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Almost Difference Set Pairs and Ideal Three-Level Correlation Binary Sequence Pairs

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ABSTRACT Almost difference set pairs are used in cryptography and coding theory, and a majority of the acknowledged almost difference set pairs are constructed by Chinese remainder theorem, cyclotomy or interleaving. In this paper, through the addition or deletion of elements from difference sets and difference set pairs, we ourselves construct almost difference set pairs and sequences thereof. Further, we obtain the ideal three-level correlation binary sequence pairs with high energy efficiency by some renown difference sets or difference set pairs, e.g. Palay-Hardmard difference sets.

INDEX TERMS Almost difference set pairs, difference sets, difference set pairs, binary correlation sequence pairs.

I. INTRODUCTION

Ideal sequences and sequence pairs with good correlation, balance and high linear complexity can be employed to a variety of technical fields including information encryption, radar, sonar, navigation, synchronization, electronic countermeasures, telemetry, and coding aperture imaging. In a real communication system, given the fact that the quantity of code sequences directly matters to the network capacity of the system and the level of spectrum utilization, the greater number of code sequences for the border choices will be a desirable result. with the purpose of obtaining address code sequences as many as possible, scholars work relentlessly to propose generalized ideal sequences, to which mismatch sequences belong. Because of the finitude of these sequences or sequence pairs we acquired from the study on difference sets or difference set pairs, we proposed the concept of almost difference set pairs, studied the properties of almost difference set pairs, and, finally, revealed the equivalence relationship between almost difference set pairs and almost optimal binary correlation sequence pairs.

For conducting the study on the more general three-level correlation binary sequence pairs, we put forward a more generalized concept of almost difference set pairs in [1], established its equivalent relationship with three-level correlation binary sequence pairs, and, we, through the method of cyclotomy and interleaving, constructed a variety of three-level correlation binary sequence pairs with out-of-phase correlation value $\{0, -4\}$ and periods $N = 4p$. The minimum difference value between the two out-of-phase correlation values of three-level correlation binary sequence pairs is 4; therefore the out-of-phase correlation values of three-level correlation binary sequence pairs with even periods are respectively $\{0, -4\}$, $\{0, 4\}$ or $\{2, 2\}$, and the minimum out-of-phase correlation values of three-level correlation binary sequence pairs with odd periods are respectively $\{3, -1\}$ or $\{1, -3\}$. A Three-level correlation binary sequence pair which meets the above-mentioned requirement is eligible for an ideal three-level correlation binary sequence pairs. At the very first, Peng *et al.* [1] constructed a great deal of binary sequence pairs with odd length and out-of-phase correlation values $\{1, -3\}$ on the basis of the difference sets and the difference set pairs. In [2], the almost difference set pair is given a new definition. A limiting condition is given on the basis of generalized almost difference set pair in order to keep the difference the two out-of-phase correlation values of the three- level correlation binary sequence pairs being unchanged as 4.

Currently, the cyclotomy is one construction method that almost difference set pairs are based on, but it rarely. produces ideal sequences. In [3], several kinds of almost difference set pairs with order pq are constructed through the application

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TABLE 1. Summary of known binary sequence pairs with ideal out-of-phase correlation values.

Length	Out-of-phase correlation values	Reference
$N = 4p,\ p\ is\ odd$	$\{0, -4\}$	[1]
$N \equiv 1(mod\ 2)$	$\{1, -3\}$	[1]
$N = pq,\ \gcd(p, q) = 1$	$\{1, -3\}$	[4]
$N = 3p,\ odd\ p > 3$	$\{1, -3\}\ or\ \{3, -1\}$	[5]
$N = 2p,\ p = 4f + 1\ f\ is\ even$	$\{2, -2\}$	[6]

of the Whiteman generalized cyclotomy. Yet, it is only when p and q are twin primes that these almost difference set pairs only correspond to ideal three-level correlation binary sequence pairs. In [4], several kinds of almost difference set pairs with out-of-phase correlation values of $\{1, -3\}$ and periods $N = np$ are constructed by using interleaving and Chinese remainder theorem. In [5], several kinds of almost difference set pairs of periods $N = 3p$ with out-of-phase correlation values of $\{3, -1\}$ or $\{1, -3\}$ are constructed respectively by using the new generalized cyclotomy based on Chinese remainder theorem. Table 1 summaries the main results currently obtained about binary correlation sequence pairs with ideal out-of-phase correlation values.In [4], several kinds of almost difference pairs with out-of-phase correlation values of 1, 3 and periods $N = np$ are constructed as the result of the utilization of interleaving and Chinese remainder theorem. In [5], a plurality of almost difference set pairs of periods $N = 3p$ with out-of-phase correlation values of $\{3, -1\}$ or $\{1, -3\}$ are respectively constructed as the result of the application of the new generalized cyclotomy on the basis of Chinese remainder theorem. Table 1 shows the major results we acquired by now on the binary correlation sequence pairs with ideal out-of-phase correlation values.

According to the energy efficiency of sequence pairs given in [8], we defined express the energy efficiency of almost difference set pairs.It can be evidently seen that the larger $e - \lambda$ results in the higher energy efficiency of almost difference set pairs is. In this paper,it is for the first time that the almost difference set pairs is constructed through the addition or reduction of elements in the difference sets and the difference to construct, and, consequently, the sequence pairs obtained is proven to be excellent in correlation. The utilization of our method enables the successful construction of a number of new almost difference set pairs with high energy efficiency, based on the known difference sets and difference set pairs with high energy efficiency, such as Paley-Hadamard difference sets and the difference set pairs in [9], [10]. Some new ideal three-level correlation binary sequence pairs from the constructed almost difference set pair were acquired when the value of $e - \lambda$ is greater than 1.

In this paper, the second section mainly refers to the related concepts and theorems. For the third section and the fourth section, we proposed eight methods for removing elements from to difference sets or difference set pairs. Finally, in accordance with theorems 1, 4 and 5,some new ideal three-level correlation binary sequence pairs with high energy efficiency, wherein the construction of sequence pairs are based on some known difference sets and difference set pairs with high energy efficiency.

II. PRELIMINARIES

Definition 1 [11]: Make D, D' be two subsets of Z_N with k and k' elements, respectively. Make $|D \cap D'| = e$, and (D, D') is called a (N, k, k', e, λ) DSP if for every nonzero $g \in Z_N^*$, g can be expressed in exactly λ ways in the form $x - y \equiv g(\text{mod } N)$, where $(x, y) \in (D, D')$. And if $D = D'$, the DSP is called a difference set, (N, k, λ) DS.

Definition 2 [2]: Make D, D' be two subsets of Z_N with k and k' elements, respectively. Let $|D \cap D'| = e$ and let T be the nonzero subset of Z_N^*, $|T| = t$, and (D, D') is called an $(N, k, k', e, \lambda, t)$ ADSP provided that the differences $d_{(D,D')}(\tau) = \lambda_1 = \lambda$, when $\tau(\text{mod } N) \in T$. And when $\tau(\text{mod } N) \in Z_N^* - T$ the differences $d_{D,D'}(\tau) = \lambda_2 = \lambda + 1$, λ is a constant integer, $d_{(D,D')}(\tau) = |D' \cap (D + \tau)|$ is difference function.

If $\lambda_1 = \lambda_2$, (D, D') is called DSP. And if $D = D'$, the ADSP is called an (N, k, λ, t) almost difference set(ADS).

Lemma 1 [1]: Make (D, D') be an $(N, k, k', e, \lambda, t)$ ADSP of Z_N, and we have the following necessary condition for the existence of an ADSP:

$$kk' = (\lambda + 1)(v - 1 - t) + e + \lambda t.$$

Lemma 2 [10]: Make (D, D') be an $(N, k, k', e, \lambda, t)$ ADSP of Z_N, and we can get that $\overline{D} = Z_N - D$, $\overline{D'} = Z_N - D'$, then

1) $(D, \overline{D'})$ is an $(N, k, N - k', k - e, k - \lambda - 1, N - t - 1)$ADSP;
2) $(\overline{D}, D')$ is an $(N, N - k, k', k' - e, k' - \lambda - 1, N - t - 1)$ADSP;
3) $(\overline{D}, \overline{D'})$ is an $(N, N - k, N - k', N + e - k - k', N + \lambda - k - k', t)$ADSP.

Lemma 3 [10]: Make (D, D') be an $(N, k, k', e, \lambda, t)$ ADSP of Z_N, and let $x = (x(0), x(1), \ldots, x(N - 1))$ and $y = (y(0), y(1), \ldots, y(N - 1))$ be two characteristic sequences of D and D', respectively. Then

$$R_{(x,y)}(\tau) = \begin{cases} N - 2(k + k') + 4e, & \tau = 0 \\ N - 2(k + k') + 4\lambda, & \tau \in T \\ N - 2(k + k') + 4(\lambda + 1), & \tau \in Z_N^* - T. \end{cases}$$

difference sets or difference set pairs so as to construct almost difference set pairs, and further correspondingly raised the corollaries of adding elements.

Definition 3: Make (D, D') be an $(N, k, k', e, \lambda, t)$ ADSP of Z_N. Then,

$$N - 2(k + k') + 4e = F$$
$$N - 2(k + k') + 4\lambda = E_1$$
$$N - 2(k + k') + 4(\lambda + 1) = E_2.$$

Make $E = \max(|E_1|, |E_2|)$. We define $\eta = \frac{F-E}{N}$ is energy efficiency of ADSP. When $e - \lambda > 1$,the energy efficiency of ADSP is deemed to be high. Let F represent the main peak value of the characteristic sequence of the difference set, and $E1, E2$ represent the out-of-phase correlation values.

III. ALMOST DIFFERENCE SET PAIRS CONSTRUCTED BY DIFFERENCE SETS

Theorem 1: Let D be the (N, k, λ) difference set of Z_N, let $d \in D$, then $(D\backslash\{d\}, D)$ is the $(N, k, k-1, k-1, \lambda-1, k-1)$ ADSP of Z_N. *Proof:* According to the definition of difference set, for any non-zero element $g \in Z_N$, the number of solutions to equation $x - y \equiv g (\mathrm{mod}\ N)$ is λ. Let $D = \{d_0, d_1, \ldots, d_{k-1}\}$ and subtract each element in D from d and we have

$$d - d_0, d - d_1, \ldots, d - d_{k-1}.$$

We have known that there are k different values. When the element d is removed from D, it is without difficulty to get number of $k-1$ different non-zero element solution in almost difference set pairs is $\lambda - 1$, and the number of the same element in $D \setminus \{d\}$ and D is $k - 1$. ■

Example 1: Let the set

$$D = \{1,2,4,5,8,9,10,11,13,16,18,19,20,21,22,23,25, 26,31,32,36,38,40,42,44,45,46,49,50,51,52,55, 62,64,65,67,72,73,76\}$$

be a $(79, 39, 19)$ difference set of Z_{79}, the value of F is 79, and the value of $E1$ is -1. According to theorem 1, by removing 1 makes us obtain the result: $D' = D \setminus \{1\}$. So far, (D, D') are a $(79, 39, 38, 38, 18, 38)$ ADSP of Z_{79}, the value of F is 77, and the value of $\{E1, E2\}$ is $\{-3, 1\}$.

Corollary 1: If D is an (N, k, λ) difference set of Z_N, and $d \in \overline{D}$, then $(D \cup \{d\}, D)$ is an $(N, k+1, k, k, \lambda, N-k-1)$ ADSP of Z_N.

Obviously, due to the symmetry of the difference set, i.e. D is a difference set, then $\overline{D}$ is also a difference set. Then the result of removing an element from D is identical with the result of adding an element to $\overline{D}$ for the construction of almost difference set pairs.

The difference set and the difference set pairs have the same symmetry, so there will be no proving process for the similar inference provided in the following.

The almost difference set pairs with $\{-3, 1\}$, $\{-1, 3\}$, $\{2, -2\}$, $\{0, -4\}$, $\{0, 4\}$ as the values of $\{E1, E2\}$ constructed by Theorem 1 correspond to three-level correlation binary sequence pairs. With $\{-3, 1\}$ as an example, $(N, k, k-1, k-1, \lambda-1, k-1)$ ADSP gets

$$\begin{cases} N - 2(k - 1 + k') + 4(\lambda - 1) = -3 \\ N - 2(k - 1 + k') + 4\lambda = 1 \end{cases}$$

from Lemma 3, then we can obtain that $E1$ of (N, k, λ) difference set is -1.

Similarly, when the out-of-phase correlation values of binary sequence pairs are $\{-1, 3\}$, $\{2, -2\}$, $\{0, -4\}$, $\{0, 4\}$, the $E1$ of difference sets are 1, 0, -2, 2. Because the value range of difference set is

$$n^2 + n + 1 \geq N \geq 4n + 1 \tag{1}$$

where $k - \lambda = n$, the minimum value of $E1$ of difference set is -1, then -2 is unable.

By $k^2 = k + (N-1)\lambda$, then:

When $E1$ of difference set is -1, $N = 4n - 1$, then

$$\begin{cases} k = 2n - 1 \\ \lambda = n - 1, \end{cases} \quad \text{or} \quad \begin{cases} k = 2n \\ \lambda = n. \end{cases}$$

Given that $(4n-1, 2n-1, n-1)$ difference set and $(4n-1, 2n, n)$ difference set are complementary, we merely take $(4n-1, 2n-1, n-1)$ difference set, which has the same parameter of Paley-Hadamard difference set.

When $E1$ of difference set is 2, there are

$$\begin{cases} N = 4n + 2 \\ k = \dfrac{1 \pm \sqrt{16n + 9}}{2} \\ \lambda = \dfrac{1 \pm \sqrt{16n + 9} - 2n}{2}. \end{cases}$$

From(1), we can get $n \in \{0, 1, 2, 3\}$, which means only the $(6, 3, 2)$ difference set meets the above-mentioned conditions;however its energy efficiency is 0.

When $E1$ of difference set is 1, there are

$$\begin{cases} N = 4n + 1 \\ k = \dfrac{4n + 1 \pm \sqrt{8n + 1}}{2} \\ \lambda = \dfrac{2n + 1 \pm \sqrt{8n + 1}}{2}. \end{cases}$$

From(1), we can further $n \geq 3$, which implies that only this difference set fulfills conditions as stated above. When $n = 3$, $(13, 4, 1)$ difference set is the singer difference set. If $n > 3$, it can be seen therefrom that the difference set does not exist, according to MannTest[12].

When $E1$ of difference set is 0, there is only one known difference set with parameters $(4, 3, 2)$, but its energy efficiency is 0.

Conclusion 1: There is a Paley-Hadamard difference set, then the almost difference set pairs with parameters $(4n-1, 2n-1, 2n-2, 2n-2, n-2, 2n-2)$ can be constructed by under Theorem 1, which the sequence pairs has a main peak value of $4n-3$, the value of $\{E1, E2\}$ of $\{-3, 1\}$ and an energy efficiency of $\frac{4n}{4n-1}$.

Conclusion 2: There is only (13, 4, 1) difference set of Z_{13} can construct (13, 4, 3, 3, 0, 3) the ideal three-level correlation binary sequence pairs by Theorem 1. The main peak value of sequence pairs is 13, the value of $\{E1, E2\}$ is $\{-1, 3\}$, and the energy efficiency is $\frac{8}{13}$.

For the next step, we used the same analysis method is used for other theorems to remove elements from difference set, so the conclusion we draw from the following similar theorems will be provided.

Theorem 2: Make D be an (N, k, λ) difference set of Z_N, let

$$d \neq d' \in D,$$
$$d - d' \notin \{d_x - d_y | d_x \in D \setminus \{d\}, d_y \in D \setminus \{d'\}\}.$$

and $(D \setminus \{d, d'\}, D)$ be an $(N, k, k-2, k-2, \lambda-1, 2(k-1))$ ADSP of Z_N. *Proof:* According to the definition of difference set,for any non-zero element $g \in Z_N$, the number of solutions to equation $x - y \equiv g (\bmod N)$ is λ. Make $D = \{d_0, d_1, \ldots, d_{k-1}\}$, take the element $d, d' \in D$ minus each element in D respectively, and we have:

$$d - d_0, d - d_1, \ldots, d - d_{k-1},$$
$$d' - d_0, d' - d_1, \ldots, d' - d_{k-1}$$

We have known that there are $2k - 1$ different values by $d - d' \notin \{d_x - d_y | d_x \in D \setminus \{d\}, d_y \in D \setminus \{d'\}\}$. When the elements d, d' are removed from D, it is easy to get that the number of $2k - 2$ different non-zero element solution pairs in almost difference set pairs is $\lambda - 1$, and the number of the same element in D and $D \setminus \{d, d'\}$ is $k - 2$. ■

Example 2: The set

$$D = \{7, 10, 11, 16, 18\}$$

is a (21, 5, 1) difference set of Z_{21}. The value of F is 21, and the value of $E1$ is 5. By removing 7 and 10, we got: $D' = \{11, 16, 18\}$. Then D and D' construct a (21, 5, 3, 3, 0, 8) ADSP of Z_{21}, for which the value of F is 17, and the value of $\{E1, E2\}$ is $\{9, 5\}$.

From (1), we can know the min value of $E1$ of ADSP constructed by Theorem 2 is 5.

Corollary 2: Let D be an (N, k, λ) difference set of Z_N, when

$$d \neq d' \in \overline{D},$$
$$d - d' \notin \{d_x - d_y | d_x, d_y \in D\},$$

$(D \cup \{d, d'\}, D)$ is an $(N, k+2, k, k, \lambda, N-2k-1)$ ADSP of Z_N.

Theorem 3: Make D be an (N, k, λ) difference set of Z_N, and

$$d \neq d' \in D,$$
$$d + d' \notin \{d_x + d_y | d_x \in D \setminus \{d\}, d_y \in D \setminus \{d'\}\},$$

then $(D \setminus \{d\}, D \setminus \{d'\})$ be an $(N, k-1, k-1, k-2, \lambda-1, 2k-3)$ ADSP of Z_N.

Proof: The proof for Theorem 3 is similar with that for Theorem 2. ■

Example 3: The set

$$D = \{1, 7, 9, 10, 12, 16, 26, 33, 34\}$$

is a (37, 9, 2) difference set of Z_{37}. The value of F is 37, and the value of $E1$ is 9. According to Theorem 3, Make $d = 1$ and $d' = 26$. At this time, $D \setminus \{d\}$ and $D \setminus \{d'\}$ are a (37, 8, 8, 7, 1, 15) ADSP of Z_{37}, for which the value of F is 33, and the value of $\{E1, E2\}$ is $\{13, 9\}$.

Corollary 3: Make D be an (N, k, λ) difference set of Z_N, when

$$d \neq d' \in \overline{D},$$
$$d + d' \notin \{d_x + d_y | d_x, d_y \in D\},$$

then $(D \cup \{d\}, D \cup \{d'\})$ is an $(N, k+1, k+1, k, \lambda, N-2k-2)$ ADSP of Z_N.

Obviously, if the conditions in theorem stated above are satisfied, there must be $k > 2, \lambda > 1$. These conditions are omitted out of the consideration for quick reading, and the same below.

IV. ALMOST DIFFERENCE SET PAIRS CONSTRUCTED BY DIFFERENCE SET PAIRS

Theorem 4: Make (D, D') be a (N, k, k', e, λ) DSP of Z_N, and let $d \in D - D'$, then $(D \setminus \{d\}, D')$ is an $(N, k-1, k', e, \lambda-1, k')$ ADSP of Z_N.

Proof: Let $D' = \{d'_0, d'_1, d'_2, \ldots, d'_{k'}\}$, then

$$d - d'_0, d - d'_1, \ldots, d - d'_{k'},$$

are k' different values. When removing an element d in $D$is moved,it is no trouble to get that the number of k' different non-zero element solution pairs in ADSP is $\lambda - 1$. ■

Example 4: (D, D') is a (69, 35, 24, 24, 12) DSP of Z_{69},

$$D = \{2,4,5,6,8, 9, 10, 11, 13, 16, 17, 18, 19, 21, 22, 23, 26, 27,33,35,36,37,38,39, 42, 43, 45, 47, 52, 53, 55, 60, 64, 66, 67\},$$
$$D' = \{4,6,9,10,13,16,18, 19, 21, 22, 27, 33, 36, 37, 39, 42, 43, 45, 52, 55, 60, 64, 66, 67\}.$$

The value of F is 47, value of $E1$ is -1. According to Theorem 4, by removing a non-common element 2 in D, we can get that $D \setminus \{2\}$ and D' is an (69, 34, 24, 11, 24) ADSP of Z_{69}, and the value of F is 49, value of $\{E1, E2\}$ is $\{-3, 1\}$.

Corollary 4: Make (D, D') be a (N, k, k', e, λ) DSP of Z_N, and $d \in D - D'$, and we can see that $(D \cup \{d\}, D')$ is an $(N, k+1, k', e+1, \lambda, N-k')$ ADSP of Z_N.

An ADSP which is constructed by Theorem 4 corresponds to an ideal three-level correlation binary sequence pairs when the value of $\{E1, E2\}$ is $\{-3, 1\}$, $\{-1, 3\}$, $\{2, -2\}$, $\{0, -4\}$, $\{0, 4\}$. When value of $\{E1, E2\}$ is $\{-3, 1\}$ and DSP with parameters $(N, k-1, k', e, \lambda-1, k')$, by the Lemma 3, we have:

$$\begin{cases} N - 2(k-1+k') + 4(\lambda-1) = 3 \\ N - 2(k-1+k') + 4\lambda = 1, \end{cases}$$

TABLE 2. Energy efficiency of different DSP.

DSP	the value of $E1$ of DSP	Condition	energy efficiency
$(N, \frac{N-1}{2}, \frac{N-1}{2}, 0, \frac{N-1}{4}), (N, \frac{N+1}{2}, \frac{N+1}{2}, 0, \frac{N+1}{4})$	1	\	low
perfect binary sequence pairs	0	$N \equiv 0(\text{mod } 4)$	low
$(tsv+ts, tv+t, sv, v, v)$	-2	$s=1, t=2, v=1$	1
$(N, \frac{N-1}{4}, \frac{N-1}{4}, 0, \frac{N-1}{16})$	2	$N=5$	low

TABLE 3. The Conclusions of Theorem 4.

DSP	Condition	energy efficiency
$(mp, \frac{mp+1}{2}, \frac{(m+1)(p+1)}{4}, \frac{(m+1)(p+1)}{4}, \frac{(m+1)(p+1)}{8})$	$m \neq p \equiv 3(\text{mod } 4)$	$\frac{(m+1)(p+1)+8}{2mp}$
	$p \equiv 3(\text{mod } 4), m \equiv 1(\text{mod } 4)$	
	$p = 4s^2+27, m \equiv 1(\text{mod } 4), m \neq p$	
	$p = 4s^2+27, m \equiv 3(\text{mod } 4), m \neq p$	
$(m(2^t-1), \frac{m(2^t-1)+1}{2}, (m+1)2^{t-2}, (m+1)2^{t-2}, (m+1)2^{t-3})$	$t \geq 1 \neq 2^t-1, m \equiv 3(\text{mod } 4) \neq 2^t-1$	$\frac{(m+1)2^{t-1}+4}{m(2^t-1)}$
	$t \geq 1, m \equiv 1\,(\text{mod } 4)$	
$(mp(p+2), \frac{(mp(p+2)+1}{2}, \frac{(m+1)(p+1)^2}{4}, \frac{(m+1)(p+1)^2}{4}, \frac{(m+1)(p+1)^2}{8})$	$p, p+2, m \equiv 3\,(\text{mod } 4)$,gcd$(m, p(p+2)) = 1$	$\frac{(m+1)(p+1)^2+8}{2mp(p+2)}$
	$p, p+2, m \equiv 1\,(\text{mod } 4)$,gcd$(m, p(p+2)) = 1$	

then the value of $\{E1, E2\}$ of DSP with parameters (N, k, k', e, λ) is $N - 2(k+k') + 4\lambda = -1$.

Similarly, when value of $\{E1, E2\}$ is $\{-1, 3\}$, $\{2, -2\}$, $\{0, -4\}$, $\{0, 4\}$, value of $E1$ of DSP with parameters (N, k, k', e, λ) is 1, 0, −2, 2.

DSP with value of $E1 = -1$ is pseud random sequence pairs, and it can construct an $(N, k-1, k', e, \lambda-1, k')$ ideal three-level correlation binary sequence pairs by Theorem 4. Among them, DSP with high energy efficiency in [10] can also construct ideal three-level correlation binary sequence pairs with high energy efficiency.

When value of $E1$ is changed, it can be shown in Table 2 that the energy efficiency of ADSP is low, there will be no further analyzation.

From the six difference set pairs in Table 3, we can effortlessly calculate the parameters, the main peak value and the out-of-phase correlation values of the ideal three-level correlation binary sequence pairs through Theorem 4, for which m, p, $p+2$, t are primes, and these ideal three-level correlation binary sequence pairs have high energy efficiency.

Theorem 5: Make (D, D') be a (N, k, k', e, λ) DSP of Z_N, and let $d \in D \cap D'$, and we can get that $(D \setminus \{d\}, D')$ is an $(N, k-1, k', e-1, \lambda-1, k'-1)$ ADSP of Z_N.

Proof: According to the definition of the difference set pairs, for any non-zero element $g \in Z_N$, the number of solutions satisfying the condition of the equation $x - y \equiv g(\text{mod } N)$ is λ. Make $D = \{d, d_1, \ldots d_k\}$, $D' = \{d, d_1', \ldots d_{k'}'\}$, each element in D' differs from element $d \in D \cap D'$, and have:

$$d - d, d - d_1', \ldots, d - d_{k'}'.$$

We have known that there are k different values, when the element d is removed from D, we can easily get that the number of $k-1$ different non-zero element solution pairs in ADSP is $\lambda - 1$. ∎

Example 5: (D, D') is a (69, 34, 24, 11, 24) DSP of Z_{69},

$$D = \{2,5,6,7,9,10,12,13,14,16, 18, 19, 20, 21, 22, 23, 25, 30, 31, 33, 36, 37, 39, 41, 48, 49, 50, 54\},$$
$$D' = \{6, 9, 14, 16, 19, 21, 31, 36, 39, 41, 49, 54\}.$$

The value of F is 23, and value of $E1$ is −1, According to Theorem 5, by removing a common element 6 in D, we obtained that $D \setminus \{6\}$ and D' is an (55, 27, 12, 11, 5, 11) ADSP of Z_{69}, the value of F is 21, and value of $\{E1, E2\}$ is $\{1, -3\}$.

Corollary 5: Make (D, D') be a (N, k, k', e, λ) DSP of Z_N, and let $d \notin D \cap D'$, and we can get that $(D \cup \{d\}, D')$ is an $(N, k+1, k', e, \lambda, N-k'-1)$ ADSP of Z_N.

Theorem 6: Let (D, D') be a (N, k, k', e, λ) DSP of Z_N, and let $d \in D \cap D'$, if $2d$ cannot be written as the sum of two distinct elements of D and D', and let $k + k' < N + 1$, then $(D \setminus \{d\}, D' \setminus \{d\})$ is an $(N, k-1, k'-1, e-1, \lambda-1, k+k'-2)$ ADSP of Z_N.

Proof: According to the definition of DSP, make:

$$D = \{d_0, d_1, d_2, \ldots, d_k\},$$
$$D' = \{d_0', d_1', d_2', \ldots, d_{k'}'\},$$

take the element $d \in D$ and each element in D' as the difference, and take the element $d' \in D'$ and each element in D as the difference to get:

$$d' - d_0, d' - d_1, \ldots, d' - d_k,$$
$$d - d_0', d - d_1', \ldots, d - d_{k'}'.$$

We have known that there are $k+k'-2$ different values by $2d$ not equal to the sum of two distinct elements of D and D'. When removing an element d in D and removing an element d' in D' are removed, there are $k+k'-2$ pairwise nonzero elements' and the number of nonzero solutions is $\lambda-1$. ■

Example 6: (D, D') is a $(31, 11, 6, 6, 2)$ DSP of Z_{31},

$$D = \{0, 11, 13, 21, 22, 26\},$$
$$D' = \{0, 7, 11, 13, 14, 19, 21, 22, 25, 26, 28\}.$$

The value of F is 21, value of $E1$ is 5. According to Theorem 6, by removing a common element 0 in D and D', we obtain $D\setminus\{0\}$ and $D'\setminus\{0\}$ is an $(31, 10, 5, 5, 1, 15)$ ADSP of Z_{31}, the value of F is 21, value of $\{E1, E2\}$ is $\{5, 9\}$.

Corollary 6: Make (D, D') be a (N, k, k', e, λ) DSP of Z_N, Make V be a universal set, if:

$$d \in V - D - D',$$
$$2d \notin \{d_x + d_y | d_x \in D, d_y \in D'\}$$

and we can get that then $(D \cup \{d\}, D' \cup \{d\})$ is an $(N, k+1, k'+1, e+1, \lambda, N-k-k'-1)$ ADSP of Z_N.

Theorem 7: Make (D, D') be a (N, k, k', e, λ) DSP of Z_N, let $d \in D, d' \in D'$ and $d \neq d'$, then

1) $(D\setminus\{d\}, D'\setminus\{d'\})$ is an $(N, k-1, k'-1, e, \lambda-1, k+k'-1)$ ADSP of Z_N, in which case
$$d, d' \notin D \cap D',$$
$$d + d' \notin \{d_x + d_y | d_x \in D\setminus\{d\}, d_y \in D'\setminus\{d'\}\}.$$
2) $(D\setminus\{d\}, D'\setminus\{d'\})$ is an $(N, k-1, k'-1, e-1, \lambda-1, k+k'-2)$ ADSP of Z_N, in which case
$$\text{either } d \text{ or } d' \in D \cap D',$$
$$d + d' \notin \{d_x + d_y | d_x \in D\setminus\{d\}, d_y \in D'\setminus\{d'\}\}.$$
3) $(D\setminus\{d\}, D'\setminus\{d'\})$ is an $(N, k-1, k'-1, e-2, \lambda-1, k+k'-3)$ ADSP of Z_N, in which case
$$d, d' \in D \cap D',$$
$$d + d' \notin \{d_x + d_y | d_x \in D\setminus\{d\}, d_y \in D'\setminus\{d'\}\}.$$

Proof: Here is how we prove 1): According to the definition of the difference set pairs, for any non-zero element $g \in Z_N$, the number of solutions satisfying the condition of the equation $x - y \equiv g(\text{mod } N)$ is λ. Make $D = \{d, d_1, \ldots d_{k-1}\}, D' = \{d', d'_1, \ldots d'_{k'-1}\}$, element $d \in D$ different from each element in D', and each element in D differ from $d' \in D'$, and we can:

$$d - d', d - d'_1, \ldots, d - d'_{k'-1},$$
$$d - d', d_1 - d', \ldots, d_{k-1} - d'.$$

Because of $d + d' \notin \{d_x + d_y | d_x \in D\setminus\{d\}, d_y \in D'\setminus\{d'\}\}$, we know that there are $k+k'+1$ different values. When the element d is removed from D and d' is removed from D', we can easily get that the number of $k+k'-1$ different non-zero element solution pairs in ADSP is $\lambda-1$.

The reason applies to the proving for 2) and 3) alike. ■

Example 7:

1) (D, D') is a $(17, 6, 3, 2, 1)$ DSP of Z_{17},
$$D = \{0, 8, 9\},$$
$$D' = \{2, 5, 8, 9, 12, 15\}.$$
The value of F is 7, valueof $E1$ is 3. According to Theorem 7(1), by removing a non-common element 0 in D and a non-common element 5 in D', we obtained $D\setminus\{0\}$ and $D'\setminus\{5\}$ is an $(17, 5, 2, 2, 0, 8)$ ADSP of Z_{17}, the value of F is 11,and value of $\{E1, E2\}$ is $\{7, 3\}$.
2) (D, D') is a $(31, 11, 6, 6, 2)$ DSP of Z_{31},
$$D = \{0, 1, 2, 4, 8, 16\},$$
$$D' = \{0, 1, 2, 4, 8, 11, 13, 16, 21, 22, 26\}.$$
The value of F is 21, value of $E1$ is 5. According to Theorem 7(2), by removing a non-common element 1 in D and a non-common element 13 in D', we obtained that $D\setminus\{1\}$ and $D'\setminus\{13\}$ is an $(31, 10, 5, 5, 1, 15)$ ADSP of Z_{31}, the value of F is 11,and value of $\{E1, E2\}$ is $\{9, 5\}$.
3) (D, D') is a $(33, 17, 12, 12, 6)$ DSP of Z_{33},
$$D = \{0,1,2,3,4,8,9,12,15,16,17, 22, 25, 27, 29, 31, 32\},$$
$$D' = \{0, 1, 3, 4, 9, 12, 15, 16, 22, 25, 27, 31\}.$$
The value of F is 23, value of $E1$ is -1, According to Theorem 7(3), by removing a non common element 0 in D and a common element 22 in D', we obtained $D\setminus\{0\}$ and $D'\setminus\{22\}$ is an $(33, 16, 11, 10, 5, 26)$ ADSP of Z_{33}, the value of F is 19, value of $\{E1, E2\}$ is $\{3, -1\}$.

Corollary 7: Make (D, D') be a (N, k, k', e, λ) DSP of Z_N, let V be a universal set, then

1) $(D \cup \{d\}, D' \cup \{d'\})$ is an $(N, k+1, k'+1, e, \lambda, N-k-k'-2)$ ADSP of Z_N, in which case
$$d \neq d' \in V - D - D',$$
$$d + d' \notin \{d_x + d_y | d_x \in D, d_y \in D'\}.$$
2) $(D \cup \{d\}, D' \cup \{d'\})$ is an $(N, k+1, k'+1, e+2, \lambda, N-k-k')$ ADSP of Z_N, in which case
$$d \in D - D',$$
$$d' \in V - D - D',$$
$$d + d' \notin \{d_x + d_y | d_x \in D, d_y \in D'\}.$$
3) $(D \cup \{d\}, D' \cup \{d'\})$ is an $(N, k+1, k'+1, e, \lambda, N-k-k'-2)$ ADSP of Z_N, in which case
$$d \in D - D',$$
$$d' \in D' - D,$$
$$d + d' \notin \{d_x + d_y | d_x \in D\setminus\{d\}, d_y \in D'\setminus\{d'\}\}.$$

Theorem 8: Let (D, D') be an (N, k, k', e, λ) DSP of Z_N, and let $d, d' \in D$, $d \neq d'$, $d - d' \notin \{d_x - d_y | d_x, d_y \in D\setminus\{d, d'\}\}$, then

1) $(D\backslash\{d, d'\}, D')$ is an $(N, k-2, k', e-2, \lambda-1, 2(k'-1))$ ADSP of Z_N, in which case

$$d, d' \in D \cap D'.$$

2) $(D\backslash\{d, d'\}, D')$ is an $(N, k-2, k', e, \lambda-1, 2k')$ ADSP of Z_N, in which case

$$d, d' \notin D \cap D'.$$

3) $(D\backslash\{d, d'\}, D')$ is an $(N, k-2, k', e-1, \lambda-1, 2k'-1)$ ADSP of Z_N, in which case

$$\textit{either } d \textit{ or } d' \in D \cap D',$$

Proof: Here is how 1)is proved. According to the definition of the difference set pairs, for any non-zero element $g \in Z_N$, the number of solutions satisfying the condition of equation $x - y \equiv g(\text{mod } N)$ is λ. Make $D = \{d, d', d_2, \ldots d_{k-1}\}$, $D' = \{d'_0, d'_1, \ldots d'_{k'-1}\}$, Make elements $d, d' \in D \cap D'$ different from each element in D', then we get:

$$d - d'_0, d - d'_1, \ldots, d - d'_{k'-1},$$
$$d' - d'_0, d' - d'_1, \ldots, d' - d'_{k'-1}.$$

Because of $d - d' \notin \{d_x - d_y | d_x, d_y \in D \setminus \{d, d'\}\}$, we know that there are $2(k' - 1)$ different values. When the elements d, d' are removed from D, we can easily get that the number of $2(k'-1)$ different non-zero element solution pairs in ADSP is $\lambda-1$. The number of the same elements in D and D' is $e-2$.

The reason applies to the proving for 2) and 3) alike. ■

Example 8:

1) (D, D') is a (31, 11, 6, 6, 2) DSP of Z_{31},

$$D = \{0, 3, 6, 12, 17, 24\},$$
$$D' = \{0, 1, 2, 3, 4, 6, 8, 12, 16, 17, 24\}.$$

The value of F is 21, and value of $\{E1, E2\}$ is 5. According to Theorem 8(1), by removing two common elements 0 and 24 in the set D', we obtained D and $D' \setminus \{0, 24\}$ is an (31, 9, 6, 4, 1, 10) ADSP of Z_{31}, the value of F is 17, value of $\{E1, E2\}$ is $\{9, 5\}$.

2) (D, D') is a (65, 33, 12, 12, 6) DSP of Z_{65},

$$D = \{0,3,6,10,11,12,13,17,19,21,22, 23, 24, 25, 27, 30,31,34,35,38,40,41,42,43,44,46, 48, 52, 53, 54, 55, 59, 62\},$$
$$D' = \{3, 12, 17, 2, 23, 27, 38, 42, 43, 48, 53, 62\}.$$

The value of F is 23, value of $E1$ is -1.According to Theorem 8(2), by removing two non-common elements 13 and 21 in the set D, we obtain $D \setminus \{13, 21\}$ and D' is an (65, 31, 12, 12, 5, 24) ADSP of Z_{65}, the value of F is 27, value of $\{E1, E2\}$ is $\{3, -1\}$.

3) (D, D') is a (35, 18, 8, 8, 4) DSP of Z_{35},

$$D = \{0,1,3,4,9,11,12,13,14,15, 16, 17, 21, 25, 27, 29, 30, 33\},$$
$$D' = \{1, 4, 9, 11, 14, 16, 21, 29\}.$$

The value of F is 15, value of $E1$ is -1. According to Theorem 8(3), by removing a non common element 30 and a common element 21 in the set D, we obtain $D \setminus \{21, 30\}$ and D' is an (35, 16, 8, 7, 3, 15) ADSP of Z_{35}, the value of F is 15, value of $\{E1, E2\}$ is $\{3, -1\}$.

Corollary 8: Make (D, D') be a (N, k, k', e, λ) DSP of Z_N, let V be a universal set, then

1) $(D \cup \{d, d'\}, D')$ is an $(N, k+2, k', e+1, \lambda, N-2k')$ ADSP of Z_N, in which case

$$d \in D - D',$$
$$d' \in V - D - D',$$
$$d - d' \notin \{d_x - d_y | d_x, d_y \in D'\}.$$

2) $(D\cup\{d, d'\}, D')$ is an $(N, k+2, k', e+2, \lambda, N-2k+1')$ ADSP of Z_N, in which case

$$d \neq d' \in D - D',$$
$$d - d' \notin \{d_x - d_y | d_x, d_y \in D' \setminus \{d, d'\}\}.$$

3) $D \setminus \{d, d'\}, D'$ is an $(N, k+2, k', e, \lambda, N-2k'-1)$ ADSP of Z_N, in which case

$$d \neq d' \in V - D - D',$$
$$d - d' \notin \{d_x - d_y | d_x, d_y \in D'\}.$$

Theorem 5, 6, 7, 8 can be analysed similarly as Theorem 4, and the conclusions are similarly as Theorem 4. Therefore, we will not repeat.

V. CONCLUSION

In this paper, for the purpose of new almost difference set pairs in a large number, we proposed eight theorems by adding or deleting elements from difference sets and difference set pairs. The relevancy between the selected difference sets as well as difference set pairs and the balance, correlation and energy efficiency of almost difference set pairs can be explicitly observed. Specifically, if the characteristics of the selected difference set and difference set pairs are great, the characteristics of the almost difference set pairs will be correspondly great.

Through computer search, only a few special examples were found among examples of constructing almost difference set pairs by difference set pairs in this paper. Our method is distinct from the construction of almost difference set pairs by difference sets with some general parametric forms. This phenomenon, we supposed, is partly because that the study of difference sets has been studied in more extensive and in-depth manner, while difference set pairs has not been yet at present. Therefore, the study on the latter remains a problem.

Currently, there are quite numerous kinds of difference sets and difference set pairs with good characteristics have been obtained. The construction methods of the computer-based exhaustively search and composite-based construction are the two construction methods lacks efficiency, thereby failing to meet the demand for sequence pairs in a large number. At present, almost difference set pairs are based on the construction method of cyclotomy, but the correlation and balance of sequence pairs obtained are not ideal. The methods currently used in this paper are efficient, with the sequence pairs obtained therefrom being well-correlated. The research in this paper makes the relation between difference sets (pairs) and almost difference set pairs established, and numerous kinds of new almost difference set pairs with good characteristics obtained. Therefore, it greatly contributes to the progress of the further research of discrete signals based on pairs.

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