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# Anti Mandelbrot Sets via Jungck-M Iteration

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**ABSTRACT** Complex fractals achieved the highest popularity in the last ten years. Researchers used the fixed point iterations to visualize the fractals and compared the image generation times to check the efficiencies of iterations. This article explores the behavior of Jungck-M iteration in the generation of anti-Mandelbrot sets. We define the orbit of Jungck-M iteration and prove its escape criteria. We establish the algorithm for anti-Mandelbrot set and visualize some graphs via proposed iteration. We calculate the image generation times for the generation of anti-Mandelbrot sets in proposed orbit and present the comparison with Jungck-CR iteration. We also discuss the variations in images at different values of the input parameters. Moreover, we present the graphs to show the escape time depends on input parameters.

**INDEX TERMS** Anti-Mandelbrot set, anti-holomorphic function, multi-corns.

## I. INTRODUCTION

The fractals theory is extensively used in differed areas of sciences and engineerings. The irregular fragments and self-similar characteristics of fractals is the main key to interest and attraction. The fractals with quadratic complex polynomial investigated in 1918, by a well known mathematician Gaston Julia [1]. He tried to iterate and sketch a complex polynomial of the form $f : z \longrightarrow z^2 + c$ where $z, c \in \mathbb{C}$ but he failed to draw it on paper. After the invention of computer, Mandelbrot started to scrutinize the work of Julia and established a sequence of iterates $\{z_k\}$ for complex polynomial $f : z \longrightarrow z^2 + c$ where $z, c \in \mathbb{C}$. He named the set of points that lay in the orbit for complex quadratic polynomial as Julia set. In 1979, Mandelbrot established another set of connected Julia set called Mandelbrot set [2]. Lakhtakia *et al.* [3] demonstrated Mandelbrot set for general complex polynomial $f : z \longrightarrow z^p + c$ where $p \geq 2$. Crowe *et al.* [4] studied in ritualistic similarity with Mandelbrot set named as Mandelbar set and analyzed its connected locus. Milnor discussed the dynamics of complex polynomial with single variable in 1990 [5]. Milnor studied the connectedness locus for anti-holomorphic complex polynomial $\bar{z}^2 + c$ and he named its graph as tri-corn. Winters explained that the boundary of the tri-corn were the smooth arc [6]. The characteristics of tri-corn and multi-corns have been analyzed by Lau *et al.* [7]. Nakane *et al.* studied various properties of tri-corns and multi-corns and demonstrated that the multi-corns were the generalized tri-corns or the tri-corns of higher degree. Multi-corns are used for commercial purposes in different industries (i.e. in hosiery and grocery) to design on coffee cups, jugs, bed sheets and shirts. The fractals found many applications in image encryption [8] or compression [9], cryptography [10], art and design [11] due to their unique and self-similar behavior. There are many applications of fractals in electrical and electronics engineering presented in [12]. These applications played a key role in the industry of security control system, radar system, capacitors, radio and antennae for wireless system [13], [14]. Furthermore, architects and civil engineers designed their ideas and project in the form of maps totally based on the logics of fractal theory [15]. Some attractive and inspiring fractals were studied in [16]–[18]. The fractals have been studied in many different ways. The most popular and most studied generalization of fractals is the fractal generation via different fixed point algorithms from the fixed point theory. Some rational and transcendental complex functions were studied in [19]. The fractals for higher powers were analyzed in [20]–[22]. Good looking, interesting and

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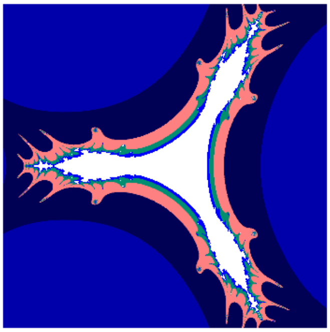

**FIGURE 1.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.1$ and generation time= $135s$.

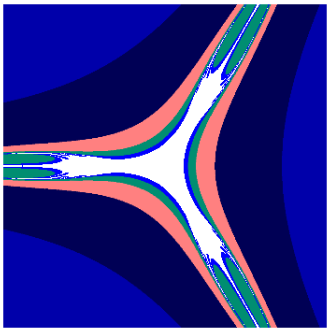

**FIGURE 2.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.1$ and generation time= $140.7s$.

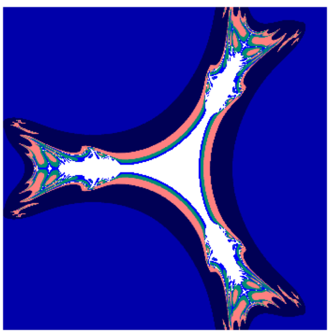

**FIGURE 3.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.2$ and generation time= $70.79s$.

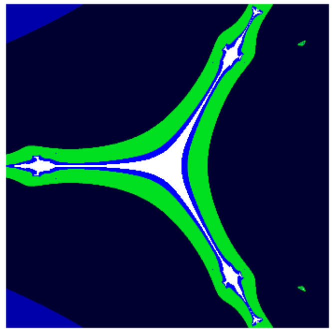

**FIGURE 4.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.2$ and generation time= $70.97s$.

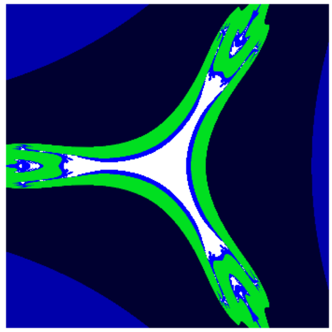

**FIGURE 5.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.3$ and generation time= $82.36s$.

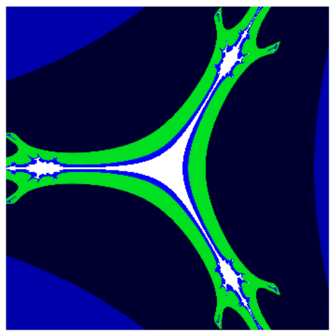

**FIGURE 6.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.3$ and generation time= $84.1s$.

attracting fractals visualized via various fixed point iterations can be found in the literature (i.e. see in [23]–[30]). Moreover, the Jungck-type iterations were used in [31]–[37]. Various iterations were also used to generate biomorphs in [38], [39] and multi-corns in [40], [41].

In this article we introduce a new Jungck-type iterative scheme in the generation of anti-Mandelbrot sets. Some beautiful and attractive tri-corns, multi-corns and anti-Mandelbrot sets visualize via Jungck-M iterative scheme which is three step iteration. We prove a theorem to establish escape criteria for Jungck-M iteration. We also discuss the graphs variation with $\alpha$. Moreover, we calculate the escape time to generate image. The paper is organized as: The section II presents some preliminaries. In section III we prove the escape criteria for Jungck-M iteration. In section IV we present some graphs of anti-Mandelbrot sets in the form of tri-corns and multi-corns via proposed iteration and Jungck-CR iteration. In the last section V, we conclude the paper.

## II. PRELIMINARIES

*Definition 1 (Julia set [1]): Assume that $P_c : \mathbb{C} \to \mathbb{C}$ be a complex map, where $c \in \mathbb{C}$. Then the set of points*

$$J_{P_c} = \{z \in \mathbb{C} : \{\left|P_c^k(z)\right|\}_{k=0}^{\infty} \text{ is bounded}\}, \tag{1}$$

*where $P_c(z)$ is the $k$-th iterate of $z$ is called the filled Julia set. The set of boundary points of $J_{P_c}$ is called simple Julia set.*

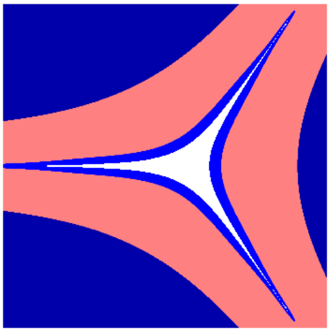

**FIGURE 7.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.4$ and generation time= $70.14s$.

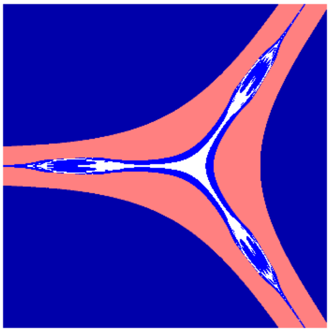

**FIGURE 8.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.4$ and generation time= $72.52s$.

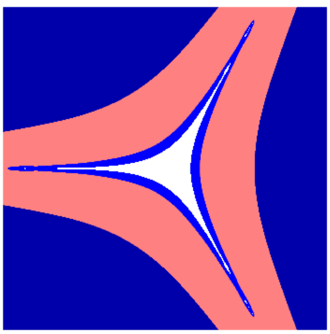

**FIGURE 9.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.5$ and generation time= $50.82s$.

*Definition 2 (Mandelbrot set [42]): Assume that $P_c : \mathbb{C} \to \mathbb{C}$ be a complex map, where $c \in \mathbb{C}$. Then the collection of complex parameters $c$ for which the corresponding Julia set $J_{P_c}$ is connected is called as Mandelbrot set $M$, i.e.,*

$$M = \{c \in \mathbb{C} : J_{P_c} \text{ is connected}\}, \tag{2}$$

equivalently Mandelbrot set can be defined as [43]:

$$M = \{c \in \mathbb{C} : \{P_c(\theta)\} \nrightarrow \infty \text{ as } k \to \infty\}, \tag{3}$$

where $\theta$ is any critical point of $P_c$, this is true for $P_c(z) = z^p + mz + c$, because its critical point is $z = 0$ and $P(0) = c$, after the first iteration we get $z_1 = c$, so we can omit the first iteration and take $c$ as the $z_0$.

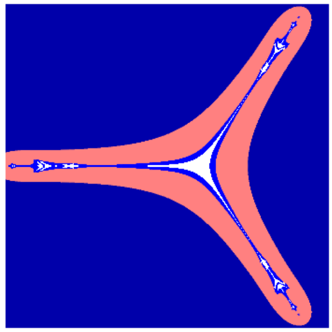

**FIGURE 10.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.5$ and generation time= $51s$.

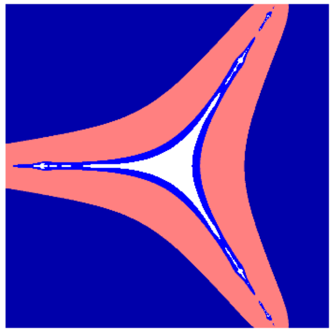

**FIGURE 11.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.6$ and generation time= $49.14s$.

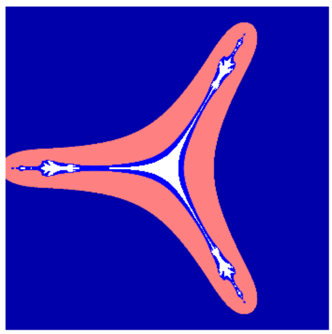

**FIGURE 12.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.6$ and generation time= $50.63s$.

*Definition 3 (Multi-corn [34]): Let $R_c(z) = \bar{z}^p + c$, where $c \in \mathbb{C}$. The multi-corn $\mathcal{M}'$ for $R_c$ is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of 0 under the action of $R_c$ is bounded, i.e.,*

$$\mathcal{M}' = \{c \in \mathbb{C} : |R_c^k(0)| \nrightarrow \infty \text{ as } k \to \infty\} \tag{4}$$

*Multi-corn for $p = 2$ is called the tri-corn.*

The M-iteration is defined as follows:

*Definition 4 (M-Iteration [44]): Let $P : \mathbb{C} \to \mathbb{C}$ be a complex map. Then for any $z_0 \in \mathbb{C}$ the M-iteration is*

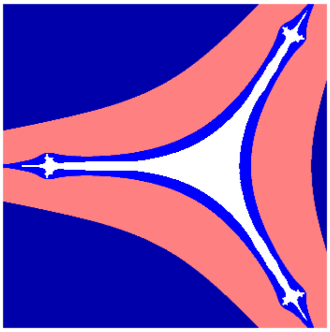

**FIGURE 13.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.7$ and generation time= 75.31*s*.

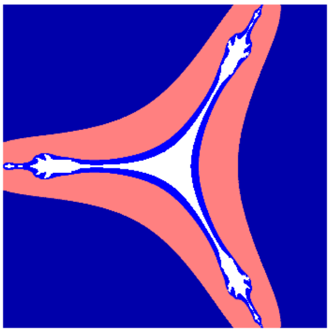

**FIGURE 14.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.7$ and generation time= 86.25*s*.

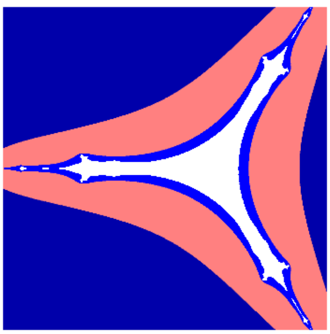

**FIGURE 15.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.8$ and generation time= 99.74*s*.

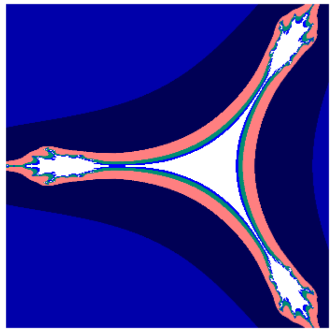

**FIGURE 16.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.8$ and generation time= 117.80*s*.

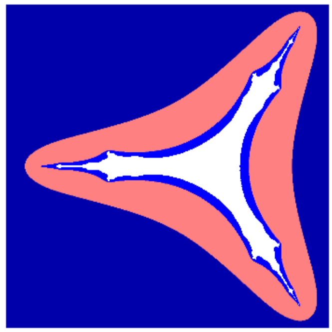

**FIGURE 17.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.9$ and generation time= 90.36*s*.

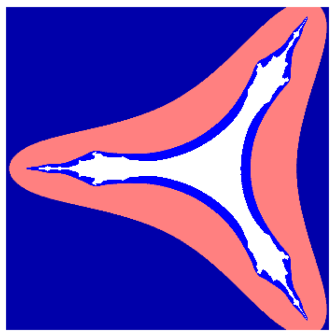

**FIGURE 18.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.9$ and generation time= 116.94*s*.

*defined as:*

$$\begin{cases} z_{k+1} = P(y_k), \\ y_k = P(x_k), \\ x_k = (1-\alpha_n)z_k + \alpha_n P(z_k), \end{cases} \tag{5}$$

*where* $\alpha_n(0, 1]$ *and* $k = 0, 1, 2, \ldots$.

*Definition 5 (Jungck-CR Iteration [35]): Let* $\mathbb{C}$ *be a nonempty complex set and* $Q, R : \mathbb{C} \to \mathbb{C}$ *be two complex mappings such that* $R$ *is a complex polynomial of degree greater than and equal to* 2 *and* $Q$ *is injective. For any* $z_0 \in \mathbb{C}$ *the Jungck-CR iteration is defined as:*

$$\begin{cases} Q(z_{k+1}) = (1-\zeta_1)Q(y_k) + \zeta_1 R(y_k) \\ Q(y_k) = (1-\zeta_2)R(z_k) + \zeta_2 R(u_k), \\ Q(u_k) = (1-\zeta_3)Q(z_k) + \zeta_3 R(z_k), \end{cases} \tag{6}$$

*where* $\alpha, \beta, \gamma \in (0, 1]$ *and* $k = 0, 1, 2, \ldots$.

*Definition 6 (Jungck M-Iteration): Let* $\mathbb{C}$ *be a nonempty complex set and* $Q, R : \mathbb{C} \to \mathbb{C}$ *be two complex mappings such that* $R$ *is a complex polynomial of degree greater than and equal to* 2 *and* $Q$ *is injective. For any* $z_0 \in \mathbb{C}$ *the*

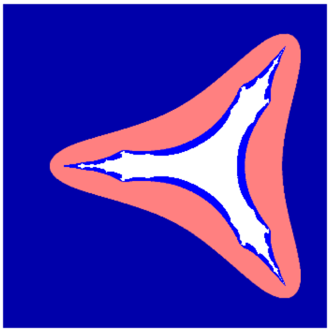

**FIGURE 19.** Quadratic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 1$ and generation time= 75.39*s*.

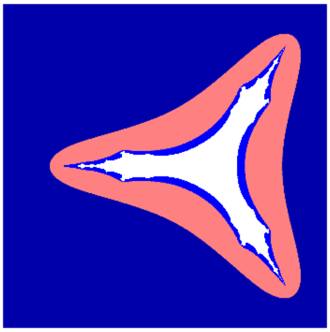

**FIGURE 20.** Quadratic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 1$ and generation time= 84.16*s*.

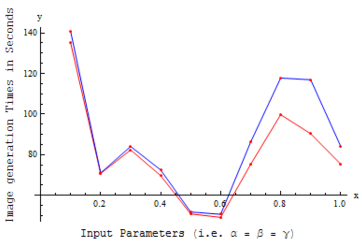


**FIGURE 21.** Time variation graph for different values of $\alpha$ for Figures 1–20.

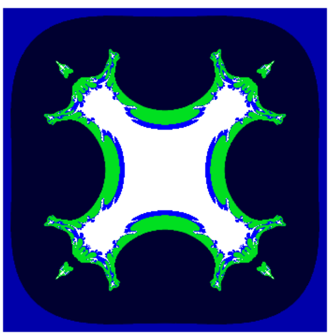

**FIGURE 22.** Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.1$ and generation time= 70.12*s*.

*Jungck-M iteration is defined as:*

$$\begin{cases} Q(z_{k+1}) = R(y_k), \\ Q(y_k) = R(x_k), \\ Q(x_k) = (1-\alpha_n)Q(z_k) + \alpha_n R(z_k), \end{cases} \tag{7}$$

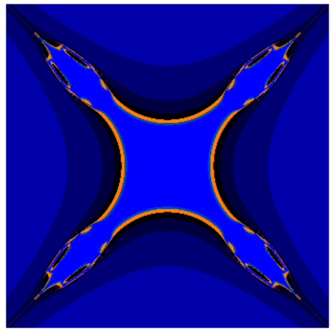

**FIGURE 23.** Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.1$ and generation time= 76.16*s*.

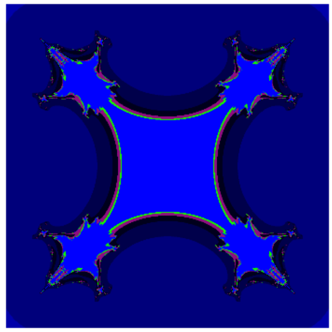

**FIGURE 24.** Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.2$ and generation time= 54.20*s*.

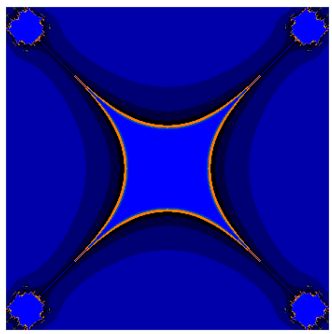

**FIGURE 25.** Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.2$ and generation time= 70.55*s*.

*where* $\alpha_n(0, 1]$ *and* $k = 0, 1, 2, \ldots$.
To generate fractals, it is necessary to define the orbit of the iteration. The orbits of our proposed iterations are defined and generalized as follows:

*Definition 7 (Orbit for Jungck-M Iteration): Assume that* $P(z_k) = (\overline{z_k}^p) + mz + c$ *be a complex polynomial with* $p \geq 2$. *Then the sequence of iterates* $\{z_k\}_{k\in\mathbb{N}}$ *from proposed iteration is called the orbit of Jungck-M iteration.*

## III. FIXED POINT RESULTS

In this section we prove some fixed point results (i.e. escape criterion or limitations) for complex function $P(z_k = \overline{z_k}^p + mz + c$ where $p \geq 2$ and $m, c \in \mathbb{C}$ via proposed-iteration.

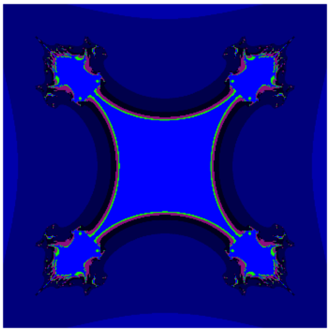

**FIGURE 26.** **Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.3$ and generation time= 52.05*s*.**

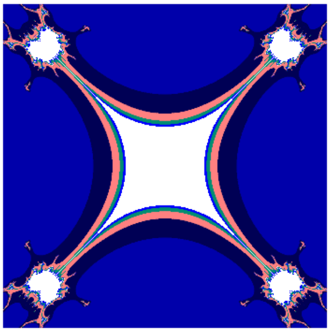

**FIGURE 27.** **Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.3$ and generation time= 117.38*s*.**

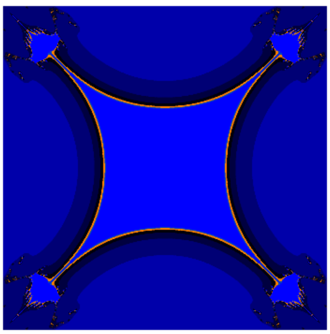

**FIGURE 28.** **Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.4$ and generation time= 53.95*s*.**

Algorithm play a key role to generate fractals and escape criteria is the basic part of algorithm. Since in Picard iterative scheme have one map while in the Jungck-M iteration we have two maps. Therefore, if we desire to replace Picard iterative scheme with the Jungck-M iteration, then we need to deal with two maps in the iteration. We handle this situation as follows. Assume that $P : \mathbb{C} \to \mathbb{C}$ be a complex polynomial. In the case of multi-corns we break $P$ into two maps $Q, R$ in the following way: $P = R - Q$, where $R(z) = \overline{z_k}^p + c$ and $Q = mz$ (i.e. injective).

*Theorem 1: Suppose that $P_c(z) = \bar{z}^p + mz + c$ where $p \geq 2$ and $m, c \in \mathbb{C}$ be a complex polynomial with $|z| \geq |c| > \left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{p-1}}$. If the sequence of iterates $\{z_k\}_{k\in\mathbb{N}}$ for Jungck*

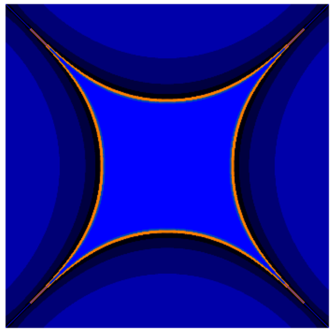

**FIGURE 29.** **Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.4$ and generation time= 73*s*.**

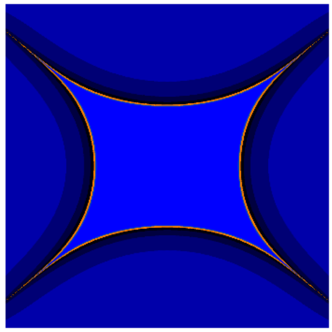

**FIGURE 30.** **Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.5$ and generation time= 43.58*s*.**

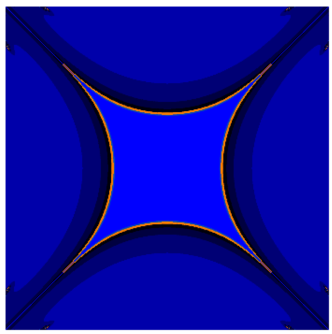

**FIGURE 31.** **Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.5$ and generation time= 72.93*s*.**

*M-iteration is defined as follows:*

$$\begin{cases} Q(z_{k+1}) = R(y_k), \\ Q(y_k) = R(x_k), \\ Q(x_k) = (1-\alpha)Q(z_k) + \alpha R(z_k), \end{cases} \tag{8}$$

*where $\alpha_n(0, 1]$ and $k = 0, 1, 2, \ldots$, then $|z_k| \longrightarrow \infty$ when $k \longrightarrow \infty$.*

*Proof:* Let $R(z) = \bar{z}^p + c$, $Q(z) = mz$, $x_0 = x$, $y_0 = y$ and $z_0 = z$, then first step of Jungck M-iteration is:

$$|Q(x_k)| = |(1-\alpha)Q(z_k) + \alpha R(z_k)|$$
$$|mx_k| = \left|(1-\alpha)mz_k + \alpha(\overline{z_k}^p) + c)\right|.$$

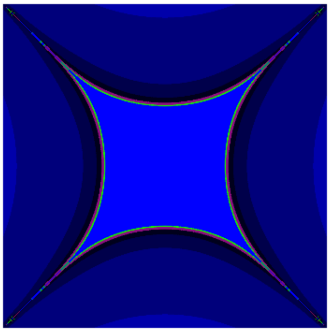

**FIGURE 32.** Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.1$ and generation time= 43.42$s$.

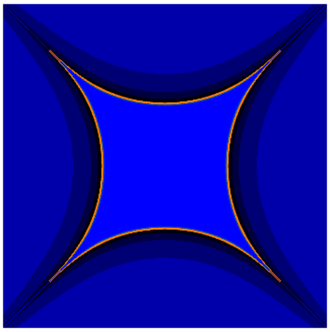

**FIGURE 33.** Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.6$ and generation time= 72.88$s$.

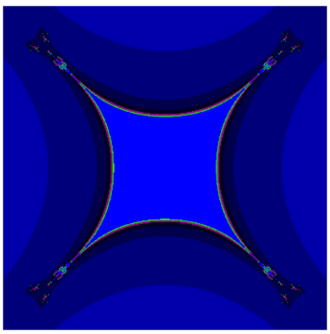

**FIGURE 34.** Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.7$ and generation time= 40.66$s$.

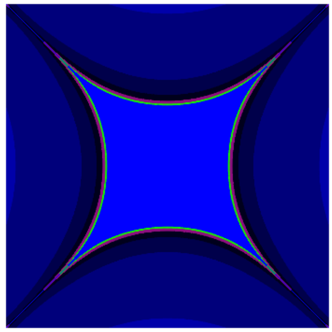

**FIGURE 35.** Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.7$ and generation time= 65.69$s$.

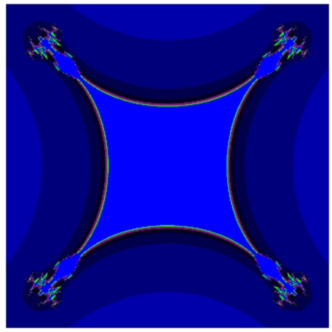

**FIGURE 36.** Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.8$ and generation time= 38.83$s$.

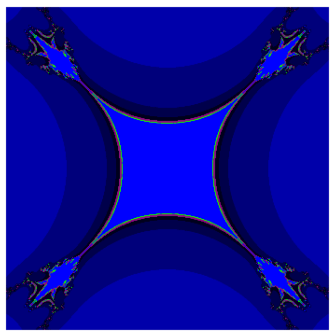

**FIGURE 37.** Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.8$ and generation time= 63.77$s$.

For $k = 0$, we have

$$\begin{aligned}
|mx_0| &= \left|(1-\alpha)mz_0 + \alpha(\overline{z_0}^p) + c)\right| . \\
&\geq \alpha\left|\overline{z}^p + c\right| - |(1-\alpha)mz| \\
&\geq \alpha\left|\overline{z}^p\right| - \alpha|c| - |mz| + \alpha|mz| \\
&\geq \alpha|z^p| - (1+|m|)|z|.
\end{aligned}$$

Because$|\overline{z}| = |z|$, $\alpha|mz| \geq 0$ and $|z| \geq \alpha|c|$. Now we have

$$\begin{aligned}
|x| &\geq |z|\left(\frac{\alpha|z|^{p-1}}{1+|m|} - 1\right) \\
|x| &\geq \alpha|z|
\end{aligned}$$

Because $|z| > \left(\frac{2(1+|m|)}{\alpha}\right)^{p-1}$, this implies $\frac{\alpha|z|^{p-1}}{1+|m|} - 1 > 1$ and $|z| > \alpha|z|$ For the second step of Jungck M-iteration, we have

$$|Q(y_k)| = |R(x_k)| .$$

For $k = 0$, we have

$$\begin{aligned}
|Q(y_0)| &= |R(x_0)| \\
|my| &= \left|\overline{x}^p + c\right| \\
&= \left|x^p + c\right| \\
&\geq |x^p| - |c| \\
&\geq \alpha|z^p| - |c|
\end{aligned}$$

$$
\begin{aligned}
&= |z|\left(\alpha|z^{p-1}| - 1\right), \because |z| \geq |c| \\
|y| &\geq |z|\left(\frac{\alpha|z|^{p-1}}{1+|m|} - 1\right) \\
|y| &\geq \alpha|z|, \ \because \alpha \leq 1.
\end{aligned}
$$

Now for the last step of proposed iteration, we have

$$|Q(z_{k+1})| = |R(y_k)| .$$

Again for $k = 0$, we have

$$
\begin{aligned}
|Q(z_1)| &= |R(y)| \\
|mz_1| &= \left|\bar{y}^p + c\right| \\
&= \left|y^p + c\right| \\
&\geq |y^p| - |c| \\
&\geq \alpha|z^p| - |c| \\
&= |z|\left(\alpha|z^{p-1}| - 1\right), \because |z| \geq |c| \\
|z_1| &\geq |z|\left(\frac{\alpha|z|^{p-1}}{1+|m|} - 1\right).
\end{aligned}
$$

Iterating upto $k^{\text{th}}$ term, we have

$$
\begin{aligned}
|z_2| &\geq |z|\left(\frac{\alpha|z|^{p-1}}{1+|m|} - 1\right)^2 \\
|z_3| &\geq |z|\left(\frac{\alpha|z|^{p-1}}{1+|m|} - 1\right)^3 \\
&\vdots \\
|z_k| &\geq |z|\left(\frac{\alpha|z|^{p-1}}{1+|m|} - 1\right)^k .
\end{aligned}
$$

Since $|z| > \left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{p-1}}$. Therefore $\frac{\alpha|z|^{p-1}}{1+|m|} - 1 > 1$. Hence $|z_k| \to \infty$ as $k \to \infty$. □

*Corollary 1: Assume that*

$$|z_n| > \max\left\{|c|, \left(\frac{2(1+|m|)}{\alpha}\right)^{\frac{1}{p-1}}\right\},$$

*for some* $n \geq 0$. *Since* $\frac{\alpha|z|^{p-1}}{1+|m|} - 1 > 1$, *therefore* $|z_{n+k}| > |z|\left(\frac{\alpha|z|^{p-1}}{1+|m|} - 1\right)^{n+k}$. *Hence* $|z_k| \to \infty$ *when* $k \to \infty$.

## IV. FRACTAL GENERATION

In this section we present some anti-Mandelbrot sets via Jungck-M iteration and Jungck-CR iteration. We need an escape criteria to visualize anti-Mandelbrot sets a criteria. We can generate anti-Mandelbrot set by using different algorithms [45]–[47] (e.g.Distance Estimator, Potential Function Algorithm, Escape Criteria etc).

In this article we use escape criteria in Algorithm 1 to fascinate the anti Julia sets in graphs. The graphs generated on computer with specifications are as following:

- Processor: Intel(R) Core(TM) i5-3320 M CPU @ 2.60GHz,
- System type: 32-bit Operating System and
- Software: Mathematica 7.0.

**Algorithm 1** Generation of Anti-Mandelbrot Set

**Input**: $R(z) = \bar{z}^p + c$ with degree $p \geq 2$ and $Q(z) = mz$–complex polynomials, $A$–covered area, $K$–the maximum number of iterations, $\alpha, \beta, \gamma \in (0, 1]$ –fixed parameters and $c \in \mathbb{C}$–complex constants, *coloursmap*$[0..M-1]$ coloursmap with $M$ colours.

**Output**: Anti-Mandelbrot set in area $A$.

**for** $c \in A$ **do**
  $EC$–escape condition
  $k = 0$
  $z_0$–initial guess from critical points of $P(z)$
  **while** $k \leq K$ **do**
    $Q(z_{k+1}) = R(y_k)$,
    $Q(y_k) = R(x_k)$,
    $Q(x_k) = (1-\alpha)Q(z_k) + \alpha R(z_k)$
    **if** $|z_{k+1}| > EC$ *with*
    **then**
      break
    $k = k + 1$
  $i = \lfloor (M-1)\frac{k}{K} \rfloor$
  colour $c$ with *coloursmap*$[i]$

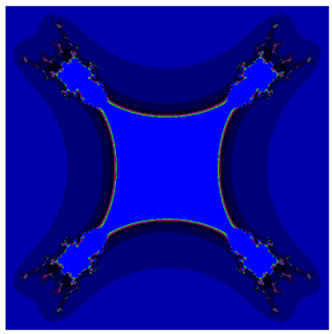

**FIGURE 38.** Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 0.9$ and generation time= 43.17$s$.

Next, we present some examples of anti-Mandelbrot sets via proposed iteration and Jungck-CR iteration for the complex polynomial $P(z) = \bar{z}^p + mz + c$ where $c \in \mathbb{C}$ a complex parameter.

### A. ANTI-MANDELBROT SETS

The Anti-Mandelbrot or multi-corn set is a generalization of Mandelbrot set which has a variety in its visuals. In this subsection we discuss some examples of anti-Mandelbrot sets of the complex polynomial $P(z) = \bar{z}^p + mz + c$ with degree $p \geq 2$ in the orbit of Jungck-M iteration. We also compare the Jungck-M iteration with Jungck-CR iteration. In all images we apply $K = 8$ (i.e. Maximum number of iterations) in Algorithm 1.

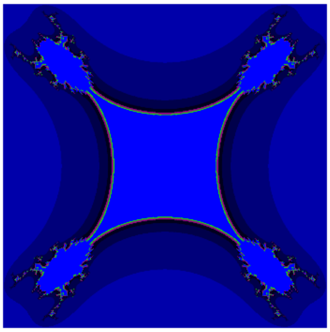

**FIGURE 39.** Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 0.9$ and generation time= 80.28*s*.

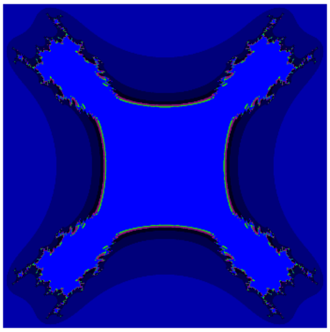

**FIGURE 40.** Cubic Anti-Mandelbrot set in Jungck-M Orbit for $\alpha = 1$ and generation time= 60.61*s*.

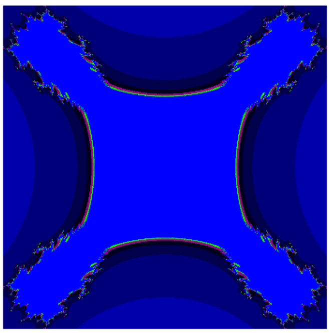

**FIGURE 41.** Cubic Anti-Mandelbrot set in Jungck-CR Orbit for $\alpha = \beta = \gamma = 1$ and generation time= 106.81*s*.

*Example 1: In this example we present quadratic anti-Mandelbrot sets for $\alpha = 0.1, 0.2, 0.3, \ldots, 1$ and $m = 0.8$ in Figures.1–20. The axis and areae for each quadratic anti-Mandelbrot set are as follows:*

- *Figure 1:* $A = [-1.5, 1.2] \times [-1.2, 1.2]$,
- *Figure 2:*$A = [-3.5, 3.5]^2$,
- *Figure 3:* $A = [-2.5, 2.5]^2$,
- *Figure 4:* $A = [-3.5, 3.5]^2$,
- *Figure 5:*$A = [-2, 2]^2$,
- *Figure 6:*$A = [-3.5, 3.5]^2$,
- *Figure 7:* $A = [-2.4, 1.8] \times [-2.5, 2.5]$,
- *Figure 8:*$A = [-6.5, 4.5] \times [-6.5, 6.5]$,

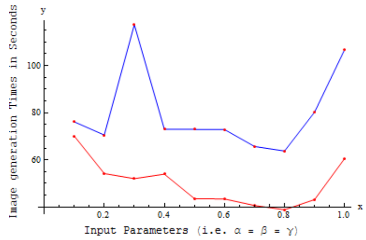


**FIGURE 42.** Time variation graph for different values of $\alpha$ for Figures 22–41.

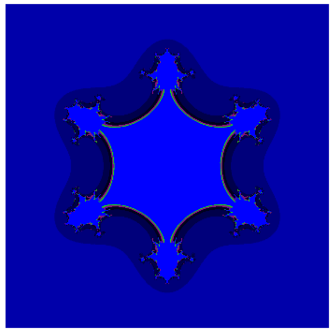

**FIGURE 43.** Anti-Mandelbrot set in Jungck-M Orbit for $p = 5, \alpha = 0.2$ and generation time= 33.77*s*.

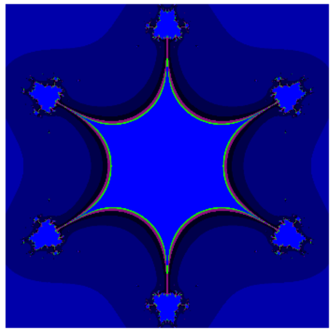

**FIGURE 44.** Anti-Mandelbrot set in Jungck-CR Orbit for $p = 5, \alpha = \beta = \gamma = 0.2$ and generation time= 55.72*s*.

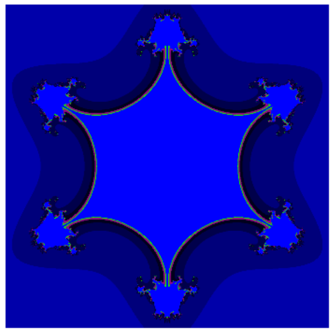

**FIGURE 45.** Anti-Mandelbrot set in Jungck-M Orbit for $p = 5, \alpha = 0.4$ and generation time= 42.48*s*.

- *Figure 9:* $A = [-2.4, 2.2] \times [-2.2, 2.2]$,
- *Figure 10:* $A = [-6.8, 4.5] \times [-6.5, 6.5]$,
- *Figure 11:* $A = [-2.4, 2.2] \times [-2.5, 2.5]$,

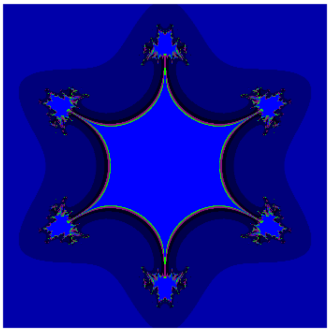

**FIGURE 46.** Anti-Mandelbrot set in Jungck-CR Orbit for $p = 5, \alpha = \beta = \gamma = 0.4$ and generation time= $55.33s$.

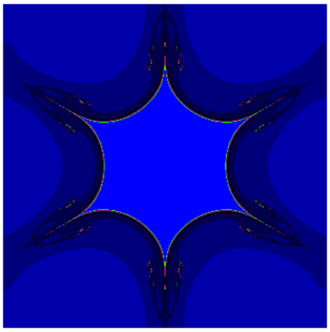

**FIGURE 47.** Anti-Mandelbrot set in Jungck-M Orbit for $p = 5, \alpha = 0.6$ and generation time= $33.89s$.

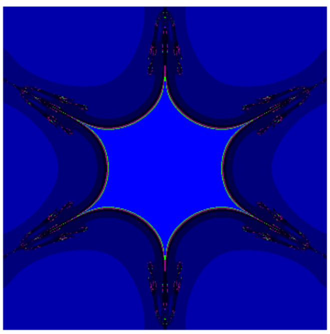

**FIGURE 48.** Anti-Mandelbrot set in Jungck-CR Orbit for $p = 5, \alpha = \beta = \gamma = 0.6$ and generation time= $51.33s$.

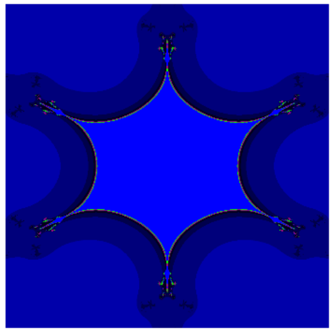

**FIGURE 49.** Anti-Mandelbrot set in Jungck-M Orbit for $p = 5, \alpha = 0.8$ and generation time= $31.84s$.

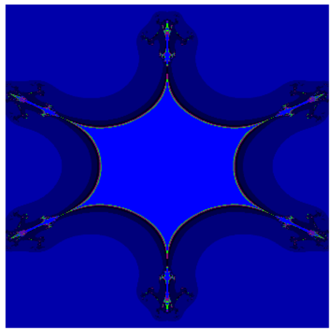

**FIGURE 50.** Anti-Mandelbrot set in Jungck-CR Orbit for $p = 5, \alpha = \beta = \gamma = 0.8$ and generation time= $51.86s$.

- *Figure 12:* $A = [-4.5, 4.5]^2$,
- *Figure 13:* $A = [-1.8, 1] \times [-1.5, 1.5]$,
- *Figure 14:* $A = [-2.8, 2.5] \times [-2.3, 2]$,
- *Figure 15:* $A = [-1.8, 1] \times [-1.5, 1.5]$,
- *Figure 16:* $A = [-1.8, 1] \times [-1.5, 1.5]$,
- *Figure 17:* $A = [-1.8, 1] \times [-1.5, 1.5]$,
- *Figure 18:* $A = [-1.8, 1] \times [-1.5, 1.5]$,
- *Figure 19:* $A = [-1.8, 1] \times [-1.5, 1.5]$,
- *Figure 20:* $A = [-1.8, 1] \times [-1.5, 1.5]$.

*In each image the main body of quadratic anti-Mandelbrot has three unique corns or branches. We observe that when the values of* $\alpha$ *for Jungck-M iteration and* $\alpha = \beta = \gamma$ *for Jungck-CR iteration move from* 0.1 *to* 0.5*, the heads of branches or corns lose their thickness and when the values of input parameters move from* 0.6 *to* 1*, then the heads of branches or corns again gain their thickness. We observe that for different input parameters areas occupied by the images are different both iterations (i.e. for Jungck-M and Jungck-CR iterations). We also analyze that all anti-Mandelbrot sets visualize in Figs. 1–20 for* $p = 2$ *are tri-corns. Moreover, we draw a graph in which we take image generation time in seconds along x–axis and* $\alpha = \beta = \gamma$ *along y–axis to present the comparison of proposed iteration with Jungck-CR iteration (i.e. graph in Fig.21). The red curve is represent the variations of times correspond to the input parameter* $\alpha$ *for Jungck-M iteration and the blue curve is represent the variations of times correspond to the input parameters* $\alpha = \beta = \gamma$ *for Jungck-CR iteration. The graph in Fig.21 clearly shows that our proposed iteration is more efficient than Jungck-CR iteration.*

*Example 2: The next example presents the cubic anti-Mandelbrot sets for* $p = 3$ *in the orbits of proposed iteration and Jungck-CR iteration. In Figures 22–41 the parameters are* $\alpha = \beta = \gamma = 0.1, 0.2, 0.3, \ldots, 1$ *and* $m = \frac{2}{3}$*. All cubic anti-Mandelbrot sets are multi-corns because the main body of each set has four branches or corns. When* $\alpha$ *for proposed iteration and* $\alpha = \beta = \gamma$ *for Jungck-CR iteration approaches to* 0.5*, the size of each corn in figures 22–31 decreases and when we increase the values of input parameters from*

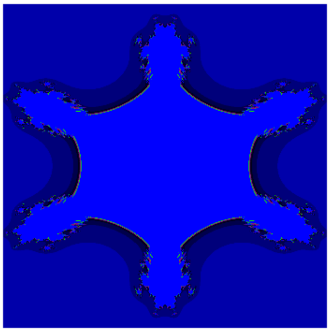

**FIGURE 51.** **Anti-Mandelbrot set in Jungck-M Orbit for $p = 5, \alpha = 1$ and generation time= 55.95$s$.**

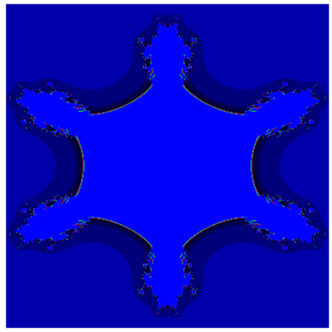

**FIGURE 52.** **Anti-Mandelbrot set in Jungck-CR Orbit for $p = 5, \alpha = \beta = \gamma = 1$ and generation time= 84.5$s$.**

*0.6 to 1 then the size of each corn increases in width. In the figure 42 we represent the graph to show the efficiency of proposed iteration, and correspondence between the image generation time and the values of input parameters for both iterations. We notice that our proposed iteration is faster than the Jungck-CR iteration in the generation of cubic anti-Mandelbrot sets.*

- *Figure 22:* $A = [-1, 1]^2$,
- *Figure 23:*$A = [-1, 1]^2$,
- *Figure 24:* $A = [-1, 1]^2$,
- *Figure 25:* $A = [-1, 1]^2$,
- *Figure 26:*$A = [-0.8, 0.8],^2$
- *Figure 27:*$A = [-1, 1]^2$,
- *Figure 28:* $A = [-0.6, 0.6]^2$,
- *Figure 29:*$A = [-0.6, 0.6]^2$,
- *Figure 30:* $A = [-0.5, 0.5] \times [-0.6, 0.6]$,,
- *Figure 31:* $A = [-0.7, 0.7]^2$,
- *Figure 32:* $A = [-0.6, 0.6]^2$,
- *Figure 33:*$A = [-0.6, 0.6]^2$,
- *Figure 34:* $A = [-0.7, 0.7]^2$,
- *Figure 35:* $A = [-0.6, 0.6]^2$,
- *Figure 36:* $A = [-0.6, 0.6]^2$,
- *Figure 37:* $A = [-0.8, 0.8]^2$,
- *Figure 38:* $A = [-0.7, 0.7]^2$,
- *Figure 39:*$A = [-0.7, 0.7]^2$,
- *Figure 40:* $A = [-0.6, 0.6]^2$,
- *Figure 41:*$A = [-0.5, 0.5]^2$,.

*Example 3: In last example we present anti-Mandelbrot sets for* $p = 5$*. In Figures 43–52 the parameters are* $0.2, 0.4, 0.6, \ldots, 1$ *and* $m = 2$*. All anti-Mandelbrot sets for* $p = 5$ *are also multi-corns because the main body of each set has six branches or corns. When input parameters approaches to* 0.5*, the corns shrink into sharp edges(see in figures 22–26). When we increase the values from* 0.6 *to* 1 *then the corns swell. The graph in the figure 53 also show that our proposed iteration is efficient that the Jungck-CR iteration.*

- *Figure 43:* $A = [-4, 4]^2$,
- *Figure 44:*$A = [-4, 4]^2$,
- *Figure 45:* $A = [-3, 3]^2$,
- *Figure 46:* $A = [-4, 4]^2$,
- *Figure 47:*$A = [-3.5, 3.5]^2$,

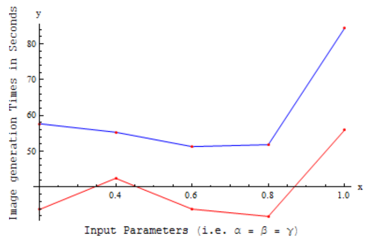


**FIGURE 53.** **Time variation graph for different values of $\alpha$ for Figures 43–52.**

- *Figure 48:*$A = [-3.8, 3.8] \times [-4, 4]$,
- *Figure 49:* $A = [-3, 3] \times [-3.5, 3.5]$,
- *Figure 50:*$A = [-3.2, 3.2] \times [-4, 4]$,
- *Figure 51:* $A = [-2.5, 2.5] \times [-3, 3]$,
- *Figure 52:* $A = [-2.5, 2.5] \times [-3, 3]$.

## V. CONCLUSION

In this article we defined a new Jungck-M iteration in the generation of anti-Mandelbrot sets. We proved escape criteria for a complex polynomial $P(z) = \bar{z}^p + mz + c$ where $p \geq 2$ and $c \in \mathbb{C}$ with $Q(z) = mz$ and $R(z) = \bar{z}^p + c$ by using Jungck-M iteration. We adjusted the established escape conditions in Algorithm 1 and visualized the anti-Mandelbrot sets in Jungck-M and Jungck-CR orbits. To show the image generation time comparison of proposed iteration with Jungck-CR iteration, we presented some examples of anti-Mandelbrot sets for $p = 2$, 3 and 5 by using the proved results. We calculated the image generation time for each anti-Mandelbrot set in Jungck-M and Jungck-CR orbit. The graphs for image time generation indicated that our proposed iteration is faster in image generation than Jungck-CR iteration. The established graphs (i.e. graphs 21, 42 and 53) also presented the relation between $\alpha, \beta, \gamma$ and image execution time. We observed the drastic changes when we varied the input parameter $\alpha$ for Jungck-M iteration and $\alpha, \beta, \gamma$ for Jungck-CR iteration. Moreover, we observed that $p+1$ corns or branches appeared on the main body of anti-Mandelbrot set for any value of $p \geq 2$.

We hope that the presented results will inspire those who are working on anti-fractals. In future, we will try to prove escape criteria for the generation of anti Mandelbrot sets with a 3D vector or with a set of quaternion.

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