}^{m}\underline{v_i}x_{ij}\leq 0,\quad j=1,2,\cdots,n\\
&\quad \sum_{r=1}^{s}\overline{u_r}y_{rj}-\theta_{\text{ADMU}}^{\text{CCR}}*\sum_{i=1}^{m}\overline{v_i}x_{ij}\geq 0,\\
&\quad j=1,2,\cdots,n\\
&\quad \sum_{r=1}^{s}\underline{u_r}y_{rj}-\sum_{i=1}^{m}\underline{v_i}x_{ij}\geq\sum_{r=1}^{s}\overline{u_r}y_{rj}-\sum_{i=1}^{m}\overline{v_i}x_{ij},\\
&\quad j=1,2,\cdots,n\\
&\quad \sum_{r=1}^{s}\overline{u_r}y_{rj}-\theta_{\text{ADMU}}^{\text{CCR}}*\sum_{i=1}^{m}\overline{v_i}x_{ij}\geq\\
&\quad \sum_{r=1}^{s}\underline{u_r}y_{rj}-\theta_{\text{ADMU}}^{\text{CCR}}*\sum_{i=1}^{m}\underline{v_i}x_{ij},\quad j=1,2,\cdots,n\\
&\quad \overline{u_r},\overline{v_i},\underline{u_r},\underline{v_i}\geq\varepsilon,\quad r=1,2,\cdots,s,i=1,2,\cdots,m
\end{aligned}\tag{16}$$

We denote by $\theta_{\text{ADMU}}^{L*}$ the optimal solution of Eq. (15). The following theorem proves $\theta_{\text{ADMU}}^{L*}$ is the lower bound of all DMUs.

*Theorem 4:* Let $\theta_0^{L*}$ be the worst relative efficiency of DMU$_0$ resulting from the interval efficiency model based on incentive compatibility. Then, we have $\theta_0^{L*}\geq\theta_{\text{ADMU}}^{L*}$.

*Proof:* Let $\theta_0^{L*}$ be the minimum solution of Eq. (11) and $\underline{u_r^{L*}}$, $\underline{v_i^{L*}}$, $\overline{u_r^{L*}}$, $\overline{v_i^{L*}}$ the optimal weights of DMU$_0$ corresponding to $\theta_0^{L*}$. Likewise, $\underline{u_r^{L*}}$, $\underline{v_i^{L*}}$, $\overline{u_r^{L*}}$, $\overline{v_i^{L*}}$ also satisfy the constraints of Eq. (15), and therefore, we have $\theta_0^{L*}=\frac{\sum_{r=1}^{s}\underline{u_r^{L*}}y_{r0}+\sum_{r=1}^{s}\overline{u_r^{L*}}y_{r0}}{\sum_{i=1}^{m}\underline{v_i^{L*}}x_{i0}+\sum_{i=1}^{m}\overline{v_i^{L*}}x_{i0}}\geq\frac{\sum_{r=1}^{s}\underline{u_r^{L*}}y_r^{\min}+\sum_{r=1}^{s}\overline{u_r^{L*}}y_r^{\min}}{\sum_{i=1}^{m}\underline{v_i^{L*}}x_i^{\max}+\sum_{i=1}^{m}\overline{v_i^{L*}}x_i^{\max}}\geq\theta_{\text{ADMU}}^{L*}$. This completes the proof. □

From Theorems 3 and 4, we can obtain the widest interval $[\theta_{\text{ADMU}}^{L*},1]$, and the efficiency of all DMUs will not exceed it. As compared with the conclusion in [13], $\theta_{\text{ADMU}}^{L*}$ is the worst relative efficiency of an ADMU, and therefore, it is more reasonable to take $\theta_{ADMU}^{L*}$ than $\theta_{\text{ADMU}}^{\text{CCR}}$ as the lower bound of all DMUs. $\theta_{\text{ADMU}}^{L*}$ is obtained by taking into account the incentive mechanism. In addition, it is higher than the result in Chen's (2014) [15] study, where also the worst relatively efficiency of the ADMU was considered the lower bound of all DMUs.

### D. IMPROVED MODEL

In the model represented by Eq. (11), we utilize the mean values of the two attitudes to set up the objective functions. However, the expectations of decision makers (DMs) from the two attitudes would be different in a real-life production environment. Now, the parameter $\eta_0$ denotes the optimistic expectation of the positive attitude. If $\eta_0>0.5$, the DMU is said to be up-type, if $\eta_0=0.5$, it is said to be neutral-type, and if $\eta_0<0.5$, it is said to be down-type. Therefore, the model can be improved by including the expectancy:

$$\begin{aligned}
&\text{Max/Min } \theta_0 = \frac{\eta_0 \sum_{r=1}^{s} \underline{u_r} y_{r0} + (1-\eta_0) \sum_{r=1}^{s} \overline{u_r} y_{r0}}{\eta_0 \sum_{i=1}^{m} \underline{v_i} x_{i0} + (1-\eta_0) \sum_{i=1}^{m} \overline{v_i} x_{i0}} \\
&\text{s.t. } \sum_{r=1}^{s} \underline{u_r} y_{rj} - \sum_{i=1}^{m} \underline{v_i} x_{ij} \le 0, \quad j = 1, 2, \cdots, n; \\
&\sum_{r=1}^{s} \overline{u_r} y_{rj} - \theta_{\text{ADMU}}^{\text{CCR}} * \sum_{i=1}^{m} \overline{v_i} x_{ij} \ge 0, \quad j = 1, 2, \cdots, n; \\
&\sum_{r=1}^{s} \underline{u_r} y_{rj} - \sum_{i=1}^{m} \underline{v_i} x_{ij} \ge \sum_{r=1}^{s} \overline{u_r} y_{rj} - \sum_{i=1}^{m} \overline{v_i} x_{ij}, \\
&j = 1, 2, \cdots, n; \\
&\sum_{r=1}^{s} \overline{u_r} y_{rj} - \theta_{\text{ADMU}}^{\text{CCR}} * \sum_{i=1}^{m} \overline{v_i} x_{ij} \ge \\
&\sum_{r=1}^{s} \underline{u_r} y_{rj} - \theta_{\text{ADMU}}^{\text{CCR}} * \sum_{i=1}^{m} \underline{v_i} x_{ij}, \\
&j = 1, 2, \cdots, n; \\
&\overline{u_r}, \overline{v_i}, \underline{u_r}, \underline{v_i} \ge \varepsilon, \quad r = 1, 2, \cdots, s, i = 1, 2, \cdots, m.
\end{aligned} \tag{17}$$

Eq. (17) can be solved by an equivalent LP model:

$$\begin{aligned}
&\text{Max/Min } \theta_0 = \frac{\eta_0 \sum_{r=1}^{s} \underline{u_r} y_{r0} + (1-\eta_0) \sum_{r=1}^{s} \overline{u_r} y_{r0}}{\eta_0 \sum_{i=1}^{m} \underline{v_i} x_{i0} + (1-\eta_0) \sum_{i=1}^{m} \overline{v_i} x_{i0}} \\
&\text{s.t. } \eta_0 \sum_{i=1}^{m} \underline{v_i} x_{i0} + (1-\eta_0) \sum_{i=1}^{m} \overline{v_i} x_{i0} = 1; \\
&\sum_{r=1}^{s} \underline{u_r} y_{rj} - \sum_{i=1}^{m} \underline{v_i} x_{ij} \le 0, \\
&j = 1, 2, \cdots, n; \\
&\sum_{r=1}^{s} \overline{u_r} y_{rj} - \theta_{\text{ADMU}}^{\text{CCR}} * \sum_{i=1}^{m} \overline{v_i} x_{ij} \ge 0, j = 1, 2, \cdots, n; \\
&\sum_{r=1}^{s} \underline{u_r} y_{rj} - \sum_{i=1}^{m} \underline{v_i} x_{ij} \ge \sum_{r=1}^{s} \overline{u_r} y_{rj} - \sum_{i=1}^{m} \overline{v_i} x_{ij}, \\
&j = 1, 2, \cdots, n; \\
&\sum_{r=1}^{s} \overline{u_r} y_{rj} - \theta_{\text{ADMU}}^{\text{CCR}} * \sum_{i=1}^{m} \overline{v_i} x_{ij} \ge \\
&\sum_{r=1}^{s} \underline{u_r} y_{rj} - \theta_{\text{ADMU}}^{\text{CCR}} * \sum_{i=1}^{m} \underline{v_i} x_{ij}, \\
&j = 1, 2, \cdots, n; \\
&\overline{u_r}, \overline{v_i}, \underline{u_r}, \underline{v_i} \ge \varepsilon, \quad r = 1, 2, \cdots, s, i = 1, 2, \cdots, m
\end{aligned} \tag{18}$$

If $\eta_0 = 1$ or $\eta_0 = 0$, the objective function considers only one attitude, which is one of the two special cases of the decision makers' preference.

## IV. RANK OF THE INTERVAL EFFICIENCY

Since the performance of a DMU is assessed as interval efficiency, we need effective and useful methods to rank and compare efficiencies. Let $A = \{A_i = [\theta_i^L, \theta_i^R] | i = 1, 2, \cdots, n\}$ be the set of the interval efficiencies of DMUs, where $A_i = [\theta_i^L, \theta_i^R]$ is the interval efficiency of the $i^{\text{th}}$ DMU. In Wang *et al.*'s (2007) study, the Hurwicz criterion approach (HCA) was used to compare and rank interval efficiencies. The definition of HCA is as follows.

*Definition 4 [13]:* Let $A_i = [\theta_i^L, \theta_i^R] i = 1, 2, \cdots, n$ be the interval efficiency and $\alpha$ the DM or assessor's level of optimism $(0 \le \alpha \le 1)$. Then, the Hurwicz index value of $A_i$ is defined as

$$H(A_i) = \alpha \theta_i^U + (1-\alpha)\theta_i^L$$

The DM or assessor's attitude toward risk can be indicated by the parameter $\alpha$. Therefore, the DM can choose different values of $\alpha$ to evaluate the interval efficiency for an overall perspective. For Definition 4.1, the size of the Hurwicz index value determines the ranking order of the interval efficiency. In other words, $A_i$ is said to be superior to $A_j$ when $H(A_i) > H(A_j)$.

For HCA, we have the following two theorems:

*Theorem 5:* Let $A_i = [\theta_i^L, \theta_i^R]$ and $A_j = [\theta_j^L, \theta_j^R]$ be two interval efficiencies. If $H(A_i) = H(A_j)$, then $\theta_i^L \le \theta_j^L < \theta_j^R \le \theta_i^R$ or $\theta_j^L \le \theta_i^L < \theta_i^R \le \theta_j^R$.

*Proof:* Supposing that the conclusion is not established, we consider the following three cases.

*Case 1:* for $\theta_j^R \le \theta_i^L or \theta_i^R \le \theta_j^L$: clearly, $H(A_i) \ne H(A_j)$, which is contradicted by the conclusion.

*Case 2:* for $\theta_i^L < \theta_j^L \le \theta_i^R < \theta_j^R$: $H(A_i) = \alpha\theta_i^R + (1-\alpha)\theta_i^L < \alpha\theta_j^R + (1-\alpha)\theta_j^L = H\left(A_j\right)$, and therefore, it is contradicted by the conclusion.

*Case 3:* for $\theta_j^L < \theta_i^L \le \theta_j^R < \theta_i^R$: the proof is similar to that of Case 2.

Therefore, we have that, if $H(A_i) = H(A_j)$, then $\theta_i^L \le \theta_j^L < \theta_j^R \le \theta_i^R$ or $\theta_j^L \le \theta_i^L < \theta_i^R \le \theta_j^R$. This completes the proof. □

*Theorem 6 [13]:* Let $A_i = [\theta_i^L, \theta_i^R]$ and $A_j = [\theta_j^L, \theta_j^R]$ be two interval efficiencies. If $\theta_i^L \le \theta_j^L$ and $\theta_i^R \le \theta_j^R$, then $H(A_i) \le H(A_j)$.

Theorem 5 shows that if $H(A_i) = H(A_j)$, then $A_i \subseteq A_j$, that is, $\theta_i^L \le \theta_j^L < \theta_j^R \le \theta_i^R$, or $A_j \subseteq A_i$, that is, $\theta_j^L \le \theta_i^L < \theta_i^R \le \theta_j^R$.

## V. NUMERICAL EXAMPLES

In this section, we provide two examples to illustrate the applications, advantages, and good discriminating power of the models presented in this paper.

*Example 1:* This example is taken from Entani *et al.* (2002) [7] and addresses the DEA efficiency evaluation of ten DMUs, where each DMU has two inputs and one output. The data set is shown in Table 1, where all inputs are normalized to one for simplicity.

Table 2 shows the CCR efficiency, the interval efficiency obtained by Wang *et al.*, and the interval efficiency obtained by the interval DEA models based on incentive compatibility. In order to indicate the priority of the DMU, the

**TABLE 1. Data for 10 decision-making units with one input and two outputs.**

| DMU | Input 1 (X1) | Output 1 (Y1) | Output2 (Y2) |
|---|---|---|---|
| $A_1$ | 1 | 1 | 8 |
| $A_2$ | 1 | 2 | 3 |
| $A_3$ | 1 | 2 | 6 |
| $A_4$ | 1 | 3 | 3 |
| $A_5$ | 1 | 3 | 7 |
| $A_6$ | 1 | 4 | 2 |
| $A_7$ | 1 | 4 | 5 |
| $A_8$ | 1 | 5 | 2 |
| $A_9$ | 1 | 6 | 2 |
| $A_{10}$ | 1 | 7 | 1 |
| ADMU | 1 | 1 | 1 |

**TABLE 2. Relative efficiencies for the 10 decision-making units with one input and two outputs.**

| DMU | Optimistic efficiency | Interval efficiency | |
|---|---|---|---|
| | | model (8) | model (14) |
| $A_1$ | 1.0000 | [0.2174, 1.0000] | [0.1524, 1.0000] |
| $A_2$ | 0.5217 | [0.2174, 0.5217] | [0.1373, 0.5217] |
| $A_3$ | 0.8235 | [0.2676, 0.8235] | [0.1913, 0.8235] |
| $A_4$ | 0.6522 | [0.2446, 0.6522] | [0.1593, 0.6522] |
| $A_5$ | 1.0000 | [0.3679, 1.0000] | [0.2632, 1.0000] |
| $A_6$ | 0.6957 | [0.2174, 0.6954] | [0.1412, 0.6954] |
| $A_7$ | 0.9565 | [0.3804, 0.9565] | [0.2474, 0.9565] |
| $A_8$ | 0.8261 | [0.2391, 0.8261] | [0.1592, 0.8261] |
| $A_9$ | 0.9565 | [0.2609, 0.9565] | [0.1767, 0.9565] |
| $A_{10}$ | 1.0000 | [0.2174, 1.0000] | [0.1462, 1.0000] |
| ADMU | 0.2174 | | [0.0815, 0.2174] |

symbols "$\sim$" and "$\succ$" are used to indicate "is identical to" and "is superior to," respectively. It is clear from Table 2 that DMU $A_1$, DMU $A_5$, and DMU $A_{10}$ are DEA efficient. The ranking of the 10 DMUs by CCR efficiencies is DMU $A_1 \sim$ DMU $A_5 \sim$ DMU $A_{10} \succ$ DMU $A_7 \sim$ DMU $A_9 \succ$ DMU $A_8 \succ$ DMU $A_3 \succ$ DMU $A_6 \succ$ DMU $A_4 \succ$ DMU $A_2$. The CCR efficiency cannot discriminate the DMUs fully when the number of efficient DMUs is more than one. Wang *et al.* [13] used the model to assess the interval efficiency of each DMU under the premise that all the efficiencies of DMUs are within the range of interval $[\theta_{\text{ADMU}}^{\text{CCR}}, 1]$. As a comparison, we used the interval DEA model based on incentive compatibility in Eq. (14) developed in this study to re-evaluate the problem. We recalculated the interval efficiency of each DMU and obtained the worst relative efficiency of the ADMU, which is considered as the lower bound efficiency of all DMUs instead of $\theta_{\text{ADMU}}^{\text{CCR}}$. From the data in Table 2, we can conclude that the interval efficiency of every DMU achieved by using the model in Eq. (12) is wider than that by using the model in Eq. (6). In particular, most of the lower bounds of interval efficiency are lower than $\theta_{\text{ADMU}}^{\text{CCR}}$. Meanwhile, we also can compute the worst relative efficiency of the ADMU as 0.0815, which is much lower than $\theta_{\text{ADMU}}^{\text{CCR}}$ (0.2174). Then, we obtain that the new widest interval efficiency of all DMUs is [0.0815, 1]. In addition, the number of DMUs with the same interval efficiency is reduced, for example DMU $A_1$ and DMU $A_{10}$ have the same interval efficiency when the model represented by Eq. (8) is implemented, but they are different from each other when that represented by Eq. (12) is implemented, and it can be found that the interval efficiencies of all DMUs are completely different when the new model is implemented, and therefore, we can differentiate and rank the DMUs better.

**TABLE 3. Hurwicz criterion approach and similarity ranking of the 10 decision-making units.**

| DMU | $\alpha = 0.5$ | | $\alpha = 0.7$ | | $\alpha = 0.36$ | |
|---|---|---|---|---|---|---|
| | Hurwicz | rank | Hurwicz | rank | Hurwicz | rank |
| $A_1$ | 0.5762 | 5 | 0.7457 | 2 | 0.4575 | 3 |
| $A_2$ | 0.3295 | 10 | 0.4064 | 10 | 0.2757 | 10 |
| $A_3$ | 0.5074 | 6 | 0.6338 | 6 | 0.4189 | 6 |
| $A_4$ | 0.4058 | 9 | 0.5043 | 9 | 0.3367 | 9 |
| $A_5$ | 0.6316 | 1 | 0.779 | 1 | 0.5285 | 1 |
| $A_6$ | 0.4183 | 8 | 0.5291 | 8 | 0.3407 | 8 |
| $A_7$ | 0.6020 | 2 | 0.7438 | 3 | 0.5026 | 2 |
| $A_8$ | 0.4927 | 7 | 0.626 | 7 | 0.3993 | 7 |
| $A_9$ | 0.5666 | 4 | 0.7226 | 5 | 0.4575 | 3 |
| $A_{10}$ | 0.5731 | 3 | 0.7438 | 3 | 0.4536 | 5 |

**TABLE 4. Input and output variable values.**

| Airline | Input | | | Output | |
|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ |
| Air Canada | 26677 | 697 | 5723 | 3239 | 2003 |
| All Nippon Airways Co | 3081 | 539 | 5895 | 4225 | 4557 |
| American Airlines. Inc. | 124055 | 1266 | 24099 | 9560 | 6267 |
| British Airways Plc | 64734 | 1563 | 13565 | 7499 | 3213 |
| Cathay Pacific Airways | 23604 | 513 | 5183 | 1880 | 783 |
| Delta Air Lines, Inc | 95011 | 572 | 19080 | 8032 | 3272 |
| IBERIA Lineas Aereas | 22112 | 969 | 4603 | 3457 | 2360 |
| Japan Airlines | 52363 | 2001 | 12097 | 6779 | 6474 |
| KLM Royal Dutch | 26504 | 1297 | 6587 | 3341 | 3581 |
| Korean Air | 19277 | 972 | 5654 | 1878 | 1916 |
| Lufthansa | 41925 | 3398 | 12559 | 8098 | 3310 |
| Quantas | 27754 | 982 | 5728 | 2481 | 2254 |
| Singapore Airlines | 31332 | 543 | 4715 | 1792 | 2485 |
| UAL Corporation | 122528 | 1404 | 22793 | 9874 | 4145 |

Table 3 shows the ranking of the DMUs when the method proposed in Section 4 is used. For $\alpha = 0.5$, the DM is risk-neutral and the ranking order of the 10 DMUs is DMU $A_5 \succ$ DMU $A_7 \succ$ DMU $A_{10} \succ$ DMU $A_9 \succ$ DMU $A_1 \succ$ DMU $A_3 \succ$ DMU $A_8 \succ$ DMU $A_6 \succ$ DMU $A_4 \succ$ DMU $A_2$; for $\alpha = 0.7$, the DM can be considered a risk-seeking type, and the ranking order of the 10 DMUs is DMU $A_5 \succ$ DMU $A_1 \succ$ DMU $A_7 \sim$ DMU $A_{10} \succ$ DMU $A_9 \succ$ DMU $A_3 \succ$ DMU $A_8 \succ$ DMU $A_6 \succ$ DMU $A_4 \succ$ DMU $A_2$; for $\alpha = 0.36$, the DM is risk-averse, and the ranking order of the 10 DMUs is DMU $A_5 \succ$ DMU $A_7 \succ$ DMU $A_1 \sim$ DMU $A_9 \succ$ DMU $A_{10} \succ$ DMU $A_3 \succ$ DMU $A_8 \succ$ DMU $A_6 \succ$ DMU $A_4 \succ$ DMU $A_2$. The above three levels of optimism reflect the different rankings of the 10 DMUs according to the different DMs' attitudes.

*Example 2:* The data in Table 4, taken from Schefczyk, M (1993) [22], covered 14 major international passenger carriers for the year 1990 (one of airlines was excluded because it transported only cargo). The variables are as follows. $x_1 =$ aircraft capacity in ton kilometers; $x_2 =$ operating cost; $x_3 =$ non-flight assets (all assets not already reflected in $x_1$), e.g., reservation systems, facilities, current assets; $y_1 =$ passenger kilometers; and $y_2 =$ non-passenger revenue.

Table 5 shows the interval efficiencies of DMUs computed by using four models. The first three of these models are those of Wang *et al.* [13], Azizi and Adjirlu [8], and

**TABLE 5.** Comparison of different interval efficiencies of the four models.

| Airline | Interval efficiency | | | |
|---|---|---|---|---|
| | Wang.et.al (2007) | Azizi, H and Adjirlu SF (2010) | Jin-Xiao Chen (2014) | Model (12) in this paper |
| Air Canada | (0.2131, 0.8684) | (0.0192, 0.8684) | (0.0056, 0.8684) | (0.1647, 0.8684) |
| All Nippon Airways Co | (0.1130, 0.3379) | (0.0102, 0.3379) | (0.0030, 0.3379) | (0.0624, 0.3379) |
| American Airlines. Inc. | (0.1410, 0.9475) | (0.0127, 0.9475) | (0.0037, 0.9475) | (0.0989, 0.9475) |
| British Airways Plc | (0.2073, 0.9581) | (0.0187, 0.9581) | (0.0055, 0.9581) | (0.1612, 0.9581) |
| Cathay Pacific Airways | (0.1853, 1.0000) | (0.0141, 1.0000) | (00049, 1.0000) | (0.1412, 1.0000) |
| Delta Air Lines, Inc | (0.1130, 0.9766,) | (0.0102, 0.9766) | (0.0030, 0.9766) | (0.0734, 0.9766) |
| IBERIA Lineas Aereas | (0.6920, 1.0000) | (0.0243, 1.0000) | (0.0071, 1.0000) | (0.2044, 1.0000) |
| Japan Airlines | (0.2578, 0.8588) | (0.0232, 0.8588) | (0.0068, 0.8588) | (0.1895, 0.8588) |
| KLM Royal Dutch | (0.2891, 0.9477) | (0.0261, 0.9477) | (0.0078, 0.9477) | (0.2206, 0.9477) |
| Korean Air | (0.2501, 1.0000) | (0.0226, 1.0000) | (0.0066, 1.0000) | (0.1837, 1.0000) |
| Lufthansa | (0.3626, 1.0000) | (0.0327, 1.0000) | (0.0096, 1.0000) | (0.2478, 1.0000) |
| Quantas | (0.2727, 1.0000) | (0.0246, 1.0000) | (0.0072, 1.0000) | (0.2084, 1.0000,) |
| Singapore Airlines | (0.1962, 1.0000) | (0.0177, 1.0000) | (0.0052, 1.0000) | (0.1479, 1.0000) |
| UAL Corporation | (0.1554, 1.0000) | (0.0140, 1.0000) | (0.0041, 1.0000) | (0.1102, 1.0000) |

**TABLE 6.** Ranking of the 14 airlines.

| Airline | HCA $\alpha = 0.5$ | Ranking | CCR ranking |
|---|---|---|---|
| Air Canada | 0.515 | 13 | 12 |
| All Nippon Airways Co | 0.200 | 14 | 14 |
| American Airlines. Inc. | 0.524 | 11 | 11 |
| British Airways Plc | 0.560 | 8 | 9 |
| Cathay Pacific Airways | 0.571 | 7 | 1 |
| Delta Air Lines, Inc | 0.524 | 10 | 8 |
| IBERIA Lineas Aereas | 0.602 | 3 | 1 |
| Japan Airlines | 0.524 | 12 | 13 |
| KLM Royal Dutch | 0.584 | 5 | 10 |
| Korean Air | 0.592 | 4 | 1 |
| Lufthansa | 0.624 | 1 | 1 |
| Quantas | 0.604 | 2 | 1 |
| Singapore Airlines | 0.574 | 6 | 1 |
| UAL Corporation | 0.555 | 9 | 1 |

Chen [15]. They are called bounded DEA, in which the efficiency of every DMU must be restricted in some range. Wang [13] considered that the range must be $[\theta_{\text{ADMU}}^{\text{CCR}}, 1]$; however, some researchers did not agree with this view, but deemed that $\theta_{\text{ADMU}}^{\text{CCR}}$ as the lower bound efficiency did not confirm the actual situation. For example, in the models of both Aziz and Adjirlu [8] and Chen [15] the relative efficiency of the IDMU and the ADMU was combined to give the new lower bound of all DMUs. As a result, the lower limit efficiency of each DMU is sometimes too small, or even close to the zero when these models are used, as shown in Table 5. Therefore, it seemed that it was not meaningful to discuss the interval efficiency.

In this study, considering the individual differences of DMUs, we set the boundary of efficiency as a relative benchmark, and propose the new interval DEA model based on incentive compatibility to make the limit of each DMU be as close as to the relative benchmark. From Table 2, we can also conclude that the lower limit of interval efficiency of every DMU is higher than that in the models of Azizi and Adjirlu [8] and Chen [15]. Meanwhile, the widest interval efficiencies of all the DMUs are as follows: (0.0102, 1.0000) for Azizi and Adjirlu's model, (0.0030, 1.0000) for Chen's model, and (0.0264, 1.0000) for the model proposed in this paper, represented by Eq. (14). This results will be more conducive to distinguish the lower efficiency limit of DMUs, and achieve reasonable sequencing.

Now, we rank the DMUs by the interval efficiency obtained by the model represented by Eq. (12) in this paper.

In Table 6, we show the ranking of the 14 major international passenger carriers fully with the interval efficiency, where the rank in the second column of the table is based on the results from HCA ($\alpha = 0.5$). Clearly, the interval efficiency is better than the CCR efficiency according to the ranking of the DMUs, and can provide an overall evaluation.

## VI. CONCLUSION

In this paper, the interval efficiency of DMUs was addressed, which can show the best and worst relative efficiency under the same constraint conditions. The interval efficiency also can evaluate DMUs more comprehensively. As compared with the traditional DEA model, for measuring the interval efficiency, the interval DEA model based on incentive compatibility proposed in this paper can give a more realistic and convincing evaluation. This is reflected in the following two aspects.

(1) the model considers two attitudes taken by the DMU during the process of evaluation, which is more in accordance with the actual situation, and second, it cites the incentive compatibility to describe the relationship of the two attitudes during the evaluation process from different perspectives. The results presented in this paper show that the lower limits of DMUs do not always attain $\theta_{\text{ADMU}}^{\text{CCR}}$.

(2) In this paper, Considering the individual differences of decision-making units, we think that the boundary of efficiency is relative, and there may be some situations in which the decision-making unit can not be realized, so we need to make it closer to the efficiency limit through certain incentive mechanism. the lower bound of the interval efficiency are computed by measuring the worst relative efficiency of the ADMU, which is between the results of bounded DEA models from the optimistic or pessimistic points of view.

Finally, the interval efficiency can allow a more comprehensive assessment of DMUs than the traditional DEA efficiency, and therefore, it is expected to be more widely applied in even more varied spheres. It is especially noteworthy that, although the interval model in this paper based on incentive compatibility is an input-oriented DEA model, it also can be constructed in output-oriented form. Furthermore, the improved model with the optimistic expectation can be considered with returns to scale. The model can also be extended with interval numbers or fuzzy numbers. Because of space limitations, this is omitted in this paper.

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