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Szegő Quadrature Kalman Filter for Oscillatory Systems

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ABSTRACT Filtering problems with oscillatory system dynamics commonly appear in real-life. However, the existing Gaussian filters, like the unscented Kalman filter (UKF), cubature Kalman filter CKF, Gauss-Hermite filter (GHF) and cubature quadrature Kalman filter (CQKF), are accurate for the nonlinear systems with a particular order of polynomials only. This manuscript introduces a new Gaussian filter, which is accurate for oscillatory systems with 2π-period of oscillation. The proposed method is named as Szegő Quadrature Kalman Filter (SQKF). The SQKF transforms the intractable integrals that appear during the filtering over a unit circle. The transformed integral is approximated using the univariate Szegő quadrature rule. The univariate quadrature rule is extended in a multivariate domain using the product rule. Simulation results reveal an improved estimation accuracy for the SQKF in an oscillatory environment. The computational burden of the SQKF is similar to the GHF and higher than the UKF, CKF and CQKF.

INDEX TERMS Nonlinear filtering, Gaussian filtering, oscillatory system, intractable integral, Szegő quadrature rule.

I. INTRODUCTION

Estimation is a stochastic method for computing the hidden states of a system from noisy measurements. A recursive implementation of estimation is called filtering. The estimation and filtering problems appear in several domains of science and technology, like the defense system [1], space technology [2], financial and biomedical modeling [3], [4], weather forecast [5], industrial diagnosis and prognosis [6], *etc*. Bayesian framework [7]–[9] is a commonly accepted filtering approach among practitioners. It interprets the estimation problem in terms of prior and posterior probability density functions (pdf).

An optimal Bayesian filter for linear systems with Gaussian noises was developed in the sixties, popularly known as Kalman filter [7], [10]. However, its application is severely restricted due to the nonlinear nature of practical problems. A derivative-based nonlinear extension of the Kalman filter, named as extended Kalman filter (EKF) [7], was developed soon after the ordinary Kalman filter. However, it suffers from several drawbacks, like the poor accuracy and stability, due to derivative computation and local linearization of nonlinear system models. Despite all the drawbacks, the EKF and its variants [11]–[13] were the only alternatives for more than three decades. Post the EKF era, the literature on derivative-free filtering can be broadly divided into two categories: particle filtering [14], [15] and Gaussian filtering [16]–[18]. The particle filter (PF) approximates the true pdfs with a large number of samples, called particles, and their associated weights. In general, it is difficult, or even impossible, to generate samples from the true pdf. Thus, an alternative pdf, known as importance density [14], [15], is selected for generating the particles. It is a highly accurate filtering method for nonlinear and non-Gaussian systems, which is the most practical scenario. However, its computational demand is significantly high due to a large number of particles. Thus, it is often inapplicable to practical filtering problems. The Gaussian filters, however, approximate the true pdfs as Gaussian and characterize them with mean and covariance. The Gaussian filters are computationally efficient and most commonly used for practical applications, though their accuracy is relatively weak compared to the particle filters. The research is ongoing to improve the estimation accuracy of the Gaussian filtering further.

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The Gaussian filters encounter intractable integrals of the form '*nonlinear function* × *Gaussian distribution*' [16]–[18]. The intractable integrals are approximated numerically, and the accuracy of a filter depends on the accuracy of numerical approximation. The literature witnesses several Gaussian filters developed by using different numerical approximation methods. Some popular Gaussian filters are unscented Kalman filter (UKF) [18], cubature Kalman filter (CKF) [16] and its extensions [19], [20], and Gauss-Hermite filter (GHF) [17] along with several extensions [21]–[23].

The UKF utilizes an unscented transformation based numerical approximation [18]. Although it attracts the practitioners significantly, the unscented transformation based numerical approximation is not a very accurate method, which leaves scope for further development. With this motivation, the CKF is developed by using spherical-radial rule [16], which is more accurate than the unscented transformation based numerical approximation. In the spherical-radial rule, the desired integral is decomposed into spherical and radial integrals. The spherical integral is approximated using spherical cubature rule and the radial integral is approximated using Gauss-Laguerre quadrature rule. The CKF is further extended in [19] and [20] by using advanced numerical approximation methods providing a higher-order approximation to the spherical and radial integrals. In a further development, the GHF is introduced, which is possibly the most accurate among all the Gaussian filters. It utilizes univariate Gauss-Hermite quadrature rule [17], [22] for approximating the intractable integrals. In the case of multivariate systems, it utilizes product rule [17] for extending the univariate quadrature rule in the multivariate domain. The GHF is further extended in [21] and [22] by replacing the product rule with computationally efficient methods of extending the univariate quadrature rule in the multivariate domain. These extensions reduce the computational burden without harming the accuracy significantly. Although the GHF and its extensions are possibly the most accurate Gaussian filters, an N-point Gauss-Hermite quadrature rule is accurate only for integration with polynomials of order up to $2N - 1$. Therefore, the GHF suffers from poor accuracy for oscillatory systems. Note that the scope of increasing N is limited for multivariate systems as the computational burden increases exponentially. Similarly, all the existing Gaussian filters are accurate for polynomial-type systems only.

The literature witnesses several extensions of Gaussian filtering, which can be applied to any of the Gaussian filters to enhance the estimation accuracy further. Some of the popular extensions are Gaussian-sum filtering [24] and iterated filtering [25]. The Gaussian-sum filtering [24] approximates the unknown pdf with multiple Gaussian components instead of a single Gaussian in the case of the ordinary Gaussian filtering. The multiple Gaussian approximates the true pdf more accurately, which results in an improved estimation accuracy. On the other hand, the iterated filtering [25] performs the crucial measurement update step recursively with a predefined termination criteria to stop the recursive process. It should be mentioned that these extensions are applicable to any of the Gaussian filters. Moreover, the accuracy ultimately depends on the accuracy of the fundamental Gaussian filters, such as the UKF, CKF and GHF, to which the extension is applied. Therefore, the inefficacy of the existing Gaussian filters for oscillatory systems remains a concern.

This paper focuses on improving the estimation accuracy of the Gaussian filtering for oscillatory systems. As discussed above, the existing filters are inaccurate due to the inefficacy of the numerical methods in approximating the integrals with oscillatory functions. This paper introduces a new numerical approximation method, named as Szegő quadrature rule, which is accurate for oscillatory functions. The Szegő quadrature rule is defined over a unit circle in a complex plane. Thus, the desired intractable integral is first transformed over a unit circle in a complex plane. The transformed integral is approximated using the Szegő quadrature rule, which generates the sample points and weights using orthogonal Szegő polynomials [26], [27]. The Gaussian filter developed with Szegő quadrature rule is named as Szegő Quadrature Kalman Filter (SQKF). The Szegő quadrature rule is again a univariate quadrature rule; hence, the product rule is used for extending the univariate rule in multivariate domain. The simulation results reveal an improved estimation accuracy for the SQKF compared to the existing Gaussian filters in the oscillatory environment. Moreover, the aforementioned extensions of the Gaussian filtering, such as the Gaussian-sum filtering and iterated filtering, can be applied over the proposed SQKF as well, to enhance the estimation accuracy further.

II. GAUSSIAN FILTERING METHOD

This section briefly reviews the Gaussian filtering method, which is simplification of Bayesian filtering framework. Consider a nonlinear system with the following state space model:

$$x_k = \phi_k(x_{k-1}) + \boldsymbol{\eta}_k, \tag{1}$$

$$y_k = \gamma_k(x_k) + \mathbf{v}_k, \tag{2}$$

where $x_k \in \Re^n$ and $y_k \in \Re^d$ are state and measurement variables at k^{th} instant, $k = \{1, 2, \cdots\}$, $\phi_k : x_{k-1} \to x_k$ and $\gamma_k : x_k \to y_k$ are nonlinear functions, $\boldsymbol{\eta}_k \in \Re^n$ and $v_k \in \Re^d$ are process and measurement noises, respectively. The process noise encounters modeling error, while the measurement noise compensates for the sensor uncertainties. $\boldsymbol{\eta}_k$ and v_k are assumed to follow zero-mean Gaussian with covariance Q_k and R_k, respectively.

The Bayesian framework consists of two steps: prediction and update. The prediction step determines the prior pdf $P(x_k|y_{1:k-1})$ using the Chapman-Kolmogorov equation. However, the update step determines the posterior pdf $P(x_k|y_{1:k})$ using Bayes rule based on the information received from y_k. In the Gaussian filtering, the conditional pdfs are approximated as Gaussian: $P(x_k|y_{1:k-1}) \approx \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})$ and $P(x_k|y_{1:k}) \approx \mathcal{N}(x_k; \hat{x}_{k|k}, P_{k|k})$, where $\hat{x}_{k|k-1}$ ($\hat{x}_{k|k}$) and $P_{k|k-1}$ ($P_{k|k}$) are mean and covariance of $x_k|y_{1:k-1}$ ($x_k|y_{1:k}$), and $\mathcal{N}$ denotes Gaussian distribution.

Subsequently, the problem of characterizing the conditional pdfs is simplified as the problem of computing the respective mean and covariance. The computation of mean and covariance involves multivariate Gaussian weighted integrals of the form

$$I^n = \int_{-\infty}^{\infty} f(x)\mathcal{N}(x;\hat{x},P)dx, \tag{3}$$

where $f : \Re^n \rightarrow \Re^n$ is a general nonlinear function. Such integrals are mostly intractable, and approximated numerically using quadrature rules. The quadrature rules are designed for $\mathcal{N}(x;0,\mathbf{I}_n)$ with $\mathbf{I}_n$ being identity matrix of dimension n. Therefore, I^n concerning $\mathcal{N}(x;0,\mathbf{I}_n)$ is approximated as

$$I_0^n = \int_{-\infty}^{\infty} f(x)\mathcal{N}(x;0,\mathbf{I}_n)dx \approx \sum_{j=1}^{N_s} W_j f(\boldsymbol{\xi}_j), \tag{4}$$

where $\boldsymbol{\xi}_j$ and $W_j \ \forall \ j \in \{1,2,\cdots,N_s\}$ are sample points and weights, respectively, while N_s is the number of sample points. The same quadrature rule can be extended for $\mathcal{N}(x;\hat{x},P)$ by transforming the sample points by mean $\hat{x}$ and covariance P, hence

$$I^n \approx \sum_{j=1}^{N_s} W_j f(\hat{x} + S\boldsymbol{\xi}_j), \tag{5}$$

where $SS^T = P$.

The Gaussian filtering method based on numerical approximation of intractable integrals can be summarized as followed [16], [20].

A. PREDICTION

$$\hat{x}_{k|k-1} = \sum_{j=1}^{N_s} W_j \boldsymbol{\xi}_{j,k|k-1}, \tag{6}$$

$$P_{k|k-1} = \sum_{j=1}^{N_s} W_j \times (\boldsymbol{\xi}_{j,k|k-1} - \hat{x}_{k|k-1})(\boldsymbol{\xi}_{j,k|k-1} - \hat{x}_{k|k-1})^T + Q \tag{7}$$

where $\boldsymbol{\xi}_{j,k|k-1} = \phi_k(\hat{x}_{k-1|k-1} + S_{k-1|k-1}\boldsymbol{\xi}_j)$.

B. UPDATE

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}), \tag{8}$$

$$P_{k|k} = P_{k|k-1} - \mathbf{K}_k P^{yy}_{k|k-1}\mathbf{K}_k^T, \tag{9}$$

where

$$\mathbf{K}_k = P^{xy}_{k|k-1}(P^{yy}_{k|k-1})^{-1}, \tag{10}$$

$$\hat{\mathbf{y}}_{k|k-1} = \sum_{i=1}^{N_s} \mathbf{W}_i \boldsymbol{\xi}^{\gamma}_{i,k|k-1}, \tag{11}$$

$$P^{yy}_{k|k-1} = \sum_{i=1}^{N_s} \mathbf{W}_i(\boldsymbol{\xi}^{\gamma}_{i,k|k-1} - \hat{\mathbf{y}}_{k|k-1})(\boldsymbol{\xi}^{\gamma}_{i,k|k-1} - \hat{\mathbf{y}}_{k|k-1})^T + \mathbf{R}_k, \tag{12}$$

$$P^{xy}_{k|k-1} = \sum_i W_i(\boldsymbol{\xi}_{i,k|k-1} - \hat{x}_{k|k-1})(\boldsymbol{\xi}^{\gamma}_{i,k|k-1} - \hat{\mathbf{y}}_{k|k-1})^T, \tag{13}$$

with $\boldsymbol{\xi}^{\gamma}_{i,k|k-1} = \gamma_k(S_{k|k-1}\boldsymbol{\xi}_i + \hat{x}_{k|k-1})$ and $\xi_{i,k|k-1} = S_{k|k-1}\boldsymbol{\xi}_i + \hat{x}_{k|k-1}$.

III. Szegő QUADRATURE KALMAN FILTER

The proposed SQKF is a Gaussian filter; hence, it follows the same filtering structure described in the previous section. However, it uses Szegő quadrature rule for generating a new set of $\boldsymbol{\xi}$ and W. The Szegő quadrature rule is accurate for oscillatory systems with 2π-period of oscillation. It transforms the desired integral over a unit circle, and approximates the transformed integral with $\boldsymbol{\xi}$ and W generated over this unit circle.

A. INTEGRAL TRANSFORMATION

The Szegő quadrature rule is defined for univariate systems. For a univariate system with zero-mean and unity covariance, I_0^n can be simplified as

$$I_0^1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}} dx,$$

where x is a univariate random variable and $f(x)$ is an oscillatory function with 2π-period. The Szegő quadrature rule transforms I_0^1 over a unit circle. Hereafter, the unit circle is denoted as $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, where $z = e^{i\theta}$ and $\mathbb{C}$ is set of all circles.

Lemma 1: For a 2π-periodic function $f(x)$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}} dx = \int_{-\pi}^{\pi} f(\theta) w(\theta) d\theta, \tag{14}$$

where θ is phase angle with respect to a point on $\mathbb{U}$, and

$$w(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-\frac{(\theta-2\pi j)^2}{2}} \tag{15}$$

is weight function on $[-\pi,\pi]$ with respect to $e^{-x^2/2}$.

Proof: For any $\theta \in [-\pi,\pi]$, it follows from [28] that

$$\sum_{j=-\infty}^{\infty} f(\theta) e^{-(\theta-2\pi j)^2/2} \leq \arg\max_{\theta\in[\pi,\pi]} |f(\theta)| \left(1 + 2\sum_{j=1}^{\infty} e^{-\frac{(2\pi(j-1))^2}{2}}\right). \tag{16}$$

It can also be observed that

$$\lim_{j\to\infty} \frac{e^{-(2\pi(j-1))^2}}{e^{-(2\pi(j-2))^2}} = 0 < 1. \tag{17}$$

Eq. (16) and (17) conclude that $\sum_{j=-\infty}^{\infty} f(\theta) e^{-(\theta-2\pi j)^2/2}$ is a convergent series. Let us now replace $w(\theta)$ from Eq. (15) into the expression in the right-hand side of Eq. (14):

$$\int_{-\pi}^{\pi} f(\theta) w(\theta) d\theta = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(\theta) \sum_{j=-\infty}^{\infty} e^{-\frac{(\theta-2\pi j)^2}{2}} d\theta.$$

As $\sum_{j=-\infty}^{\infty} f(\theta)e^{-(\theta-2\pi j)^2/2}$ a convergent series, the summation and integration can be interchanged. Therefore,

$$\int_{-\pi}^{\pi} f(\theta)w(\theta)d\theta = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} \int_{-\pi}^{\pi} f(\theta)e^{-\frac{(\theta-2\pi j)^2}{2}} d\theta.$$

Substituting $x = \theta - 2\pi j$ in the right-hand side, we get

$$\int_{-\pi}^{\pi} f(\theta)w(\theta)d\theta = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} \int_{-\pi-2\pi j}^{\pi-2\pi j} f(x+2\pi j)e^{-\frac{x^2}{2}} dx.$$

As $f(x)$ is 2π-periodic, the above equation can be further simplified as

$$\int_{-\pi}^{\pi} f(\theta)w(\theta)d\theta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-\frac{x^2}{2}} dx.$$

□

B. INTEGRAL APPROXIMATION USING Szegő QUADRATURE RULE

The Szegő quadrature rule is defined over unit circle for approximating the integrals of the form $\int_{-\pi}^{\pi} f(\theta)w(\theta)d\theta$ using Szegő polynomials. Subsequently, it approximates the desired univariate integral I_0^1. The Szegő polynomials $\{\vartheta_m(z)\}_{m=1}^{\infty}$ are orthonormal on $\mathbb{U}$ with respect to a weight function $\omega(\theta)$, *i.e.*

$$\langle \vartheta_m, \vartheta_{\underline{m}} \rangle_\omega = \int_{-\pi}^{\pi} \vartheta_m(z)\overline{\vartheta_{\underline{m}}(z)}\omega(\theta)d\theta = \delta_{m,\underline{m}},$$

where $\langle \vartheta_m, \vartheta_{\underline{m}} \rangle_\omega$ represents the inner product of $\vartheta_m(z)$ and $\vartheta_{\underline{m}}(z)$ induced by $\omega(\theta)$, $\overline{\vartheta_{\underline{m}}(z)}$ is conjugate of $\vartheta_{\underline{m}}(z)$, and δ represents Kronecker delta. Explicit expressions of $\vartheta_m(z)$ are available only for exceptional $\omega(\theta)$. They are generally constructed from their monic sequence obtained from iterative solution of Szegő forward recurrence relation. If $\{\rho_m(z)\}_{m=1}^{\infty}$ denotes the monic sequence of $\vartheta_m(z)$, then the recurrence relation is given as [26], [27]

$$\begin{bmatrix} \rho_m(z) \\ \rho_m^*(z) \end{bmatrix} = \begin{bmatrix} z & \alpha_m \\ \overline{\alpha_m} z & 1 \end{bmatrix} \begin{bmatrix} \rho_{m-1}(z) \\ \rho_{m-1}^*(z) \end{bmatrix}; \quad m \geq 1,$$

where $\rho_{\underline{m}}^*(z)$ is reciprocal of $\rho_m(z)$ computed as $\rho_m^*(z) = z^m\overline{\rho_m(1/\overline{z})}$, and $\alpha_m = \rho_m(0)\ \forall\ m \geq 0$ is Verblunsky coefficient for $\omega(\theta)$. The initial condition is taken as $\rho_0(z) = \rho_0^*(z) = 1$. Moreover, $\alpha_0 = 1$ and $|\alpha_m| < 1\ \forall\ m \leq 1$. The author refers to [26], [27] for a detailed discussion on $\rho_m(z)$, α_m and the resultant polynomials. The solution of recurrence relation concerning $\omega(\theta) = w(\theta)$ generates the Szegő polynomials of our interest.

The Szegő quadrature rule is based on an extended form of Szegő polynomials, where $\rho_m(z)$ is generated using q-binomial coefficient. This extended form is also known as Rogers-Szegő polynomial [29]. The q-binomial coefficient for $0 < q < 1$ is defined as

$$\begin{cases} (m)_q = \prod_{j=1}^{m}(1-q^j), \\ \begin{pmatrix} m \\ j \end{pmatrix}_q = \dfrac{(m)_q}{(j)_q(m-j)_q} = \dfrac{\prod_{k=m-j+1}^{m}(1-q^k)}{\prod_{k=1}^{j}(1-q^k)}, \end{cases} \tag{18}$$

where

$$(0)_q = \begin{pmatrix} m \\ 0 \end{pmatrix}_q = \begin{pmatrix} m \\ m \end{pmatrix}_q = 1.$$

An explicit form of monic and Rogers-Szegő polynomials concerning the weight function $w(\theta)$ is derived in [28] as

$$\rho_m(z) = \sum_{j=0}^{m}(-1)^{m-j} \begin{pmatrix} m \\ j \end{pmatrix}_s s^{\frac{m-j}{2}} z^j \tag{19}$$

and

$$\vartheta_m(z) = \frac{1}{\sqrt{(m)_s}}\rho_m(z), \tag{20}$$

where $s = e^{-1}$ is the parameter q for $\omega(\theta) = w(\theta)$.

For the above monic and Rogers-Szegő polynomials, the univariate quadrature points and associated weights are derived in [28]. If $\xi_j\ \forall\ j \in \{1, 2, \cdots, N_s\}$ represent the univariate quadrature points with N_s be the desired number of quadrature points, then ξ_j can be obtained as roots of

$$\sum_{j=0}^{N_s} C_j \left(1 + (-1)^{N_s} s^{(j-N_s/2)}\right) z^j = 0, \tag{21}$$

where the coefficient C_j is given as

$$C_j = (-1)^{N_s-j} \begin{pmatrix} N_s \\ j \end{pmatrix}_s s^{\frac{N_s-j}{2}}.$$

The weight associated with a univariate point ξ_j is given as

$$W_j = \frac{(s)_{N_s}}{2\mathrm{Re}\left[\xi_j\rho_m'(\xi_j)\overline{\rho_m(\xi_j)}\right] - N_s|\rho_m(\xi_j)|^2}, \tag{22}$$

where $\mathrm{Re}[\cdot]$ represents the real part of complex number and $\rho_m'(z)$ represents the first derivative of $\rho_m(z)$.

The roots of Eq. (21), i.e. ξ_j, are obtained on a unit circle, and represented in a complex plane. Note that the desired quadrature points are analogous to angle, which can be obtained as phase angle of ξ_j. Subsequently, the desired quadrature points are: $\zeta_j = \angle\xi_j\ \forall\ j \in \{1, 2, \cdots, N_s\}$ *i.e.* $\zeta_j \in [-\pi, \pi]$. As ξ_j is a point on a unit circle, $|\xi_j| = 1$. The values of univariate quadrature points and weights, ζ_j and W_j, are shown in Table 1 for $N_s = 10$. Based on ζ_j and W_j, the integral I_0^1 can be approximated as

$$\begin{aligned} I_0^1 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-\frac{x^2}{2}} d \\ &= \int_{-\pi}^{\pi} f(\theta)w(\theta)d\theta \approx \sum_{j=0}^{N_s} W_j f(\zeta_j). \end{aligned} \tag{23}$$

C. MULTIVARIATE EXTENSION OF Szegő QUADRATURE RULE

The Szegő quadrature rule approximates the univariate integral I_0^1. However, the desired integral I_0^n is multivariate extension of I_0^1. The SQKF utilizes product rule for extending the univariate rule in multivariate domain. With ζ_j and W_j ($\forall\ j \in \{1, 2, \cdots, N_s\}$) being the univariate quadrature

TABLE 1. $N_s = 10$: Univariate quadrature points in complex plane (ξ_j), Univariate quadrature points in terms of phase angle in radian (ζ_j), and the weights (W_j) associated with ξ_j and ζ_j.

ξ_j	ζ_j	W_j
$-0.896681 \pm 0.442678i$	± 2.6830094808	0.00900598
$-0.418797 \pm 0.90808i$	± 2.0029165946	0.0335071
$0.166392 \pm 0.98606i$	± 1.4036268647	0.0865554
$0.672324 \pm 0.740257i$	± 0.8334519570	0.158343
$0.962015 \pm 0.272995i$	± 0.2765054275	0.212589

points and weights, respectively, the product rule approximates I_0^n as

$$I_0^n \approx \sum_{j_1=1}^{N_s}\sum_{j_2=1}^{N_s}\cdots\sum_{j_n=1}^{N_s} f\left(\left[\zeta_{j_1}, \zeta_{j_2}, \cdots, \zeta_{j_n}\right]^T\right) W_{j_1} W_{j_2} \cdots W_{j_n}.$$

Subsequently, the desired multivariate intractable integral I^n with respect to $\mathcal{N}(x; \hat{x}, P)$ can be approximated as

$$I^n \approx \sum_{j_1=1}^{N_s}\sum_{j_2=1}^{N_s}\cdots\sum_{j_n=1}^{N_s} f\left(\hat{x} + S\left[\zeta_{j_1}, \zeta_{j_2}, \cdots, \zeta_{j_n}\right]^T\right) \times W_{j_1} W_{j_2} \cdots W_{j_n}. \quad (24)$$

Therefore, the multivariate quadrature points and weights can be obtained as: $\boldsymbol{\xi} = \left[\zeta_{j_1}, \zeta_{j_2}, \cdots, \zeta_{j_n}\right]^T$ and $W = W_{j_1} W_{j_2} \cdots W_{j_n} \forall j_1, j_2, \cdots, j_n \in \{1, 2, \cdots, N_s\}$. Note that the number of multivariate quadrature points are $(N_s)^n$. The proposed SQKF can be implemented by using the steps shown in Eq. (6) to (13), with $\boldsymbol{\xi}$ and W as derived above.

The existing Gaussian filters, such as the UKF, CKF, and GHF, utilize the numerical approximation methods accurate only for polynomials up to particular orders. Subsequently, they suffer from poor estimation accuracy for the oscillatory systems. Note that the Szegő quadrature rule approximates the integrals of the form $\int_{-\pi}^{\pi} f(\theta) w(\theta) d\theta$. As derived in Lemma 1, this form of integral also represents the univariate integral $I_0^1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}} dx$ (for zero mean and unity covariance systems), if $f(x)$ is an oscillatory function. Moreover, applying the product rule and the transformation of x with mean $\hat{x}$ and covariance P, the same rule can be extended to approximate the desired integral $I^n = \int_{-\infty}^{\infty} f(x) \mathcal{N}(x; \hat{x}, P) dx$ for an oscillatory form of $f(x)$. Thus, the Szegő quadrature rule can approximate the desired integrals for oscillatory systems with better accuracy. Subsequently, the proposed SQKF is expected to outperform the existing filters in an oscillatory environment.

It should be mentioned that the Szegő quadrature rule is specifically designed for oscillatory systems. Therefore, the SQKF does not apply to the systems without periodic oscillation. Moreover, it should also be mentioned that a wide range of nonlinear filtering problems involve oscillation. Thus, the practical applicability of the SQKF remains wide.

Remark 1: Number of multivariate quadrature points required for implementing the SQKF is $(N_s)^n$. Therefore, the computational burden of the SQKF increases exponentially with increasing dimension. Moreover, the computational burden is similar to the GHF, and higher than the UKF and CKF.

Remark 2: Smolyak rule [21] and adaptive sparse-grid rule [22], used in SGHF and ASGHF, respectively, can replace the product rule to reduce the computational burden.

IV. SIMULATION AND RESULTS

In this section, the proposed SQKF is simulated for two nonlinear filtering problems with oscillatory state dynamics. The simulation is performed in Matlab over a personal computer with Intel Core i7, 1.89 GHz processor, 8 GB RAM, and 64-bit operating system. The performance of SQKF is compared with the existing nonlinear Gaussian filters, UKF, CKF, GHF and cubature quadrature filter (CQKF), as well as the PF and related techniques. The performance analysis is based on root mean square error (RMSE).

A. PROBLEM 1

The first problem is a parameter estimation problem concerning Duffing oscillator [30], [31]. The state dynamics follows a nonlinear second-order differential equation of form

$$\ddot{x} + 2b\epsilon\dot{x} + \epsilon^2 x + ax^3 = u(t), \quad (25)$$

where x is displacement, a and b are stiffness coefficient and damping coefficient, respectively, and ϵ is angular velocity. $u(t) = A cos\lambda t$ is harmonic input, where A is amplitude and λ is frequency. From Eq. (25), we can formulate the following state dynamics [30], [31]

$$\begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= (-2\, b\epsilon x_2 - \epsilon^2\, x_1 + x_3 x_1^3 + A cos\lambda t)dt + \sigma\, dB_1(t), \\ dx_3 &= dB_2(t), \end{aligned}$$

where x_1, x_2 and x_3 are displacement, velocity and the stiffness coefficient a, respectively, while B_1 and B_2 are independent Brownian motions. The discretized form of above state dynamics is shown in Eq. (26), as shown at the bottom of the next page, where

$$c_2(k) = -2\, b\epsilon x_2(k) - \epsilon^2\, x_1(k) - x_3(k) x_1(k)^3 + A\cos\lambda t_k,$$

and $[I_1, I_{10}, I_{100}, I_2]^T$ is zero-mean Gaussian with covariance

$$\Sigma = \begin{bmatrix} T & \frac{T^2}{2} & \frac{T^3}{6} & 0 \\ \frac{T^2}{2} & \frac{T^3}{3} & \frac{T^4}{8} & 0 \\ \frac{T^3}{6} & \frac{T^4}{8} & \frac{T^5}{20} & 0 \\ 0 & 0 & 0 & T \end{bmatrix}.$$

Please note the sufficiently large value of Σ over a given sampling interval T to introduce a sufficiently large noise. The measurement equation follows $y_k = x_k + \sigma_m I_m = [x_1(k)\ x_2(k)\ x_3(k)]^T + \sigma_m I_m$.

A simulated dataset of x is generated by assuming the true initial state as $x_0 = [0.5, -1, 1]^T$. The filtering is performed

over 200 time-steps with the initial estimate generated from $\mathcal{N}(x_0, P_{0|0})$ with $P_{0|0} = I_n$ being the initial error covariance. For the given initial condition and state dynamics, a sufficiently large input is taken by considering the input sinusoidal amplitude as $A = 10$. A small damping coefficient is taken as $b = 0.04$ to cause high oscillation, which further causes higher nonlinearity to make the filtering more challenging. The angular velocity of the oscillator is assigned with a reasonable value, $\epsilon = 1.15\pi$. The angular velocity of the input is considered to be smaller than the angular velocity of the oscillator, and assigned with a value $\lambda = .75\epsilon$. The sampling time is considered to be small enough to characterize the oscillation, and taken as $T = \pi/(80\lambda)$. Considering that the noise coefficients should be sufficiently large to characterize the practical difficulties of filtering, they are chosen as $\sigma = A/10$, $\sigma_\alpha = 0.00158$ and $\sigma_m = 0.23$.

Hereafter, SQKF_{N_s} denotes the proposed SQKF with N_s-point univariate quadrature rule. The UKF is generally implemented with a positive integer value of κ to ensure that all the weights are non-negative. Thus, it is taken as $\kappa = 2$ for the implementation. Other values can also be chosen, however, the estimation accuracy may not change significantly. The number of multivariate points for the GHF is exponential to the number of univariate quadrature points. Therefore, the number of univariate quadrature points is generally chosen to be small, such as 3 to 4, to restrict the computational burden. Thus, a 4-point GHF (with 4 univariate quadrature points) is considered for simulation. For a fair comparison, the proposed method is also applied with 4-point univariate Szegő quadrature rule, i.e. SQKF_4. Moreover, the performance of SQKF is separately compared for varying number of univariate quadrature points. Please note the non-oscillatory nature of the measurement equation, where the SQKF applies ordinary Gauss-Hermite quadrature rule.

Fig 1 shows the true variation of velocity against the position, and the same estimated from SQKF_4. The estimated trajectory closely matches the true values, which interprets a successful estimation for the SQKF. Furthermore, Fig. 2 shows RMSE plots for the SQKF (implemented with $N_s = 4$) and the existing filters, UKF, CKF, GHF and CQKF. The RMSE for the SQKF is lower than the existing filters, which indicates an improved estimation accuracy. The relative computation time is observed as 1, 0.94, 8.64, 1.82 and 8.76 for the UKF, CKF, GHF, CQKF and SQKF, respectively. It indicates an increased computational burden for the SQKF compared to the UKF, CKF and CQKF, though similar to

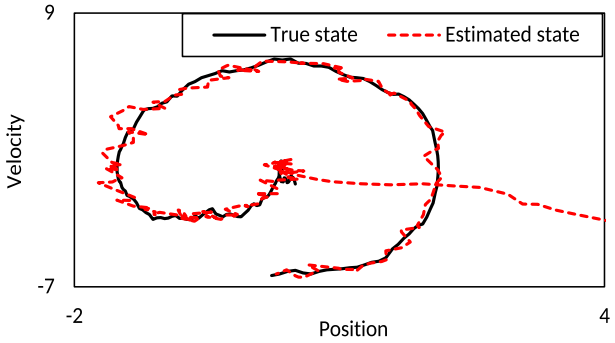

FIGURE 1. Problem 1: True and estimated plots of velocity against the position.

TABLE 2. Problem 1: Average RMSE of SQKF with number of univariate quadrature points.

	SQKF_2	SQKF_4	SQKF_6	SQKF_8
State 1	0.1401	0.1238	0.1191	0.1190
State 2	0.1604	0.1582	0.1587	0.1589
State 3	0.1652	0.1267	0.1276	0.1282

the GHF. The average RMSE obtained from the SQKF for various N_s is shown in Table 2. The table concludes a decreasing RMSE, so that an improving estimation accuracy, as the number of univariate quadrature points increases up to 6. However, the RMSE increases as the univariate quadrature points increase further from 6 to 8. This may be due to several possible reasons, such as the saturation of numerical approximation method, an inaccurate Gaussian being a closer approximation of the true pdf and numerical instability of the quadrature rule [28]. The relative computational time for N_s being 2, 4, 6 and 8 are 1, 7.36, 25.71 and 55.53, respectively, which shows an exponentially increasing computational time as N_s increases.

B. PROBLEM 2

For the second problem, the dynamic state space model of the system is given as

$$x_k = |20\cos(x_{k-1})| + \boldsymbol{\eta}_k, \tag{27}$$

$$y_k = \sqrt{(1 + x_k^T x_k)} + \mathbf{v}_k. \tag{28}$$

For the simulation purpose, the system dimension is considered as 2. The initial state is taken as $x_0 = [1, 2]^T$. The initial estimate and covariance are considered as $\hat{x}_{0|0} = [3, 4]^T$ and $P_{0|0} = I_n$, respectively. Note that the peak values of x

$$\begin{aligned}
x_1(k+1) &= x_1(k) + x_2(k)T + c_2(k)\frac{T^2}{2} - A\lambda\sin\lambda t_k\frac{T^3}{6} - \epsilon^2 x_2(k)\frac{T^3}{6} - x_3(k)x_1(k)^2x_2(k)\frac{T^3}{2} + \sigma I_{10} + 2b\epsilon\sigma I_{100},\\
x_2(k+1) &= x_2(k) + \frac{A}{\lambda}(\sin\lambda t_{k+1} - \sin\lambda t_k) - 2b\epsilon\left(x_2(k)T + c_2(k)\frac{T^2}{2}\right) - \epsilon^2\left(x_1(k)T + x_2(k)\frac{T^2}{2}\right) - x_3(k)\Big(x_1(k)^3T\\
&\quad + 3x_1(k)^2x_2(k)\frac{T^2}{2}\Big) + \sigma I_1 - 2b\epsilon\sigma I_{10},\\
x_3(k+1) &= x_3(k) + \sigma_\alpha I_2.
\end{aligned} \tag{26}$$

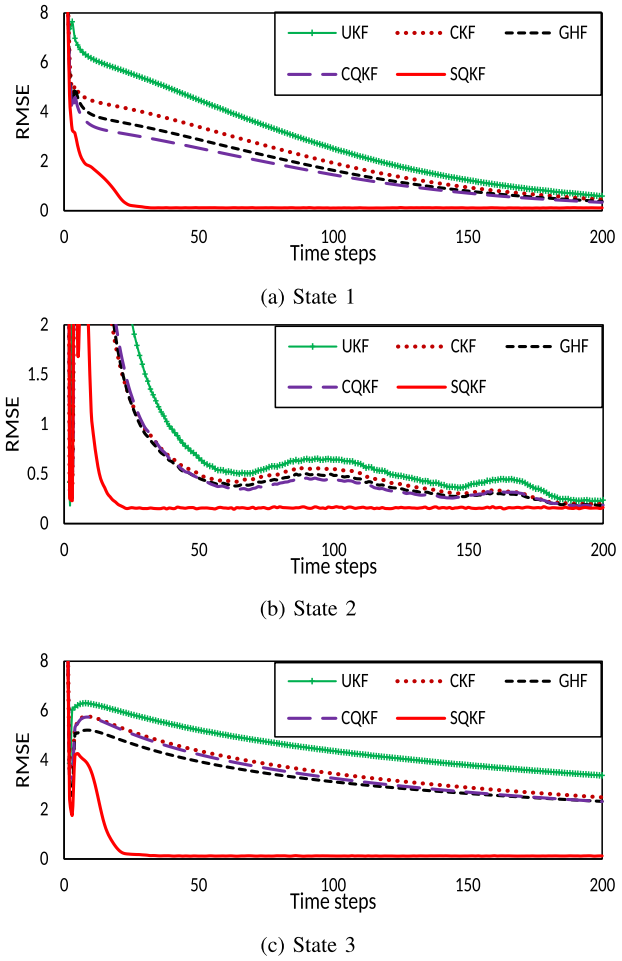

FIGURE 2. Problem 1: RMSE plots for the SQKF and the existing filters, UKF, CKF and GHF. The SQKF is implemented with 4-points univariate quadrature rule.

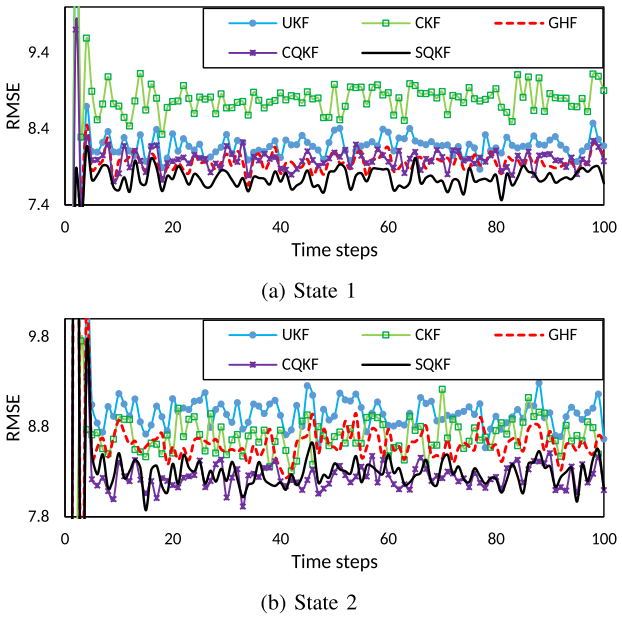

FIGURE 3. Problem 2: RMSE plots for the SQKF and the existing filters, UKF, CKF and GHF. The SQKF is implemented with 4-points univariate quadrature rule.

TABLE 3. Problem 2: Average RMSE of SQKF with number of univariate quadrature points.

	$SQKF_2$	$SQKF_4$	$SQKF_6$	$SQKF_8$
State 1	8.8888	7.6941	7.4014	7.5334
State 2	9.5060	8.2808	7.6254	8.1083

and y are 20 and $20\sqrt{2}$ (approximated value for $n = 2$), respectively. The noise covariances are assigned to restrict the noises mostly within 10% of the peak values. Subsequently, they are taken as: $Q_{ij} = 0.5\ \forall\ i = j$ and 0.05 otherwise, and $R = 1$. The parameters of the UKF, GHF and SQKF are assigned with the same values as the previous problem.

The RMSE plots for the SQKF and the existing Gaussian filters are shown in Fig. 3. The RMSE obtained from the SQKF is lowest, which concludes an improved accuracy for the SQKF compared to the existing filters. The relative computational time is observed as similar to the Problem 1. Furthermore, the effect of increasing the number of univariate quadrature points is studied in Table 3. The table concludes a decreasing RMSE for an increasing number of univariate quadrature points, except for 8 univariate quadrature points which observed an increased average RMSE similar to the Problem 1. The possible reasons for this increased RMSE are the same as those discussed in Problem 1. Please note that the RMSE of the proposed SQKF is similar to the CQKF for the second state. However, Table 3 shows that the RMSE of the SQKF further reduces with increased univariate quadrature points. Thus, the claim that the proposed SQKF is the most accurate for oscillatory systems among all the considered Gaussian filters remains intact.

The relative computational time of the SQKF is observed as 1, 2.44, 4.71 and 8.01 for N_s being 2, 4, 6 and 8, respectively. Note that the relative increase in computational time with every increment in N_s is lesser for this problem compared to Problem 1 due to the smaller system dimension.

C. COMPARISON WITH PARTICLE FILTER AND RELATED TECHNIQUES

The performance of the proposed SQKF is further compared with PF and related techniques, unscented PF (UPF) and cubature PF (CPF). The PF and related techniques generally give high accuracy but require a large computational load. Thus, the comparison is based on the relative changes in RMSE for PF and related techniques (compared to the SQKF) with relative changes in computational time. The relative changes are shown in Table 4 for Problem 1 and Table 5 for Problem 2. As the computational burden of the SQKF increases exponentially, $SQKF_4$ is chosen for Problem 1 (gives 64 three-dimensional points); however, $SQKF_6$ is implemented for Problem 2 (gives only 36 two-dimensional points). The comparison is limited to 500 particles for the PF

TABLE 4. Problem 1: Relative RMSE and computational time for SQKF, PF, UPF and CPF.

Specified metrics for comparison		$SQKF_4$	50 particles			100 particles			200 particles			500 particles		
			PF	UPF	CPF	PF	UPF	CPF	PF	UPF	CPF	PF	UPF	CPF
Relative average RMSE	State 1	1	1.67	1.63	1.95	1.26	1.28	1.53	1.22	1.04	1.22	0.65	0.85	0.99
	State 2	1	1.6	1.49	1.65	1.31	1.26	1.36	1.18	1.11	1.18	1.02	1	1.03
	State 3	1	2.04	1.62	1.95	1.56	1.25	1.51	1.19	1	1.19	0.9	0.84	0.97
Relative computational time		1	0.83	7.07	6.09	1.46	13.6	11.5	2.93	27.1	25.4	7.36	66.6	63.6

TABLE 5. Problem 2: Relative RMSE and computational time for SQKF, PF, UPF and CPF.

Specified metrics for comparison		$SQKF_6$	50 particles			100 particles			200 particles			500 particles		
			PF	UPF	CPF	PF	UPF	CPF	PF	UPF	CPF	PF	UPF	CPF
Relative average RMSE	State 1	1	1.13	1.03	1.01	1.13	1.03	1.01	1.12	1.02	1	1.09	1.01	0.99
	State 2	1	1.05	0.96	0.95	1.04	0.95	0.94	1.02	0.95	0.93	1	0.95	0.93
Relative computational time		1	1.35	10.9	9.69	2.55	20.2	18.1	5.04	40.5	33.7	13.1	100	86.2

and related techniques, as the computational time becomes extensively large (compared to the SQKF) for this limit itself.

Table 4 shows an increased RMSE, i.e. poor accuracy, for the PF and related techniques with up to 200 particles in the case of Problem 1. The computational time for 200 particles itself is increased almost three times for the PF and more than 25 times for the UPF and CPF compared to the SQKF. The RMSE for the PF and related techniques improves when using 500 particles. However, the computational burden, in this case, becomes extensively larger than the SQKF.

Table 5 shows that the ordinary PF gives a higher RMSE, i.e. poor accuracy, in all cases, even as the computational burden increases thirteen times (for 500 particles). The UPF and CPF underperform the SQKF for the first state and outperform for almost all the chosen particles for the second state. The overall performance (collectively for the first and second states) of the UPF and CPF may be concluded as similar or marginally improved than the SQKF even as the computational burden increases as high as 100 times of the SQKF.

The overall analysis of Table 4 and 5 concludes that the proposed SQKF outperforms the PF and related techniques if the computational time of the two is comparable. The performance of the SQKF remains better even as the computational time of the PF and related techniques increase a few times to that of the SQKF. Moreover, the PF and related techniques may outperform the SQKF when their computational time is extensively larger than that of the SQKF. It should be mentioned that the sampling interval is usually small in the filtering with oscillatory systems to appropriately characterize the fast swings in the state dynamics due to the oscillation. Thus, a large computational time may not be ensured for PF and related techniques. Therefore, the SQKF may become a better choice than the PF and related techniques for the practical applications involving oscillation.

V. DISCUSSION AND CONCLUSION

Nonlinear estimation and filtering problems appear in several domains of science and technology, like target tracking, stochastic modeling, diagnosis and prognosis of industrial equipment, signal characterization, *etc*. Many of these problems are based on oscillating system dynamics. However, the existing Gaussian filters are accurate only for the systems represented by polynomials of particular orders. Subsequently, the development of an advanced Gaussian filter, accurate for the oscillatory systems, is a timely research problem. The inefficacy of the existing Gaussian filters for the oscillatory systems is mainly due to the inefficacy of their numerical approximation methods for the oscillatory functions. The proposed SQKF uses Szegő quadrature rule, which is accurate for Gaussian weighted integrals with oscillatory functions. Subsequently, it offers high estimation accuracy when filtering with oscillatory systems. The simulation results conclude the improved accuracy of the SQKF compared to the existing Gaussian filters in the oscillatory environment. The computational burden of the SQKF is similar to the GHF, which is higher than the UKF, CKF and CQKF. Moreover, the PF and related techniques perform poorer or comparable to the SQKF unless their computational time is not extensively larger than the SQKF. With an efficient solution to the filtering problems with oscillatory system dynamics, the SQKF can be a key Gaussian filter for future applications.

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