IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received September 16, 2020, accepted October 21, 2020, date of publication October 26, 2020, date of current version November 6, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3033929

Quasi-Tree Graphs With Extremal General Multiplicative Zagreb Indices

JIANWEI DU AND XIAOLING SUN
School of Science, North University of China, Taiyuan 030051, China
Corresponding authors: Jianwei Du (jianweidu@nuc.edu.cn) and Xiaoling Sun (sunxiaoling@nuc.edu.cn)

This work was supported by the Shanxi Province Science Foundation for Youths under Grant 201901D211227.

ABSTRACT Zagreb indices and their modified versions of a molecular graph are important molecular descriptors which can be applied in characterizing the structural properties of organic compounds from different aspects. In this article, by exploring the structures of the quasi-tree graphs with given different parameters (order, perfect matching and number of pendant vertices) and using the properties of the general multiplicative Zagreb indices, we determine the minimal and maximal values of general multiplicative Zagreb indices on quasi-tree graphs with given order, with perfect matchings, and with given number of pendant vertices. Furthermore, we present the minimal and maximal values of general multiplicative Zagreb indices on trees with perfect matchings.

INDEX TERMS General multiplicative Zagreb indices, quasi-tree graph, tree, perfect matching, pendant vertex.

I. INTRODUCTION

Topological molecular descriptors are mathematical invariants reflecting some biological and physico-chemical properties of organic compounds on the chemical graph, and they play a substantial role in materials science, chemistry and pharmacology, etc. (see [6], [7], [15]). The famous Zagreb indices is one of the first topological molecular descriptors. They are first introduced by Gutman and Trinajstić [8] and used to examine the structure dependence of total π-electron energy on molecular orbital. The first and second Zagreb indices M_1 and M_2, respectively, of a graph G are defined as:

$$M_1(G) = \sum_{v\in V(G)} d_G(v)^2,\ M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v),$$

where $d_G(v)$ is the degree of vertex v.

These two classical topological molecular descriptors (M_1 and M_2) and their modified versions have been applied in studying heterosystems, ZE-isomerism, chirality and complexity of molecule, etc. Todeschini *et al.* [16] had presented a version of Zagreb indices which nowadays are called multiplicative Zagreb indices, and they are expressed by

$$\Pi_1(G) = \prod_{v\in V(G)} d_G(v)^2,$$
$$\Pi_2(G) = \prod_{uv\in E(G)} d_G(u)d_G(v) = \prod_{v\in V(G)} d_G(v)^{d_G(v)}.$$

Many investigations of mathematical properties on the multiplicative Zagreb indices have been obtained, recent results see [10], [19]–[21], [23] and the references cited therein.

Recently, Vetrík and Balachandran [18] defined the first general multiplicative Zagreb index P_1^α and the second general multiplicative Zagreb index P_2^α, respectively, of a graph G as

$$P_1^\alpha(G) = \prod_{v\in V(G)} d_G(v)^\alpha,\ P_2^\alpha(G) = \prod_{v\in V(G)} d_G(v)^{\alpha d_G(v)} \quad (1)$$

for any real number $\alpha \neq 0$. Vetrík and Balachandran [18] determined the minimal and maximal general multiplicative Zagreb indices for trees with fixed order or segments or branching vertices or number of pendant vertices, and they also identified the extremal trees. Other relevant conclusions on general multiplicative Zagreb indices can be found in [1]–[3], [17].

The mathematical properties of general multiplicative Zagreb indices deserve further study since they can be used

The associate editor coordinating the review of this manuscript and approving it for publication was Yilun Shang.

to detect the chemical compounds which may have desirable properties. Namely, if one can find some properties well-correlated with these two descriptors for some value of α, then the extremal graphs should correspond to compounds with minimum or maximum value of that property. Furthermore, one such property has already been found for multiplicative Zagreb indices. Since general multiplicative Zagreb indices for some value of α can correlate with biological, physico-chemical and other properties of chemical compounds, we use graph theory to characterize these chemical structures. The vertices and edges of graphs represent the atoms and the chemical bonds of a compound, respectively.

We only deal with the simple connected graphs in this work. Let $G = (V(G), E(G))$ be the graph having vertex set $V(G)$ and edge set $E(G)$. Let $G + xy$ and $G - xy$ be the graphs obtained from G by adding the edge $xy \notin E(G)$ ($x, y \in V(G)$) and removing the edge $xy \in E(G)$, respectively. Denoted by $G - u$ the graph obtained from G by removing the vertex u ($u \in V(G)$) with its incident edges. Let us denote the number of vertices of degree i by n_i and the neighbourhoods of a vertex v in G by $N_G(v)$, respectively. If there exists $u \in V(G)$ such that $G - u$ is a tree, we call G a quasi-tree graph and u a quasi vertex. Suppose H_1 and H_2 are two vertex disjoint graphs. Let $H_1 \vee H_2$ be the graph having vertex set $V(H_1 \vee H_2) = V(H_1) \cup V(H_2)$ and edge set $E(H_1 \vee H_2) = E(H_1) \cup E(H_2) \cup \{v_1v_2 | v_1 \in V(H_1), v_2 \in V(H_2)\}$. As usual, let us denote the n-vertex cycle, n-vertex complete graph, n-vertex star and n-vertex path by C_n, K_n, S_n and P_n, respectively. We can refer [4] for other terminologies and notations.

In this work, by exploring the structures of the quasi-tree graphs with given different parameters (order, perfect matching and number of pendant vertices) and using the properties of the general multiplicative Zagreb indices, the minimal and maximal values of general multiplicative Zagreb indices on quasi-tree graphs with given order, with perfect matchings, and with given number of pendant vertices are determined. Furthermore, the minimal and maximal values of general multiplicative Zagreb indices on trees with perfect matchings are presented.

II. PRELIMINARIES

By simple calculation, we can get the following Lemmas 1 and 2.

Lemma 1: $f_1(x) = \frac{x+c}{x}$ is strictly decreasing for $x \geq 1$, where $c \geq 1$ is a constant.

Lemma 2: $f_2(x) = \frac{(x+c)^{x+c}}{x^x}$ is strictly increasing for $x \geq 1$, where $c \geq 1$ is a constant.

By the definition of P_i^α ($i = 1, 2$), the Lemma 3 is immediate.

Lemma 3: Suppose $G = (V, E)$ is a simple connected graph, for $e = xy \notin E(G)$, $x, y \in V(G)$, then

(i) $P_i^\alpha(G) < P_i^\alpha(G + e)$, $P_i^\alpha(G) > P_i^\alpha(G - e)$ ($i = 1, 2$) for $\alpha > 0$;

(ii) $P_i^\alpha(G) > P_i^\alpha(G + e)$, $P_i^\alpha(G) < P_i^\alpha(G - e)$ ($i = 1, 2$) for $\alpha < 0$.

Lemma 4: Let

$$g(m) = \frac{2^{2(m-2)}(m+1)^{m+1}}{3^{3(m-1)}},$$

where $m \geq 3$. Then $g(m) > 1$.

Proof: Let $h(m) = \ln g(m) = 2(m-2)\ln 2 + (m+1)\ln(m+1) - 3(m-1)\ln 3$. Then

$$\begin{aligned} h'(m) &= 2\ln 2 + \ln(m+1) + 1 - 3\ln 3 \\ &\geq \ln 4 + \ln 4 + 1 - \ln 27 = \ln \frac{16e}{27} > 0. \end{aligned}$$

So $h(m)$ is strictly increasing for $m \geq 3$. Therefore, $g(m) \geq g(3) = \frac{2^2 4^4}{3^6} = \frac{1024}{729} > 1$. □

III. GENERAL MULTIPLICATIVE ZAGREB INDICES OF QUASI-TREE GRAPHS

Let G be a quasi-tree graph and $u \in V(G)$ be a quasi vertex such that $G - u$ is a tree. If $d_G(u) = 1$, G is a tree with extremal general multiplicative Zagreb indices, that had been determined in [18]. Therefore, in what follows, we only discuss the case of $d_G(u) \geq 2$. Let

$\mathbf{QT}(n) = \{G | G$ is an n-vertex quasi-tree graph with $d_G(u) \geq 2$, where u is a quasi vertex of $G\}$.

Denoted by $\mathscr{C}_n$ the graph obtained from C_3 by adding $n-3$ pendant edges on its one vertex.

Lemma 5: Suppose $G \in \mathbf{QT}(n)$ such that G has the largest value of P_i^α for $\alpha > 0$, $i = 1, 2$. Let $u \in V(G)$ be a quasi vertex such that $G - u$ is a tree. Then $d_G(u) = n - 1$.

Proof: If $d_G(u) \leq n - 2$, then G contains a vertex w with $uw \notin V(G)$. Obviously, $G + uw \in \mathbf{QT}(n)$. By Lemma 3, $P_i^\alpha(G + uw) > P_i^\alpha(G)$ for $\alpha > 0$, $i = 1, 2$, a contradiction. Hence $d_G(u) = n - 1$. □

Theorem 1: Suppose G is in $\mathbf{QT}(n)$ with $n \geq 3$, then

$$4^\alpha(n-1)^\alpha \leq P_1^\alpha(G) \leq 4^\alpha(n-1)^\alpha 3^{\alpha(n-3)}$$

for $\alpha > 0$, the left (resp. right) equality holds only when $G \cong \mathscr{C}_n$ (resp. $G \cong P_{n-1} \vee K_1$).

Proof: By induction on n. For $n = 3$, $G \cong C_3$ and the theorem is true. Suppose $n > 3$ and the theorem is still valid for $\mathbf{QT}(n-1)$.

To begin with we determine the minimal $P_1^\alpha(G)$ for $\alpha > 0$. If G contains no pendant vertex, it can be seen that there exists $x \in V(G)$ with $d_G(x) = 2$ for $G \in \mathbf{QT}(n)$. Denote $N_G(x) = \{y_1, y_2\}$. If $y_1y_2 \notin E(G)$, set $G' = G - x + y_1y_2$. Then $G' \in \mathbf{QT}(n-1)$. By induction hypothesis and (1), it follows that

$$\begin{aligned} P_1^\alpha(G) &= 2^\alpha \prod_{y \in V(G)\setminus\{x\}} d_G(y)^\alpha = 2^\alpha P_1^\alpha(G') \\ &\geq 2^\alpha \cdot 4^\alpha(n-2)^\alpha = 4^\alpha(2(n-2))^\alpha \\ &> 4^\alpha(n-1)^\alpha \end{aligned}$$

for $\alpha > 0$.

If $y_1y_2 \in E(G)$, set $G'' = G - x$, then $G'' \in \mathbf{QT}(n-1)$. By induction hypothesis and Lemma 1, it follows that

$$\begin{aligned} P_1^\alpha(G) &= 2^\alpha d_G(y_1)^\alpha d_G(y_2)^\alpha \prod_{y \in V(G)\setminus\{x,y_1,y_2\}} d_G(y)^\alpha \\ &= 2^\alpha \frac{d_G(y_1)^\alpha}{(d_G(y_1)-1)^\alpha} \frac{d_G(y_2)^\alpha}{(d_G(y_2)-1)^\alpha} P_1^\alpha(G'') \\ &\geq 2^\alpha \left(\frac{n-1}{n-2}\right)^\alpha \left(\frac{n-1}{n-2}\right)^\alpha 4^\alpha (n-2)^\alpha \\ &> 4^\alpha (2(n-2))^\alpha > 4^\alpha (n-1)^\alpha \end{aligned}$$

for $\alpha > 0$.

Otherwise, G contains pendant vertices. Suppose $v \in V(G)$ with $d_G(v) = 1$. Clearly, $G - v \in \mathbf{QT}(n-1)$. We use w to denote the neighbour of v. By induction hypothesis and Lemma 1, it follows that

$$\begin{aligned} P_1^\alpha(G) &= d_G(w)^\alpha \prod_{y \in V(G)\setminus\{v,w\}} d_G(y)^\alpha \\ &= \frac{d_G(w)^\alpha}{(d_G(w)-1)^\alpha} P_1^\alpha(G-v) \\ &\geq \left(\frac{n-1}{n-2}\right)^\alpha 4^\alpha (n-2)^\alpha \\ &= 4^\alpha (n-1)^\alpha \end{aligned}$$

for $\alpha > 0$. With equality only when $d_G(w) = n-1$ and $G - v \cong \mathscr{C}_{n-1}$. This implies $G \cong \mathscr{C}_n$.

Next, we determine the maximal $P_1^\alpha(G)$ for $\alpha > 0$. Choose $G \in \mathbf{QT}(n)$ with the maximum value of $P_1^\alpha(G)$ for $\alpha > 0$. Let $u \in V(G)$ be a quasi vertex such that $G - u$ is a tree. By Lemma 5, $d_G(u) = n-1$. Thus G contains $y \in V(G)$ with $d_G(y) = 2$ for $G \in \mathbf{QT}(n)$. Denote $N_G(y) \setminus \{u\} = \{z\}$. Clearly, $d_G(z) \geq 3$ and $G - y \in \mathbf{QT}(n-1)$. By induction hypothesis and Lemma 1, it follows that

$$\begin{aligned} P_1^\alpha(G) &= d_G(y)^\alpha d_G(u)^\alpha d_G(z)^\alpha \prod_{x \in V(G)\setminus\{y,z,u\}} d_G(x)^\alpha \\ &= 2^\alpha \frac{(n-1)^\alpha}{(n-2)^\alpha} \frac{d_G(z)^\alpha}{(d_G(z)-1)^\alpha} P_1^\alpha(G-y) \\ &\leq 2^\alpha \left(\frac{n-1}{n-2}\right)^\alpha \left(\frac{3}{2}\right)^\alpha 4^\alpha (n-2)^\alpha 3^{\alpha(n-4)} \\ &= 4^\alpha (n-1)^\alpha 3^{\alpha(n-3)} \end{aligned}$$

for $\alpha > 0$. With equality holds only if $d_G(z) = 3$ and $G - y \cong P_{n-2} \vee K_1$. This implies $G \cong P_{n-1} \vee K_1$. □

Theorem 2: Suppose G is in $\mathbf{QT}(n)$ with $n \geq 3$, then

$$2^{2\alpha n} \leq P_2^\alpha(G) \leq 2^{2\alpha(n-2)}(n-1)^{2\alpha(n-1)}$$

for $\alpha > 0$, the left (resp. right) equality holds only when $G \cong C_n$ (resp. $G \cong S_{n-1} \vee K_1$).

Proof: By induction on n. For $n = 3$, $G \cong C_3$ and the theorem is true. Suppose $n > 3$ and the theorem is still valid for $\mathbf{QT}(n-1)$.

To begin with we determine the minimal $P_2^\alpha(G)$ for $\alpha > 0$. If G contains no pendant vertex, it can be seen that there exists $x \in V(G)$ with $d_G(x) = 2$ for $G \in \mathbf{QT}(n)$. Denote $N_G(x) = \{y_1, y_2\}$. If $y_1y_2 \notin E(G)$, set $G' = G - x + y_1y_2$. Then $G' \in \mathbf{QT}(n-1)$. By induction hypothesis and (1), it follows that

$$\begin{aligned} P_2^\alpha(G) &= 2^{2\alpha} \prod_{y \in V(G)\setminus\{x\}} d_G(y)^{\alpha d_G(y)} \\ &= 2^{2\alpha} P_2^\alpha(G') \\ &\geq 2^{2\alpha} 2^{2\alpha(n-1)} = 2^{2\alpha n} \end{aligned}$$

for $\alpha > 0$. With equality holds only when $G' \cong C_{n-1}$. This implies that $G \cong C_n$.

If $y_1y_2 \in E(G)$, set $G'' = G - x$, then $G'' \in \mathbf{QT}(n-1)$. By induction hypothesis and Lemma 2, it follows that

$$\begin{aligned} P_2^\alpha(G) &= 2^{2\alpha} d_G(y_1)^{\alpha d_G(y_1)} d_G(y_2)^{\alpha d_G(y_2)} \\ &\quad \cdot \prod_{y \in V(G)\setminus\{x,y_1,y_2\}} d_G(y)^{\alpha d_G(y)} \\ &= 2^{2\alpha} \frac{d_G(y_1)^{\alpha d_G(y_1)}}{(d_G(y_1)-1)^{\alpha(d_G(y_1)-1)}} \\ &\quad \cdot \frac{d_G(y_2)^{\alpha d_G(y_2)}}{(d_G(y_2)-1)^{\alpha(d_G(y_2)-1)}} P_2^\alpha(G'') \\ &> 2^{2\alpha} 2^{2\alpha} 2^{2\alpha} 2^{2\alpha(n-1)} > 2^{2\alpha n} \end{aligned}$$

for $\alpha > 0$.

Otherwise, G contains pendant vertices. Suppose $v \in V(G)$ with $d_G(v) = 1$. Clearly, $G - v$ is in $\mathbf{QT}(n-1)$. Denoted by $N_G(v) = \{w\}$. It is evident that $d_G(w) \geq 2$. For $d_G(w) = 2$, we have $G - v \ncong C_{n-1}$ since $G \in \mathbf{QT}(n)$. By induction hypothesis and (1), we have

$$P_2^\alpha(G) = 2^{2\alpha} P_2^\alpha(G-v) > 2^{2\alpha} 2^{2\alpha(n-1)} = 2^{2\alpha n}$$

for $\alpha > 0$.

If $d_G(w) \geq 3$, by (1), induction hypothesis and Lemma 2, we have

$$\begin{aligned} P_2^\alpha(G) &= \frac{d_G(w)^{\alpha d_G(w)}}{(d_G(w)-1)^{\alpha(d_G(w)-1)}} P_2^\alpha(G-v) \\ &\geq \frac{3^{3\alpha}}{2^{2\alpha}} 2^{2\alpha(n-1)} = \left(\frac{27}{16}\right)^\alpha 2^{2\alpha n} > 2^{2\alpha n} \end{aligned}$$

for $\alpha > 0$.

Next, we determine the maximal $P_2^\alpha(G)$ for $\alpha > 0$. Choose $G \in \mathbf{QT}(n)$ with the maximum value of $P_2^\alpha(G)$ for $\alpha > 0$. In view of Lemma 5, we can see that G contains a vertex y with $d_G(y) = 2$ since $G \in \mathbf{QT}(n)$. From Lemma 5, we have $G - y \in \mathbf{QT}(n-1)$. Let $N_G(y) = \{z_1, z_2\}$. If $d_G(z_1) = d_G(z_2) = n-1$, it follows that $G - \{z_1, z_2\}$ has no edges since $G \in \mathbf{QT}(n)$. And this implies $G \cong S_{n-1} \vee K_1$. If one of $d_G(z_1)$ and $d_G(z_2)$ is less than $n-1$, then $G \ncong S_{n-1} \vee K_1$. By (1), induction hypothesis and Lemma 2, it follows that

$$\begin{aligned} P_2^\alpha(G) &= 2^\alpha d_G(z_1)^{\alpha d_G(z_1)} d_G(z_2)^{\alpha d_G(z_2)} \\ &\quad \cdot \prod_{x \in V(G)\setminus\{y,z_1,z_2\}} d_G(x)^{\alpha d_G(x)} \\ &= 2^\alpha \frac{d_G(z_1)^{\alpha d_G(z_1)}}{(d_G(z_1)-1)^{\alpha(d_G(z_1)-1)}} \\ &\quad \cdot \frac{d_G(z_2)^{\alpha d_G(z_2)}}{(d_G(z_2)-1)^{\alpha(d_G(z_2)-1)}} P_2^\alpha(G-y) \end{aligned}$$

$$< 2^{\alpha}\Big(\frac{(n-1)^{n-1}}{(n-2)^{n-2}}\Big)^{\alpha}\Big(\frac{(n-1)^{n-1}}{(n-2)^{n-2}}\Big)^{\alpha} \cdot 2^{2\alpha(n-3)}(n-2)^{2\alpha(n-2)} = 2^{2\alpha(n-2)}(n-1)^{2\alpha(n-1)}$$

for $\alpha > 0$.

The proof is completed. □

In [18], Vetrík and Balachandran determined the minimum and maximum general multiplicative Zagreb indices on trees of order n. The results for $\alpha > 0$ are as follows.

Theorem 3 [18]*:* Let T be a tree on n vertices and $T \ncong P_n, S_n$. Then

$$P_1^{\alpha}(S_n) < P_1^{\alpha}(T) < P_1^{\alpha}(P_n),$$
$$P_2^{\alpha}(P_n) < P_2^{\alpha}(T) < P_2^{\alpha}(S_n).$$

for $\alpha > 0$.

Therefore, by Theorem 3, one can expand the conclusions for $G \in \mathbf{QT}(n)$ to the whole quasi-tree graphs, as described next.

Theorem 4: Suppose G is a quasi-tree graph with n vertices, then

$$(n-1)^{\alpha} \leq P_1^{\alpha}(G) \leq 3^{\alpha(n-3)}4^{\alpha}(n-1)^{\alpha}$$

for $\alpha > 0$, with the left (resp. right) equality only when $G \cong S_n$ (resp. $G \cong P_{n-1} \vee K_1$).

Theorem 5: Suppose G is a quasi-tree graph with n vertices, then

$$2^{2\alpha(n-2)} \leq P_2^{\alpha}(G) \leq 2^{2\alpha(n-2)}(n-1)^{2\alpha(n-1)}$$

for $\alpha > 0$, with the left (resp. right) equality only when $G \cong P_n$ (resp. $G \cong S_{n-1} \vee K_1$).

IV. GENERAL MULTIPLICATIVE ZAGREB INDICES OF TREES WITH A PERFECT MATCHING

Let $\boldsymbol{T}_{2m}$ be the tree of order $2m$ arisen from S_{m+1} by adding one pendant edge to its $m-1$ pendant vertices.

Theorem 6: Suppose T is an $2m$-vertex tree with a perfect matching, where $m \geq 2$, then

$$P_1^{\alpha}(T) \geq 2^{\alpha(m-1)}m^{\alpha} \text{ and } P_2^{\alpha}(T) \leq 2^{2\alpha(m-1)}m^{\alpha m}$$

for $\alpha > 0$, with equalities only when $T \cong \boldsymbol{T}_{2m}$.

Proof: If $m = 2$, $T \cong P_4 = \boldsymbol{T}_4$, the theorem holds.

If $m \geq 3$, choose T such that T has the minimum P_1^{α} (maximum P_2^{α}) for $\alpha > 0$. Let M be a perfect matching of T. Suppose $v \in V(T)$ is a vertex with the maximum degree of T.

Claim 1. $d_T(x) \leq 2$ for each vertex $x \in V(T)$ different from v.

On the contrary suppose that there exists $v' \in V(T)\backslash\{v\}$ such that $d_T(v') \geq 3$. Denote $N_T(v) = \{x_1, x_2, \cdots, x_r\}$ and $N_T(v') = \{y_1, y_2, \cdots, y_s\}$, where $r \geq s \geq 3$. Suppose P is the only path from v to v' in T. Without loss of generality, assume that $x_1, y_1 \in V(P)$ (perhaps $x_1 = v'$ or $y_1 = v$). Note that $|M \cap \{y_2v', y_3v', \cdots, y_sv'\}| \leq 1$. We assume that $y_3v', \cdots, y_sv' \notin M$. Set $T' = T - \{y_3v', \cdots, y_sv'\} + \{y_3v, \cdots, y_sv\}$. Obviously, T is also an $2m$-vertex tree with a perfect matching. By (1) and Lemmas 1,2 ($c = s-2$), for $\alpha > 0$, it follows that

$$\frac{P_1^{\alpha}(T')}{P_1^{\alpha}(T)} = \frac{(r+s-2)^{\alpha}2^{\alpha}}{s^{\alpha}r^{\alpha}} = \Big(\frac{\frac{r+s-2}{r}}{\frac{2+s-2}{2}}\Big)^{\alpha} < 1,$$

and

$$\frac{P_2^{\alpha}(T')}{P_2^{\alpha}(T)} = \frac{(r+s-2)^{\alpha(r+s-2)}2^{2\alpha}}{s^{\alpha s}r^{\alpha r}} = \Big(\frac{\frac{(r+s-2)^{r+s-2}}{r^r}}{\frac{s^s}{2^2}}\Big)^{\alpha} > 1,$$

a contradiction to the choose of T.

By Claim 1, T is a tree having some pendant paths attached to v.

Claim 2. $d_T(v) > 2$.

To the contrary suppose that $d_T(v) \leq 2$. Since v is the maximum degree vertex of T, then $d_T(v) \geq 2$. Hence $d_T(v) = 2$ and $T \cong P_{2m}$. Since $m \geq 3$, then $\boldsymbol{T}_{2m} \ncong P_{2m}$. By Theorem 3, $P_1^{\alpha}(\boldsymbol{T}_{2m}) < P_1^{\alpha}(P_{2m})$ and $P_2^{\alpha}(\boldsymbol{T}_{2m}) > P_2^{\alpha}(P_{2m})$, a contradiction with the choose of T.

We use $P_1, P_2, \cdots, P_l$ ($l \geq 3$) to denote the paths attached to v in T.

Claim 3. The length of P_i ($i = 1, 2, \cdots, l$) is of equal or less than 2 in T.

On the contrary, if there exists P_i ($i \in \{1, 2, \cdots, l\}$) with length greater than 2 in T, assume without loss of generality that $|E(P_1)| \geq 3$ in T. Let $P_1 = v_1v_2\cdots v_r$, where $v_1 = v$ and $r \geq 4$. Then P_1 contains at least one edge v_jv_{j+1} with $v_jv_{j+1} \notin M$ and $j \in \{2, 3, \cdots, r-1\}$. Set $T'' = T - v_jv_{j+1} + vv_{j+1}$. Clearly, T'' is also an $2m$-vertex tree with a perfect matching. By (1) and Lemmas 1,2, for $l \geq 3$ and $\alpha > 0$, we have

$$\frac{P_1^{\alpha}(T'')}{P_1^{\alpha}(T)} = \frac{(d_T(v)+1)^{\alpha}}{2^{\alpha}d_T(v)^{\alpha}} = \Big(\frac{l+1}{2l}\Big)^{\alpha} = \Big(\frac{\frac{l+1}{l}}{\frac{2}{1}}\Big)^{\alpha} < 1,$$

and

$$\frac{P_2^{\alpha}(T'')}{P_2^{\alpha}(T)} = \frac{(d_T(v)+1)^{\alpha(d_T(v)+1)}}{d_T(v)^{\alpha d_T(v)}2^{2\alpha}} = \Big(\frac{(l+1)^{l+1}}{l^l 2^2}\Big)^{\alpha} = \Big(\frac{\frac{(l+1)^{l+1}}{l^l}}{\frac{2^2}{1^1}}\Big)^{\alpha} > 1,$$

a contradiction again.

Denote $V_1 = \{x \in V(T)|d_T(x) = 1, xv \in E(G)\}$. Since T has a perfect matching, from Claim 3, it can be concluded that $|V_1| = 1$. This implies $T \cong \boldsymbol{T}_{2m}$. □

In view of Theorem 3, the following Theorem 7 is immediate.

Theorem 7: Let T be a tree on $2m$ vertices with a perfect matching, where $m \geq 2$. Then

$$P_1^{\alpha}(T) \leq 2^{2\alpha(m-1)} \text{ and } P_2^{\alpha}(T) \geq 2^{4\alpha(m-1)}$$

for $\alpha > 0$, with equalities only when $T \cong P_{2m}$.

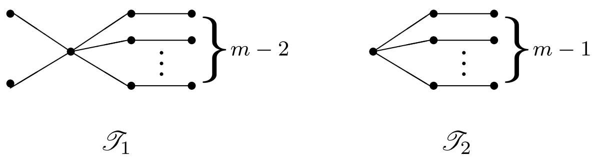

FIGURE 1. The graphs $\mathscr{T}_1$ and $\mathscr{T}_2$.

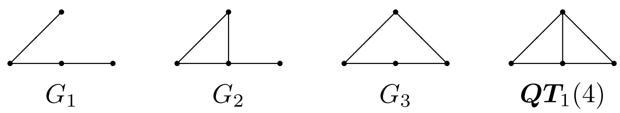

FIGURE 2. The graphs G_1, G_2, G_3 and $\boldsymbol{QT}_1(4)$.

V. GENERAL MULTIPLICATIVE ZAGREB INDICES OF QUASI-TREE GRAPHS WITH A PERFECT MATCHING

Suppose G is a quasi-tree graph with a perfect matching and $u \in V(G)$ is a quasi vertex such that $G-u$ is a tree. In view of Lemma 3, if G has the minimal value of P_i^α $(i = 1, 2)$ for $\alpha > 0$, then $d_G(u) = 1$. When $d_G(u) = 1$, then G is a tree with minimum P_i^α $(\alpha > 0)$ for $i = 1, 2$, that had been presented in Theorems 6 and 7. Thus, in this section, we only find the maximal value of P_i^α $(i = 1, 2)$ of G.

Let $\mathscr{T}_1$ be the tree on $2m-1$ vertices obtained from S_{m+1} by attaching one pendant edge to its $m-2$ pendant vertices, see Figure 1. Let $\mathscr{T}_2$ be the tree on $2m-1$ vertices obtained from S_m by attaching one pendant edge to its each pendant vertex, see Figure 1. Let $\boldsymbol{QT}_1(2m) = \mathscr{T}_1 \vee K_1$ and $\boldsymbol{QT}_2(2m) = \mathscr{T}_2 \vee K_1$.

Lemma 6: Suppose $m \geq 3$ is a positive integer, then

$$P_2^\alpha(\boldsymbol{QT}_1(2m)) > P_2^\alpha(\boldsymbol{QT}_2(2m)) \tag{2}$$

for $\alpha > 0$.

Proof: By (1) and Lemma 2, it follows that

$$\begin{aligned}
&\frac{P_2^\alpha(\boldsymbol{QT}_1(2m))}{P_2^\alpha(\boldsymbol{QT}_2(2m))}\\
&= \frac{2^{2\alpha m}3^{3\alpha(m-2)}(m+1)^{\alpha(m+1)}(2m-1)^{\alpha(2m-1)}}{2^{2\alpha(m-1)}3^{3\alpha(m-1)}m^{\alpha m}(2m-1)^{\alpha(2m-1)}}\\
&= \left(\frac{2^2(m+1)^{m+1}}{3^3 m^m}\right)^\alpha\\
&\geq \left(\frac{2^2 4^4}{3^3 3^3}\right)^\alpha = \left(\frac{1024}{729}\right)^\alpha > 1
\end{aligned}$$

for $\alpha > 0$ and $m \geq 3$. □

Theorem 8: Suppose G is an $2m$-vertex quasi-tree graph with a perfect matching, where $m \geq 2$, then

$$P_1^\alpha(G) \leq (2m-1)^\alpha 3^{\alpha(2m-3)} 2^{2\alpha}$$

for $\alpha > 0$, with equality only when $G \cong P_{2m-1} \vee K_1$.

Proof: If $m = 2$, $G \in \{G_1, G_2, G_3, \boldsymbol{QT}_1(4)\}$ (as displayed in Figure 2). From Lemma 3, it follows that $P_1^\alpha(\boldsymbol{QT}_1(4)) > P_1^\alpha(G_i)$ for $\alpha > 0$, i=1,2,3.

If $m \geq 3$, choose G with the maximal value of P_1^α for $\alpha > 0$. We suppose that there is a quasi vertex $u \in V(G)$ such that $T = G - u$ is a tree since G is a quasi-tree graph. Suppose $v \in V(T)$ is a vertex with the maximum degree of T.

Claim 1. $uy \in E(G)$ for each $y \in V(T)$.

A similar proof of Lemma 5, omitted.

Claim 1 implies that $G \cong T \vee K_1$.

Claim 2. $d_T(v) = 2$.

On the contrary assume that $d_T(v) \geq 3$. Denote $N_T(v) = \{x_1, x_2, \cdots, x_r\}$, $r \geq 3$. Since $2m \geq 6$, then there exist x_i $(i \in \{1, 2, \cdots, r\})$ such that $d_T(x_i) \geq 2$ and $vx_i \notin M$. Assume without loss of generality that $d_T(x_1) \geq 2$ and $vx_1 \notin M$. Suppose z is a pendant vertex in T such that x_1 is not belongs to the vertices of the path from z to v of T. Set $T' = T - x_1v + zx_1$. Then $G' = T' \vee K_1$ is also a quasi-tree graph of order $2m$ with a perfect matching. In view of (1) and Lemma 1, it can be concluded that

$$\frac{P_1^\alpha(G')}{P_1^\alpha(G)} = \frac{(r)^\alpha 3^\alpha}{(r+1)^\alpha 2^\alpha} = \left(\frac{\frac{3}{2}}{\frac{r+1}{r}}\right)^\alpha > 1$$

for $\alpha > 0$, which contradicts the choose of G. So T is a path on $2m-1$ vertices. □

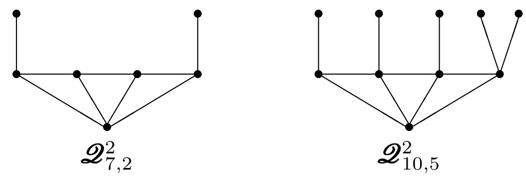

FIGURE 3. The graphs $\mathscr{Q}^2_{7,2}$ and $\mathscr{Q}^2_{10,5}$.

Theorem 9: Suppose G is an $2m$-vertex quasi-tree graph with a perfect matching, where $m \geq 2$, then

$$P_2^\alpha(G) \leq 2^{2\alpha m} 3^{3\alpha(m-2)} (m+1)^{\alpha(m+1)} (2m-1)^{\alpha(2m-1)}$$

for $\alpha > 0$, with equality only when $G \cong \boldsymbol{QT}_1(2m)$.

Proof: If $m = 2$, $G \in \{G_1, G_2, G_3, \boldsymbol{QT}_1(4)\}$ (as depicted in Figure 3). In view of Lemma 3, it follows that $P_2^\alpha(\boldsymbol{QT}_1(4)) > P_2^\alpha(G_i)$, i=1,2,3.

If $m \geq 3$, choose G with the maximum value of P_2^α for $\alpha > 0$. Let M be a perfect matching of G. We suppose that there is a quasi vertex $u \in V(G)$ such that $T = G - u$ is a tree since G is a quasi-tree graph. Suppose $v \in V(T)$ is a vertex with the maximum degree of T.

Claim 1. $uy \in E(G)$ for every $y \in V(T)$.

A similar proof of Lemma 5, omitted.

Claim 2. $d_T(x) \leq 2$ for each $x \in V(T)$ except v.

On the contrary suppose that there exists $v' \in V(T)\backslash\{v\}$ such that $d_T(v') \geq 3$. Denote $N_T(v) = \{x_1, x_2, \cdots, x_r\}$ and $N_T(v') = \{y_1, y_2, \cdots, y_s\}$, where $3 \leq s \leq r$. From Claim 1, we have $d_G(v) = r + 1$ and $d_G(v') = s + 1$. Assume that P is the only path from v to v' in T. Without loss of generality, suppose that $x_1, y_1 \in V(P)$ (perhaps $v' = x_1$ or $v = y_1$). Note that $|M \cap \{y_2v', y_3v', \cdots, y_sv'\}| \leq 1$. Assume without loss of generality that $y_3v', \cdots, y_sv' \notin M$. Let $G' = G - \{y_3v', \cdots, y_sv'\} + \{vy_3, \cdots, vy_s\}$. Obviously, G' is also a quasi-tree graph on $2m$ vertices with a perfect matching. By (1) and Lemma 2 ($c = s-2$), it follows that

$$\frac{P_2^\alpha(G')}{P_2^\alpha(G)} = \frac{3^{3\alpha}(r+s-1)^{\alpha(r+s-1)}}{(r+1)^{\alpha(r+1)}(s+1)^{\alpha(s+1)}} = \left(\frac{\frac{(r+s-1)^{r+s-1}}{(r+1)^{r+1}}}{\frac{(s+1)^{s+1}}{3^3}}\right)^\alpha > 1$$

for $\alpha > 0$, which contradicts the choice of G.

By Claim 2, T is a tree with some pendant paths attached to v.

Claim 3. $d_T(v) > 2$.

To the contrary suppose that $d_T(v) \leq 2$. Since v is the maximum degree vertex of T, then $d_T(v) \geq 2$. Hence $d_T(v) = 2$ and T is a path of order $2m-1$. In view of Claim 1, we have $G \cong P_{2m-1} \vee K_1$. One can easily come to the conclusion that $P_2^\alpha(G) = (2m-1)^{\alpha(2m-1)}3^{3\alpha(2m-3)}2^{4\alpha}$. By Lemma 4, for $m \geq 3$ and $\alpha > 0$, it follows that

$$\frac{P_2^\alpha(\boldsymbol{QT}_1(2m))}{P_2^\alpha(G)} = \frac{(2m-1)^{\alpha(2m-1)}(m+1)^{\alpha(m+1)}3^{3\alpha(m-2)}2^{2\alpha m}}{(2m-1)^{\alpha(2m-1)}3^{3\alpha(2m-3)}2^{4\alpha}} = \left(\frac{(m+1)^{m+1}2^{2(m-2)}}{3^{3(m-1)}}\right)^\alpha > 1,$$

a contradiction to the choose of G.

We use $P_1, P_2, \cdots, P_l$ ($l \geq 3$) to denote the paths attached to v in T.

Claim 4. The length of P_i ($i = 1, 2, \cdots, l$) is of equal or less than 2 in T.

On the contrary, if there exists P_i ($i \in \{1, 2, \cdots, l\}$) with length greater than 2 in T, without loss of generality, assume that $|E(P_1)| \geq 3$ in T. Let $P_1 = v_1v_2\cdots v_r$, where $v_1 = v$ and $r \geq 4$. Then P_1 contains at least one edge v_jv_{j+1} ($j \in \{2, 3, \cdots, r-1\}$) such that $v_jv_{j+1} \notin M$. Set $G'' = G - v_jv_{j+1} + vv_{j+1}$. Obviously, G'' is also a quasi-tree graph of order $2m$ with a perfect matching. By (1) and Lemma 2, for $l \geq 3$ and $\alpha > 0$, we have

$$\frac{P_2^\alpha(G'')}{P_2^\alpha(G)} = \frac{2^{2\alpha}(d_G(v)+1)^{\alpha(d_G(v)+1)}}{3^{3\alpha}d_G(v)^{\alpha d_G(v)}} = \left(\frac{(l+2)^{l+2}2^2}{(l+1)^{l+1}3^3}\right)^\alpha = \left(\frac{\frac{(l+2)^{l+2}}{(l+1)^{l+1}}}{\frac{3^3}{2^2}}\right)^\alpha > 1,$$

a contradiction again.

Denote $V_1 = \{x \in V(T) | d_T(x) = 1, xv \in E(G)\}$. Since G has a perfect matching, in view of Claim 4, it can be concluded that $|V_1| = 0$ or $|V_1| = 2$.

If $|V_1| = 0$, then $G \cong \boldsymbol{QT}_2(2m)$. If $|V_1| = 2$, then $G \cong \boldsymbol{QT}_1(2m)$. By (2), for $\alpha > 0$ and $m \geq 3$, $P_2^\alpha(\boldsymbol{QT}_1(2m)) > P_2^\alpha(\boldsymbol{QT}_2(2m))$. Thus, $G \cong \boldsymbol{QT}_1(2m)$. □

VI. GENERAL MULTIPLICATIVE ZAGREB INDICES OF QUASI-TREE GRAPHS WITH GIVEN PENDANT VERTICES

Let G be an n-vertex quasi-tree graph having k pendant vertices and $u \in V(G)$ be a quasi vertex such that $G - u$ is a tree. If $k = n-1$, G is a star. If $k = n-2$, G is a double star which belongs to trees and trees having $n-2$ pendant vertices with extremal P_i^α had been obtained in [18]. Furthermore, by Lemma 3, if G has the minimum P_i^α ($i = 1, 2$) for $\alpha > 0$, then $d_G(u) = 1$. For $d_G(u) = 1$, G is a tree with minimum P_i^α for $i = 1, 2$, that had been presented in [18]. Thus, we only consider the maximum P_i^α ($i = 1, 2$) of G with $1 \leq k \leq n-3$ in this section.

Let $\mathscr{Q}_{n,k}^1$ be the graph arisen from S_{n-1} and an isolated vertex u by adding edges to connecting u with the centeral vertex and $n-k-2$ pendant vertices of S_{n-1}.

Let $\mathscr{Q}_{n,k}^2$ be the n-vertex quasi-tree graphs having k pendant vertices and degree sequence $(n-k-1, \underbrace{t+2, \cdots, t+2}_{2n-k-4-t(n-k-1)}, \underbrace{t+1, \cdots, t+1}_{t(n-k-1)-n+3}, \underbrace{1, \cdots, 1}_{k})$, where $t = \lfloor \frac{2n-k-4}{n-k-1} \rfloor$. In Figure 3, we have drawn $\mathscr{Q}_{7,2}^2$ and $\mathscr{Q}_{10,5}^2$. Obviously, sometimes the graph $\mathscr{Q}_{n,p}^2$ is not unique.

Theorem 10: Let G be a quasi-tree graph on n vertices with k pendant vertices, where $1 \leq k \leq n-3$, then

$$P_1^\alpha(G) \leq (n-k-1)^\alpha (t+2)^{\alpha[2n-k-4-t(n-k-1)]} \cdot (t+1)^{\alpha[t(n-k-1)-n+3]}$$

for $\alpha > 0$, where $t = \lfloor \frac{2n-k-4}{n-k-1} \rfloor$, with equality only when $G \cong \mathscr{Q}_{n,k}^2$.

Proof: Choose G with the maximum value of P_1^α for $\alpha > 0$. We suppose that there is a quasi vertex $u \in V(G)$ such that $T = G - u$ is a tree since G is a quasi-tree graph. Clearly, u is not adjacent to the pendant vertices of G, otherwise $G - u$ is unconnected. Denoted by $\{y_1, y_2, \cdots, y_k\}$ the pendant vertices of G.

Claim 1. $uy \in E(G)$ for each vertex $y \in (V(T) \setminus \{y_1, y_2, \cdots, y_k\})$.

Similar to the proof of lemma 5, omitted.

Claim 2. If $d_T(v), d_T(w) \geq 2$, then $|d_T(v) - d_T(w)| \leq 1$.

Suppose that T contains two vertices v, w satisfying $|d_T(v) - d_T(w)| \geq 2$. Assume $d_T(v) - 2 = r - 2 \geq d_T(w) = s \geq 2$. Choose $v' \in N_T(v)$ such that v' is not on the path from v to w in T. Set $G' = G - vv' + wv'$. Then G' is also a quasi-tree graph on n vertices with k pendant vertices. By Lemma 1, we have

$$\frac{P_1^\alpha(G')}{P_1^\alpha(G)} = \frac{(s+1)^\alpha(r-1)^\alpha}{r^\alpha s^\alpha} = \left(\frac{\frac{s+1}{s}}{\frac{r}{r-1}}\right)^\alpha > 1.$$

A contradiction to the choose of G.

By Claims 1 and 2, one can get that G has degree 1, i or $i+1$, $n-k-1$, where $i \geq 2$. Hence

$$1 + k + n_i + n_{i+1} = n. \tag{3}$$

Since $\sum_{y\in V(G)} d_G(y) = 2|E(G)|$, then

$$k + in_i + (i+1)n_{i+1} + n - k - 1 = 2(n-k-1+n-2) = 2(2n-k-3). \tag{4}$$

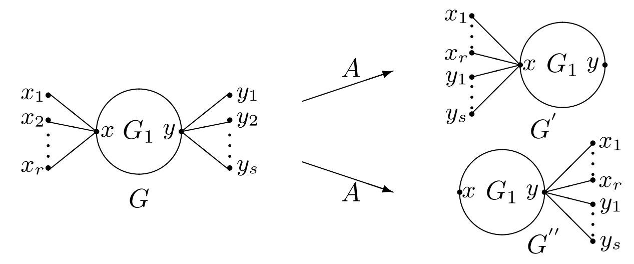

FIGURE 4. Transformation *A*.

By (3) and (4), we have $i = \frac{2n-k-4}{n-k-1} + \frac{n_i}{n-k-1}$. By (3), $n_i \leq n-k-1$, hence $i = \lfloor \frac{2n-k-4}{n-k-1} \rfloor + 1$.

Furthermore, we have $n_i = (n-k-1)\lfloor \frac{2n-k-4}{n-k-1} \rfloor - n + 3$, $n_{i+1} = 2n-k-4-(n-k-1)\lfloor \frac{2n-k-4}{n-k-1} \rfloor$.

Therefore we can get the degree sequence of G, that is

$$\begin{aligned}&(n-k-1, \underbrace{i+1, \cdots, i+1}_{n_{i+1}}, \underbrace{i, \cdots, i}_{n_i}, \underbrace{1, \cdots, 1}_{k})\\ &\quad = (n-k-1, \underbrace{t+2, \cdots, t+2}_{2n-k-4-t(n-k-1)}, \underbrace{t+1, \cdots, t+1}_{t(n-k-1)-n+3},\\ &\qquad \underbrace{1, \cdots, 1}_{k}),\end{aligned}$$

where $t = \lfloor \frac{2n-k-4}{n-k-1} \rfloor$. □

Transformation A: Let G be the graph as shown in Figure 4, and $x, y \in V(G_1)$, where $x_1, x_2, \cdots, x_r$ are pendant vertices adjacent to x, and $y_1, y_2, \cdots, y_s$ are pendant vertices adjacent to y. Set $G' = G - \{yy_1, yy_2, \cdots, yy_s\} + \{xy_1, xy_2, \cdots, xy_s\}$, $G'' = G - \{xx_1, xx_2, \cdots, xx_r\} + \{yx_1, yx_2, \cdots, yx_r\}$, as shown in Figure 4.

Lemma 7: Suppose G', G'' and G are the graphs in Figure 4, then either $P_2^\alpha(G') < P_2^\alpha(G)$ or $P_2^\alpha(G'') < P_2^\alpha(G)$ for $\alpha < 0$, and either $P_2^\alpha(G') > P_2^\alpha(G)$ or $P_2^\alpha(G'') > P_2^\alpha(G)$ for $\alpha > 0$.

Proof: By (1), it follows that

$$\begin{aligned}\frac{P_2^\alpha(G')}{P_2^\alpha(G)} &= \left(\frac{(d_G(x)+s)^{d_G(x)+s}(d_G(y)-s)^{d_G(y)-s}}{d_G(x)^{d_G(x)}d_G(y)^{d_G(y)}}\right)^\alpha,\\ \frac{P_2^\alpha(G'')}{P_2^\alpha(G)} &= \left(\frac{(d_G(y)+r)^{d_G(y)+r}(d_G(x)-r)^{d_G(x)-r}}{d_G(x)^{d_G(x)}d_G(y)^{d_G(y)}}\right)^\alpha.\end{aligned}$$

If $d_G(x) > d_G(y) - s$, by Lemma 2, we have $\frac{(d_G(x)+s)^{d_G(x)+s}(d_G(y)-s)^{d_G(y)-s}}{d_G(x)^{d_G(x)}d_G(y)^{d_G(y)}} > \frac{d_G(y)^{d_G(y)}}{(d_G(y)-s)^{d_G(y)-s}} \cdot \frac{(d_G(y)-s)^{d_G(y)-s}}{d_G(y)^{d_G(y)}} = 1$; otherwise, $d_G(x) \leq d_G(y)-s$, we have $d_G(y) \geq d_G(x)+s > d_G(x) - r$, so $\frac{(d_G(y)+r)^{d_G(y)+r}(d_G(x)-r)^{d_G(x)-r}}{d_G(x)^{d_G(x)}d_G(y)^{d_G(y)}} > \frac{(d_G(x)-r)^{d_G(x)-r}}{d_G(x)^{d_G(x)}} \cdot \frac{d_G(x)^{d_G(x)}}{(d_G(x)-r)^{d_G(x)-r}} = 1$. □

Theorem 11: Let G be a quasi-tree graph on n vertices with k pendant vertices, where $1 \leq k \leq n-3$, then

$$P_2^\alpha(G) \leq (n-1)^{\alpha(n-1)}(n-k-1)^{\alpha(n-k-1)}2^{2\alpha(n-k-2)}$$

for $\alpha > 0$, with equality only when $G \cong \mathscr{Q}_{n,k}^1$.

Proof: Choose G with the maximum value of P_2^α for $\alpha > 0$. We suppose that there is a quasi vertex $u \in V(G)$ such that $T = G - u$ is a tree since G is a quasi-tree graph. Denoted by $\{y_1, y_2, \cdots, y_k\}$ the pendant vertices of G.

Claim 1. $uy \in E(G)$ for each vertex $y \in (V(T) \setminus \{y_1, y_2, \cdots, y_k\})$.

Similar to the proof of lemma 5, omitted.

By Lemma 7, the following Claim 2 can be obtained immediately.

Claim 2. $y_1, y_2, \cdots, y_k$ are attached to the same vertex, say v, in G.

Claim 3. Every vertex of $V(T) \setminus \{v, y_1, y_2, \cdots, y_k\}$ is the neighbor of v in T.

On the contrary, suppose that there exists a vertex $z \in (V(T) \setminus \{v, y_1, y_2, \cdots, y_k\})$ such that the path $P = vz_1z_2 \cdots z_tz$ from v to z is of length greater than 1 in T, where $t \geq 1$. If $d_G(v) \geq d_G(z_t)$, let $G' = G - zz_t + vz$. Then G' is also a quasi-tree graph on n vertices with k pendant vertices. By (1) and Lemma 2, for $\alpha > 0$, we have

$$\begin{aligned}\frac{P_2^\alpha(G')}{P_2^\alpha(G)} &= \frac{(d_G(z_t)-1)^{\alpha(d_G(z_t)-1)}(d_G(v)+1)^{\alpha(d_G(v)+1)}}{d_G(v)^{\alpha d_G(v)}d_G(z_t)^{\alpha d_G(z_t)}}\\ &= \left(\frac{\frac{(d_G(v)+1)^{d_G(v)+1}}{d_G(v)^{d_G(v)}}}{\frac{d_G(z_t)^{d_G(z_t)}}{(d_G(z_t)-1)^{d_G(z_t)-1}}}\right)^\alpha > 1,\end{aligned}$$

a contradiction with the choose of G. If $d_G(v) < d_G(z_t)$, let $G'' = G - \{y_1, y_2, \cdots, y_k\} + \{z_ty_1, z_ty_2, \cdots, z_ty_k\}$. Then G'' is also a quasi-tree graph on n vertices with k pendant vertices. By (1) and Lemma 2, for $\alpha > 0$, we have

$$\begin{aligned}\frac{P_2^\alpha(G'')}{P_2^\alpha(G)} &= \frac{(d_G(v)-k)^{\alpha(d_G(v)-k)}(d_G(z_t)+k)^{\alpha(d_G(z_t)+k)}}{d_G(v)^{\alpha d_G(v)}d_G(z_t)^{\alpha d_G(z_t)}}\\ &= \left(\frac{\frac{(d_G(z_t)+k)^{d_G(z_t)+k}}{d_G(z_t)^{d_G(z_t)}}}{\frac{d_G(v)^{d_G(v)}}{(d_G(v)-k)^{d_G(v)-k}}}\right)^\alpha > 1,\end{aligned}$$

a contradiction again.

By Claims 1, 2 and 3, it can be concluded that $G \cong \mathscr{Q}_{n,k}^1$. □

VII. BOUNDS OF GENERAL MULTIPLICATIVE ZAGREB INDICES ON QUASI-TREE GRAPHS FOR $\alpha < 0$

Let b be a positive constant. Then we have the following two facts:

(i) If $0 < b < 1$ and $\alpha < 0$, then $b^\alpha > 1$,

(ii) If $b > 1$ and $\alpha < 0$, then $0 < b^\alpha < 1$.

By these two facts and similar proof of Theorems 1,2,6–11, the conclusions of general multiplicative Zagreb indices on quasi-tree graphs for $\alpha < 0$ can be determined.

Theorem 12: Let G be in $\mathbf{QT}(n)$ with $n \geq 3$. Then

$$3^{\alpha(n-3)}4^\alpha(n-1)^\alpha \leq P_1^\alpha(G) \leq 4^\alpha(n-1)^\alpha$$

for $\alpha < 0$, with the left (resp. right) equality only when $G \cong P_{n-1} \vee K_1$ (resp. $G \cong \mathscr{C}_n$).

Theorem 13: Let G be in $\mathbf{QT}(n)$ with $n \geq 3$. Then

$$(n-1)^{2\alpha(n-1)}2^{2\alpha(n-2)} \leq P_2^\alpha(G) \leq 2^{2\alpha n}$$

for $\alpha < 0$, with the left (resp. right) equality only when $G \cong S_{n-1} \vee K_1$ (resp. $G \cong C_n$).

Theorem 14: Suppose G is an $2m$-vertex quasi-tree graph with a perfect matching, where $m \geq 2$, then

$$(2m-1)^{\alpha} 3^{\alpha(2m-3)} 2^{2\alpha} \leq P_1^{\alpha}(G) \leq m^{\alpha} 2^{\alpha(m-1)}$$

for $\alpha < 0$, with the left (resp. right) equality only when $G \cong P_{2m-1} \vee K_1$ (resp. $G \cong \boldsymbol{T}_{2m}$).

Theorem 15: Suppose G is an $2m$-vertex quasi-tree graph with a perfect matching, where $m \geq 2$, then

$$P_2^{\alpha}(G) \geq (m+1)^{\alpha(m+1)} 2^{2\alpha m} 3^{3\alpha(m-2)} (2m-1)^{\alpha(2m-1)}$$

for $\alpha < 0$, with equality only when $G \cong \boldsymbol{QT}_1(2m)$; and

$$P_2^{\alpha}(G) \leq 2^{2\alpha(2m-2)}$$

for $\alpha < 0$, with equality only when $G \cong P_{2m}$.

Theorem 16: Suppose G is an n-vertex quasi-tree graph having k pendant vertices, where $1 \leq k \leq n-3$, then

$$P_1^{\alpha}(G) \geq (n-k-1)^{\alpha} (t+2)^{\alpha[2n-k-4-t(n-k-1)]} \cdot (t+1)^{\alpha[t(n-k-1)-n+3]}$$

for $\alpha < 0$, where $b = \lfloor \frac{2n-k-4}{n-k-1} \rfloor$, with equality only when $G \cong \mathscr{Q}_{n,k}^2$.

Theorem 17: Suppose G is an n-vertex quasi-tree graph having k pendant vertices, where $1 \leq k \leq n-3$, then

$$P_2^{\alpha}(G) \geq (n-1)^{\alpha(n-1)} (n-k-1)^{\alpha(n-k-1)} 2^{2\alpha(n-k-2)}$$

for $\alpha < 0$, with equality only when $G \cong \mathscr{Q}_{n,k}^1$.

VIII. MULTIPLICATIVE ZAGREB INDICES OF QUASI-TREE GRAPHS

For a (molecular) graph G, it follows that $P_1^2(G) = \Pi_1(G)$ and $P_2^1(G) = \Pi_2(G)$. By Theorems 1,2,6–11, one can get the following corollaries.

Corollary 1: Let G be in $\mathbf{QT}(n)$, where $n \geq 3$. Then

$$16(n-1)^2 \leq \Pi_1(G) \leq 16 \cdot 3^{2(n-3)} (n-1)^2,$$

with the left (resp. right) equality only when $G \cong \mathscr{C}_n$ (resp. $G \cong P_{n-1} \vee K_1$).

Corollary 2: Let G be in $\mathbf{QT}(n)$, where $n \geq 3$. Then

$$2^{2n} \leq \Pi_2(G) \leq 2^{2(n-2)} (n-1)^{2(n-1)},$$

with the left (resp. right) equality only when $G \cong C_n$ (resp. $G \cong S_{n-1} \vee K_1$).

Corollary 3: Suppose G is an $2m$-vertex quasi-tree graph with a perfect matching, where $m \geq 2$, then

$$m^2 2^{2(m-1)} \leq \Pi_1(G) \leq 16(2m-1)^2 3^{2(2m-3)},$$

with the left (resp. right) equality only when $G \cong \boldsymbol{T}_{2m}$ (resp. $G \cong P_{2m-1} \vee K_1$).

Corollary 4: Suppose G is an $2m$-vertex quasi-tree graph with a perfect matching, where $m \geq 2$, then

$$2^{2(2m-2)} \leq \Pi_2(G) \leq 2^{2m} 3^{3(m-2)} (2m-1)^{(2m-1)} (m+1)^{(m+1)},$$

with the left (resp. right) equality only when $G \cong P_{2m}$ (resp. $G \cong \boldsymbol{QT}_1(2m)$).

Corollary 5: Suppose G is an n-vertex quasi-tree graph having k pendant vertices, where $1 \leq k \leq n-3$. Then

$$\Pi_1(G) \leq (n-k-1)^2 (t+2)^{2[2n-k-4-t(n-k-1)]} \cdot (t+1)^{2[t(n-k-1)-n+3]},$$

where $b = \lfloor \frac{2n-k-4}{n-k-1} \rfloor$, with equality only when $G \cong \mathscr{Q}_{n,k}^2$.

Corollary 6: Suppose G is an n-vertex quasi-tree graph having k pendant vertices, where $1 \leq k \leq n-3$. Then

$$\Pi_2(G) \leq 2^{2(n-k-2)} (n-k-1)^{(n-k-1)} (n-1)^{(n-1)},$$

with equality only when $G \cong \mathscr{Q}_{n,k}^1$.

IX. CONCLUSION

The extremal value of many topological indices on quasi-tree graphs have been studied extensively, such as [5], [9], [11]–[14], [22]. However, there are few results on the topological indices of quasi-tree graphs with given graph parameters. In our work, we obtained the extremal value of the general multiplicative Zagreb indices on quasi-tree graphs with given order, with perfect matchings, and with fixed pendant vertices. The methods of this article studying quasi-tree graphs with perfect matchings and quasi-tree graphs with fixed pendant vertices can also be used to study some other topological indices (belongs to the class $\varphi_1(G) = \sum_{v \in V(G)} f(d_G(u))$ or $\varphi_2(G) = \prod_{v \in V(G)} f(d_G(u))$ for a graph G, such as the variable sum exdeg index, the zeroth-order general Randić index, the first Zagreb index and its modified versions, etc.) of quasi-tree graphs with given graph parameters.

REFERENCES

[1] M. R. Alfuraidan, M. Imran, M. K. Jamil, and T. Vetrik, "General multiplicative Zagreb indices of graphs with bridges," *IEEE Access*, vol. 8, pp. 118725–118731, Jul. 2020.
[2] M. R. Alfuraidan, T. Vetrík, and S. Balachandran, "General multiplicative Zagreb indices of graphs with a small number of cycles," *Symmetry*, vol. 12, no. 4, p. 514, Apr. 2020.
[3] S. Balachandran and T. Vetrík, "General multiplicative Zagreb indices of trees with given independence number," *Quaestiones Mathematicae*, pp. 1–10, Jul. 2020, doi: 10.2989/16073606.2020.1740817.
[4] J. A. Bondy and U. S. R. Murty, *Graph Theory With Applications*. New York, NY, USA: Elsvier, 1976.
[5] T. Dehghan-Zadeh and A. R. Ashrafi, "Atom–bond connectivity index of quasi-tree graphs," *Rendiconti del Circolo Matematico di Palermo*, vol. 63, no. 3, pp. 347–354, Sep. 2014.
[6] I. Gutman and B. Furtula, Eds., *Novel Molecular Structure Descriptors—Theory and Applications I*. Kragujevac, Serbia: Univ. Kragujevac, 2010.
[7] I. Gutman and B. Furtula, Eds., *Novel Molecular Structure Descriptors—Theory and Applications II*. Kragujevac, Serbia: Univ. Kragujevac, 2010.
[8] I. Gutman and N. Trinajstić, "Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons," *Chem. Phys. Lett.*, vol. 17, no. 4, pp. 535–538, Dec. 1972.
[9] S. Li, X. Li, and W. Jing, "On the extremal Merrifield–Simmons index and Hosoya index of quasi-tree graphs," *Discrete Appl. Math.*, vol. 157, no. 13, pp. 2877–2885, Jul. 2009.
[10] J.-B. Liu, C. Wang, S. Wang, and B. Wei, "Zagreb indices and multiplicative Zagreb indices of Eulerian graphs," *Bull. Malaysian Math. Sci. Soc.*, vol. 42, no. 1, pp. 67–78, Jan. 2019.
[11] M. Lu and J. Gao, "On the Randić index of quasi-tree graphs," *J. Math. Chem.*, vol. 42, pp. 297–310, Oct. 2007.

[12] S. Qiao, "On the Zagreb index of quasi-tree graphs," *Appl. Math. E-Notes*, vol. 10, pp. 147–150, 2010.
[13] S. Qiao, "On zeroth-order general Randić index of quasi-tree graphs containing cycles," *Discrete Optim.*, vol. 7, no. 3, pp. 93–98, Aug. 2010.
[14] X. Sun, Y. Gao, and J. Du, "The harmonic index of two-trees and quasi-tree graphs," *J. Math. Inequalities*, vol. 13, pp. 479–493, Jun. 2019.
[15] R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors*. Weinheim, Germany: Wiley-VCH, 2000.
[16] R. Todeschini and V. Consonni, "New local vertex invariants and molecular descriptors based on functions of the vertex degrees," *MATCH Commun. Math. Comput. Chem.*, vol. 64, no. 2, pp. 359–372, 2010.
[17] T. Vetrík and S. Balachandran, "General multiplicative Zagreb indices of graphs with given clique number," *Opuscula Math.*, vol. 39, no. 3, pp. 433–446, 2019.
[18] T. Vetrík and S. Balachandran, "General multiplicative Zagreb indices of trees," *Discrete Appl. Math.*, vol. 247, pp. 341–351, Oct. 2018.
[19] S. Wang, C. Wang, L. Chen, and J.-B. Liu, "On extremal multiplicative Zagreb indices of trees with given number of vertices of maximum degree," *Discrete Appl. Math.*, vol. 227, pp. 166–173, Aug. 2017.
[20] S. Wang, C. Wang, and J.-B. Liu, "On extremal multiplicative Zagreb indices of trees with given domination number," *Appl. Math. Comput.*, vol. 332, pp. 338–350, Sep. 2018.
[21] S. Wang and B. Wei, "Multiplicative Zagreb indices of k-trees," *Discrete Appl. Math.*, vol. 180, pp. 168–175, Jan. 2015.
[22] K. Xu, J. Wang, and H. Liu, "The Harary index of ordinary and generalized quasi-tree graphs," *J. Appl. Math. Comput.*, vol. 45, nos. 1–2, pp. 365–374, Jun. 2014.
[23] X. Zhang, H. M. Awais, M. Javaid, and M. K. Siddiqui, "Multiplicative Zagreb indices of molecular graphs," *J. Chem.*, vol. 2019, Dec. 2019, Art. no. 5294198.

JIANWEI DU received the B.S. degree in computational mathematics from Shanxi University, in 2002, the M.S. degree in applied mathematics from Shanxi University, in 2006, and the Ph.D. degree in signal and information processing from the North University of China, in 2018. Since 2006, he has been a member of the School of Science, North University of China. He is currently a Lecturer with the North University of China and a member of the Shanxi Key Laboratory of Combinatorics. His research interests include graph theory and its applications, and matrix theory.

XIAOLING SUN received the M.S. degree in applied mathematics from Shanxi University, in 2006, and the Ph.D. degree in precision instrument and mechanics from the North University of China, in 2019. Since 2006, she has been a member of the School of Science, North University of China. She is currently a Lecturer with the North University of China and a member of the Shanxi Key Laboratory of Combinatorics. Her research interests include graph theory and its applications, especially chemical graph theory, and extremal graph theory.