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Static Output Feedback Control for Fuzzy Systems With Stochastic Fading Channel and Actuator Faults

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ABSTRACT This work focuses on the issue of static output feedback control for Takagi-Sugeno (T-S) fuzzy systems in the discrete-time domain. Both the fading channel and actuator faults are determined by a set of stochastic variables. Specifically, an actuator fault is described by a nonhomogeneous Markov chain, and fading channels are characterized by the l-order Rician fading model. Using the Lyapunov-Krasovskii function, sufficient conditions are established, and controller gains are developed. Finally, the practical mass-spring-damping model is expressed to verify the practicability of the theoretical results.

INDEX TERMS T-S fuzzy system, fading channel, static output feedback control, actuator fault.

I. INTRODUCTION

In recent decades, comprehensive physical applications have encouraged an increase in the amount of attention given to nonlinear systems. Compared with linear systems, nonlinear systems are generally difficult to explore. To address this, many efficient methodologies have been proposed. Among them, the T-S fuzzy model has been extensively recognized. Because of its influential approximation capacity, the T-S fuzzy model has been used to approximate many complex nonlinear plants [1]–[5]. In fact, in addition to this approach, nonlinear plants can be divided into finite weighted sums of linear subsystems. Note that by adopting the T-S fuzzy framework, the sophisticated theories and approaches for linear systems can be extended to the analysis of nonlinear systems. In addition, by means of parallel distributed compensation (PDC), the controllers/filters can also be solved. In light of the aforementioned discussion, the analysis and synthesis of T-S fuzzy systems have attracted much attention [6]–[10].

In networked control systems (NCSs), control techniques have been widely studied and investigated. As an efficient tool in NCSs, the state feedback control (SFC) strategy plays a significant role in modeling NCSs. In this strategy, the states are required to be constantly accessible. However, for uncertain/unknown state conditions, SFC seems to be unrealistic. To tackle this situation, a static output feedback control (SOFC) scheme has been developed for NCSs [11]–[18]. Compared with other control laws, the SOFC has been widely applied due to its unique structure. In light of the above observations, applying SOFCs for T-S fuzzy systems is more realistic, which partly inspires this work.

In addition, the signals are transmitted via shared communication channels, which may result in multiple path fading and other unanticipate factors, such as fading channel (FC), data collision, dropout loss, *et al.* Among them, the FC is more general, where the received signals are modeled by various paths with different probability distributions. In fact, because of its probabilistic sequences, FC models are more general than unideal measurements governed by Bernoulli sequences. In this regard, FCs abate the performance and quality of plants. To eliminate the drawbacks, filtering/control issues for NCSs with FCs have been addressed [19]–[23]. However, much attention has been devoted to normal channels, and the reported results offer few insights into

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SOFCs for fuzzy systems. Following this trend, this study seeks to solve this problem.

Based on the above observations, we aim to establish an SOFC for T-S fuzzy systems with FCs and actuator faults. The major contribution of this work is as follows. (1) To characterize the property of incomplete measurements, the phenomena of FCs are covered, all of which are obeyed by probabilistic variables. By referring to a Rician fading model, the missing measurement can be settled. (2) The actuator faults are described by a nonhomogeneous Markov chain, where time-varying transition probabilities are polytope structured. (3) To better model reality, randomly occurring uncertainties are revealed. (4) By establishing a Lyapunov-Krasovskii function, a mode-dependent SOFC design is formed.

Notations: In this research, $\mathbb{R}^n$ symbolizes the n-dimensional Euclidean. $\mathscr{E}\{\cdot\}$ indicates the expectation operator. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are, respectively represent the maximum and minimum eigenvalue. diag$\{\cdots\}$ means the block diagonal matrix. Sym$\{Z\}$ indicates $Z^\top + Z$. $L_2[0,\infty)$ means the square integrable space on $[0,\infty)$.

II. PRELIMINARY

Consider the fuzzy systems depicted by IF-THEN rules:

Rule p: IF $\zeta_1(k)$ is M_{p1}, and $\cdots$ and $\zeta_r(k)$ is M_{pr}, Then

$$\begin{cases} \delta(k+1) = (A_p + \alpha(k)\Delta A_p)\delta(k) + B_p u(k) + D_p\omega(k), \\ y(k) = C_p\delta(k), \\ z(k) = F_p\delta(k) + H_p u(k), \end{cases} \tag{1}$$

where $M_{pq}(p = 1, 2, \cdots, t, q = 1, 2, \cdots, r)$ are fuzzy sets, r is the number of "IF-THEN" rules. $\zeta(k) = [\zeta_1(k), \zeta_2(k), \cdots, \zeta_r(k)]$ indicates the premise variable. $\delta(k) \in \mathbb{R}^{n_y}$, $y(k) \in \mathbb{R}^{n_y}$, $z(k) \in \mathbb{R}^{n_z}$, $u(k) \in \mathbb{R}^{n_u}$ are, respectively, the state vector, measured output, controlled output signal and control input. $\omega(k) \in \mathbb{R}^{n_w}$ means the disturbance input that residing in $l_2[0,\infty)$. A_p, B_p, C_p, D_p, F_p and H_p are known matrices with proper dimensions. The matrix ΔA_p is characterized by

$$\Delta A_p = M_p\Delta_p(k)N_p, \tag{2}$$

where M_p and N_p are known matrices. $\Delta_p(k)$ is an unknown matrix with the form of $\Delta_p^\top(k)\Delta_p(k) \le I$. The stochastic variable (SV) $\alpha(k)$ is utilized to depict the parameter uncertainties in a probabilistic way. Here, for SV $\alpha(k)$, it is obvious that

$$\Pr\{\alpha(k) = 1\} = \overline{\alpha}, \quad \Pr\{\alpha(k) = 0\} = 1 - \overline{\alpha},$$

where $\overline{\alpha} \in [0, 1]$, with mathematical expectation $\mathscr{E}\{\alpha(k) - \alpha\} = 0$ and $\mathscr{E}\{(\alpha(k) - \alpha)^2\} = \overline{\alpha}(1 - \overline{\alpha}) = \alpha^*$.

Remark 1: Most of the literature concerns parameter uncertainty with time-varying scalar structure (TVSS), i.e., $\Delta A_p(k) = \pi(k)$. *Although an easier SOFC design can be acquired for system (1), the addressed results may be conservative due to neglect of the cross term. In this work, probabilistic parameter uncertainty is applied. In contrast to TVSS, norm-bounded uncertainty achieves less conservative results.*

Inspired by [11], [12], the T-S fuzzy system (1) can be reestablished as:

$$\begin{cases} \delta(k+1) = \sum_{p=1}^{t} \hbar_p(\zeta(k))[(A_p + \alpha(k)\Delta A_p)\delta(k) \\ \qquad\qquad + B_p u(k) + D_p\omega(k)], \\ y(k) = \sum_{p=1}^{t} \hbar_p(\zeta(k))C_p\delta(k), \\ z(k) = \sum_{p=1}^{t} \hbar_p(\zeta(k))(F_p\delta(k) + H_p u(k)), \end{cases} \tag{3}$$

where $\mu_p(\zeta(k)) = \prod_{q=1}^{r} M_{pq}(\zeta_q(k)) \ge 0$, $\hbar(\zeta(k)) = \frac{\mu_p(\zeta(k))}{\sum_{q=1}^{t}\mu_q(\zeta(k))} > 0$, $\sum_{p=1}^{t}\hbar_p(\zeta(k)) = 1$, and $M_{pq}(\zeta_q(k))$ symbolizes the grade of membership of $\zeta_q(k)$ in set M_{pq}.

For simplification, one denotes

$$A_\hbar = \sum_{p=1}^{t}\hbar_p(\zeta(k))A_p, \quad B_\hbar = \sum_{p=1}^{t}\hbar_p(\zeta(k))B_p,$$
$$C_\hbar = \sum_{p=1}^{t}\hbar_p(\zeta(k))C_p, \quad D_\hbar = \sum_{p=1}^{t}\hbar_p(\zeta(k))D_p,$$
$$F_\hbar = \sum_{p=1}^{t}\hbar_p(\zeta(k))F_p, \quad H_\hbar = \sum_{p=1}^{t}\hbar_p(\zeta(k))H_p.$$

Accordingly, system (3) can be rewritten as

$$\begin{cases} \delta(k+1) = (A_\hbar + \alpha(k)\Delta A_\hbar)\delta(k) \\ \qquad\qquad + B_\hbar u(k) + D_\hbar\omega(k), \\ y(k) = C_\hbar\delta(k), \\ z(k) = F_\hbar\delta(k) + H_\hbar u(k). \end{cases} \tag{4}$$

In limited source circumstances, all signals are transformed through a shared communication network. The phenomena of fading channels (FCs) cannot be neglected in many circumstances, which may lead to different transmission rates among channels. To reveal the different capacities of a channel, the lth-order Rice FC model [20]–[23] is expressed as follows:

$$\bar{y}(k) = \sum_{s=0}^{l_k}\beta_s(k)y(k-s) + \nu(k), \tag{5}$$

where $l_k = \min\{l, k\}$, $\nu(k)$ is an external disturbance, and $\beta_s(k)$ symbolize the channel coefficients that are mutually independent SVs satisfying

$$\Pr\{\beta_s(k) = 1\} = \overline{\beta}_s, \quad \Pr\{\beta_s(k) = 0\} = 1 - \overline{\beta}_s.$$

Note that $\beta_s(k) \in [0, 1]$ and variances $\mathscr{E}\{(\beta_s(k) - \beta_s)^2\} = \beta_s^*$.

Remark 2: To describe the phenomena of FCs, induced delays, and data losses in an unreliable network, a l*th-order Rician fading model is applied. Different from the conventional single Gauss distribution, mutually independent coefficients that have values in the set* $[0, 1]$ *are addressed.*

In the following, a more general actuator fault (AA) with the property of time-varying is developed, it yields

$$u(k) = \Phi(\vartheta_k)\overline{u}(k), \tag{6}$$

where $\Phi(\vartheta_k) = \text{diag}\{\phi_1(\vartheta_t), \phi_2(\vartheta_t), \cdots, \phi_{n_u}(\vartheta_t)\}$ and $\phi_l(\vartheta_t) \in [0, 1], (l = 1, 2, \cdots, n_n)$. $\{\vartheta_k, k \geq 0\} \in \mathcal{M} = \{1, 2, \cdots, M\}$ is recognized as a discrete-time nonhomogeneous Markov chain (MC) with the transition probability (TP) matrix $\Pi(k) = [\varpi_{\mu\nu}]$:

$$\Pr\{\vartheta_{k+1} = \nu \mid \vartheta_k = \mu\} = \varpi_{\mu\nu}(k), \quad \forall \mu, \nu \in \mathcal{M},$$

where $\varpi_{\mu\nu}(k)$ is the time-varying TP, and $\varpi_{\mu\nu}(k) \geq 0$, $\sum_{\nu\in\mathcal{M}} \varpi_{\mu\nu}(k) = 1$. To characterize the TP matrix $\Pi(k)$ form of polytope structure:

$$\Pi(k) = \Pi(\varphi^n(k)) = \sum_{n=1}^{N} \varphi^n(k)\Pi^n, \tag{7}$$

where $\varphi^n(k) \geq 0$ and $\sum_{n=1}^{N} \varphi^n(k) = 1$. For $n = 1, 2, \cdots, N$ with N symbolizes the vertices number of $\Pi(k)$, Π^n are the vertex matrices. For $\vartheta_k = \mu \in \mathcal{M}$, the system AA coefficient matrix of the μ-th mode is symbolized by Φ_μ.

Specifically, $\phi_l(\vartheta_k) = 0$, $\phi_l(\vartheta_k) \in (0, 1)$ and $\phi_l(\vartheta_k) = 1$, respectively, imply the fault, partial fault, and no fault.

In light of system (1), a fuzzy-based static output feedback controller (SOFC) is constructed.

Rule p: IF $\zeta_1(k)$ is M_{p1}, and $\cdots$ and $\zeta_r(k)$ is M_{pr}, Then

$$\overline{u}(k) = K_p\overline{y}(k), \tag{8}$$

where K_p is the corresponding controller gain to be solved. By resorting to T-S fuzzy strategies [3-6], (8) implies

$$\overline{u}(k) = K_\hbar\overline{y}(k). \tag{9}$$

where $K_\hbar = \sum_{p=1}^{t} \hbar_p(\zeta(k))K_p$.

In terms of (4)-(6) and (9), we obtain the closed-loop fuzzy system as below:

$$\begin{cases} \delta(k+1) = (A_\hbar + \overline{\alpha}\Delta A_\hbar + \overline{\beta}_0 B_\hbar\Phi_\mu K_\hbar C_\hbar)\delta(k) \\ \qquad + D_\hbar\omega(k) + \widetilde{\alpha}(k)\Delta A_\hbar\delta(k) \\ \qquad + \widetilde{\beta}_0(k)B_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k) \\ \qquad + \sum_{s=1}^{l_k} \overline{\beta}_s B_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k-s) \\ \qquad + \sum_{s=1}^{l_k} \widetilde{\beta}_s(k)B_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k-s)z \\ \qquad + B_\hbar\Phi_\mu K_\hbar\nu(k), \\ z(k) = (F_\hbar + \overline{\beta}_0 H_\hbar\Phi_\mu K_\hbar C_\hbar)\delta(k) \\ \qquad + \widetilde{\beta}_0(k)H_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k) \\ \qquad + H_\hbar\Phi_\mu K_\hbar\nu(k) \\ \qquad + \sum_{s=1}^{l_k} \overline{\beta}_s H_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k-s) \\ \qquad + \sum_{s=1}^{l_k} \widetilde{\beta}_s(k)H_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k-s), \end{cases} \tag{10}$$

where $\widetilde{\alpha}(k) = \alpha(k) - \overline{\alpha}$, $\widetilde{\beta}(k) = \beta(k) - \overline{\beta}$.

Remark 3: In reality, one may experience bias faults, especially for physical plants with multiple actuators. Recently, mode-independent AA has been fully studied in To remove the mode-independent restriction and reflect the stochastic occurrence of AA, a homogeneous MC is utilized in [21]. In fact, it is unrealistic that the TP of a homogeneous MC remains time invariant. To eliminate this drawback, a non-homogeneous MC accounting for the characterization of the time-varying TP is adopted.

Before proceeding further, the necessary definition and lemmas are introduced.

Definition 1 [20]: The system (10) with $\omega(k) = 0$ and $\nu(k) = 0$ is said to be stochastic stable (SS), if for any (δ_0, ϑ_0), such that

$$\mathscr{E}\left\{\sum_{k=0}^{\infty} \| \delta(k) \|^2 | \delta_0, \vartheta_0\right\} < \infty.$$

The object of this work is to explore the $\mathcal{H}_\infty$ fuzzy control problem for system (10) such that the following conditions satisfied:

(i) System (10) is SS in mean square;

(ii) Under zero initial condition and a performance level γ, the controlled output $z(k)$ satisfies

$$\sum_{k=0}^{\infty} \mathscr{E}\left\{\| \delta(k) \|^2\right\} < \gamma^2 \sum_{k=0}^{\infty} \mathscr{E}\left\{\| \varrho(k) \|^2\right\}.$$

Lemma 1 [16]: For compatible matrices $\mathscr{X} = \mathscr{X}^\top$, $\mathscr{Y}$ and $\mathscr{Z}$, $\mathscr{U}$ satisfying $\mathscr{U}^\top\mathscr{U} \leq I$, then $\mathscr{X} + \mathscr{Y}\mathscr{U}\mathscr{Z} + \mathscr{Z}^\top\mathscr{U}^\top\mathscr{Y}^\top < 0$, if and only if there exists $\varepsilon > 0$ such that $\mathscr{X} + \varepsilon^{-1}\mathscr{Y}\mathscr{Y}^{-1} + \mu\mathscr{Z}^\top\mathscr{Z} < 0$.

Lemma 2 [18]: If there exist a scalar ε and matrices $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ and $\mathscr{U}$ satisfying

$$\begin{bmatrix} \mathscr{X} & \mathscr{Z} + \varepsilon\mathscr{U}^\top \\ * & -\varepsilon Sym\{\mathscr{Y}\} \end{bmatrix} < 0,$$

such that

$$\mathscr{X} + \mathscr{Z}\mathscr{Y}^{-1}\mathscr{U} + \mathscr{U}^\top\mathscr{Y}^{-\top}\mathscr{Z}^\top < 0.$$

In what follows, sufficient conditions are attained such that the system (10) is SS in mean square.

III. MAIN RESULTS

Theorem 1: For given scalars $\overline{\alpha}$, α^*, $\overline{\beta}_s$ and β_s^* $(s = 1, 2, \cdots, l_k)$, if there exists matrices $P_\mu^n > 0$ and $Q_s > 0$ $(s = 1, 2, \cdots, l_k)$ such that

$$\Gamma_{ppnl\mu} < 0, \quad (1 \leq p \leq t) \tag{11}$$

$$\Gamma_{pqnl\mu} + \Gamma_{qpnl\mu} < 0, \quad (1 \leq p < q \leq t) \tag{12}$$

where

$$\Gamma_{pqnl\mu} = \begin{bmatrix} \Gamma_{n\mu}^1 & \Gamma_{pq\mu}^2 & \Gamma_{pq\mu}^3 \\ * & -(\mathcal{P}_{nl\mu})^{-1} & 0 \\ * & * & -\mathcal{I} \end{bmatrix},$$

$$\mathcal{I} = \text{diag}\{I, I, \mathscr{I} \otimes I\}, \quad \mathscr{I} = \text{diag}\{\underbrace{I, I, \cdots, I}_{l_k}\},$$

$$\Gamma^1_{n\mu} = \text{diag}\left\{-P^n_\mu + \sum_{j=1}^{l_k} Q_j,\ -\mathcal{Q},\ -\gamma^2 I,\ -\gamma^2 I\right\},$$
$$\Gamma^2_{pq\mu} = \left[\Sigma^{1\top}_{pq\mu}\ \Sigma^{2\top}_{pq\mu}\ \Sigma^{3\top}_{pq\mu}\ \Sigma^{4\top}_{pq\mu}\right],$$
$$\Gamma^3_{pq\mu} = \left[\Sigma^{5\top}_{pq\mu}\ \Sigma^{6\top}_{pq\mu}\ \Sigma^{7\top}_{pq\mu}\right],$$
$$\Sigma^{(1)}_{pq\mu} = [A_p + \overline{\alpha}\Delta A_p + \overline{\beta}_0 B_p\Phi_\mu K_q C_p\ \ \Psi\otimes B_p\Phi_\mu K_q C_p\ \ D_p\ \ B_p\Phi_\mu K_q],\quad \Sigma^{(2)}_{pq\mu} = \left[\sqrt{\alpha^*}\Delta A_p\ 0\ 0\ 0\right],$$
$$\Sigma^{(3)}_{pq\mu} = \left[\sqrt{\beta^*_0}B_p\Phi_\mu K_q C_p\ 0\ 0\ 0\right],\quad \Sigma^{(4)}_{pq\mu} = [0\ \Upsilon\otimes B_p\Phi_\mu K_q C_p\ 0\ 0],\quad \mathcal{Q} = \text{diag}\{Q_1, Q_2, \cdots, Q_{l_k}\},$$
$$\Sigma^{(5)}_{pq\mu} = [F_p + \overline{\beta}_0 H_p\Phi_\mu K_q C_p\ \ \Psi\otimes H_p\Phi_\mu K_q C_p\ \ 0\ \ H_p\Phi_\mu K_q],\quad \Sigma^{(6)}_{pq\mu} = \left[\sqrt{\beta^*_0}H_p\Phi_\mu K_q C_p\ 0\ 0\ 0\right],$$
$$\Sigma^{(7)}_{pq\mu} = [0\ \Upsilon\otimes H_p\Phi_\mu K_q C_p\ 0\ 0],$$
$$\Psi = [\overline{\beta}_1 I\ \overline{\beta}_2 I\ \cdots\ \overline{\beta}_{l_k} I],$$
$$\Upsilon = \text{diag}\{\sqrt{\beta^*_1}I,\ \sqrt{\beta^*_2}I, \cdots, \sqrt{\beta^*_{l_k}}I\},$$
$$\mathcal{P}_{nl\mu} = \text{diag}\left\{\sum_{\nu\in\mathcal{M}}\varpi^n_{\mu\nu}P^l_\nu,\ \sum_{\nu\in\mathcal{M}}\varpi^n_{\mu\nu}P^l_\nu,\ \sum_{\nu\in\mathcal{M}}\varpi^n_{\mu\nu}P^l_\nu,\ \mathscr{I}\otimes\sum_{\nu\in\mathcal{M}}\varpi^n_{\mu\nu}P^l_\nu\right\}.$$

Proof: In consideration of (11)-(12), it is clear that

$$\begin{aligned}\Gamma_{\hbar nl\mu} &= \sum_{p=1}^{t}\hbar_p(\zeta(k))\sum_{q=1}^{t}\hbar_q(\zeta(k))\Gamma_{ppnl\mu}\\ &= \sum_{p=1}^{t}\hbar_p(\zeta(k))\Gamma_{ppnl\mu} + \sum_{p=1}^{t-1}\hbar_p(\zeta(k))\\ &\quad\times\sum_{q=p+1}^{t}\hbar_q(\zeta(k))(\Gamma_{ijnl\mu}+\Gamma_{jinl\mu}) < 0, \quad (13)\end{aligned}$$

where

$$\Gamma_{\hbar nl\mu} = \begin{bmatrix}\Gamma^1_{n\mu} & \Gamma^2_{\hbar\mu} & \Gamma^3_{\hbar\mu}\\ * & -(\mathcal{P}_{nl\mu})^{-1} & 0\\ * & * & -\mathcal{I}\end{bmatrix},$$
$$\Gamma^2_{\hbar\mu} = \left[\Sigma^{1\top}_{\hbar\mu}\ \Sigma^{2\top}_{\hbar\mu}\ \Sigma^{3\top}_{\hbar\mu}\ \Sigma^{4\top}_{\hbar\mu}\right],\quad \Gamma^3_{\hbar\mu} = \left[\Sigma^{5\top}_{\hbar\mu}\ \Sigma^{6\top}_{\hbar\mu}\ \Sigma^{7\top}_{\hbar\mu}\right],$$
$$\Sigma^{(1)}_{\hbar\mu} = [A_\hbar + \overline{\alpha}\Delta A_\hbar + \overline{\beta}_0 B_\hbar\Phi_\mu K_\hbar C_\hbar\ \ \Psi\otimes B_\hbar\Phi_\mu K_\hbar C_\hbar\ \ D_\hbar\ \ B_\hbar\Phi_\mu K_\hbar],\quad \Sigma^{(2)}_{\hbar\mu} = \left[\sqrt{\alpha^*}\Delta A_\hbar\ 0\ 0\ 0\right],$$
$$\Sigma^{(3)}_{\hbar\mu} = \left[\sqrt{\beta^*_0}B_\hbar\Phi_\mu K_\hbar C_\hbar\ 0\ 0\ 0\right],\quad \Sigma^{(4)}_{\hbar\mu} = [0\ \Upsilon\otimes B_\hbar\Phi_\mu\times K_\hbar C_\hbar\ 0\ 0],\quad \Sigma^{(5)}_{\hbar\mu} = [F_\hbar + \overline{\beta}_0 H_\hbar\Phi_\mu K_\hbar C_\hbar\ \ \Psi\otimes H_\hbar\Phi_\mu K_\hbar C_\hbar\ \ 0\ \ H_\hbar\Phi_\mu K_\hbar],$$
$$\Sigma^{(6)}_{\hbar\mu} = \left[\sqrt{\beta^*_0}H_\hbar\Phi_\mu K_\hbar C_\hbar\ 0\ 0\ 0\right],\quad \Sigma^{(7)}_{\hbar\mu} = [0\ \Upsilon\otimes H_\hbar\Phi_\mu K_\hbar C_\hbar\ 0\ 0].$$

Applying the Schur complement to (13), the inequalities (14) and (15) can be obtained:

$$\widehat{\Gamma}^1_{\hbar\mu} + \sum_{f=1}^{4}\widehat{\Sigma}^{(f)\top}_{\hbar\mu}\sum_{\nu\in\mathcal{M}}\varpi^n_{\mu\nu}P^l_\nu\widehat{\Sigma}^{(f)\top}_{\hbar\mu} < 0, \quad (14)$$
$$\Gamma^1_{\hbar\mu} + \sum_{f=1}^{4}\Sigma^{(f)\top}_{\hbar\mu}\sum_{\nu\in\mathcal{M}}\varpi^n_{\mu\nu}P^l_\nu\Sigma^{(f)\top}_{\hbar\mu} + \sum_{f=5}^{7}\Sigma^{(f)\top}_{\hbar\mu}\Sigma^{(f)\top}_{\hbar\mu} < 0, \quad (15)$$

where

$$\widehat{\Gamma}^1_{\hbar\mu} = \text{diag}\left\{-P^n_\mu + \sum_{j=1}^{l_k}Q_j,\ -\mathcal{Q}\right\}.$$

Next, formulating the Lyapunov functional in the form of:

$$V(\delta_k, \vartheta_k) = V_1(\delta_k, \vartheta_k) + V_2(\delta_k, \vartheta_k), \quad (16)$$

where

$$V_1(\delta_k, \vartheta_k) = \delta^\top(k)\sum_{n=1}^{N}\varpi^n(k)P^n_\mu\delta(k),$$
$$V_2(\delta_k, \vartheta_k) = \sum_{j=1}^{l_k}\sum_{i=k-j}^{k-1}\delta^\top(i)Q_i\delta(i).$$

By calculating the difference of $V(\delta_k, \vartheta_k)$, it yields

$$\mathscr{E}\{\Delta V(\delta_k, \vartheta_k)\} = \mathscr{E}\{V(\delta_{k+1}, \vartheta_{k+1} = \nu \mid \delta_k, \vartheta_k = \mu)\} - V(\delta_k, \vartheta_k).$$

Recalling (10), $\mathscr{E}\{\Delta V_1(\delta_k, \vartheta_k)\}$ renders that

$$\begin{aligned}&\Delta V_1(\delta_k, \vartheta_k)\\ &= \mathscr{E}\left\{\delta^\top(k+1)\sum_{n=1}^{N}\varphi^n(k+1)P^n_\nu\delta(k+1) \mid \delta_k, \vartheta_k = \mu\right\} - \delta^\top(k)\sum_{n=1}^{N}\varphi^n(k)P^n_\mu\delta(k)\\ &= \mathscr{E}\Big\{\Big[(A_\hbar + \overline{\alpha}\Delta A_\hbar + \overline{\beta}_0 B_\hbar\Phi_\mu K_\hbar C_\hbar)\delta(k) + \sum_{s=1}^{l_k}\overline{\beta}_s B_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k-s) + D_\hbar\omega(k)\\ &\quad + B_\hbar\Phi_\mu K_\hbar\nu(k)\Big]^\top\sum_{\nu\in\mathcal{M}}\varpi_{\mu\nu}(k)\sum_{n=1}^{N}\varphi^n(k+1)P^n_\nu\\ &\quad\times\Big[(A_\hbar + \overline{\alpha}\Delta A_\hbar + \overline{\beta}_0 B_\hbar\Phi_\mu K_\hbar C_\hbar)\delta(k) + D_\hbar\omega(k) + \sum_{s=1}^{l_k}\overline{\beta}_s B_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k-s) + B_\hbar\Phi_\mu K_\hbar\nu(k)\Big]\\ &\quad + \overline{\alpha}(1-\overline{\alpha})\delta^\top(k)\Delta A^\top_\hbar\sum_{\nu\in\mathcal{M}}\varpi_{\mu\nu}(k)\sum_{n=1}^{N}\varphi^n(k+1)\\ &\quad\times P^n_\nu\Delta A_\hbar\delta(k) + \widetilde{\beta}^2_0(k)(B_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k))^\top\end{aligned}$$

$$\times \sum_{\nu\in\mathcal{M}} \varpi_{\mu\nu}(k)\sum_{n=1}^{N}\varphi^n(k+1)P_\nu^n B_\hbar \Phi_\mu K_\hbar C_\hbar \delta(k) + \left(\sum_{s=1}^{l_k}\widetilde{\beta}_s(k)B_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k-s)\right)^\top \times \sum_{\nu\in\mathcal{M}} \varpi_{\mu\nu}(k)\sum_{n=1}^{N}\varphi^n(k+1)P_\nu^n \times \left(\sum_{s=1}^{l_k}\widetilde{\beta}_s(k)B_\hbar\Phi_\mu K_\hbar C_\hbar\delta(k-s)\right)\Bigg\} - \delta^\top(k)\sum_{n=1}^{N}\varphi^n(k)P_\mu^n\delta(k). \quad (17)$$

Added by the variances of FCs, (17) can be reformulated as

$$\begin{aligned}\Delta V_1(\delta_k,\vartheta_k) &= \mathscr{E}\left\{\theta^\top(k)\sum_{f=1}^{4}\left[\Sigma_{\hbar\mu}^{(f)\top}\sum_{\nu\in\mathcal{M}}\varpi_{\mu\nu}(k) \times \sum_{n=1}^{N}\varphi^n(k+1)P_\nu^n\Sigma_{\hbar\mu}^{(f)\top}\right]\theta(k)\right\} - \delta^\top(k)\sum_{n=1}^{N}\varphi^n(k)P_\mu^n\delta(k)\\ &= \mathscr{E}\left\{\theta^\top(k)\sum_{f=1}^{4}\left[\Sigma_{\hbar\mu}^{(f)\top}\sum_{\nu\in\mathcal{M}}\sum_{n=1}^{N}\varphi^n(k)\varpi_{\mu\nu}^n \times \sum_{n=1}^{N}\varphi^n(k+1)P_\nu^n\Sigma_{\hbar\mu}^{(f)\top}\right]\theta(k)\right\} - \delta^\top(k)\sum_{n=1}^{N}\varphi^n(k)P_\mu^n\delta(k), \quad (18)\end{aligned}$$

where

$$\delta_{l_k}^\top(k) = \left[\delta^\top(k-1)\ \delta^\top(k-2)\ \cdots\ \delta^\top(k-l_k)\right],$$
$$\theta^\top(k) = \left[\delta^\top(k)\ \delta_{l_k}^\top(k)\ \omega^\top(k)\ \nu^\top(k)\right].$$

Set $\varphi^n(k+1)=\chi^l(k)$, it holds that

$$\sum_{n=1}^{N}\varphi^n(k+1)P_\nu^n = \sum_{l=1}^{N}\chi^l(k)P_\nu^l, \quad (19)$$

where $\sum_{l=1}^{N}\chi^l(k)=1$ and $\chi^l(k)\geq 0$.

By combining the inequalities (13)-(19), $\Delta V_1(\delta_k,\vartheta_k)$ can be rewritten as

$$\Delta V_1(\delta_k,\vartheta_k) = \mathscr{E}\left\{\theta^\top(k)\sum_{f=1}^{4}\left[\Sigma_{\hbar\mu}^{(f)\top}\sum_{\nu\in\mathcal{M}}\sum_{n=1}^{N}\varphi^n(k)\varpi_{\mu\nu}^n \times \sum_{l=1}^{N}\chi^l(k)P_\nu^l\Sigma_{\hbar\mu}^{(f)\top}\right]\theta(k)\right\} - \delta^\top(k)\sum_{n=1}^{N}\varphi^n(k)P_\mu^n\delta(k), \quad (20)$$

On the other hand, for term $V_2(\delta_k,\vartheta_k)$, we further have

$$\begin{aligned}\Delta V_2(\delta_k,\vartheta_k) &= \mathscr{E}\left\{\sum_{j=1}^{l_k}\sum_{p=k-j+1}^{k}\delta^\top(k)Q_i\delta(i) - \sum_{j=1}^{l_k}\sum_{i=k-j}^{k-1}\delta^\top(i)Q_i\delta(i)\right\}\\ &= \delta^\top(k)\sum_{j=1}^{l_k}Q_j\delta(k) - \sum_{j=1}^{l_k}\delta^\top(k-j) \times Q_j\delta(k-j). \quad (21)\end{aligned}$$

In what follows, the SS of the system (10) with $\omega(k)=0$ and $\nu(k)=0$ will be verified. Recalling the inequalities (20)-(21), it can be obtained that

$$\begin{aligned}&\mathscr{E}\{\Delta V(\delta_k,\vartheta_k)\}\\ &= \mathscr{E}\left\{\widehat{\theta}^\top(k)\sum_{f=1}^{4}\left[\widehat{\Sigma}_{\hbar\mu}^{(f)\top}\sum_{\nu\in\mathcal{M}}\sum_{n=1}^{N}\varphi^n(k)\varpi_{\mu\nu}^n \times \sum_{l=1}^{N}\chi^l(k)P_\nu^l\widehat{\Sigma}_{\hbar\mu}^{(f)\top}\right]\widehat{\theta}(k)\right\}\\ &\quad + \delta^\top(k)\sum_{n=1}^{N}\varphi^n(k)\left(-P_\mu^n+\sum_{j=1}^{l_k}Q_j\right)\delta(k) - \sum_{j=1}^{l_k}\delta^\top(k-j)Q_j\delta(k-j)\\ &= \widetilde{\theta}^\top(k)\sum_{n=1}^{N}\varphi^n(k)\sum_{l=1}^{N}\chi^l(k)\widehat{\Gamma}_{\hbar\mu}\widetilde{\theta}(k), \quad (22)\end{aligned}$$

where

$$\widehat{\Gamma}_{\hbar\mu} = \mathrm{diag}\left\{-P_\mu^n+\sum_{j=1}^{l_k}Q_j,\ -\mathcal{Q}\right\} + \sum_{f=1}^{4}\widehat{\Sigma}_{\hbar\mu}^{(f)\top}\sum_{\nu\in\mathcal{M}}\varpi_{\mu\nu}^n P_\nu^l\widehat{\Sigma}_{\hbar\mu}^{(f)\top},$$
$$\widehat{\Sigma}_{\hbar\mu}^{(1)} = \left[A_\hbar+\overline{\alpha}\Delta A_\hbar+\overline{\beta}_0 B_\hbar\Phi_\mu K_\hbar C_\hbar\ \ \Psi\otimes B_\hbar\Phi_\mu K_\hbar C_\hbar\right],$$
$$\widehat{\Sigma}_{\hbar\mu}^{(2)} = \left[\sqrt{\alpha^*}\Delta A_\hbar\ 0\right],\quad \widehat{\Sigma}_{\hbar\mu}^{(3)} = \left[\sqrt{\beta_0^*}B_\hbar\Phi_\mu K_\hbar C_\hbar\ 0\right],$$
$$\widehat{\Sigma}_{\hbar\mu}^{(4)} = \left[0\ \Upsilon\otimes B_\hbar\Phi_\mu K_\hbar C_\hbar\right],\quad \widehat{\theta}^\top(k) = \left[\delta^\top(k)\ \delta_{l_k}^\top(k)\right].$$

Letting $\xi = \max_{\mu\in\mathcal{M}}\{\lambda_{\max}(\widetilde{\Gamma}_{\hbar\mu})\}$, one gets $\xi<0$, (22) gives rise to

$$\mathscr{E}\{V(\delta_{k+1},\vartheta_{k+1})\} - \mathscr{E}\{V(\delta_k,\vartheta_k)\} < \xi \parallel \delta(k)\parallel^2. \quad (23)$$

For integer $T > 0$, it can be achieved from (23) that

$$\mathscr{E}\{V(\delta_T, \vartheta_T)\} - \mathscr{E}\{V(\delta_0, \vartheta_0)\} \le \xi \sum_{k=0}^{T} \mathscr{E}\{\| \delta(k) \|^2\},$$

which means

$$\sum_{k=0}^{T} \mathscr{E}\{\| \delta(k) \|^2\} \le -\frac{1}{\xi} \mathscr{E}\{V(\delta_0, \vartheta_0)\}. \tag{24}$$

Consequently, the following condition is achieved:

$$\lim_{k\to\infty} \mathscr{E}\{\| \delta(k) \|^2\} < \infty. \tag{25}$$

Therefore, the system (10) is SS with $\omega(k) = 0$ and $\nu(k) = 0$.

To analyze the $\mathcal{H}_\infty$ performance for the system (10) under zero initial condition, the following index function is adopted:

$$\mathscr{J}(T) = \mathscr{E}\left\{\sum_{k=0}^{T} z^\top(k)z(k) - \gamma^2 \varrho^\top(k)\varrho(k)\right\}, \tag{26}$$

where

$$\varrho(k) = [\omega^\top(k)\ \nu^\top(k)]^\top.$$

By utilizing the inequalities (21)-(22), $\mathscr{J}(T)$ can be further improved as below:

$$\begin{aligned}
\mathscr{J}(T) &= \mathscr{E}\left\{\sum_{k=0}^{T}\left[z^\top(k)z(k) - \gamma^2 \varrho^\top(k)\varrho(k)\right]\right\} \\
&\le \mathscr{E}\left\{\sum_{k=0}^{T}\left[z^\top(k)z(k) - \gamma^2(\omega^\top(k)\omega(k)\right.\right. \\
&\qquad \left.\left. + \nu^\top(k)\nu(k)) + \Delta V(\delta_k, \vartheta_k)\right]\right\} \\
&= \sum_{k=0}^{T} \mathscr{E}\left\{\theta^\top(k) \sum_{n=1}^{N} \varphi^n(k) \sum_{l=1}^{N} \chi^l(k)\right. \\
&\quad \times \left[\Gamma^1_{\hbar\mu} + \sum_{f=1}^{4} \Sigma^{(f)\top}_{\hbar\mu} \sum_{\nu\in\mathcal{M}} \varpi^n_{\mu\nu} P^l_\nu \Sigma^{(f)\top}_{\hbar\mu}\right. \\
&\quad \left.\left. + \sum_{f=5}^{7} \Sigma^{(f)\top}_{\hbar\mu} \Sigma^{(f)\top}_{\hbar\mu}\right]\theta(k)\right\}.
\end{aligned} \tag{27}$$

By resorting to the inequalities (15) and (27) such that

$$\mathscr{J}(T) < 0, \tag{28}$$

Letting $T \to \infty$, (28) results in

$$\sum_{k=0}^{\infty} \mathscr{E}\{\| z(k) \|^2\} \le \gamma^2 \sum_{k=0}^{\infty} \mathscr{E}\{\| \varrho(k) \|^2\}. \tag{29}$$

Therefore, we conclude that the system (10) is SS with $\mathcal{H}_\infty$ performance index γ. This completes the proof. □

Next, in Theorem 2, the control design scheme is elicited.

Theorem 2: For given scalars $\overline{\alpha}$, α^*, $\overline{\beta}_s$ and β^*_s $(s = 1, 2, \cdots, l_k)$, if there exists matrices $P^n_\mu > 0$, $Q_s > 0$ $(s = 1, 2, \cdots, l_k)$, and matrices X_q, Y_q $(q = 1, 2, \cdots, r)$ such that

$$\Theta_{ppnl\mu} < 0, \quad (1 \le p \le r) \tag{30}$$

$$\Theta_{pqnl\mu} + \Theta_{qpnl\mu} < 0, \quad (1 \le p < q \le r) \tag{31}$$

where

$$\begin{aligned}
\Theta_{pqnl\mu} &= \begin{bmatrix} \Theta^1_{pqnl\mu} & \Theta^2_{pq\mu} & \rho\Lambda^1_p & \Lambda^2_p \\ * & \Theta^3_q & 0 & 0 \\ * & * & -\rho I & 0 \\ * & * & * & -\rho I \end{bmatrix}, \\
\Theta^1_{pqnl\mu} &= \begin{bmatrix} \Gamma^1_{n\mu} & \overline{\Gamma}^2_{pq\mu} & \overline{\Gamma}^3_{pq\mu} \\ * & \mathcal{P}_{nl\mu} & 0 \\ * & * & -\mathcal{I} \end{bmatrix}, \\
\Theta^2_{pq\mu} &= \begin{bmatrix} \Theta^2_{1pq\mu} & \Theta^2_{2pq\mu} & \cdots & \Theta^2_{(l_k+3)pq\mu} \end{bmatrix}, \\
\Theta^2_{spq\mu} &= \begin{bmatrix} \epsilon_1 \mathcal{Z}^s_{pq} & \mathcal{W}^{s\top}_{pq\mu} \end{bmatrix}, \quad (s = 1, 2, \cdots, l_k + 3) \\
\Theta^3_q &= \operatorname{diag}\{-\epsilon_1 \operatorname{Sym}\{X_q\}, -\epsilon_2 \operatorname{Sym}\{X_q\}, \cdots, \\
&\quad -\epsilon_{l_k+3}\operatorname{Sym}\{X_q\}\}, \quad \Lambda^1_p = \begin{bmatrix} N_p & 0_{l_k} & 0 \cdots 0 \end{bmatrix}, \\
\Lambda^2_p &= \begin{bmatrix} 0\ 0_{l_k}\ 0\ 0\ \overline{\alpha} M^\top_p G^\top\ \sqrt{\alpha^*} M^\top_p G^\top\ 0 \cdots 0 \end{bmatrix}, \\
\mathcal{W}^1_{pq\mu} &= \left[0\ 0\ 0\ 0\ \overline{\beta}_0(GB_p\Phi_\mu - B_p\Phi_\mu X_q)^\top\ 0\right. \\
&\quad \sqrt{\beta^*_0}(GB_p\Phi_\mu - B_p\Phi_\mu X_q)^\top\ 0\ \overline{\beta}_0(H_p\Phi_\mu \\
&\quad \left. - H_p\Phi_\mu X_q)^\top\ \sqrt{\beta^*_0}(H_p\Phi_\mu - H_p\Phi_\mu X_q)^\top\ 0\right]^\top, \\
\mathcal{W}^2_{pq\mu} &= \left[0\ 0_{l_k}\ 0\ 0\ (GB_p\Phi_\mu - B_p\Phi_\mu X_q)^\top\ 0\ 0\right. \\
&\quad \left. 0\ (H_p\Phi_\mu - H_p\Phi_\mu X_q)^\top\ 0\ 0\right]^\top, \\
\mathcal{W}^3_{pq\mu} &= \left[0\ 0_{l_k}\ 0\ 0\ (GB_p\Phi_\mu - B_p\Phi_\mu X_q)^\top\ 0\ 0\right. \\
&\quad \left. 0\ (H_p\Phi_\mu - H_p\Phi_\mu X_q)^\top\ 0\ 0\right]^\top, \\
\mathcal{W}^{3+s}_{pq\mu} &= \left[0\ 0_{l_k}\ 0\ 0\ 0\ 0\ 0\ 0\ \underbrace{0\ \cdots\ 0}_{s-1}\ \sqrt{\beta^*_s}(GB_p\Phi_\mu\right. \\
&\quad -B_p\Phi_\mu X_q)^\top\ \underbrace{0\ \cdots\ 0}_{l_k-s}\ 0\ 0\ \underbrace{0\ \cdots\ 0}_{s-1} \\
&\quad \left.\sqrt{\beta^*_s}(H_p\Phi_\mu - H_p\Phi_\mu X_q)^\top\ \underbrace{0\ \cdots\ 0}_{l_k-s}\right]^\top, \\
\mathcal{Z}^1_{pq} &= \begin{bmatrix} Y_qC_p & 0_{l_k} & 0 & 0 \end{bmatrix}, \ \mathcal{Z}^2_{pq} = \begin{bmatrix} 0 & \Psi\otimes Y_qC_p & 0 & 0 \end{bmatrix}, \\
\mathcal{Z}^3_{pq} &= \begin{bmatrix} 0 & 0_{l_k} & 0 & Y_q \end{bmatrix}, \\
\mathcal{Z}^{3+s}_{pq} &= \left[0\ \underbrace{0\ \cdots\ 0}_{s-1}\ Y_qC_p\ \underbrace{0\ \cdots\ 0}_{l_k-s}\ 0\ 0\right],
\end{aligned}$$

$$\Gamma^2_{pq\mu} = \begin{bmatrix} (GA_p + \overline{\beta}_0 B_p \Phi_\mu Y_q C_p)^\top & 0 & \sqrt{\beta_0^*}(B_p\Phi_\mu Y_q C_p)^\top & 0 \\ (\Psi \otimes B_p \Phi_\mu Y_q C_p)^\top & 0 & 0 & (\Upsilon \otimes B_p\Phi_\mu Y_q C_p)^\top \\ (GD_p)^\top & 0 & 0 & 0 \\ (B_p\Phi_\mu Y_q)^\top & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma^3_{n\mu} = \begin{bmatrix} (F_p + \overline{\beta}_0 H_p \Phi_\mu Y_q C_p)^\top & \sqrt{\beta_0^*}(H_p\Phi_\mu Y_q C_p)^\top & 0 \\ (\Psi \otimes H_p \Phi_\mu Y_q C_p)^\top & 0 & (\Upsilon \otimes H_p\Phi_\mu Y_q C_p)^\top \\ 0 & 0 & 0 \\ (H_p\Phi_\mu Y_q)^\top & 0 & 0 \end{bmatrix},$$

$$\overline{\mathcal{P}}_{nl\mu} = \mathrm{diag}\Big\{ \sum_{\nu\in\mathcal{M}} \varpi^n_{\mu\nu} P^l_\nu - G - G^\top, \sum_{\nu\in\mathcal{M}} \varpi^n_{\mu\nu} P^l_\nu - G - G^\top, \sum_{\nu\in\mathcal{M}} \varpi^n_{\mu\nu} P^l_\nu - G - G^\top, \mathscr{I} \otimes \sum_{\nu\in\mathcal{M}} \varpi^n_{\mu\nu} P^l_\nu - G - G^\top \Big\}.$$

Meanwhile, the controller gains are formulated by

$$K_q = X_q^{-1} Y_q. \tag{32}$$

Proof: Applying Schur complement to (30)-(31), the following inequality can be obtained:

$$\begin{bmatrix} \Theta^1_{pqnl\mu} & \Theta^2_{pq\mu} \\ * & \Theta^3_q \end{bmatrix} + \rho \Lambda_p^{1\top}\Lambda_p^1 + \rho^{-1}\Lambda_p^{2\top}\Lambda_p^2 < 0. \tag{33}$$

By resorting to Lemma 1, (33) is described by

$$\begin{bmatrix} \Theta^1_{pqnl\mu} & \Theta^2_{pq\mu} \\ * & \Theta^3_q \end{bmatrix} + \Lambda_p^{1\top}\Delta_p^\top(k)\Lambda_p^2 + \Lambda_p^{2\top}\Delta_p(k)\Lambda_p^1 < 0. \tag{34}$$

Obviously, (34) infers

$$\begin{bmatrix} \vec{\Theta}^1_{pqnl\mu} & \Theta^2_{pq\mu} \\ * & \Theta^3_q \end{bmatrix} < 0. \tag{35}$$

Benefit from the Lemma 2, it can be obtained that

$$\vec{\Theta}^1_{pqnl\mu} + \sum_{s=1}^{l_k+3} \mathcal{W}^s_{pq\mu} X_q^{-1} \mathcal{Z}^s_{pq} + \sum_{s=1}^{l_k+3} \mathcal{Z}^{s\top}_{pq} X_q^{-\top} \mathcal{W}^{s\top}_{pq\mu} < 0, \tag{36}$$

where

$$\vec{\Theta}^1_{pqnl\mu} = \begin{bmatrix} \Gamma^1_{n\mu} & \vec{\Gamma}^2_{n\mu} & \Gamma^3_{n\mu} \\ * & \overline{\mathcal{P}}_{nl\mu} & 0 \\ * & * & -\mathcal{I} \end{bmatrix},$$

$$\vec{\Gamma}^2_{n\mu} = \begin{bmatrix} (GA_p + \overline{\alpha} G\Delta A_p + \overline{\beta}_0 B_p\Phi_\mu Y_q C_p)^\top & \sqrt{\alpha^*}\Delta A_p & \sqrt{\beta_0^*}(B_p\Phi_\mu Y_q C_p)^\top & 0 \\ (\Psi \otimes B_p\Phi_\mu Y_q C_p)^\top & 0 & 0 & (\Upsilon \otimes B_p\Phi_\mu Y_q C_p)^\top \\ (GD_p)^\top & 0 & 0 & 0 \\ (B_p\Phi_\mu Y_q)^\top & 0 & 0 & 0 \end{bmatrix}.$$

Additionally, (36) can be further reformulated as

$$\begin{bmatrix} \Gamma^1_{n\mu} & \Gamma^2_{pq\mu} G^\top & \Gamma^3_{pq\mu} \\ * & \mathcal{P}_{nl\mu} & 0 \\ * & * & -\mathcal{I} \end{bmatrix} < 0. \tag{37}$$

We acquire the fact that

$$\mathcal{P}_{nl\mu} - G - G^\top \geq -G(\mathcal{P}_{nl\mu})^{-1}G^\top, \tag{38}$$

which together with (37) leads to

$$\begin{bmatrix} \Gamma^1_{n\mu} & \Gamma^2_{pq\mu} G^\top & \Gamma^3_{pq\mu} \\ * & -G(\mathcal{P}_{nl\mu})^{-1}G^\top & 0 \\ * & * & -\mathcal{I} \end{bmatrix} < 0. \tag{39}$$

Pre- and post-multiplying (39) by $\mathrm{diag}\{I, I_{l_k}, I, I, G, G, G, G, I, I, I\}$ and its transpose, it can be concluded that the inequalities (30) and (31) are satisfied from (11) and (12). □

IV. A NUMERICAL EXAMPLE

To substantiate the effectiveness of the established results, the practical mass-spring-damping model (MSDM) from [25] is provided, and its dynamic equation is depicted as:

$$\mathscr{M}\ddot{x} + \mathscr{F}_f + \mathscr{F}_s = u(t), \tag{40}$$

where

$\mathscr{M}$: the mass
$\mathscr{F}_f$: the friction force
$\mathscr{F}_s$: the restoring force
$u(t)$: the control input

Furthermore, $\mathscr{F}_f$ and $\mathscr{F}_s$ are, respectively, formed by $\mathscr{F}_f = c\dot{x}$ and $\mathscr{F}_s = k(1 + a^2x^2)x$, where x means the displacement of point, scalars $c > 0$, k and a. Accordingly, equation (40) is inferred as

$$\mathscr{M}\ddot{x} + c\dot{x} + kx + ka^2x^3 = u(t).$$

In general, defining $x(t) = [x_1^\top(t)\ x_2^\top(k)]^\top = [x^\top\ \dot{x}^\top]^\top$, letting $x_1(t) \in [-2, 2]$, $M = 2kg$, $c = 2N \cdots n/s$, $k = 5N/m$, $a = 0.2m^{-1}$. Here, $T = 0.03s$ is sampling time. Thus, the MSDM can be approximated by the following fuzzy model:

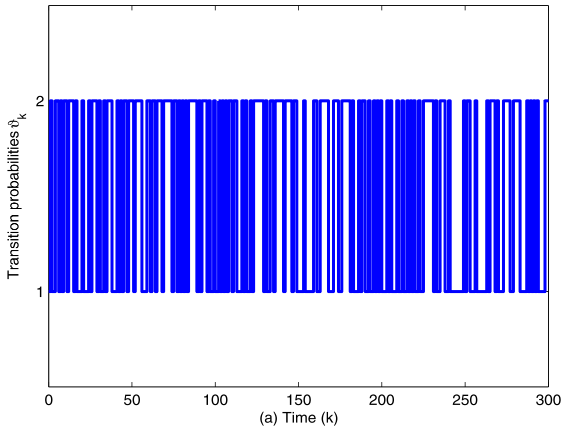

FIGURE 1. The nonhomogeneous MC ϑ_k.

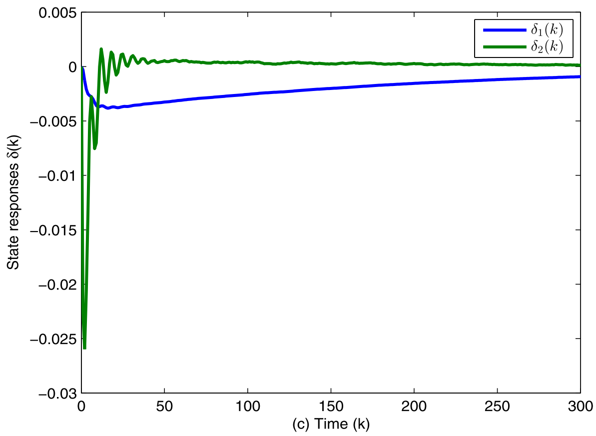

FIGURE 2. State response of $\delta(k)$.

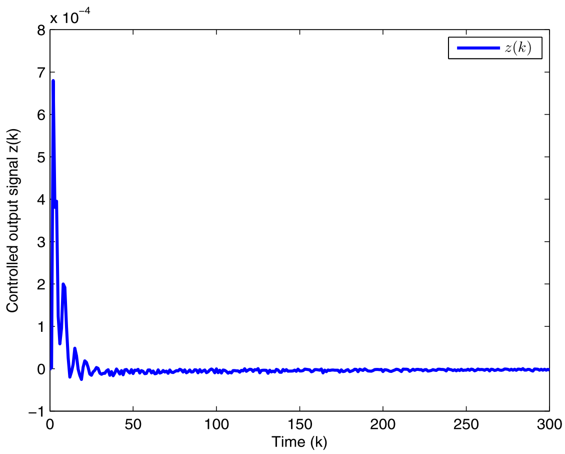

FIGURE 3. Controlled output signal $z(k)$.

Rule p: IF $\zeta_1(k)$ is M_{p1}, and $\cdots$ and $\zeta_r(k)$ is M_{pr}, Then

$$\begin{cases} \delta(k+1) = (A_p + \alpha(k)\Delta A_p)\delta(k) + B_p u(k) + D_p\omega(k), \\ y(k) = C_p\delta(k), \\ z(k) = F_p\delta(k) + H_p u(k), \end{cases}$$

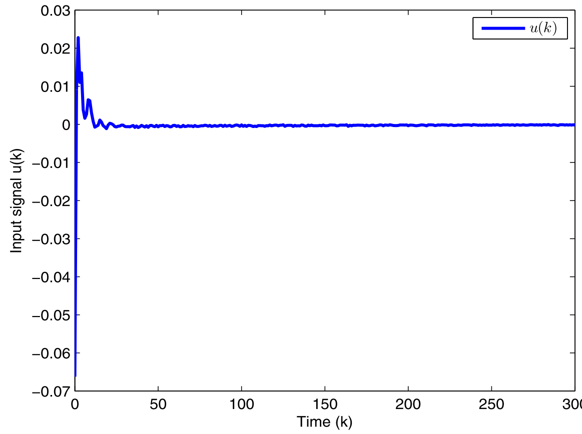

FIGURE 4. Input signal $u(k)$.

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ \dfrac{-k - kax_1^2(k)}{\mathscr{M}} & -\dfrac{c}{\mathscr{M}} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ \dfrac{-k}{\mathscr{M}} & -\dfrac{c}{\mathscr{M}} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}.$$

More specifically, the membership functions are chosen as $h_1(\delta_1(k)) = (\delta_1^2/4)$, $h_2(\delta_1(k)) = 1 - h_1(\delta_1(k))$. The aforementioned matrices are improved:

$$\begin{aligned} D_1 &= \begin{bmatrix} 0.0042 \\ -0.0036 \end{bmatrix}, \quad C_1 = [-0.0826\ 1.00], \\ F_1 &= [0.0017\ -0.0058], \quad H_1 = -0.0033, \\ D_2 &= \begin{bmatrix} 0.0004 \\ -0.0080 \end{bmatrix}, \quad C_2 = [-0.0502\ 1.00], \\ F_2 &= [-0.0020\ -0.0061], \quad H_2 = 0.0228, \\ &M_p[0.02\ 0.01]^\top, \quad N_p = [0.01\ 0.01],\ (p = 1, 2). \end{aligned}$$

Other scalars are chosen as $\overline{\alpha} = 0.5$, $(\alpha^*)^2 = 0.25$, $\overline{\beta}_0 = 0.8$, $\overline{\beta}_1 = 0.6$, $\overline{\beta}_2 = 0.3$, $(\beta_0^*)^2 = 0.012$, $(\beta_1^*)^2 = 0.026$, $(\beta_2^*)^2 = 0.03$. Meanwhile, the TP matrices Π^s $(s = 1, 2)$ are taken as

$$\Pi^1 = \begin{bmatrix} 0.4 & 0.6 \\ 0.85 & 0.15 \end{bmatrix}, \quad \Pi^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}.$$

By solving the LMIs in Theorem 2, one gets the parameters of desired fuzzy controller gains:

$$K_1 = -1.0264, \quad K_2 = -1.0001.$$

Letting $x(0) = [0\ 0]^\top$, $\omega(k) = \exp(-0.1k)\sin(k)$ and $\nu(k) = 0.2\exp(-k^2)$, the nonhomogeneous MC ϑ_k is depicted in Fig. 1. Added by the aforementioned controller gains, simulation results are revealed in Figs. 2-4. The resulting state trajectory $\delta(k)$ is shown by Fig. 2, the control output signal is given in Fig. 3 and control input signal is presented in Fig. 4.

V. CONCLUSION

In this study, an SOFC for nonlinear systems with FCs and AAs has been explored using the T-S fuzzy model. Both FCs and AAs are determined by a set of stochastic variables. An AA is described by a nonhomogeneous MC that accounts for the characterization of time-varying TPs. FCs are characterized by the l-order Rician fading model. By applying the Lyapunov-Krasovskii function and PDC technique, sufficient conditions are established, and controller gains are developed. Finally, the practical MSDM is expressed to verify the practicability of the theoretical results. In the near future, our attention will be shifted to sliding mode control scheme for complexity with FC [27], [28].

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