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# Subspace Codes Based on Partial Injective Maps of Vector Spaces Over Finite Fields

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**ABSTRACT** Subspace codes are widely used in error corrections of random network coding. In this article, subspace codes based on partial injective maps of vector spaces over finite fields are considered. Several bounds of the subspace codes $(n, M, 2b, e)_q$ based on $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ are presented. The anticode bound and Ahlswede-Aydinian bound of the subspace codes $(n, M, 2b, e)_q$ are obtained by using the EKR theorem for $e$-partial injective maps of $\mathbb{F}_q^{(n)}$. Finally, we show that the $(n, M, 2b, e)_q$ subspace codes based on $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ reach the Wang-Xing-Safavi-Naini bound if and only if they are certain Steiner structures in $I_e^n$.

**INDEX TERMS** Subspace codes, partial injective maps, bounds, EKR theorem, finite fields.

## I. INTRODUCTION

Let $\mathbb{F}_q$ be a finite field with $q$ elements, where $q$ is a prime power and $\mathbb{F}_q^{(n)}$ is the $n$-dimensional row vector space over $\mathbb{F}_q$. The collection of all the subspaces with dimension $e$ of $\mathbb{F}_q^{(n)}$ is known as the Grassmannian space over $\mathbb{F}_q$, denoted by $\mathcal{G}_q(n, e)$, where $0 \leq e \leq n$. The set of all the subspaces of $\mathbb{F}_q^{(n)}$ is called the projective space of order $n$, denoted by $\mathcal{P}_q(n)$. For any subspaces $U, V \in \mathcal{P}_q(n)$, define the distance function between $U$ and $V$ as $d(U, V) = \dim U + \dim V - 2\dim(U \cap V)$. The function is showed to be a metric (see [1]), thus $\mathcal{P}_q(n)$ is a metric space. A nonempty collection $\mathbb{C}$ of the projective space $\mathcal{P}_q(n)$ is called a subspace code. The minimum distance of a subspace code $\mathbb{C}$ is $d(\mathbb{C}) = \min\{d(U, V) | U \neq V,\ U, V \in \mathbb{C}\}$. Different from the classical coding theory, of which every codeword is only a vector, here every codeword of $\mathbb{C}$ is itself a subspace. A subspace code $\mathbb{C}$ is denoted by $(n, M, d)_q$ if it has $M$ codewords and $d(\mathbb{C}) = d$. Moreover, if $\mathbb{C} \subseteq \mathcal{G}_q(n, e)$, $\mathbb{C}$ can be denoted by $(n, M, d, e)_q$.

Subspace codes become the natural objects in some applications, for instance, in non-coherent linear network coding [2] and linear authentication [3]. Koetter and Kschischang [2] defined an operator channel when they studied random network coding. Meanwhile, they showed that the errors and erasures could be corrected by a subspace code $(n, M, d, e)_q$ over the operator channel if the sum of errors and erasures is less than $\frac{d}{2}$. These research results motivate many domestic and overseas scholars′ great interest in subspace codes (see [4]–[8]).

Bounds on subspace codes in the projective space are considered in recent years. About subspace codes in the projective space, a generalization of the Singleton bound are offered in [1]. Barg and Nogin [9] derived the Gilbert-Varshamov and Hamming bounds for packings of spheres (codes) in the Grassmann manifolds over $\mathbb{R}$ and $\mathbb{C}$. Bachoc [10] developed the linear programming method to get bounds for the cardinality of Grassmannian codes endowed with the chordal distance. Henkel [11] had a research about the Sphere-packing bounds, which used the Riemann geometric machinery of volume evaluates according to the curvature. Gao and Wang [12] given several bounds about the subspace code $(n + r, M, d, (e, 1))_q$ based on the subspaces of type $(e, 1)$ in singular linear space $\mathbb{F}_q^{(n+r)}$ over $\mathbb{F}_q$.

In this article, a pair $(V, f)$ is called an $e$-partial injective map of $\mathbb{F}_q^{(n)}$ over $\mathbb{F}_q$, where $0 \leq e \leq n$, if $V \in \mathcal{G}_q(n, e)$ and $f : V \to \mathbb{F}_q^{(n)}$ is an injective map. Denote the collection of all the $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ over $\mathbb{F}_q$ by $I_e^n$, where $0 \leq e \leq n$. For any two $e$-partial injective maps $(V, f), (L, j) \in I_e^n$, we give a definition about the natural distance function between $(V, f)$ and $(L, j)$ in $I_e^n$ as below

$$d((V, f), (L, j)) = 2e - 2\dim((V, f) \wedge (L, j)), \qquad (1)$$

where $(V, f) \wedge (L, j) = (D, h)$ with $D$ is the maximum element of the set $\{D \subseteq V \cap L | f|_D = j|_D\}$ and $h = f|_D = j|_D$. The function above is a metric on $I_e^n$ (see the Section 2). A non empty aggregate $\mathbb{C}$ of $I_e^n$ is called an $(n, M, d, e)_q$ code if

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the size of codewords is $M$ and $d(\mathbb{C}) = d$. The maximum number of codewords in an $(n, M, d, e)_q$ code is denoted by $A_q(n, d, e)$. The aim of the paper is to determine the bounds of $A_q(n, d, e)$.

In the theory of subspace coding, one of the important questions is to determine the largest possible size of codewords with a given minimum distance and to construct the optimal subspace codes. In the previous literature, sphere-packing bound, Wang-Xing-Safavi-Naini bound, Singleton bound, Johnson bound, Gilbert-Varshamov bound, anticode bound and Ahlswede-Aydinian bound have been given for constant dimensional subspace codes (CDCs for short), whose codewords are some $e$-dimensional subspaces of $\mathbb{F}_q^{(n)}$ with the following subspace distance

$$d(U, V) = 2e - 2\dim(U \cap V), \quad U, V \in \mathcal{G}_q(n, e).$$

Let $I_e^n$ denote the collection of all the $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ over $\mathbb{F}_q$, where $0 \le e \le n$. Motivated by the results on constant dimensional subspace codes, we define the distance on $I_e^n$ as

$$d((V,f),(L,j)) = 2e - 2\dim((V,f) \wedge (L,j)),$$

where $(V,f) \wedge (L,j) = (D,h)$ with $D$ is the maximum element of the set $\{D \subseteq V \cap L | f|_D = j|_D\}$ and $h = f|_D = j|_D$. Then we provide the sphere-packing bound, Wang-Xing-Safavi-Naini bound, Singleton bound, Johnson bound, Gilbert-Varshamov bound, anticode bound and Ahlswede-Aydinian bound for subspace codes based on $e$-partial injective maps of $\mathbb{F}_q^{(n)}$, which are generalizations of bounds for constant dimensional subspace codes, in order to increase the number of codewords in constant dimensional subspace codes with a given minimum distance and to construct more optimal subspace codes.

Finally, we obtain some optimal subspace codes based on $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ from the Steiner structures in $I_e^n$.

The rest of this article is designed as below. In Section 2, the anzahl formulas about $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ are presented. In Section 3, several bounds on the subspace codes $(n, M, d, e)_q$ based on $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ are presented. In Section 4, The anticode bound and Ahlswede-Aydinian bound of the subspace codes $(n, M, 2b, e)_q$ are obtained using the Erdős-Ko-Rado theorem for $e$-partial injective maps of $\mathbb{F}_q^{(n)}$. In Section 5, we give a summary of the whole paper.

## II. PRELIMINARIES

In this section, we introduce the distance defined on $I_e^n$ and some anzahl theorems about $e$-partial injective maps are given.

For the sake of simplicity, we introduce the use of Gaussian coefficient [13],

$$\begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q = \frac{\prod_{t=m_2-m_1+1}^{m_2} (q^t - 1)}{\prod_{t=1}^{m_1} (q^t - 1)}$$

by convenience $\begin{bmatrix} m_2 \\ 0 \end{bmatrix}_q = 0$ and $\begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q = 0$ whenever $m_1 < 0$ and $m_2 < m_1$.

Firstly, we have the following Lemma 2.1 about $I_e^n$ clearly.

*Lemma 2.1:* Let $1 \le e \le n$, then

$$|I_e^n| = \begin{bmatrix} n \\ e \end{bmatrix}_q q^{\frac{e(e-1)}{2}} \cdot \prod_{a=n-e+1}^{n} (q^a - 1).$$

*Proof:* The lemma follows clearly from that $I_e^n$ denotes the collection of all the $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ over $\mathbb{F}_q$.

For any two elements $(V,f)$, $(L,j) \in I_e^n$, we say that $(L,j)$ includes $(V,f)$, denoted by $(V,f) \preceq (L,j)$, if $V \subseteq L$ and $f = j|_V$. For a fixed $\sigma$-partial injective map $(L,j) \in I_\sigma^n$, let $(L,j)^{(e)}$ denote the set of all the $e$-partial injective maps $(V,f)$ included in $(L,j)$.

*Lemma 2.2:* For $1 \le e \le \sigma \le n$, then

$$\left|(L,j)^{(e)}\right| = \begin{bmatrix} \sigma \\ e \end{bmatrix}_q.$$

*Proof:* The lemma follows from the inclusion relation that $(L,j)$ includes $(V,f)$ if $V \subseteq L$ and $f = j|_V$. For a fixed $e$-partial injective map $(V,f) \in I_e^n$, let ${}^r(V,f)$ denote the collection of all the $r$-partial injective maps $(D,h)$ including $(V,f)$.

*Lemma 2.3:* For $1 \le e \le r \le n$, then

$$\left|{}^r(V,f)\right| = \begin{bmatrix} n-e \\ r-e \end{bmatrix}_q \times (q^n - q^e)(q^n - q^e - 1) \times (q^n - q^e - 2) \cdots (q^n - q^e + 1).$$

*Proof:* The lemma follows from the inclusion relation that $(L,j)$ includes $(V,f)$ if $V \subseteq L$ and $f = j|_V$.

The following theorem shows that the function (1) defined on $I_e^n$ is indeed a metric on $I_e^n$.

*Theorem 2.4:* The function defined on $I_e^n$, i.e., $d((V,f), (L,j)) = \dim V + \dim L - 2\dim((V,f) \wedge (L,j))$, is indeed a metric on $I_e^n$.

*Proof:* For any $(V,f)$, $(L,j)$, $(Y,h) \in I_e^n$, the function (1) should satisfy the following three properties:

(i) non-negativity: $d((V,f),(L,j)) \ge 0$ with equality if and only if $(V,f) = (L,j)$.

(ii) symmetry: $d((V,f),(L,j)) = d((L,j),(V,f))$.

(iii) triangle inequality:

$$d((V,f),(L,j)) \le d((V,f),(Y,h)) + d((Y,h),(L,j)).$$

The first two conditions for the function (1) are satisfied clearly and we only focus on the third condition, i.e., triangle inequality.

$$\begin{aligned}
&\frac{1}{2}(d((V,f),(L,j)) - d((V,f),(Y,h)) - d((Y,h),(L,j))) \\
&= \dim((V,f) \wedge (Y,h)) + \dim((Y,h) \wedge (L,j)) \\
&\quad - \dim((V,f) \wedge (L,j)) - \dim Y \\
&= \dim((V,f)^{(1)} \cap (Y,h)^{(1)}) + \dim((Y,h)^{(1)} \cap (L,j)^{(1)}) \\
&\quad - \dim((V,f)^{(1)} \cap (L,j)^{(1)}) - \dim Y
\end{aligned}$$

$$
\begin{aligned}
&= \dim(((V,f)^{(1)} \cup (L,j)^{(1)}) \cap (Y,h)^{(1)}) \\
&\quad + \dim((V,f)^{(1)} \cap (L,j)^{(1)} \cap (Y,h)^{(1)}) \\
&\quad - \dim((V,f)^{(1)} \cap (L,j)^{(1)}) - \dim Y \\
&= -\dim((V,f)^{(1)} \cap (L,j)^{(1)}) - \dim Y \leq 0.
\end{aligned}
$$

Thus, the proof process completed.

Let $(V,f) \in I_e^n$, denote the number of $(L,j) \in I_e^n$ such that $\dim(V \cap L) = e-a$, $\dim((V,f) \wedge (L,j)) = e-b$, where $0 \leq a \leq \min\{e, n-e\}$, $a \leq b \leq e$ by $n_{(a,b-a)}$. The following Lemma 2.5 gives the exact value of $n_{(a,b-a)}$.

*Lemma 2.5:* Let $0 \leq a \leq \min\{e, n-e\}$, $a \leq b \leq e$, then

$$
n_{(a,b-a)} = q^{a^2} \begin{bmatrix} e \\ a \end{bmatrix}_q \begin{bmatrix} n-e \\ a \end{bmatrix}_q \begin{bmatrix} e-a \\ e-b \end{bmatrix}_q \times (q^n - q^{e-a}) \prod_{\alpha=1}^{q^e - q^{e-b} - 1} (q^n - q^{e-b} - \alpha).
$$

*Proof:* Let $(V,f) \in I_e^n$ be a fixed $e$-partial injective map of $\mathbb{F}_q^{(n)}$. Firstly we know that the number of $e$-dimensional subspaces $L$ such that $\dim(V \cap L) = e-a$ is $q^{a^2} \begin{bmatrix} e \\ a \end{bmatrix}_q \begin{bmatrix} n-e \\ a \end{bmatrix}_q$. Once the $e$-dimensional subspace $L$ is determined, next we calculate the possible partial injective map $j$ satisfying that $\dim((V,f) \wedge (L,j)) = e-b$. Note that the $e$-dimensional subspace $L$ is a disjoint union of the sets $\{y \in V \cap L | f(y) = j(y)\}$, $\{y \in V \cap L | f(y) \neq j(y)\}$, $L \backslash V \cap L$. For the first set $\{y \in V \cap L | f(y) = j(y)\}$, the choices of $j$ are $\begin{bmatrix} e-a \\ e-b \end{bmatrix}_q$. For the second set $\{y \in V \cap L | f(y) \neq j(y)\}$, here $f(y) \neq j(y)$ and $j$ is an injective map, so the number of the choices of $j$ is $(q^n - q^{e-b} - 1)(q^n - q^{e-b} - 2) \cdots (q^n - q^{e-a})$. For the last set $L \backslash V \cap L$, there are $(q^n - q^{e-a})(q^n - q^{e-a} - 1) \cdots (q^n - q^e + 1)$ choices of $j$ because $j$ is an injective map. Therefore,

$$
n_{(a,b-a)} = q^{a^2} \begin{bmatrix} e \\ a \end{bmatrix}_q \begin{bmatrix} n-e \\ a \end{bmatrix}_q \begin{bmatrix} e-a \\ e-b \end{bmatrix}_q \times (q^n - q^{e-a}) \prod_{\alpha=1}^{q^e - q^{e-b} - 1} (q^n - q^{e-b} - \alpha).
$$

## III. BOUNDS ON SUBSPACE CODES BASED ON PARTIAL INJECTIVE MAPS

Denote the largest number of codewords in an $(n, M, d, e)_q$ subspace code based on $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ by $A_q(n, d, e)$. Since the distance of any two codewords in an $(n, M, d, e)_q$ subspace code is an even number by (1), we just need to consider $A_q(n, d, e)$ in the case of even $d = 2b$.

The following is the definition of sphere in $I_e^n$.

*Definition 3.1:* The sphere $\mathcal{S}(2\alpha, e; (V,f))$ of radius $\alpha$ centered at an $e$-partial injective map $(V,f)$ of $\mathbb{F}_q^{(n)}$ is defined to be the set of all the $e$-partial injective maps $(L,j)$ in $I_e^n$ whose distance from $(V,f)$ is less than or equal to $2\alpha$. Namely,

$$
\mathcal{S}(2\alpha, e; (V,f)) = \left\{ (L,j) \in I_e^n | d((V,f),(L,j)) \leq 2\alpha \right\}.
$$

The following Lemma 3.2 gives the size of the sphere $\mathcal{S}(2\alpha, e; (V,f))$ in $I_e^n$.

*Theorem 3.2:* The number of all the $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ in $\mathcal{S}(2\alpha, e; (V,f))$ is independent of the choice of $(V,f)$ and

$$
\begin{aligned}
&|\mathcal{S}(2\alpha, e; (V,f))| \\
&= \sum_{b=0}^{\alpha} \sum_{a=0}^{\min\{b,n-e\}} q^{a^2} \begin{bmatrix} e \\ a \end{bmatrix}_q \begin{bmatrix} n-e \\ a \end{bmatrix}_q \\
&\quad \times \begin{bmatrix} e-a \\ e-b \end{bmatrix}_q (q^n - q^{e-a}) \prod_{s=1}^{q^e - q^{e-b} - 1} (q^n - q^{e-b} - s),
\end{aligned}
$$

where $(K,f) \in I_e^n$ is a fixed $e$-partial injective map of $\mathbb{F}_q^{(n)}$.

*Proof:* By Definition 3.1 and (1), we get

$$
\mathcal{S}(2\alpha, e; (V,f)) = \left\{ (L,j) \in I_e^n | d((V,f),(L,j)) \leq 2\alpha \right\},
$$

that is,

$$
\mathcal{S}(2\alpha, e; (V,f)) = \left\{ (L,j) \in I_e^n | \dim((V,f) \wedge (L,j)) \geq e - \alpha \right\}.
$$

Then by Lemma 2.5, we can get the theorem quickly.

The following Theorem 3.3 is the sphere-packing bound of subspace code $(n, M, 2b, e)_q$ in $I_e^n$, which is analog of the bound in the projective space.

*Theorem 3.3 (Sphere-Packing Bound):* Let $\alpha = \lfloor (b-1)/2 \rfloor$, thus

$$
\begin{aligned}
A_q(n, 2b, e) \leq &\begin{bmatrix} n \\ e \end{bmatrix}_q q^{\frac{e(e-1)}{2}} \prod_{a=n-e+1}^{n} (q^a - 1) \\
&\div \Big( \sum_{l=0}^{\alpha} \sum_{a=0}^{\min\{r,n-e\}} q^{a^2} \begin{bmatrix} e \\ a \end{bmatrix}_q \begin{bmatrix} n-e \\ a \end{bmatrix}_q \begin{bmatrix} e-a \\ e-b \end{bmatrix}_q \\
&\times (q^n - q^{e-a}) \prod_{s=1}^{q^e - q^{e-r} - 1} (q^n - q^{e-r} - s) \Big).
\end{aligned}
$$

*Proof:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, M, 2b, e)_q$ code, hence let $\alpha = \lfloor (b-1)/2 \rfloor$. Thus the spheres with radius $\alpha$ and centered with each codeword of $\mathbb{C}$ are disjoint, in addition each of these spheres includes $\beta$ $e$-partial injective maps of $\mathbb{F}_q^{(n)}$, where

$$
\begin{aligned}
\beta = &\sum_{l=0}^{\alpha} \sum_{a=0}^{\min\{r,n-e\}} q^{a^2} \begin{bmatrix} e \\ a \end{bmatrix}_q \begin{bmatrix} n-e \\ a \end{bmatrix}_q \begin{bmatrix} e-a \\ e-b \end{bmatrix}_q \\
&\times (q^n - q^{e-a}) \prod_{s=1}^{q^e - q^{e-r} - 1} (q^n - q^{e-r} - s).
\end{aligned}
$$

The number of $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ contained in all the spheres with radius $\alpha$ centered at each codeword of $\mathbb{C}$ cannot exceed the total number of $e$-partial injective maps of $\mathbb{F}_q^{(n)}$. Owing to the totality of $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ is

$$
\begin{bmatrix} n \\ e \end{bmatrix}_q q^{\frac{e(e-1)}{2}} \prod_{a=n-e+1}^{n} (q^a - 1).
$$

Thus we get the theorem quickly.

The following theorem is the Wang-Xing-Safavi-Naini bound of subspace code $(n, M, 2b, e)_q$ in $I_e^n$, which is analogy of the bound in the projective space.

*Theorem 3.4 (Wang-Xing-Safavi-Naini Bound):* Let $b \le e$, then

$$A_q(n, 2b, e) \le \frac{\begin{bmatrix} n \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+b}^{n} (q^a - 1)}{\begin{bmatrix} e \\ e-b+1 \end{bmatrix}_q}.$$

*Proof:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, M, 2b, e)_q$ code. On the one hand, by Lemma 2.2, every codeword of $\mathbb{C}$ includes exactly $\begin{bmatrix} e \\ e-b+1 \end{bmatrix}_q$ $(e-b+1)$-partial injective maps of $\mathbb{F}_q^{(n)}$. On the other hand, a given $(e-b+1)$-partial injective map of $\mathbb{F}_q^{(n)}$ must not be included in two different codewords of $\mathbb{C}$, or else, if $\dim((V, f) \wedge (L, j)) \ge e-b+1$ for arbitrary two different codewords $(V, f)$ and $(L, j)$ of $\mathbb{C}$, by (1), $d((V, f), (L, j)) = 2e - 2\dim((V, f) \wedge (L, j)) \le 2b - 2$, which is contradictious to the minimum distance $2b$ of $\mathbb{C}$. The totality of $(e-b+1)$-partial injective maps of $\mathbb{F}_q^{(n)}$ is

$$\begin{bmatrix} n \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+b}^{n} (q^a - 1),$$

so

$$M \le \frac{\begin{bmatrix} n \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+b}^{n} (q^a - 1)}{\begin{bmatrix} e \\ e-a+1 \end{bmatrix}_q}.$$

Next a puncturing operation on $(n, M, d, e)_q$ code in $I_e^n$ is defined. Suppose $\mathbb{C} \subseteq I_e^n$ is an $(n, M, d, e)_q$ code. For $(V, f) \in \mathbb{C}$, $(Y, h) \in I_{n-1}^n$, define

$$\mathcal{H}_{m-1}((V, f) \wedge (Y, h)) = \begin{cases} (V, f) \wedge (Y, h), & \textit{if}\, (V, f) \wedge (Y, h) \in I_{e-1}^n, \\ (D, f|_D), & \text{otherwise}, \end{cases}$$

where $D$ is an $(e-1)$-dimensional subspace of $V$.

Then we can get a punctured code $\mathbb{C}'$. Although this kind of puncturing operation usually does not produce a unique code, we mark each such punctured code as $\mathbb{C}' = \mathbb{C}|_{(Y,h)}$ and it is an $(n-1, M, d', e-1)_q$ code. We can get the Lemma 3.5.

*Lemma 3.5:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, M, d, e)_q$ code such that $d > 2$. Then for $(Y, h) \in I_{n-1}^n$, $\mathbb{C}' = \mathbb{C}|_{(Y,h)}$ should be an $(n-1, M, d', e-1)_q$ code such that $d' \ge d - 2$.

*Proof:* According to the definition of code $\mathbb{C}'$, only the cardinality and the minimum distance of code $\mathbb{C}'$ need to be considered. First of all, we show that $d' \ge d - 2$. Let $(V, f), (L, j)$ be arbitrary two different codewords of code $\mathbb{C}$. Suppose that $(V', f') = \mathcal{H}_{e-1}((V, f) \wedge (Y, h))$ and $(L', j') = \mathcal{H}_{e-1}((L, j) \wedge (Y, h))$ are the corresponding codewords in code $\mathbb{C}'$ according to the definition of code $\mathbb{C}'$. Evidently, $(V', f') \preceq (V, f)$ and $(L', j') \preceq (L, j)$, then we have $((V', f') \wedge (L', j')) \preceq ((V, f) \wedge (L, j))$, so that we get

$$\dim((V', f') \wedge (L', j')) \le \dim((V, f) \wedge (L, j)). \tag{2}$$

By (1) and (2), we have

$$\begin{aligned} d((V', f'), (L', j')) &= 2(e-1) - 2\dim((V', f') \wedge (L', j')) \\ &\ge 2e - 2 - 2\dim((V, f) \wedge (L, j)) \\ &= d((V, f), (L, j)) - 2 \\ &\ge d - 2 > 0, \end{aligned}$$

which implies that $d' \ge d - 2$. By $d > 2$, $d((V', f'), (L', j')) \ge d - 2 > 0$, so $(V', f')$ and $(L', j')$ are different, which shows that code $\mathbb{C}'$ defined above has the same number of codewords as code $\mathbb{C}$.

The following bound is established according to the above puncturing operation on $(n, M, d, e)_q$ code in $I_e^n$, which is analog of the bound in the projective space.

*Theorem 3.6 (Singleton Bound):* Let $e \ge b$, then

$$A_q(n, d, e) \le \begin{bmatrix} n-b+1 \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+1}^{n-b+1} (q^a - 1).$$

*Proof:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, M, d, e)_q$ code. As Lemma 3.5 is punctured $(b-1)$ times repeatedly, an $(n-b+1, M, d', e-b+1)_q$ code with that size $M$ may be not greater than $\left|I_{e-b+1}^{n-b+1}\right|$ can be obtained. We all known that

$$\left|I_{e-b+1}^{n-b+1}\right| = \begin{bmatrix} n-b+1 \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+1}^{n-b+1} (q^a - 1).$$

The theorem follows immediately.

*Theorem 3.7 (Johnson Bound I):* Let $b \le e - 1$, then

$$A_q(n, 2b, e) \le \frac{(q^n - 1)^2}{q^e - 1} A_q(n-1, 2b, e-1).$$

*Proof:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, M, 2b, e)_q$ code, and which reaches the value of $A_q(n, 2b, e)$, i.e., $M = A_q(n, 2b, e)$. Each codeword ($e$-partial injective map of $\mathbb{F}_q^{(n)}$) of $\mathbb{C}$ contains $\frac{q^e-1}{q-1}$ 1-partial injective maps of $\mathbb{F}_q^{(n)}$. There are in total $\left|I_1^n\right| = \frac{(q^n-1)^2}{q-1}$ 1-partial injective maps of $\mathbb{F}_q^{(n)}$. Hence, there exists a 1-partial injective map of $\mathbb{F}_q^{(n)}$ $(L, j)$ satisfying that the 1-partial injective map $(L, j)$ is included in at least $\frac{M \cdot (q^e-1)}{(q^n-1)^2}$ code words of $\mathbb{C}$.

Let $\mathbb{F}_q^{(n)} = L \oplus Q$, where $Q$ is a subspace with $(n-1)$-dimension of $\mathbb{F}_q^{(n)}$. Let $(Q, h)$ be an $(n-1)$-partial injective map of $\mathbb{F}_q^{(n)}$. Define the following set

$$\mathbb{C}' = \{(Q, h) \wedge (V, f) | (V, f) \in \mathbb{C},\ (L, j) \preceq (V, f)\}.$$

Clearly $\mathbb{C}'$ consists of $(e-1)$-partial injective maps of $\mathbb{F}_q^{(n-1)}$, so $\mathbb{C}'$ can be considered as an $(n-1, M', 2b', e-1)_q$ code with the size $H'$ and the minimum distance $2b'$, and $M' \ge \frac{M \cdot (q^e-1)}{(q^n-1)^2}$.

Let $(P', s')$ and $(Y', \alpha')$ be arbitrary two code words of $\mathbb{C}'$, thus it must be have matching code words $(P, s)$ and $(Y, \alpha)$ of $\mathbb{C}$ with $(P', s') = (Q, h) \wedge (P, s)$ and $(Y', \alpha') = (Q, h) \wedge (Y, \alpha)$ with $(L, j) \preceq (P, s)$ and $(L, j) \preceq (Y, \alpha)$. Noting that $(P', s') \wedge (Y', \alpha') = (Q, h) \wedge [(P, s) \wedge (Y, \alpha)]$, we get

$$\begin{aligned} \dim((P', s') \wedge (Y', \alpha')) &= \dim((Q, h) \wedge [(P, s) \wedge (Y, \alpha)]) \\ &= \dim(W) + \dim((P, s) \wedge (Y, \alpha)) - n \\ &= \dim((P, s) \wedge (Y, \alpha)) - 1 \end{aligned}$$

and the third equal sign is based on the fact that $(P, s) \wedge (Y, \alpha)$ contains $(L, j)$. Hence,

$$\dim((P', s') \wedge (Y', \alpha')) = \dim((P, s) \wedge (Y, \alpha)) - 1. \quad (3)$$

By (1) and (3), $d((P', s'), (Y', \alpha')) = d((P, s), (Y, \alpha))$ from which $b = b'$ follows. To sum up, we get

$$A_q(n, 2b, e) \leq \frac{(q^n - 1)^2}{q^e - 1} A_q(n-1, 2b, e-1).$$

The theorem follows immediately.

*Theorem 3.8 (Johnson Bound II):* Let $b \leq e \leq n-1$, then $A_q(n, 2b, e)$

$$\leq \frac{(q^n - 1) q^{\frac{(n-1)(n-2)}{2}} \prod_{a=2}^{n} (q^a - 1)}{(q^{n-e} - 1) \prod_{a=0}^{q^{n-1} - q^e - 1} (q^n - q^e - i)} A_q(n-1, 2b, e).$$

*Proof:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, M, 2b, e)_q$ code. For any $(L, j) \in I_{n-1}^n$, define the following set

$$\mathbb{C}_{(L,j)} = \{(V, f) \in \mathbb{C} | (V, f) \preceq (L, j)\}.$$

Clearly, $\mathbb{C}_{(L,j)}$ is an $(n-1, \left|\mathbb{C}_{(L,j)}\right|, 2b, e)_q$ code, where $\left|\mathbb{C}_{(L,j)}\right| \leq A_q(n-1, 2b, e)$. It is know that

$$\sum_{(L,j) \in I_{n-1}^n} \left|\mathbb{C}_{(L,j)}\right| = M \cdot \begin{bmatrix} n-e \\ n-1-e \end{bmatrix}_q \prod_{a=0}^{q^{n-1} - q^e - 1} (q^n - q^e - a).$$

Note that

$$\begin{aligned} &\sum_{(L,j) \in I_{n-1}^n} \left|\mathbb{C}_{(L,j)}\right| \\ &\leq \left|I_{n-1}^n\right| \cdot A_q(n-1, 2b, e) \\ &= \begin{bmatrix} n \\ n-1 \end{bmatrix}_q q^{\frac{(n-1)(n-2)}{2}} \prod_{a=2}^{n} (q^a - 1) \cdot A_q(n-1, 2b, e). \end{aligned}$$

We obtain

$$M \leq \frac{(q^n - 1) q^{\frac{(n-1)(n-2)}{2}} \cdot \prod_{a=2}^{n} (q^a - 1)}{(q^{n-e} - 1) \cdot \prod_{a=0}^{q^{n-1} - q^e - 1} (q^n - q^e - a)} A_q(n-1, 2b, e).$$

The theorem follows immediately.

*Theorem 3.9 (Gilbert-Varshamov Bound):* Let $b \leq e$, then

$$\begin{aligned} A_q(n, 2b, e) \geq &\begin{bmatrix} n \\ e \end{bmatrix}_q \cdot q^{\frac{e(e-1)}{2}} \prod_{a=n-e+1}^{n} (q^a - 1) \\ &\div \Big( \sum_{r=0}^{b-1} \sum_{a=0}^{\min\{r, n-e\}} q^{a^2} \begin{bmatrix} e \\ a \end{bmatrix}_q \begin{bmatrix} n-e \\ a \end{bmatrix}_q \begin{bmatrix} e-a \\ e-b \end{bmatrix}_q \\ &\times (q^n - q^{e-a}) \prod_{s=1}^{q^e - q^{e-r} - 1} (q^n - q^{e-r} - s) \Big). \end{aligned}$$

*Proof:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, A_q(n, 2b, e), 2b, e)_q$ code. There is no $e$-partial injective map $(L, j) \in I_e^n \backslash \mathbb{C}$ satisfying $d((V, f), (L, j)) \geq 2b$ for any $(V, f) \in \mathbb{C}$. Or else, through increasing the $e$-partial injective map $(L, j)$ to $\mathbb{C}$, we can obtain a new $(n, A_q(n, 2b, e) + 1, 2b, e)_q$ code, this will be in contradiction with the size of codewords $A_q(n, 2b, e)$.

**TABLE 1.** The explicit bounds for fixed parameters.

| $(b, e)$ | $(2, 2)$ | $(1, 2)$ | $(2, 1)$ |
|---|---|---|---|
| Sphere-packing bound | $\frac{(q^{n-1}-1)^2(q^n-1)^2 q}{(q-1)(q^2-1)(q^n-q^2)}$ | $\frac{(q^{n-1}-1)^2(q^n-1)^2 q}{(q^2-1)^2(q^n-q^2)}$ | $\frac{(q^n-1)^2}{(q^n-q)(q-1)}$ |
| Wang-Xing-Safavi-Naini bound | $\frac{(q^n-1)^2}{q^2-1}$ | $\frac{(q^{n-1}-1)^2(q^n-1)^2 q}{(q-1)(q^2-1)}$ | nonexist |
| Singleton bound | $\frac{(q^{n-1}-1)^2}{q-1}$ | $\frac{(q^{n-1}-1)^2(q^n-1)^2 q}{(q-1)(q^2-1)}$ | nonexist |
| Gilbert-Varshamov bound | Omitted(complex) | $\frac{(q^{n-1}-1)^2(q^n-1)^2 q}{(q^2-1)^2(q^n-q^2)}$ | nonexist |

**TABLE 2.** The explicit bounds for fixed parameters.

| $(b, e)$ | $(2, 2)$ | $(1, 2)$ | $(2, 1)$ |
|---|---|---|---|
| Sphere-packing bound | $\frac{(q+1)(q-1)(q^2+q+1)^2}{q}$ | $\frac{(q-1)(q^2+q+1)^2}{q}$ | $\frac{(q^2+q+1)^2}{q(q+1)}$ |
| W-X-S-N bound | $\frac{(q-1)(q^2+q+1)^2}{q+1}$ | $(q+1)(q-1)^2 \times (q^2+q+1)^2 q$ | nonexist |
| Singleton bound | $(q+1)^2(q-1)$ | $(q+1)(q-1)^2 \times (q^2+q+1)^2 q$ | nonexist |

Thus, for arbitrary $(V, f) \in \mathbb{C}$,

$$\bigcup_{(V,f) \in \mathbb{C}} \mathcal{S}(2(b-1), e; (V, f)) \supseteq I_e^n,$$

which implies that

$$\sum_{(V,f) \in \mathbb{C}} |\mathcal{S}(2(b-1), e; (V, f))| \geq \left|I_e^n\right|.$$

We obtain $V_q(n, 2b, e) \cdot |\mathcal{S}(2(b-1), e; (V, f))| \geq \left|I_e^n\right|$. The theorem follows immediately by Lemma 2.1 and Theorem 3.2.

*Remark 3.10:* Let $n \geq 3$. The bounds listed in Theorem 3.3, Theorem 3.4, Theorem 3.6, Theorem 3.9 by choosing fixed parameters $b, e$ are provided as below in table 1.

From the second column of the table 1, we obtain $V_q(n, 2, 2) = \frac{(q^{n-1}-1)^2(q^n-1)^2 q}{(q^2-1)^2(q^n-q^2)}$.

Let $n = 3$, the bounds listed in Theorem 3.3, Theorem 3.4, Theorem 3.6, Theorem 3.9 by choosing fixed parameters $b, e$ are given in the following table 2.

From Table 2, we obtain $(3, (q+1)^2(q-1), 4, 2)_q$, $(3, \frac{(q-1)(q^2+q+1)^2}{q}, 2, 2)_q$, $(3, \frac{(q^2+q+1)^2}{q(q+1)}, 4, 1)_q$ are optimal codes.

## IV. ANTICODE BOUND AND AHLSWEDE-AYDINIAN BOUND FROM E-K-R THEOREM

Erdős-Ko-Rado theorem theorem is a classical result in extremal combinatorics. The Erdős-Ko-Rado theorem theorem gives a bound on the size of a family of $k$-subsets of a set that every pairwise intersection has size at least $t$ and describes exactly which families meet this bound.

*Theorem 4.1 [14]:* Let $n \geq (k-t+1)(t+1)$ and $\mathcal{F}$ be a family of $k$-subsets of an $n$-set $X$ such that $\left|S \cap S'\right| \geq t$ for any $S, S' \in \mathcal{F}$. Then $|\mathcal{F}| \leq \binom{n-t}{k-t}$ holds. If $n > (k-t+1)(t+1)$ and $|\mathcal{F}| = \binom{n-t}{k-t}$, then $\mathcal{F}$ consists of all $k$-subsets containing a fixed $t$-subset of $X$.

An anticode $\mathcal{A}(2\alpha)$ of diameter $2\alpha$ in $I_e^n$ is any subset of $I_e^n$ such that $d((V,f),(L,j)) \leq 2\alpha$ for all $(V,f)$,$(L,j) \in \mathcal{A}(2\alpha)$. The maximum anticode of diameter $2(b-1)$ in $I_e^n$ is provided in [15]. A family $\mathcal{F} \subseteq I_e^n$ is known as $\alpha$-intersecting if $\dim((V,f) \wedge (L,j)) \geq \alpha$ for all $(V,f),(L,j) \in \mathcal{F}$. The Lemma 4.2 which is shown below is the EKR theorem for $e$-partial injective maps of $\mathbb{F}_q^{(n)}$.

*Lemma 4.2 [15]:* Presume that $\mathcal{F} \subseteq I_e^n$ is an $(e-b+1)$-intersecting family with $0 < e-b+1 < e$. Assume that either $b = 2$ and $(q+1)(q^e-1) < (q^{n-e+1}-1)(q^n-q^{e-1})$ or $b > 2$ and

$$\begin{bmatrix} e \\ b-1 \end{bmatrix}_q^2 < \begin{bmatrix} n-e+b-1 \\ b-1 \end{bmatrix}_q \cdot \prod_{a=e-b+1}^{e-1} (q^n - q^a),$$

then

$$|\mathcal{F}| \leq \begin{bmatrix} n-e+b-1 \\ b-1 \end{bmatrix}_q \cdot \prod_{a=e-b+1}^{e-1} (q^n - q^a).$$

Ahlswede and Aydinian [16] obtained the following Lemma 4.3 based on vertex transitive graphs, which is used in the sequel.

*Lemma 4.3 [16]:* Let $\Gamma = (R, J)$ be a figure that satisfies a transitive group of automorphisms $Aut(\Gamma)$, if $V$, $L$ be any subsets of the vertex set $R$. Thus it must be have some $j \in Aut(\Gamma)$ satisfy $\frac{|j(V)\cap L|}{|L|} \geq \frac{|V|}{|R|}$.

By Lemma 4.3, we give the result as below.

*Lemma 4.4:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, M, 2b, e)_q$ code. For an arbitrary subset $\mathcal{L} \subseteq I_e^n$, there must be an $(n, M^*, 2b, e)_q$ code $\mathbb{C}^* \subseteq \mathcal{L}$ with $\frac{M^*}{|\mathcal{L}|} \geq \frac{M}{|I_e^n|}$.

*Proof:* Define a figure with vertex set $I_e^n$ and two vertices $(V,f)$ and $(L,j)$ of vertex set $I_e^n$ are adjacent if $\dim((V,f) \wedge (L,j)) = e-1$. Then $\Gamma$ accepts a transitive group of automorphisms $GL_n(\mathbb{F}_q)$. By Lemma 4.3, there must be some $T \in GL_n(\mathbb{F}_q)$ with that $\frac{|\{ET|E\in\mathbb{C}\}\cap\mathcal{L}|}{|\mathcal{L}|} \geq \frac{M}{|I_e^n|}$.

Let $\mathbb{C}^* = \{ET|E \in \mathbb{C}\} \cap \mathcal{L}$. For any $R^*$, $Q^* \in \mathbb{C}^*$, there exist $R$, $Q \in \mathbb{C}$ such that $R^* = RT$ and $Q^* = QT$. We have $\dim(R^* \wedge Q^*) = \dim((R \wedge Q)T) = \dim(R \wedge Q)$ which means that $d(R^*, Q^*) = d(R, Q)$. According to that $\mathbb{C}^* \subseteq \mathcal{L}$ is an $(n, M^*, 2b, e)_q$ code with $M^* = |\{ET|E \in \mathbb{C}\} \cap \mathcal{L}|$. We can get the theorem fastly.

*Theorem 4.5:* Presume that either $b = 2$ and $(q+1)(q^e - 1) < (q^{n-e+1}-1)(q^n-q^{e-1})$ or $b > 2$ and

$$\begin{bmatrix} e \\ b-1 \end{bmatrix}_q^2 < \begin{bmatrix} n-e+b-1 \\ b-1 \end{bmatrix}_q \cdot \prod_{a=e-b+1}^{e-1} (q^n - q^a).$$

Then

$$A_q(n, 2b, e) \leq \frac{\begin{bmatrix} n \\ e \end{bmatrix}_q q^{\frac{e(e-1)}{2}} \cdot \prod_{a=n-e+1}^{n} (q^a - 1)}{\begin{bmatrix} n-e+b-1 \\ b-1 \end{bmatrix}_q \cdot \prod_{a=e-b+1}^{e-1} (q^n - q^a)}.$$

*Proof:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, M, 2b, e)_q$ code and $\mathcal{A}(2(b-1))$ be the largest anticode in $I_e^n$. Then $\mathbb{C}^* \cap \mathcal{A}(2(b-1))$ has at most one element. By Lemma 4.4,

$$A_q(n, 2b, e) \leq \frac{|I_e^n|}{|\mathcal{A}(2(b-1))|} = \frac{\begin{bmatrix} n \\ e \end{bmatrix}_q q^{\frac{e(e-1)}{2}} \cdot \prod_{a=n-e+1}^{n} (q^a - 1)}{\begin{bmatrix} n-e+b-1 \\ b-1 \end{bmatrix}_q \cdot \prod_{a=e-b+1}^{e-1} (q^n - q^a)}.$$

The following Theorem 4.6 is the Ahlswede and Aydinian bound of $(n, M, 2b, e)_q$ code in $I_e^n$, which is provided by Ahlswede and Aydinian [16] on the size of codes in the projective space.

*Theorem 4.6 (Ahlswede and Aydinian Bound):* Let $0 \leq a \leq b \leq e$, then

$$\begin{aligned} A_q(n, 2b, e) \leq{}& \begin{bmatrix} n \\ e \end{bmatrix}_q q^{\frac{e(e-1)}{2}} \prod_{a=n-e+1}^{n} (q^a - 1) \\ & \div \Big( \sum_{\alpha=e-a}^{e} q^{(e-\alpha)^2} \begin{bmatrix} e \\ \alpha \end{bmatrix}_q \begin{bmatrix} n-e \\ e-\alpha \end{bmatrix}_q \begin{bmatrix} \alpha \\ e-b \end{bmatrix}_q \\ & \times (q^n - q^\alpha) \prod_{s=1}^{q^e - q^{e-b} - 1} (q^n - q^{e-b} - s) \Big). \end{aligned}$$

*Proof:* Let $\mathbb{C} \subseteq I_e^n$ be an $(n, M, 2b, e)_q$ code and $(V,f) \in I_e^n$ be a fixed $e$-partial injective map of $\mathbb{F}_q^{(n)}$. Define $\mathcal{L} = \{(L,j) \in I_e^n | \dim(V \cap L) \geq e-a, \dim((L,j) \wedge (V,f)) = e-b\}$. By Lemma 2.5, the size of set $\mathcal{L}$ is

$$\begin{aligned} |\mathcal{L}| ={}& \sum_{\alpha=e-a}^{e} n_{(e-\alpha, b-e+\alpha)} \\ ={}& \sum_{\alpha=e-a}^{e} q^{(e-\alpha)^2} \begin{bmatrix} e \\ \alpha \end{bmatrix}_q \begin{bmatrix} n-e \\ e-\alpha \end{bmatrix}_q \begin{bmatrix} \alpha \\ e-b \end{bmatrix}_q (q^n - q^\alpha) \\ & \times \prod_{s=1}^{q^e - q^{e-b} - 1} (q^n - q^{e-b} - s). \end{aligned} \tag{4}$$

Define $\mathbb{C}_1 = \{E_1 = \mathcal{H}_{e-a}((L,j) \wedge (V,f)) | (L,j) \in \mathbb{C}^*\}$, where $\mathcal{H}_{e-a}((L,j) \wedge (V,f)) = (L,j) \wedge (V,f)$, if $\dim((L,j) \wedge (V,f)) = e-a$; otherwise $\mathcal{H}_{e-a}((L,j) \wedge (V,f))$ is some $(e-a)$-partial injective map of $\mathbb{F}_q^{(n)}$.

Let $(L,j)$ and $(Y,h)$ be arbitrary two different codewords of code $\mathbb{C}^*$, then

$$d((L,j),(Y,h)) = 2e - 2\dim((L,j) \wedge (Y,h)) \geq 2b,$$

which implies that $\dim((L,j) \wedge (Y,h)) \leq e-b$. For $E_1 = \mathcal{H}_{e-a}((L,j) \wedge (V,f))$ and $Q_1 = \mathcal{H}_{e-a}((Y,h) \wedge (V,f))$, we have

$$\begin{aligned} d(E_1, Q_1) &= 2(e-a) - 2\dim(E_1 \wedge Q_1) \\ &\geq 2(e-a) - 2\dim((L,j) \wedge (Y,h)) \\ &\geq 2(e-a) - 2(e-b) = 2(b-a). \end{aligned}$$

Then, $\mathbb{C}_1$ is an $(e, M^*, 2(b-a), (e-a))_q$ code with $M^* \leq A_q(e, 2(b-a), e-a)$. Therefore,

$$M \leq \frac{|I_e^n|}{|\mathcal{L}|} A_q(e, 2(b-a), e-a).$$

By Lemma 2.1 and (4), the theorem follows immediately.

## V. STEINER STRUCTURE BASED ON PARTIAL INJECTIVE MAPS

In this section, we will give the Steiner structure based on $e$-partial injective maps.

*Definition 5.1:* Let $1 \leq r \leq e \leq n$. A Steiner structure $\mathbb{S}_q[r, e; n]$ in $I_e^n$ is a collection $\mathbb{S}$ of elements from $I_e^n$ satisfying that each $r$-partial injective map in $I_r^n$ is included in exactly one $e$-partial injective map of $\mathbb{S}$. The $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ in $\mathbb{S}$ are known as blocks of the Steiner structure $\mathbb{S}_q[r, e; n]$.

*Lemma 5.2:* The total number of blocks in the Steiner structure $\mathbb{S}_q[r, e; n]$ is

$$\frac{\begin{bmatrix} n \\ r \end{bmatrix}_q q^{\frac{r(r-1)}{2}} \prod_{a=n-r+1}^{n} (q^a - 1)}{\begin{bmatrix} e \\ r \end{bmatrix}_q}.$$

*Proof:* By Lemma 2.2, each block ($e$-partial injective map of $\mathbb{F}_q^{(n)}$) of the Steiner structure $\mathbb{S}_q[r, e; n]$ contains $\begin{bmatrix} e \\ r \end{bmatrix}_q$ $r$-partial injective maps of $\mathbb{F}_q^{(n)}$. Through the definition 5.1, we can know that every $r$-partial injective map of $\mathbb{F}_q^{(n)}$ is included in just one block of $\mathbb{S}$. The totality of $r$-partial injective of $\mathbb{F}_q^{(n)}$ is $|I_r^n| = \begin{bmatrix} n \\ r \end{bmatrix}_q q^{\frac{r(r-1)}{2}} \prod_{a=n-r+1}^{n} (q^a - 1)$. Thus the totality of blocks in the $\mathbb{S}_q[r, e; n]$ is

$$\frac{\begin{bmatrix} n \\ r \end{bmatrix}_q q^{\frac{r(r-1)}{2}} \prod_{a=n-r+1}^{n} (q^a - 1)}{\begin{bmatrix} e \\ r \end{bmatrix}_q}.$$

*Theorem 5.3:* A Steiner structure $\mathbb{S}_q[r, e; n]$ is an $(n, M, 2b, e)_q$ code with the size

$$M = \frac{\begin{bmatrix} n \\ r \end{bmatrix}_q q^{\frac{r(r-1)}{2}} \prod_{a=n-r+1}^{n} (q^a - 1)}{\begin{bmatrix} e \\ r \end{bmatrix}_q}$$

and $b = e-r+1$.

*Proof:* Through Definition 5.1 and Lemma 5.2, it will be enough to show that $b = e-r+1$ is correct. First of all, for arbitrary two different blocks $(V, f), (L, j) \in \mathbb{S}_q[r, e; n]$, we can get

$$\dim((V, f) \wedge (L, j)) \leq r - 1, \tag{5}$$

otherwise there at least exists a $r$-partial injective map of $\mathbb{F}_q^{(n)}$ contained in $(V, f)$ and $(L, j)$ which is contradictious to the definition of the Steiner structure $\mathbb{S}_q[r, e; n]$. Thus, by (1) and (5),

$$\begin{aligned} d((V, f), (L, j)) &= 2e - 2\dim((V, f) \wedge (L, j)) \\ &\geq 2e - 2(r - 1) = 2(e-r + 1). \end{aligned}$$

Moreover, $2b$ is the minimum distance of the code $(n, M, 2b, e)_q$, then we have $b \geq e-r + 1$.

Conversely, let $(Y, h)$ be a fixed $(r-1)$-partial injective map of $\mathbb{F}_q^{(n)}$ and we can choose two $r$-partial injective maps $(V_1, f_1)$ and $(L_1, j_1)$ of $\mathbb{F}_q^{(n)}$ such that $(Y, h) = (V_1, f_1) \wedge (L_1, j_1)$. According to the definition of the Steiner structure $\mathbb{S}_q[r, e; n]$, there is a unique corresponding blocks $(V, f)$ and $(L, j)$ in $\mathbb{S}_q[r, e; n]$ satisfying $(V_1, f_1) \preceq (V, f)$ and $(L_1, j_1) \preceq (L, j)$, respectively. Then, $(Y, h) = (V_1, f_1) \wedge (L_1, j_1) \subseteq (V, f) \wedge (L, j)$ from which we can get

$$\dim((V, f) \wedge (L, j)) \geq r - 1. \tag{6}$$

By (1) and (6), $d((V, f), (L, j)) = 2e - 2\dim((V, f) \wedge (L, j)) \leq 2e - 2(r - 1) = 2(e-r + 1)$. Moreover, $2b$ is the minimum distance of the code $(n, M, 2b, e)_q$, then we have $b \leq e-r + 1$.

From the above two aspects, we obtain $b = e - r + 1$.

*Theorem 5.4:* An $(n, M, 2b, e)_q$ code $\mathbb{C}$ achieves the Wang-Xing-Safavi-Naini bound, namely,

$$M = \frac{\begin{bmatrix} n \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+b}^{n} (q^a - 1)}{\begin{bmatrix} e \\ e-b+1 \end{bmatrix}_q}$$

when and only when $\mathbb{C}$ is a Steiner structure $\mathbb{S}_q[e-b+1, e; n]$.

*Proof:* According to Theorem 3.4 and Theorem 5.3, we can get that in case of $\mathbb{C}$ is a Steiner structure $\mathbb{S}_q[e-b+1, e; n]$, thus $\mathbb{C}$ should be an $(n, M, 2b, e)_q$ code, where

$$M = \frac{\begin{bmatrix} n \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+b}^{n} (q^a - 1)}{\begin{bmatrix} e \\ e-b+1 \end{bmatrix}_q}.$$

Obviously, the code reaches the Wang-Xing-Safavi-Naini bound.

Conversely, suppose $\mathbb{C}$ is an $(n, M, 2b, e)_q$ reaching the Wang-Xing-Safavi-Naini bound, that is,

$$M = \frac{\begin{bmatrix} n \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+b}^{n} (q^a - 1)}{\begin{bmatrix} e \\ e-b+1 \end{bmatrix}_q}.$$

Firstly, we have that any $(e-b+1)$-partial injective map of $\mathbb{F}_q^{(n)}$ cannot be included in two different codewords $(V, f)$ and $(L, j)$ of $\mathbb{C}$, or else, by (1),

$$\begin{aligned} d((V, f), (L, j)) &= 2e - 2\dim((V, f) \wedge (L, j)) \\ &\leq 2e - 2(e-b + 1) = 2b - 2, \end{aligned}$$

it is in contradiction with the minimum distance $2b$ of $\mathbb{C}$.

Then, from the fact that each codeword of $\mathbb{C}$ includes $\begin{bmatrix} e \\ e-b+1 \end{bmatrix}_q$ different $(e-b+1)$-partial injective maps of $\mathbb{F}_q^{(n)}$, we realize that all the codewords of $\mathbb{C}$ include altogether

$$H\begin{bmatrix} e \\ e-b+1 \end{bmatrix}_q = \begin{bmatrix} n \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+b}^{n} (q^a - 1)$$

distinct $(e-b+1)$-partial injective maps of $\mathbb{F}_q^{(n)}$. And there are altogether

$$\begin{bmatrix} n \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+b}^{n} (q^a - 1)$$

distinct $(e-b+1)$-partial injective maps of $\mathbb{F}_q^{(n)}$, this means that every $(e-b+1)$-partial injective map of $\mathbb{F}_q^{(n)}$ is included in exactly one code word of $\mathbb{C}$. So, when we regard the codewords of $\mathbb{C}$ as blocks, code $\mathbb{C}$ is a Steiner structure $\mathbb{S}_q\,[e-b+1, e; n]$.

*Corollary 5.5:*

$$A_q(n, 2b, e) = \frac{\begin{bmatrix} n \\ e-b+1 \end{bmatrix}_q q^{\frac{(e-b+1)(e-b)}{2}} \prod_{a=n-e+b}^{n} (q^a - 1)}{\begin{bmatrix} e \\ e-b+1 \end{bmatrix}_q}$$

when and only when the Steiner structure $\mathbb{S}_q\,[e-b+1, e; n]$ exists.

## VI. CONCLUSION

In the article, the subspace code $(n, M, 2b, e)_q$ based on $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ over $\mathbb{F}_q$ are considered. Several bounds of the subspace code $(n, M, 2b, e)_q$ based on $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ are presented. The anticode bound and Ahlswede-Aydinian bound of the subspace code $(n, M, 2b, e)_q$ are obtained using the EKR theorem for $e$-partial injective maps of $\mathbb{F}_q^{(n)}$. Finally, we show that the $(n, M, 2b, e)_q$ codes based on $e$-partial injective maps of $\mathbb{F}_q^{(n)}$ reach the Wang-Xing-Safavi-Naini bound when and only when they are certain Steiner structures in $I_e^n$.

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