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Infinite and Finite Time-Frequency Interval Based Variants of Second-Order Balanced Truncation for Stable and Unstable Systems

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ABSTRACT The first half of this manuscript deals with a review of different variants for second-order balanced truncation techniques for stable second-order form systems. Later, multiple nonexisting extensions of model order reduction techniques for stable and unstable second-order form systems are proposed. The framework for gramians based frequency or time (or both) limited model order reduction is presented. Comments on the preservation of large order system properties in the reduced-order model (ROM) that include structure and stability are discussed, and consequent conditions for the preservation of these properties in ROM are stated. The proposed techniques are tested on multiple benchmark/self-generated examples for successful validation of presented developments. The superiority of proposed extensions over existing techniques is also validated. The presented study can be utilized as a resource for infinite and finite interval model order reduction applications of continuous or discrete, stable, or unstable second-order form systems.

INDEX TERMS Second-order systems, model order reduction, frequency-limited gramians, time-limited gramians, unstable systems, Hankel singular values.

I. INTRODUCTION

Mathematical modeling of the physical system plays a pivotal role in the analysis and design of dynamical systems. Researchers often encounter complex and large-scale second-order form systems (SOSs) in applications like nanosystems, electric circuits, electro-mechanical systems [1]–[5], etc. The number of state equations and state variables for such applications is often large (in thousands or millions) that makes it difficult or even impossible to handle system complexity for real-time practical applications due to computational, cost, or storage limitations. Here comes the need for model order reduced (MOR) techniques that approximate the behavior of a large system with lower-order manageable models called reduced-order models (ROMs). MOR techniques ensure the preservation of certain properties of the original system like passivity, stability, structure, etc. A review of MOR schemes for standard state space systems for an infinite interval, frequency-limited, or time-limited intervals is provided in [6]. Moreover, a review regarding MOR techniques based on Krylov and Arnoldi methods have been presented in [7] for standard state-space forms. Balanced truncation yields stable ROM as well as provides a *priori* error bounds but it needs the original system (or portion of the original system) to be stable [8]–[11]. Moreover, in balanced truncation, the unique solution of Lyapunov equations is only possible when the original system is stable [12].

Regarding stable SOSs, the concept of second-order balanced truncation (SOBT) was presented in [10]. [10] proposed the reduction scheme by transforming SOS representation into first-order generalized form with the requirement of structure preservation in ROM. If generalized balanced truncation techniques do not preserve SOS structure, the obtained ROM loses physical interpretation [13], [14], and the simulation tools available for SOSs simulation, analysis, or design of SOSs can not be applied.

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Reference [10] ensured the structure preservation in ROM by partitioning gramians into position and velocity portions. The position gramian caters for the first derivative state(s) and velocity gramian take care of retention of the corresponding second derivative state(s). Reference [10] introduced four schemes for structure-preserving MOR of SOSs by balancing position and velocity gramian with a certain combination. Apart from [10], multiple other articles that discussed the balanced truncation schemes for generalized systems include [14]–[19]. Later, [20] and [21] extended SOBT techniques of [10] for frequency-limited applications of continuous and discrete-time SOSs. For frequency or time-limited applications like filter design, controller design, signal reconstruction, etc., the ROM performance is emphasized for the required interval only and ROM performance for the remaining interval is not of interest. Frequency-limited gramians and corresponding frequency-limited continuous-time algebraic Lyapunov equations (ALEs) were proposed and conditions for stability preservation in ROM were stated in [20]. This framework applicable to continuous time SOSs, therefore, frequency-limited model reduction techniques for discrete-time SOSs were presented in [21] as an extension. Further, for time-limited interval applications of continuous-time SOSs, [22] presented the extension to time-limited gramians, time-limited continuous ALEs, and conditions for stability preservation in ROM. All the limited interval techniques discussed above were certified for correct development for multiple illustrative examples that included large order benchmark examples from [23] or self-generated examples. Similar examples for reviewed techniques as well as for our proposed extensions have been utilized in this article.

Prior discussed methods can be applied to stable SOSs. As mentioned earlier, when the original system is unstable, the balanced truncation process halts, and the solution of algebraic Lyapunov equations becomes non-unique. In this context, multiple MOR techniques for unstable standard and descriptor system forms have been presented in [15], [24]–[30], etc. for unstable systems, but these techniques do not apply to unstable SOSs, because SOSs require structure preservation and its dynamics are differently interpreted. Moreover, existing frequency-limited MOR techniques are not directly applicable to unstable SOSs due to the unsolvability of ALEs. Therefore to cater to these constraints, we propose multiple techniques for MOR of unstable SOSs in the second part of this article. The first proposed technique decomposes the unstable system into stable and unstable parts. Balanced truncation is applied to the stable part and the reduced system is augmented with an unstable part to obtain the overall ROM. The ROM obtained from this technique does not provide structure preservation that results in a blind approximation of original unstable SOS. To avoid misinterpretation, secondly, two structure-preserving SOBT techniques for unstable SOSs are proposed. The system is first stabilized using the Bernoulli feedback stabilization procedure and gramians are computed for the stabilized system. As the second technique retains second order structure, as well as involves, stabilized system dynamics, this technique far closely approximates original system behavior. Finally, the extension of these MOR techniques for frequency or time (or both) limited MOR applications of unstable SOSs are proposed. Proposed techniques are tested on multiple systems and results certify the stated superiority and correct development respectively. The paper is organized as follows: Section II presents a review on stable continuous-time SOSs, frequency/time, or combined time-frequency gramians and corresponding continuous ALEs followed by a similar discussion for discrete-time SOSs in section III. Section IV discusses SOBT techniques and section V presents proposed MOR schemes for unstable SOSs for infinite interval and frequency/time (or both) limited interval applications. Section VI presents the results and discussions and the paper is concluded.

II. CONTINUOUS-TIME STABLE SECOND-ORDER SYSTEMS

The continuous linear time-invariant SOS is given in (1).

$$\begin{aligned} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) &= B_2u(t) \\ C_2\dot{x}(t) + C_1x(t) &= y(t) \end{aligned} \quad (1)$$

where $D \in \mathbb{R}^{n\times n}, M \in R^{n\times n}, K \in \mathbb{R}^{n\times n}, B_2 \in \mathbb{R}^{n\times m}$, C_2 and $C_1 \in \mathbb{R}^{p\times n}, x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$, n is system order, m is number of input(s), p is number of output(s) of the system.

The coefficients M and D represent the second and first-order derivative of state(s), respectively. The existence of this pair of the first and second-order state make system structure second order. ROM form of system (1) is given in (2).

$$\begin{aligned} M_r\ddot{x}_r(t) + D_r\dot{x}_r(t) + K_rx_r(t) &= B_{2r}u(t) \\ C_{2r}\dot{x}_r(t) + C_{1r}x_r(t) &= y_r(t) \end{aligned} \quad (2)$$

where $M_r \in \mathbb{R}^{r\times r}, D_r \in \mathbb{R}^{r\times r}, K_r \in \mathbb{R}^{r\times r}, B_{2r} \in \mathbb{R}^{r\times m}$, C_{1r} and $C_{2r} \in \mathbb{R}^{p\times r}$, $x_r(t) \in \mathbb{R}^r, u(t) \in \mathbb{R}^m$ and $y_r(t) \in \mathbb{R}^p$, and $r \ll n$.

Generalized first-order state-space form of (1) can be re-written as (3).

$$\begin{aligned} E\dot{q} &= Aq(t) + Bu(t) \\ y(t) &= Cq(t) \end{aligned} \quad (3)$$

with

$$q(t) = \begin{bmatrix} x(t)^T & \dot{x}(t)^T \end{bmatrix}, \quad E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix},$$
$$A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \; C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

and the corresponding ROM (2) in the generalized arrangement becomes as (4).

$$\begin{aligned} E_r\dot{q}_r &= A_rq_r(t) + B_ru(t) \\ y_r(t) &= C_rq_r(t) \end{aligned} \quad (4)$$

For SOS (1), the transfer function is given in (5).

$$G(s) = (sC_2 + C_1)(s^2M + sD + K)^{-1}B_2 \quad (5)$$

or written in short as $G = [M, D, K, B_2, C_1, C_2]$.

System (1) is stable if all the zeros of $P(\lambda) = \lambda^2 M + \lambda D + K$ have negative real parts (i.e lie in the left half-plane) or unstable if any of zeros of $P(\lambda) = \lambda^2 M + \lambda D + K$ have positive real parts (i.e lie in the right half-plane). The conditions of observability as well as controllability for system (1) and (3) are described in [10] in detail.

Remark: In first order form (3), augmentation of matrices require the arrangement of matrices M, D, K to be preserved in ROM (4). If matrix entries in the above-given matrices get mess up, the system loses SOS interpretation that results in a blind and poor approximation of the original system.

If system (1) is stable, observability and controllability gramians for (3) given by:

$$G_{c\Omega} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(j\Omega E - A)^{-1}BB^T(-j\Omega E - A)^{-T}d\Omega \quad (6)$$
$$G_{o\Omega} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(-j\Omega E - A)^{-T}C^TC(j\Omega E - A)^{-1}d\Omega \quad (7)$$

are the positive semidefinite, symmetric and unique solutions of the generalized continuous-time ALEs (8) and (9).

$$EG_{c\Omega}A^T + AG_{c\Omega}E^T = -BB^T \quad (8)$$
$$E^TG_{o\Omega}A + A^TG_{o\Omega}E = -C^TC \quad (9)$$

Equations (8) and (9) yield the positive definite solution for a stable system when the right side of the equations is positive definite. These equations are the frequency domain representation of the gramians. As we further discuss frequency/time-limited versions of SOSs for a limited interval applications, therefore it seems suitable to present the time domain representation of the same gramians.

Equivalently, if system (1) is stable, the gramians for system (3) given by:

$$G_{ct} = \int_0^{\infty} F_s(t)BB^TF_s^T(t)dt \quad (10)$$
$$G_{ot} = \int_0^{\infty} F_s^T(t)C^TCF_s(t)dt \quad (11)$$

are the unique, positive semidefinite and symmetric solutions of the continuous generalized ALEs:

$$EG_{ct}A^T + AG_{ct}E^T = -BB^T \quad (12)$$
$$E^TG_{ot}A + A^TG_{ot}E = -C^TC \quad (13)$$

where $F_s(t) = e^{E^{-1}At}E^{-1}$ is the fundamental solution matrix of (3). For time or frequency (or both) limited applications of stable SOSs, the limited interval gramians and corresponding ALEs are defined next.

A. FREQUENCY-LIMITED GRAMIANS

To emphasize MOR of SOS (1) in the desired frequency interval, the continuous-time frequency-limited observability and controllability gramians are given in (14) and (15) [20].

$$G_{c\delta\Omega} = \frac{1}{2\pi}\int_{\delta\Omega}(j\Omega E - A)^{-1}BB^T(-j\Omega E - A)^{-T}d\Omega \quad (14)$$
$$G_{o\delta\Omega} = \frac{1}{2\pi}\int_{\delta\Omega}(-j\Omega E - A)^{-T}C^TC(j\Omega E - A)^{-1}d\Omega \quad (15)$$

where $\delta\Omega = [-\Omega_2, -\Omega_1]U[\Omega_1, \Omega_2]$ is the continuous symmetric frequency interval that guarantees the frequency-limited gramians $G_{c\delta\Omega}$ and $G_{o\delta\Omega}$ are real, symmetric and positive definite. When $\Omega_1 = -\infty$ and $\Omega_2 = +\infty$, gramians in (14) and (15) become same as in (6) and (7), respectively.

Take note in (14) and (15) that integrals are evaluated for the required interval instead of infinite interval. This establishes the optimization of ROM performance over the required interval. (Same is true for time or combined time-frequency interval scenario). The frequency-limited gramians (14) and (15) are the unique, positive semidefinite and symmetric solutions of the frequency-limited continuous ALEs (16) and (17), respectively.

$$EG_{c\delta\Omega}A^T + AG_{c\delta\Omega}E^T = -ES_cBB^T - BB^TS_c^TE^T \quad (16)$$
$$E^TG_{o\delta\Omega}A + A^TG_{o\delta\Omega}E = -E^TS_c^TC^TC - C^TCS_cE \quad (17)$$

where $E,\ A,\ B$ and C possess second-order structure and

$$S_c = \frac{1}{2\pi}\int_{\delta\Omega}(j\Omega E - A)^{-1}d\Omega$$

B. TIME-LIMITED GRAMIANS

To emphasize the performance of ROM in the required time interval for time-limited applications, time-limited controllability and observability gramians are defined as [22]:

$$G_{c\delta T} = \int_{\delta T} F_s(t)BB^TF_s^T(t)dt = \int_0^{\infty} F_s(t)X_cF_s^T(t)dt \quad (18)$$
$$G_{o\delta T} = \int_{\delta T} F_s^T(t)C^TCF_s(t)dt = \int_0^{\infty} F_s^T(t)Y_oF_s(t)dt \quad (19)$$

where [22]

$$X_c = F_s(t_1)BB^TF_s^T(t_1) - F_s(t_2)BB^TF_s^T(t_2)$$
$$Y_o = F_s^T(t_1)C^TCF_s(t_1) - F_s^T(t_2)C^TCF_s(t_2),$$

$\delta T = [t_1, t_2]$, $t_2 > t_1 \geq 0$, $G_{c\delta T}$ and $G_{o\delta T} \geq 0$ if $t_2 \geq t_1$, and

$$G_{c\delta T} = G_{ct_1} - G_{ct_2}$$
$$G_{o\delta T} = G_{ot_1} - G_{ot_2}$$

The time-limited gramians $G_{c\delta T}$ and $G_{o\delta T}$ of (18) and (19) are respectively the unique, positive semidefinite and symmetric solutions of the continuous ALEs (20) and (21), respectively.

$$EG_{c\delta T}A^T + AG_{c\delta T}E^T + X_c = 0 \quad (20)$$
$$E^TG_{o\delta T}A + A^TG_{o\delta T}E + Y_o = 0 \quad (21)$$

C. COMBINED TIME AND FREQUENCY INTERVAL GRAMIANS (PROPOSED EXTENSION-I)

For applications where ROM response optimization for limited frequency and time interval is simultaneously required, the concept of combined time-frequency interval gramians is utilized. The limited time-frequency intervals controllability gramian $G_{c(\delta T\Omega)}$ in the intervals $\delta\Omega = [\Omega_1, \Omega_2]$ and $\delta T = [t_1, t_2]$ is the solution of the algebraic Lyapunov equation (22).

$$EG_{c(\delta T\Omega)}A^T + AG_{c(\delta T\Omega)}E^T + X_{c(\delta T\Omega)} = 0 \tag{22}$$

where .* represent the conjugate and

$$\begin{aligned} X_{c(\delta T\Omega)} &= X_c(t_1, \delta\Omega) - X_c(t_2, \delta\Omega) \\ X_c(t, \delta\Omega) &= X_c(t, \Omega_2) - X_c(t, \Omega_1) \\ X(t, \Omega) &= X(t)S^*(\Omega) + S(\Omega)X(t) = S(t)X(\Omega)S^*(t) \\ X(\Omega) &= S(\Omega)BB^T + BB^T S^*(\Omega) \\ X(t) &= S(t)BB^T S^*(t) \\ S(\Omega) &= \frac{j}{2\pi}\ln((j\Omega I + A)(-j\Omega I + A)^{-1}) \\ S(t) &= e^{At} \end{aligned}$$

Note in (22) that gramians not only involve emphasis in frequency but also in time simultaneously that lead to our first non-existing proposition for SOSs. Similarly, the limited time-frequency intervals observability gramian $G_{o(\delta T\Omega)}$ in the intervals $\delta\Omega = [\Omega_1, \Omega_2]$ and $\delta T = [t_1, t_2]$ is the solution of the algebraic Lyapunov equation:

$$E^T G_{o(\delta T\Omega)}A + A^T G_{o(\delta T\Omega)}E + Y_{o(\delta T\Omega)} = 0 \tag{23}$$

where

$$\begin{aligned} Y_{o(\delta T\Omega)} &= Y_o(t_1, \delta\Omega) - Y_o(t_2, \delta\Omega) \\ Y_o(t, \delta\Omega) &= Y_o(t, \Omega_2) - Y_o(t, \Omega_1) \\ Y_o(t, \Omega) &= Y_o(t)S(\Omega) + S^*(\Omega)Y_o(t) = S^*(t)Y_o(\Omega)S(t) \\ Y_o(\Omega) &= S^*(\Omega)BB^T + BB^T S^(\Omega) \\ Y_o(t) &= S^*(t)BB^T S(t) \end{aligned}$$

III. DISCRETE-TIME STABLE SECOND-ORDER SYSTEMS

The discrete-time SOS is given by (24).

$$\begin{aligned} M_d x[k+2] + D_d x[k+1] + K_d x[k] &= B_{d2}u[k] \\ C_{d2}x[k+1] + C_{d1}x[k] &= y[k] \end{aligned} \tag{24}$$

and the generalized SOS can be written as:

$$\begin{aligned} E_d q[k+1] &= A_d q[k] + B_d u[k] \\ y[k] &= C_d q[k] \end{aligned} \tag{25}$$

and its reduced form can be written as:

$$\begin{aligned} E_{dr} q_r[k+1] &= A_{dr} q_r[k] + B_{dr} u[k] \\ y_r[k] &= C_{dr} q_r[k] \end{aligned} \tag{26}$$

where k is the discrete-time index. For a stable system (24) (having all the eigenvalues of the pencil $\lambda E - A_d$ within the unit circle), gramians given by [21]:

$$W_{c\omega} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{j\omega}E_d - A_d)^{-1}B_d B_d^T(e^{-j\omega}E_d - A_d)^{-T}d\omega \tag{27}$$

$$W_{o\omega} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-j\omega}E_d - A_d)^{-T}C_d^T C_d(e^{j\omega}E_d - A_d)^{-1}d\omega \tag{28}$$

are the positive semidefinite, unique and symmetric solutions of the generalized discrete-time ALEs:

$$A_d W_{c\omega} A_d^T - E_d W_{c\omega} E_d^T = -B_d B_d^T \tag{29}$$
$$A_d^T W_{o\omega} A_d - E_d^T W_{o\omega} E_d = -C_d^T C_d \tag{30}$$

Similarly, the discrete-time gramians in time domain representation written as (31) and (32):

$$W_{ck} = \sum_{k=0}^{\infty} F_d^k B_d B_d^T (F_d^T)^k \tag{31}$$

$$W_{ok} = \sum_{k=0}^{\infty} (F_d^T)^k C_d^T D_d F_d \tag{32}$$

are the positive semidefinite, unique and symmetric solutions of the generalized discrete-time ALEs:

$$A_d W_{ck} A_d^T - E W_{ck} E_d^T = -B_d B_d^T \tag{33}$$
$$A_d^T W_{ot} A_d - E_d^T W_{ok} E_d = -C_d^T C_d \tag{34}$$

where $F_d = e^{E_d^{-1}A_{dk}}E_d^{-1}$ is the fundamental solution matrix of (26).

A. DISCRETE FREQUENCY-LIMITED GRAMIANS

Discrete-time frequency-limited gramians for (24) given by (35) and (36) [21].

$$W_{c\delta\omega} = \frac{1}{2\pi}\int_{-\omega}^{\omega}(e^{j\omega}E_d - A_d)^{-1}B_d B_d^T(e^{-j\omega}E_d - A_d)^{-T}d\omega \tag{35}$$

$$W_{o\delta\omega} = \frac{1}{2\pi}\int_{-\omega}^{\omega}(e^{-j\omega}E_d - A_d)^{-T}C_d^T C_d(e^{j\omega}E - F)^{-1}d\omega \tag{36}$$

are the solution of discrete-time ALEs (37) and (38).

$$A_d W_{c\delta\omega} A_d^T - E_d W_{c\delta\omega} E_d^T = -B_d B_d^T S_d^* - S_d B_d B_d^T \tag{37}$$
$$A_d^T W_{o\delta\omega} A_d - E_d^T W_{o\delta\omega} E_d = -C_d^T C_d S_d - S_d^* C_d^T C_d \tag{38}$$

where $S_d = \frac{-(\omega_2-\omega_1)I}{4\pi} + E_d S_{d1}$ and $S_{d1} = \frac{1}{2\pi}\int_{\delta\omega}(E_d - A_d e^{-j\omega})^{-1}d\omega$, $\delta\omega = [-\omega_2, -\omega_1]U[\omega_1, \omega_2]$ is the discrete symmetric frequency interval.

B. DISCRETE TIME-LIMITED GRAMIANS

The discrete time-limited controllability and observability gramians are defined as (39) and (40).

$$W_{c\delta k} = \sum_{i=k_1}^{k_2} F_d^i B_d B_d^T (F_d^T)^i = \sum_{i=0}^{\infty} F_d^i X_c {F_d^T}^i \tag{39}$$

$$W_{o\delta k} = \sum_{i=k_1}^{k_2} (F_d^T)^i C_d^T C_d F_d^i = \sum_{i=0}^{\infty} {F_d^T}^i Y_o F_d^i \tag{40}$$

The controllability gramian $W_{c\delta k}$ and observability gramian $W_{o\delta k}$ are the solutions of time-limited ALEs (41) and (42).

$$A_d W_{c\delta k} A_d^T - E_d W_{c\delta k} E_d^T + X_c = 0 \quad (41)$$
$$A_d W_{o\delta k} A_d^T - E_d W_{o\delta k} E_d^T + Y_o = 0 \quad (42)$$

where $X_c = F_d^{k_1} B_d B_d^T (F_d^T)^{k_1} - F_d^{k_2} B_d B_d^T (F_d^T)^{k_2}$ and $Y_o = (F_d^{k1})^T C_d^T C_d (F_d^{k1}) - (F_d^{k2})^T C_d^T C_d (F_d^{k2})$.

C. COMBINED DISCRETE TIME-FREQUENCY INTERVAL GRAMIANS (PROPOSED EXTENSION-II)

For discrete-time SOS applications involving ROM performance optimization over both time and frequency interval simultaneously, the time-frequency limited control-lability gramian $W_c(\delta k\omega)$ in the intervals $\delta\omega = [\omega_1, \omega_2]$ and $\delta k = [k_1, k_2]$ is the solution of the algebraic Lyapunov equation:

$$A_d W_c(\delta k\omega) A_d^T - E_d W_c(\delta k\omega) E_d^T + X_c(\delta k\omega) = 0 \quad (43)$$

where

$$\begin{aligned}
X_c(\delta k\omega) &= X_c(k_1, \delta\omega) - X_c(k_2, \delta\omega) \\
X_c(k, \delta\Omega) &= X_c(k, \omega_2) - X_c(k, \omega_1) \\
X(k, \omega) &= X(k)S^*(\omega) + S(\omega)X(k) = S(k)X(\omega)S^*(k) \\
X(\omega) &= S(\omega)B_d B_d^T + B_d B_d^T S^*(\omega) \\
X(k) &= S(k)B_d B_d^T S^*(k) \\
S(\omega) &= \frac{j}{2\pi} \ln((j\omega E_d + A_d)(-j\omega E_d + A_d)^{-1}) \\
S(t) &= e^{A_d k}
\end{aligned}$$

Note in (43) that gramians not only involve emphasis in frequency but also in time simultaneously that lead to our second non-existing proposition for SOSs. Similarly, the time-frequency interval observability gramian $W_o(\delta k\omega)$ in the intervals $\delta\omega = [\omega_1, \omega_2]$ and $\delta k = [k_1, k_2]$ is the solution of the algebraic Lyapunov equation .

$$A_d^T W_o(\delta k\omega) A_d - E_d^T W_o(\delta k\omega) E_d + Y_o(\delta k\omega) = 0 \quad (44)$$

where

$$\begin{aligned}
Y_o(\delta k\omega) &= Y_o(k_1, \delta\omega) - Y_o(k_2, \delta\omega) \\
Y_o(k, \delta\omega) &= Y_o(k, \omega_2) - Y_o(k, \omega_1) \\
Y_o(k, \omega) &= Y_o(k)S(\omega) + S^*(\omega)Y_o(k) = S^*(k)Y_o(\omega)S(k) \\
Y_o(\omega) &= S^*(\omega)B_d B_d^T + B_d B_d^T S(\omega) \\
Y_o(k) &= S^*(k)B_d B_d^T S(k)
\end{aligned}$$

IV. BALANCED TRUNCATION FOR STABLE SECOND-ORDER SYSTEMS

Multiple balanced truncation techniques for MOR of stable SOS have been presented in [10] that preserve SOS structure in ROM by partitioning the gramians into position and velocity portions are given below.

$$G_c = \begin{bmatrix} G_{pc} & G_{12c} \\ G_{12c}^T & G_{vc} \end{bmatrix}, \quad G_o = \begin{bmatrix} G_{po} & G_{12o} \\ G_{12o}^T & G_{vo} \end{bmatrix}$$

where G_{vo}, G_{po}, G_{vc} and G_{pc} are velocity and position observability and controllability gramians, respectively.

(**Note**: A generic notation for the continuous system is used in this section, and the same relations are valid for a discrete-time as well.)

Definition 1: Consider a stable SOS.

1. The eigenvalues of product $G_{pc}G_{po}$ when square rooted, yields the position Hankel singular values (HSVs).
2. The eigenvalues of product $G_{vc}G_{vo}$ when square rooted, yields the velocity HSVs.
3. The eigenvalues of product $G_{pc}G_{vo}$ when square rooted, yield position-velocity HSVs.
4. The eigenvalues of product $G_{vc}G_{po}$ when square rooted, yields the velocity-position HSVs.

The product of position and velocity gramians yields the following definition.

Definition 2: The stable SOS is called:

1. Position balanced if $G_{pc} = G_{po} = diag(\zeta_1^p, \ldots, \zeta_n^p)$.
2. Velocity balanced if $G_{vc} = G_{vo} = diag(\zeta_1^v, \ldots, \zeta_n^v)$.
3. Position-velocity balanced if $G_{pc} = G_{vo} = diag(\zeta_1^{pv}, \ldots, \zeta_n^{pv})$.
4. Velocity-position balanced if $G_{vc} = G_{po} = diag(\zeta_1^{vp}, \ldots, \zeta_n^{vp})$.

where ζ represents either velocity or position or both velocity-position (position-velocity) HSVs arranged in descending order.

Complete theoretical background, derivation and proofs of the above statements and forthcoming algorithms can be found in [10]. Balanced transformations for Definition 2 have been derived in [10] that yield HSVs, whose magnitudes depict the level of involvement of position and velocity states in system dynamics. States corresponding to small magnitudes are least involved and are truncated at a small reduction error cost. To elaborate the truncation steps in brief, in following, two algorithms are presented. In Algorithm 1, the transformation matrix is selected such that:

$$G_{pc} = G_{po} = G_{vo} = diag(\zeta_1^p, \ldots, \zeta_r^p)$$

and in Algorithm 2, position controllability gramian is balanced with velocity observability gramian.

Remark: The balanced transformation (P_l, P_r) used in Algorithm 1 and 2 for frequency or time-limited applications are called double-sided transformation which may make right-side terms of frequency/time (or both) limited ALEs indefinite. Thus, the stability of ROM is not guaranteed in these cases. However, ROM stability is guaranteed for infinite interval applications [10].

To assure stability of the ROM, single-sided transformation may be used that ensures the right-hand side terms of limited interval ALEs to remain positive definite. In single-sided transformation, one of the gramians for balancing is selected over a limited interval, while the other gramian is selected for infinite interval as discussed below.

V. BALANCED TRUNCATION FOR UNSTABLE SECOND-ORDER SYSTEMS (PROPOSED TECHNIQUES)

For unstable SOSs, obtaining a solution for gramians using ALEs becomes impossible as ALEs get unsolvable for

Algorithm 1 Position Balanced SOBT (SOBTp)**

Input: Given a stable large scale SOS (1).
Output: The ROM (2).
1. Calculate the gramians G_{pc}, G_{po}, G_{vc} and G_{vo} using ALEs.
2. Calculate the Cholesky factors R_p, R_v, L_p and L_v of gramians as in (11).
3. Compute singular value decomposition (SVD) for the products:

$$R_p^T L_p = \begin{bmatrix} U_{p1} & U_{p2} \end{bmatrix} \begin{bmatrix} \Sigma_{p1} & 0 \\ 0 & \Sigma_{p2} \end{bmatrix} \begin{bmatrix} V_{p1} & V_{p2} \end{bmatrix}^T$$

$$R_v^T M^T L_v = \begin{bmatrix} U_{v1} & U_{v2} \end{bmatrix} \begin{bmatrix} \Sigma_{v1} & 0 \\ 0 & \Sigma_{v2} \end{bmatrix} \begin{bmatrix} V_{v1} & V_{v2} \end{bmatrix}^T$$

where the matrices $\begin{bmatrix} U_{p1} & U_{p2} \end{bmatrix}$, $\begin{bmatrix} V_{p1} & V_{p2} \end{bmatrix}$, $\begin{bmatrix} U_{v1} & U_{v2} \end{bmatrix}$ and $\begin{bmatrix} V_{v1} & V_{v2} \end{bmatrix}$ are orthogonal and

$$\Sigma_{p1} = diag(\zeta_1^p, \ldots, \zeta_r^p) \quad \Sigma_{p2} = diag(\zeta_{r+1}^p, \ldots, \zeta_n^p)$$
$$\Sigma_{v1} = diag(\zeta_1^v, \ldots, \zeta_r^v) \quad \Sigma_{v2} = diag(\zeta_{r+1}^v, \ldots, \zeta_n^v)$$

4. Calculate the ROM

$$M_r = P_l^T M P_r \quad D_r = P_l^T D P_r \quad K_r = P_l^T K P_r$$
$$B_{2r} = P_l^T B_2 \quad C_{1r} = C_1 P_r \quad C_{2r} = C_2 P_r$$

where $P_l = L_v V_{v1} \Sigma_{p1}^{-1/2}$ and $P_r = R_p U_{p1} \Sigma_{p1}^{-1/2}$.

Algorithm 2 Position-Velocity Balanced SOBT (SOBTpv)**

Input: Given a stable large scale SOS (1).
Output: The ROM (2).
1. Compute the gramians G_{pc} and G_{vo} utilizing ALEs.
2. Compute the Cholesky factors R_p and L_v of gramians G_{pc} and G_{vo}.
3. Calculate SVD for the products:

$$R_p^T M^T L_v = \begin{bmatrix} U_{pv1} & U_{pv2} \end{bmatrix} \begin{bmatrix} \Sigma_{pv1} & 0 \\ 0 & \Sigma_{pv2} \end{bmatrix} \begin{bmatrix} V_{pv1} & V_{pv2} \end{bmatrix}^T$$

where $\begin{bmatrix} U_{pv1} & U_{pv2} \end{bmatrix}$ and $\begin{bmatrix} V_{pv1} & V_{pv2} \end{bmatrix}$ are orthogonal and

$$\Sigma_{pv1} = diag(\zeta_1^{pv}, \ldots, \zeta_r^{pv})$$
$$\Sigma_{pv2} = diag(\zeta_{r+1}^{pv}, \ldots, \zeta_n^{pv})$$

4. Calculate the ROM

$$M_r = I_n \quad D_r = P_l^T D P_r \quad K_r = P_l^T K P_r$$
$$B_{2r} = P_l^T B_2 \quad C_{1r} = C_1 P_r \quad C_{2r} = C_2 P_r$$

where $P_l = L_v V_{pv1} \Sigma_{pv1}^{-1/2}$ and $P_r = R_p U_{pv1} \Sigma_{pv1}^{-1/2}$

unstable SOSs that halt the reduction process. To avoid unsolvability of ALEs, in this section, multiple non-existing techniques for MOR of these systems are proposed.

Algorithm 3 Stability Preserving Limited Interval SOBTp

Input: Given a stable large scale SOS.
Output: The ROM $G_r = [M_r, D_r, K_r, B_{2r}, C_{1r}, C_{2r}]$.
1. Calculate any of the pairs of gramians (G_{pc} and G_{po}) or (G_{pc} and G_{po}) using respective ALEs.
2. Compute single-sided transformation (P_{lsst}, P_{rsst}) by balancing:

$$G_{pc\delta} = G_{po} = diag(\zeta_1^{psst}, \ldots, \zeta_r^{psst})$$

or

$$G_{pc} = G_{po\delta} = diag(\zeta_1^{psst}, \ldots, \zeta_r^{psst})$$

3. Perform steps 1-3 of Algorithm 1.

Algorithm 4 Stability Preserving Limited Interval SOBTpv

Input: Given a stable large scale SOS.
Output: The ROM $G_r = [M_r, D_r, K_r, B_{2r}, C_{1r}, C_{2r}]$.
1. Calculate any of the pairs of gramians (G_{pc} and G_{vo}) or (G_{pc} and G_{vo}) using respective ALEs.
2. Compute single-sided transformation (P_{lsst}, P_{rsst}) by balancing:

$$G_{pc\delta} = G_{vo} = diag(\zeta_1^{pvsst}, \ldots, \zeta_r^{pvsst})$$

or

$$G_{pc} = G_{vo\delta} = diag(\zeta_1^{pvsst}, \ldots, \zeta_r^{pvsst})$$

3. Perform steps 1-3 of Algorithm 2.

A. REDUCTION THROUGH DECOMPOSITION (NON STRUCTURE PRESERVING BALANCED TRUNCATION)

At first, to make ALEs solvable, large order unstable system is decomposed into stable and unstable portions, and ROM for the stable portion is computed. Then obtained ROM is augmented with the unstable portion to yield the required ROM as discussed next.

1) DECOMPOSITION INTO STABLE AND UNSTABLE PORTIONS

Unstable System (3) can be written as (45).

$$\begin{aligned} H_t &= \begin{bmatrix} U^T E U, U^T A U, U^T B, C U, 0 \end{bmatrix} \\ &= \left[\begin{bmatrix} E_{t11} & E_{t12} \\ 0 & E_{t22} \end{bmatrix}, \begin{bmatrix} A_{t11} & A_{t12} \\ 0 & A_{t22} \end{bmatrix}, \begin{bmatrix} B_{t1} \\ B_{t2} \end{bmatrix}, \begin{bmatrix} C_{t1} & C_{t2} \end{bmatrix}, 0 \right] \end{aligned} \tag{45}$$

where U is the orthogonal transformation. In (45), the terms E_{t12} and A_{t12} represent the coupling between the states. Let $X_t = WX$ be a transformation that transforms a system into a decoupled form.

$$\begin{aligned} H_d &= \begin{bmatrix} W^{-1} E_t W, W^{-1} A_t W, W^{-1} B_t, C_t W, 0 \end{bmatrix} \\ &= \left[\begin{bmatrix} E_{11} & 0 \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_{1t} \\ B_{2t} \end{bmatrix}, \begin{bmatrix} C_{1t} & C_{2t} \end{bmatrix}, 0 \right] \end{aligned} \tag{46}$$

The decoupling transformation W allows the linear system decomposition into independent reduced-order subsystems. The complete derivation of different variants of this transformation can be found in [31]. Further, the decomposition of system (46) into stable and unstable portions yields (47).

$$\begin{aligned} H_d &= H_s(\text{Stable System}) + H_u(\text{Unstable System}) \\ &= \left[E_{11}, A_{11}, B_{1d}, C_{1d}, 0\right] + \left[E_{22}, A_{22}, B_{2t}, C_{2t}, 0\right] \end{aligned} \quad (47)$$

2) REDUCTION OF STABLE PORTION

In reduction process of stable subsystem H_s, controllability and observability gramians G_{cs} and G_{os} are obtained for stable portion H_s by solving ALEs. The obtained gramians are invoked in (48) to obtain HSVs.

$$\zeta_i = \sqrt{\lambda_i(G_{cs}G_{os})} \quad (48)$$

The truncation based on magnitudes of these HSVs is applied on stable portion H_s to obtain ROM (H_{rs}) for stable portion.

3) OVERALL REDUCED MODEL FOR UNSTABLE SOS

Reduced stable model and unstable portions are combined to obtain the required ROM.

$$H_r = H_{rs} + H_u \quad (49)$$

Remarks:

1. The unstable system (3) requires the structure preservation in ROM, but the ROM (49) obtained from the proposed technique of Section V-A does not provide structure preservation that results in a blind approximation of original SOS. The second-order dynamics get misinterpreted, and tools designed specifically for SOSs do not apply to this ROM.

2. The augmentation of unstable dynamics in ROM also degrade ROM (49) performance. Also, if an unstable part of the system is large, it becomes impossible to avoid large order ROM, because of the augmentation of the unstable portion.

3. As for limited interval applications, gramians are invoked only for the stable portion. Therefore, the emphasis of interval on overall ROM (49) is not well implemented.

4. The decomposition of system into stable and unstable portions pose a constraint on reduction order, r e.g. the following 9^{th} order unstable discrete-time single input single output (SISO) SOS

$$D = -\begin{bmatrix} 0.8 & 1.2 & 0.7 & -41 & 0.7 & 598.4 & 393.4 & 1442.6 & 1426.2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2450.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$C_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad M = I_9$$

has only one stable eigenvalue. Thus, this system cannot be reduced to any order between $1 < r \leq n$. As a result, an approximation that limited to $r = 1$ exhibits poor ROM performance.

5. If all the eigenvalues of the system are unstable, this technique halts the reduction process. Same is true for benchmark piezoelectric system [32] when considered in continuous domain.

B. REDUCTION THROUGH STABILIZATION (STRUCTURE PRESERVING SOBT)

To make ALEs solvable for unstable SOS (3), Bernoulli feedback stabilization can be applied to stabilize the system, and the stabilized system is used to compute gramians. Thereafter, gramians are partitioned into position and velocity portions to achieve structure preservation, and ROM is computed using SOBT. Bernoulli's feedback solution (theoretic development and proof can be found in [9]) and gramians are computed from (50)-(53).

$$(A - BK_c)G_{cstab}E^T + EG_{cstab}(A - BK_c)^T = -BB^T \quad (50)$$
$$(A - K_oC)^T G_{ostab}E + E^T G_{ostab}(A - K_oC) = -C^T C \quad (51)$$

where $K_c = B^T X_{cs} E$ and $K_o = EX_{os}C^T$ are known as Bernoulli stabilizing feedback matrices, due to the fact that the matrices X_{cs} and X_{os} are the respective stabilizing solutions of the generalized algebraic Bernoulli equations (52) and (53).

$$E^T X_{cs}A + A^T X_{cs}E = A^T X_{cs}BB^T X_{cs}E \quad (52)$$
$$AX_{os}E^T + AX_{os}A^T = EX_{os}C^T CX_{os}E^T \quad (53)$$

Remarks:

1. The gramians are now computed from (50) and (51) using a complete portion of the stabilized system in the proposed technique of Section V-B. Moreover, partitioning of gramians into position and velocity portions after computation ensures the structure preservation in ROM in contrast to the technique of Section V-A. Also, in this technique, there is no need to augment any large order unstable portion.

2. The techniques of section V can be extended to discrete-time systems as described in Section III.

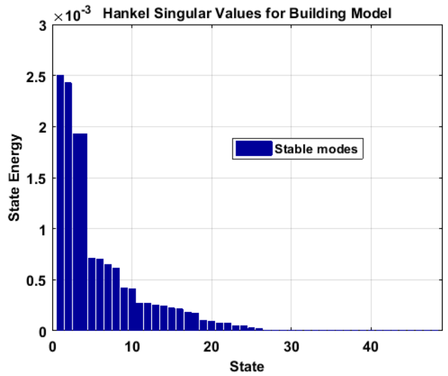

FIGURE 1. Hankel singular values plot for building model.

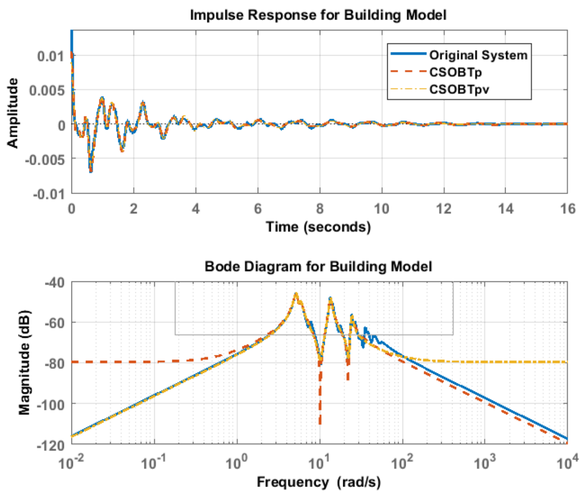

FIGURE 2. Responses for building model for an infinite range.

VI. ILLUSTRATIVE EXAMPLES

The techniques of Sections I-V are applied to multiple benchmark/self-generated examples and results are validated for stable and unstable systems in continuous/discrete domain as discussed below.

A. CONTINUOUS-TIME STABLE SYSTEM

The MOR techniques are first applied to stable continuous 24^{th} order system of building structure available at Slicot database [33]. At first, a guess of suitable reduction order is made by sketching the HSVs plot of this model as shown in Figure 1. By inspecting Figure 1, it is decided to reduce building model to $r = 8$ using techniques continuous SOBTp and continuous SOBTpv for infinite interval. The responses for ROMs are depicted in Figure 2. Figure 2 shows that ROMs comprehensively approximate the original building model response. However, for example, ROM responses in frequency interval, $\delta\Omega = [100, 1000]$ rad/sec are not catching the large scale system response well as it was not emphasized. When required, to emphasize ROM in this frequency interval, frequency-limited continuous SOBTp and SOBTpv techniques are utilized to obtain ROMs. The ROMs responses for frequency-limited techniques are shown in Figure 3.

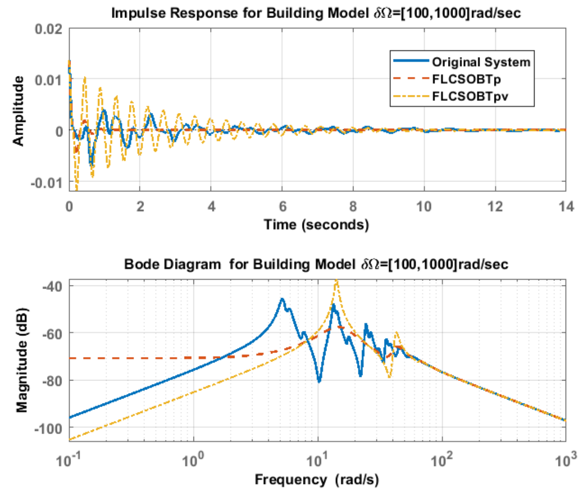

FIGURE 3. Responses for building model for $\delta\Omega = [100, 1000]$ rad/sec.

Figure 3 shows that response in $\delta\Omega = [100, 1000]$ rad/sec now has been well emphasized as compared to Figure 2. Further note in Figure 3 that although responses have been emphasized in $\delta\Omega = [100, 1000]$ rad/sec, but at the same time, responses in time interval $\delta T = [0, 5]$ sec are not up to the mark as simultaneous emphasis in frequency-time was not performed. Therefore, we utilize time-limited continuous SOBTp and SOBTpv techniques to emphasize this time interval and corresponding ROM responses are depicted in Figure 4. Figure 4 shows that the required time interval $\delta T = [0, 5]$ sec has been comprehensively emphasized but correspondingly, the response in frequency interval $\delta\Omega = [100, 1000]$ rad/sec has been degraded again (as simultaneous emphasis in frequency-time was not performed). Finally, to achieve emphasis in both time and frequency intervals simultaneously, combined time-frequency continuous SOBTp and SOBTpv techniques are applied and results are shown in Figure 5. Figure 5 shows that now the responses have been properly emphasized both in required time and frequency intervals. These results show that these proposed structure preserving MOR techniques can be used as a strong tool for MOR of continuous-time stable SOSs for infinite and finite interval applications.

B. DISCRETE-TIME STABLE SYSTEM

Similarly, the HSVs plot for 6^{th} order discrete-time stable system of Example 1 (given below) is shown in Figure 6. The model is reduced to $r = 3$ in the infinite range in the first step using discrete-time SOBTp and SOBTpv techniques and responses are shown in Figure 7.

Example 1: A stable 6^{th} order discrete-time SOS:

$$D_d = \begin{bmatrix} 0.01 & 0.02 & 0.07 & 0.1 & -1 & 0.9 \\ 0.6 & 0.2 & 0.2 & 0.1 & -0.1 & 0.1 \\ 0.08 & 0.2 & 0.02 & 0.3 & 0.01 & 0.1 \\ 0.001 & 0.4 & 1 & 0.2 & 0.4 & 0.07 \\ 0.5 & 0 & 0.01 & 02 & 0.35 & 0.22 \\ 0.0366 & 0.09 & 0.5 & 0.1 & 0.02 & 0.120 \end{bmatrix},$$

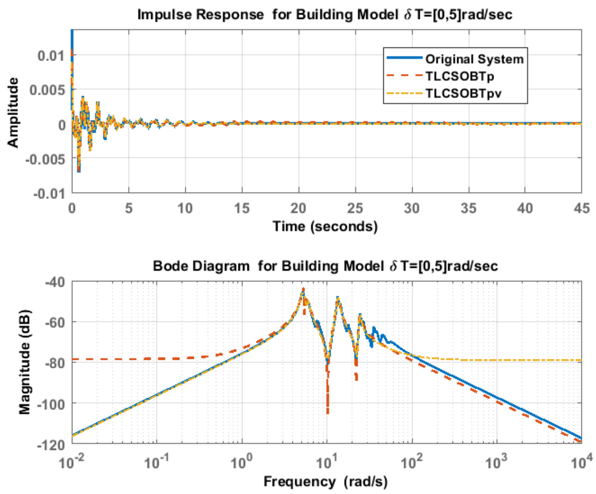

FIGURE 4. Responses for building model for $\delta T = [0, 5]$ sec.

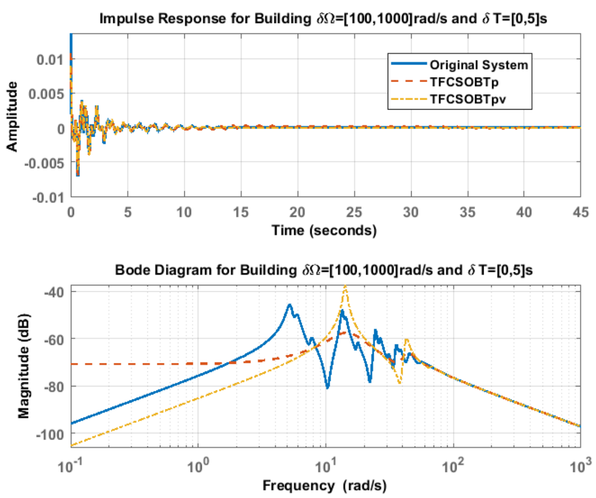

FIGURE 5. Responses for building model for $\delta T = [0, 5]$ sec and $\delta\Omega = [100, 1000]$ rad/sec.

$$K_d = \begin{bmatrix} 0.212 & 0.02 & 0.80 & 0.1 & 0.18 & 0 \\ 0.25 & 0.05 & 0.065 & 0.001 & 0.47 & 0.098 \\ 0.2 & 0.08 & 0.04 & 0.098 & 0 & 0.45 \\ 0.01 & 0.35 & 0.09 & 0.24 & 0.35 & 0.05 \\ 0.187 & 0.35 & 0.01 & 0.12 & 0.055 & 0.2 \\ 0.18 & 0.065 & 0.032 & 0.212 & 0.082 & 0.42 \end{bmatrix},$$

$$B_{d2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$C_{d1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_{d2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_d = I_6$$

Figure 7 shows that ROMs again comprehensively approximate the original discrete model response. However, ROM responses in normalized frequency interval $\delta\omega = [0.4, 0.6]$ rad/sec is not catching the original system response well as emphasis in this frequency range was not applied. To emphasize ROM response in this frequency interval, frequency-limited discrete SOBTp and SOBTpv techniques are utilized to obtain ROMs. The ROM responses for these frequency-limited techniques are shown in Figure 8. Figure 8 shows that response in $\delta\omega = [0.4, 0.6]$ rad/sec has been well emphasized as compared to Figure 7. Further, in Figure 8, although responses have been emphasized in interval $\delta\omega = [0.4, 0.6]$ rad/sec, but at the same time, responses in time interval $\delta k = [0, 0.1]$ sec are not up

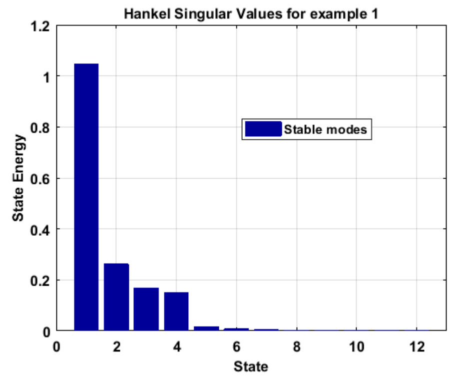

FIGURE 6. Hankel singular values plot for Example 1.

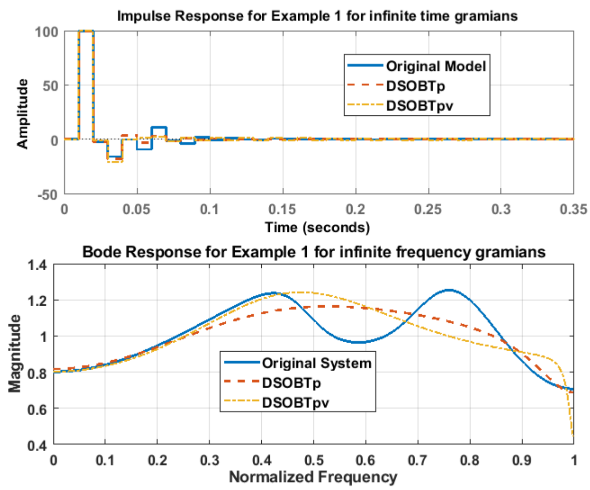

FIGURE 7. Responses for Example 1 for an infinite range.

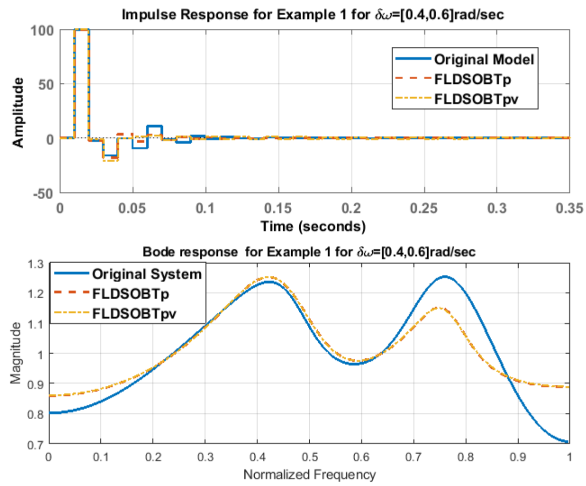

FIGURE 8. Responses for Example 1 for $\delta\Omega = [0.4, 0.6]$ rad/sec.

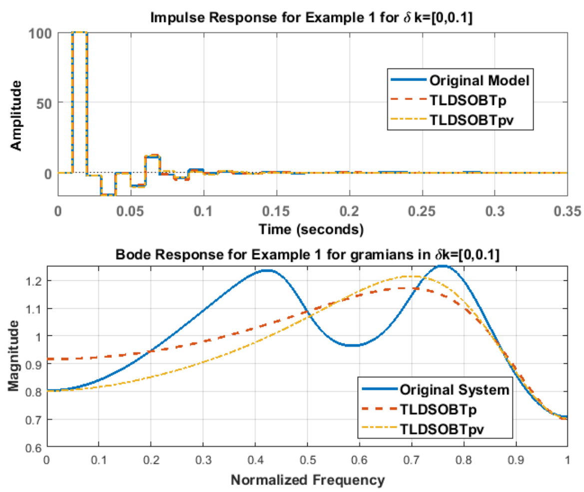

FIGURE 9. **Responses for Example 1 for $\delta k = [0, 0.1]$ sec.**

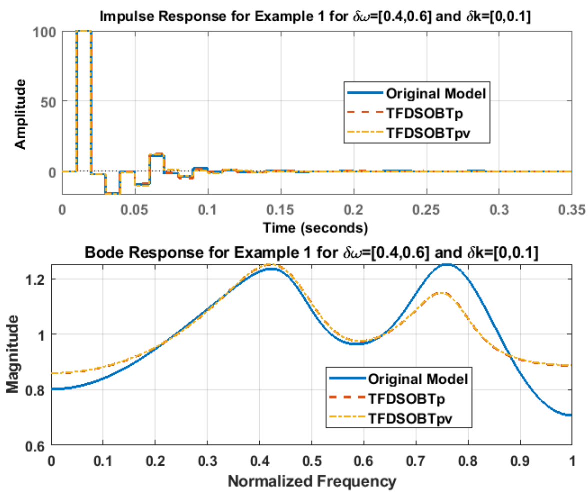

FIGURE 10. **Responses for Example 1 for $\delta k = [0, 0.1]$ sec and $\delta\Omega = [0.4, 0.6]$ rad/sec.**

to the mark (as simultaneous emphasis in frequency-time was not applied). Therefore, we utilize time-limited discrete SOBTp and SOBTpv techniques to emphasize this time interval and corresponding ROMs responses are depicted in Figure 9. Figure 9 shows that the required time interval $\delta k = [0, 0.1]$ sec has been comprehensively emphasized but correspondingly, the response in frequency interval $\delta\omega = [0.4, 0.6]$ rad/sec is degraded again (as again simultaneous emphasis in frequency-time was not applied). Finally, to achieve emphasis in both time and frequency intervals simultaneously, combined time-frequency discrete SOBTp and SOBTpv techniques are applied and results are shown in Figure 10. Figure 10 shows that now the responses have been properly emphasized, both in the required time and frequency intervals simultaneously. These results again show that these techniques can be used for MOR of discrete-time stable SOSs for infinite and finite interval applications.

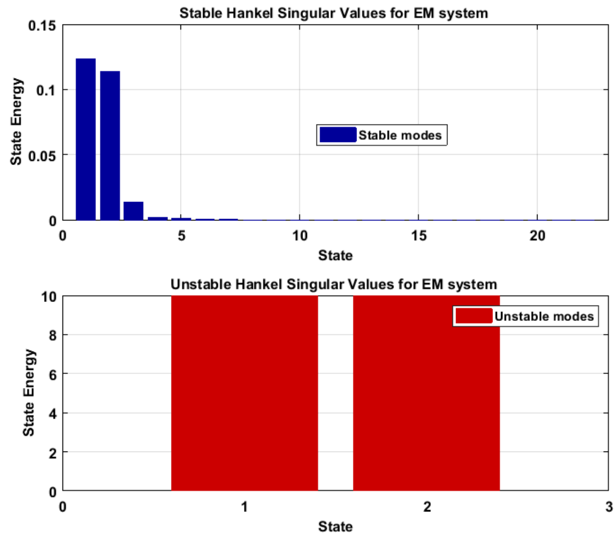

FIGURE 11. **HSVs plot for EMS model.**

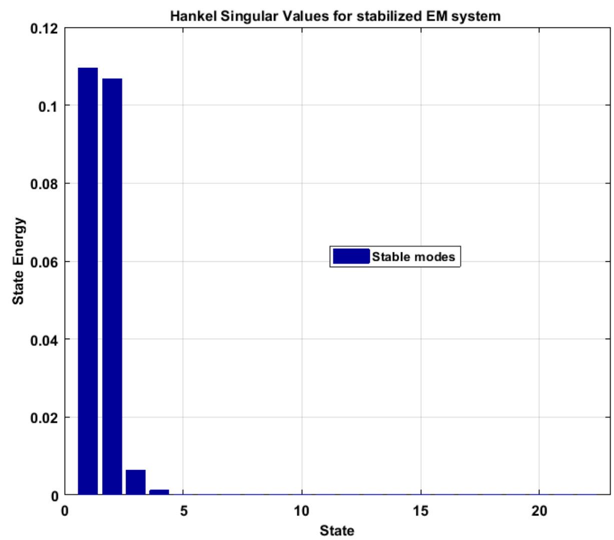

FIGURE 12. **HSVs plot for stabilized EMS model.**

C. CONTINUOUS-TIME UNSTABLE SYSTEM

The proposed techniques are applied to unstable continuous electromagnetic SOS (EMS) [34] (extended to twelve stages/orders), and HSVs for this system is shown in Figure 11. We see those unstable modes are present, and therefore we separate stable and unstable modes and apply non-structure-preserving continuous balanced truncation technique. On the other end, we stabilize the model using Bernoulli stabilization and apply structure-preserving SOBT. The HSVs plot for the stabilized electromagnetic model is shown in Figure 12, and ROM responses (for $r = 2$) for these techniques are shown in Figure 13.

Figure 13 shows that the structure-preserving ROM responses precisely approximate the original model response. The impulse response for non-structure-preserving continuous balanced truncation is unstable in due to augmentation of the unstable dynamics (and therefore is omitted in

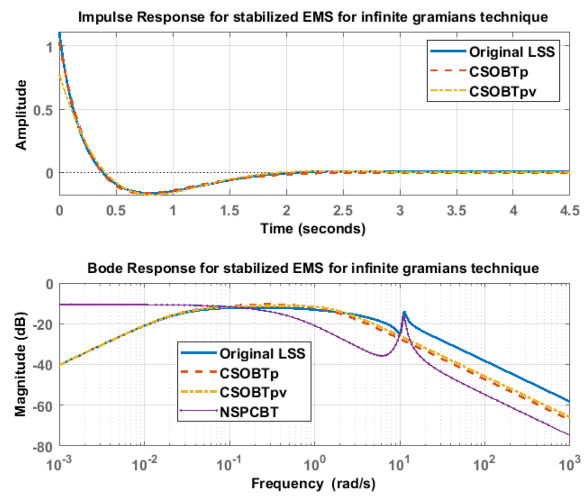

FIGURE 13. Responses for EMS model for an infinite range.

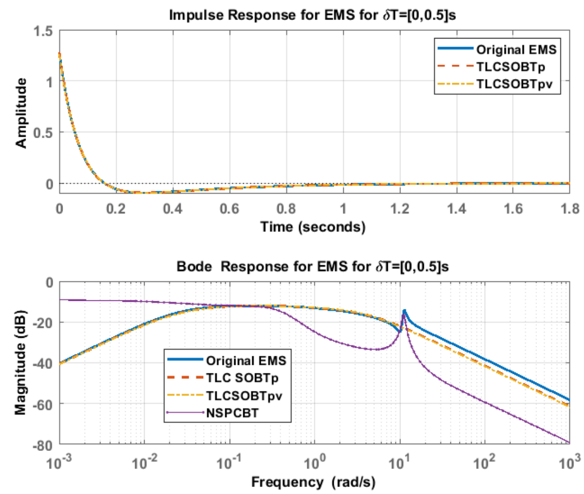

FIGURE 15. Responses for EMS model for $\delta T = [0, 5]$ sec.

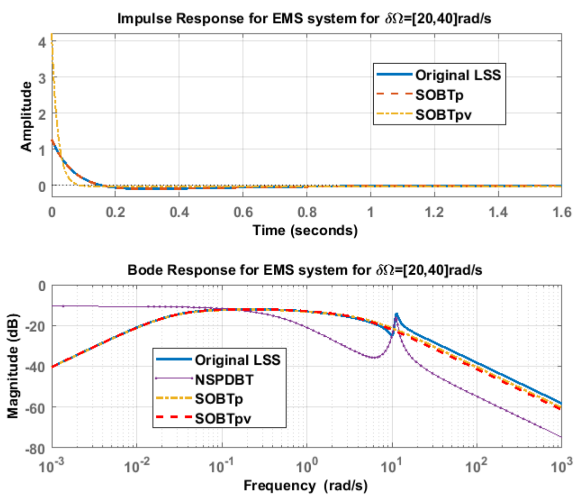

FIGURE 14. Responses for EMS model for $\delta\Omega = [20, 40]$ rad/sec.

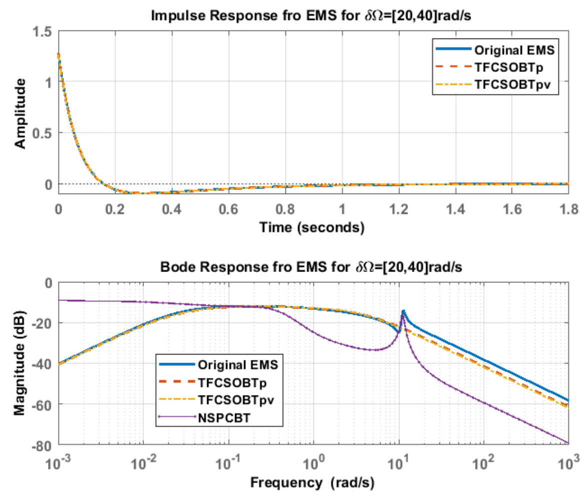

FIGURE 16. Responses for EMS model for $\delta T = [0, 5]$ sec and $\delta\Omega = [20, 40]$ rad/sec.

this article). The bode response of ROM obtained from non-structure-preserving continuous balanced truncation technique that blindly approximates the original model behavior is evident in Figure 13. Figure 13 also shows that ROM frequency response for this system in interval above 20 rad/sec is not up to the mark. Therefore, frequency interval emphasis is enforced for interval $\delta\Omega = [20, 40]$ rad/sec and results are depicted in Figure 14.

Figure 14 shows that the required frequency interval has now been comprehensively emphasized as compared to Figure 13, however, respective impulse response in transient phase, say $\delta T = [0, 0.5]$ sec is not catching the original system impulse response. To achieve emphasis in $\delta T = [0, 0.5]$ sec, structure-preserving time-limited SOBTp and SOBTpv techniques for stabilized system are applied and results are shown in Figure 15.

Figure 15 shows that the required time interval has been well emphasized but consequent frequency interval $\delta\Omega = [20, 40]$ rad/sec has been degraded. To achieve emphasis on these time and frequency intervals simultaneously, combined time-frequency SOBT techniques are applied to this system and results are depicted in Figure 16. Figure 16 shows that both the required time and frequency intervals have now been emphasized simultaneously. It is noteworthy that results for non-structure-preserving balanced truncation technique produce unstable ROMs and degraded bode responses. These results once more proved that structure-preserving MOR techniques can be used for MOR of continuous-time unstable SOSs for infinite and finite interval applications with confidence.

D. DISCRETE-TIME UNSTABLE SYSTEM

Finally, the MOR techniques are applied to unstable discrete piezoelectric (PZT) SOSs [32] (extended to twelve stages) and HSVs plot for this system is shown in Figure 17. It is observed that all the unstable modes are present. Thus, we first separate stable and unstable modes and apply non-structure-preserving discrete balanced truncation technique.

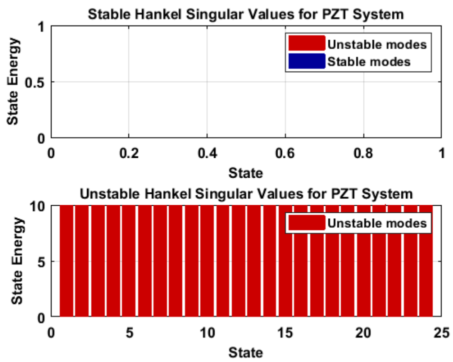

FIGURE 17. HSVs for PZT system (All modes are unstable).

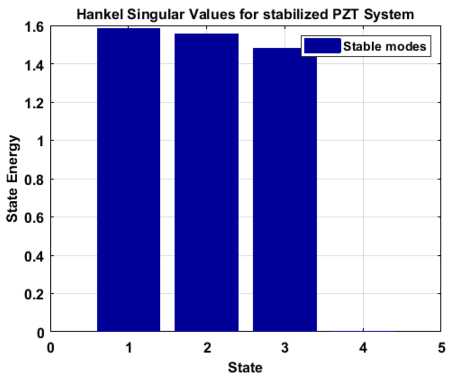

FIGURE 18. HSVs for stabilized PZT system.

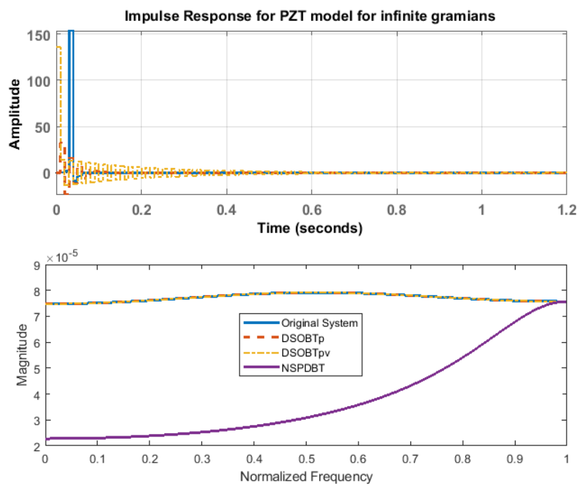

FIGURE 19. Responses for PZT system for an infinite range.

On the other end, we stabilize the model using Bernoulli stabilization (by using discrete-time versions of stabilization solutions) and apply structure-preserving SOBT. The HSVs plot for the stabilized electromagnetic model is shown in Figure 18 and ROM responses (for $r = 2$) for these techniques are shown in Figure 19.

Figure 19 shows that structure-preserving ROM responses precisely approximate the original model response. The impulse response for non-structure-preserving discrete balanced truncation is unstable due to augmentation of the

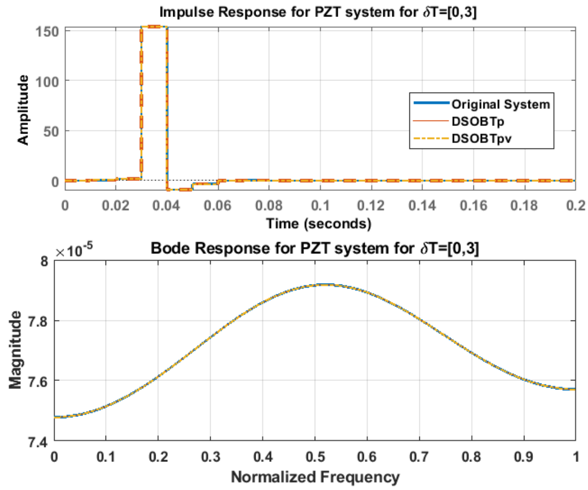

FIGURE 20. Responses for PZT system for $\delta k = [0, 3]$ sec.

unstable dynamics (also omitted in this article). As was earlier mentioned in the remark, the bode response of ROM obtained from non-structure-preserving discrete balanced truncation technique blindly approximates the original model behavior as evident in Figure 19. Moreover, structure-preserving ROM responses well emphasize any frequency interval at this time. However, initial time interval $\delta k = [0, 3]$ sec needs emphasis. Time-limited versions of these techniques are applied and results are shown in Figure 20 (poor results for non-structure-preserving discrete balanced truncation technique have been omitted in both subplots of Figure 20). Figure 20 shows that the required time interval has been emphasized and any corresponding frequency interval is already emphasized. These results once again certify that structure-preserving MOR techniques can be trustworthily used for MOR of discrete-time unstable SOSs for infinite and finite interval applications.

VII. CONCLUSION

A review of existing structure-preserving and non-structure-preserving second-order balanced truncation for second-order systems was presented in this article. Several extensions of these techniques for unstable SOSs were proposed. The frameworks for infinite and finite, frequency/time (or both) limited gramians and its corresponding algebraic Lyapunov equations were presented. Structure-preserving and non-structure-preserving techniques for a generalized form of stable and unstable continuous or discrete-time second order systems were discussed. Conditions for the preservation of stability and structure in ROMs were stated and techniques to obtain stable ROMs were also proposed. The presented developments were tested on continuous and discrete systems. Results certified the correct development of the proposed techniques and claimed superiorities. The proposed techniques can be utilized for model order reduction applications of continuous/discrete, stable/unstable second-order systems in infinite or finite time/frequency (or both) intervals.

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