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On Mixed Metric Dimension of Some Path Related Graphs

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ABSTRACT A vertex $k \in V_G$ determined two elements (vertices or edges) $\ell, m \in V_G \cup E_G$, if $d_G(k, \ell) \neq d_G(k, m)$. A set R_{m} of vertices in a graph G is a mixed metric generator for G, if two distinct elements (vertices or edges) are determined by some vertex set of R_{m}. The least number of elements in the vertex set of R_{m} is known as mixed metric dimension, and denoted as $dim_m(G)$. In this article, the mixed metric dimension of some path related graphs is obtained. Those path related graphs are P_n^2 the square of a path, $T(P_n)$ total graph of a path, the middle graph of a path $M(P_n)$, and splitting graph of a path $S(P_n)$. We proved that these families of graphs have constant and unbounded mixed metric dimension, respectively. We further presented an improved result for the metric dimension of the splitting graph of a path $S(P_n)$.

INDEX TERMS Mixed metric dimension, metric dimension, edge metric dimension, path related graphs.

I. INTRODUCTION AND PRELIMINARY RESULTS

Let for a graph $G = (V_G, E_G)$, where V_G, interpret the vertices, and E_G the edges of a graph. A vertex $k \in V_G$, resolve two elements (vertices or edges) $\ell, m \in E_G \cup V_G$, if $d_G(k, \ell) \neq d_G(k, m)$. All the graphs studied here are simple, connected, and finite. A set R_{m}, is said to be the mixed metric generator of a graph G, if every two distinct elements (vertices or edges) are determined by some vertex set of R_{m}. The mixed metric dimension is the least cardinality of a mixed metric generator; the notion used here to represent is $dim_m(G)$. This idea is put forward in [9], and recently it is investigated in [18], where the authors studied some rotationally symmetric graphs, and the mixed metric dimension of $P(n, 2)$ is studied in [19]. The author in [4] computed mixed metric dimension of flower snarks J_n, and wheel graphs W_n. Also, in [5] presented some lower bounds for this invariant. The mixed metric dimension is the combination of a well studied metric and edge metric dimension.

The concept of metric dimension was put forward by Slater [21], where it was expressed as locating sets, and later by Harary and Melter [7] called it as a metric dimension where the metric generators were termed as resolving sets. There are numerous metric dimension applications, such as identifying an intruder in a network, robotics navigation, chemistry, and pattern recognition or image processing; for further studies related to this invariant, some of the references are, see, for instance, [2], [3], [12]. Some of the recent studies on the metric dimension are in [8], [13], [20].

When some vertex set of a graph resolves the graphs' vertices, the authors called it the metric dimension. When the edges of graphs are resolved by some vertex set of a graph, the author in [10] termed it as the edge metric dimension. Mathematically it can be written as, for $k \in V_G$, and for some $e = \ell m$.

$$d_G(k, e) = min(d_G(k, \ell), d_G(k, m))$$

Now, the vertex $k \in V_G$, distinguish two edges say e_1, and e_2, if the condition holds that is $d_G(k, e_1) \neq d_G(k, e_2)$. When it comes to defining the edge metric dimension, it is similar to the metric dimension; the only difference is that the resolving set say R_e, resolves the edges of the graph. The least cardinality of the resolving set for this invariant is called edge metric dimension, and usually, the notation used is dim_e. After the first introductory article [10], many authors

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investigated this invariant. Some of the latest articles for the reader's convenience are as [11], [14], [15], [22], [23].

A vertex $k \in V_G$ determined two elements (vertices or edges) $\ell, m \in V_G \cup E_G$, if $d_G(k, \ell) \neq d_G(k, m)$. A set R_{m} of vertices in a graph G is a mixed metric generator for G, if two distinct elements (vertices or edges) are determined by some vertex set of R_{m}. The least number of elements in the vertex set of R_{m} is known as mixed metric dimension, and denoted as $dim_m(G)$.

II. KNOWN RESULTS

The mixed metric generator is the combination of metric, and edge metric generator, as shown in the following relationship;

Observation: [9]

$$dim_m(G) \geq max\{dim(G), dim_e(G)\}$$

While discussing the mixed metric dimension structure, it is imperative to note that a graph with a pendant vertex cannot form a mixed metric generator.

Proposition 1 [9]: For any graph G, $2 \leq dim_m(G) \leq n$.

Proposition 2 [10]: For a path graph, $dim(P_n) = dim_e(P_n) = 1$.

Proposition 3 [9]: For a path graph P_n of order $n \geq 4$, $dim_m(P_n) = 2$.

Proof: The proof is quite apparent from Proposition1, and the mixed metric generator must contain both end vertices of a path graph. Recently the invariant of mixed metric dimension is studied for some general families of graphs, as mentioned in the following results.

Theorem 1 [18]: (i) Let $\mathcal{D}_n$ be the prism graph, then;

$$dim_m(\mathcal{D}_n) = \begin{cases} 3, & \text{n is even;} \\ 4, & \text{n is odd.} \end{cases}$$

(ii) Let $\mathcal{A}_n$ be an anti-prism graph, then;

$$dim_m(\mathcal{A}_n) = \begin{cases} 4, & \text{n is even;} \\ 5, & \text{n is odd.} \end{cases}$$

(iii) Let $\mathcal{R}_n$ be a graph of convex polytope, then; $dim_m(\mathcal{R}_n) = 5$.

Theorem 2 [19]: For the Petersen graph $P(n, 2)$, we have

$$dim_m(P(n, 2)) = \begin{cases} 4, & n \equiv 0, 2(mod 4); \\ 5, & n \equiv 1, 3(mod 4). \end{cases}$$

Theorem 3 [4]: (i) For odd n, we have for J_n;

$$dim_m(J_n) = \begin{cases} 5, & n = 5; \\ 4, & n \geq 7. \end{cases}$$

(ii) For wheel graph W_n, we have;

$$dim_m(W_n) = \begin{cases} 4, & n = 3; \\ n, & n \geq 4. \end{cases}$$

Recently, the lower bound for mixed metric dimension of the general graphs is presented;

Theorem 4 [5]: Let G be an r regular graph, the lower bound is;

$$dim_m \geq 1 + \lceil \log_2(1 + r) \rceil$$

The authors in [5] studied the exact value of the mixed metric dimension by using the Theorem 4 for the family of torus graphs. As the torus graph is the regular graph of degree 4.

Theorem 5 [5]: For $m, n \geq 3$, for the family of torus graphs, we have;

$$dim_m(T_{m,n}) = 4$$

The paper discuss the families of graphs related to path graphs. Four families are studied which are, square of path graphs P_n^2, total graph of paths $T(P_n)$, middle graph of paths $M(P_n)$, and splitting of path graphs $S(P_n)$. The families of P_n^2, and $T(P_n)$ have constant mixed metric dimension. The families of $M(P_n)$, and $S(P_n)$ have unbounded mixed metric dimension. Furthermore, we improved the result for the metric dimension of $S(P_n)$ studied in [17].

Remark 1: The graphs studied in this article are related to the path graphs, so the mixed metric generator for these families of graphs must contain both end-vertices.

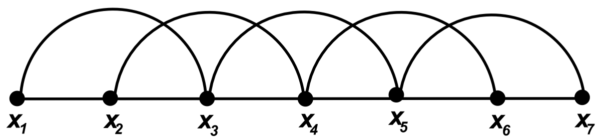

FIGURE 1. The graph of P_7^2.

III. MAIN RESULTS

Now we present the main results of this article.

IV. CONSTANT MIXED METRIC DIMENSION

A. SQUARE OF PATH GRAPHS P_n^2

The square of path graph is constructed by joining the every pair of vertices distance 2 in a path. The square of path graph P_7^2 is shown Figure 1. For the graph of P_n^2, we have $deg(x_1) = deg(x_n) = 2$, $deg(x_2) = deg(x_{n-1}) = 3$, and $deg(x_i) = 4$ $(3 \leq i \leq n-2)$. Mathematically the vertex and edge set of P_n^2 are,

$$V(P_n^2) = \{x_i | 1 \leq i \leq n\}$$
$$E(P_n^2) = \{x_i x_{i+1} | 1 \leq i \leq n-1\} \cup \{x_i x_{i+2} | 1 \leq i \leq n-2\}$$

It is also noted that $|V(P_n^2)| = n$, and $|E(P_n^2)| = 2n - 3$. The metric dimension of P_n^2 is presented as follows,

Theorem 6 [1]: For a graph of P_n^2, $dim(P_n^2) = 2$.

We extend our study to the mixed metric dimension of P_n^2; it is imperative to consider that the choice of basis vertices is the core of the problem.

Lemma 1: If $n \equiv 0(mod 2)$, and P_n^2 is the square of path graphs, we have $dim_m(P_n^2) \leq 4$.

Proof: Now let us write, when $n = 2j, j \geq 3$. Let $R_{\mathrm{m}} = \{x_1, x_2, x_{n-1}, x_n\}$ be the mixed resolving set for the square of a path graphs P_n^2. The distinct representation of vertices and edges with respect to R_{m} is presented in the following tables.

TABLE 1. Representation of vertices of P_n^2.

$r(v\|R_m)$	x_1	x_2	x_{n-1}	x_n	conditions
x_{1+2i}	0	1	$j-1$	j	$i=0$
	i	i	$j-i-1$	$j-i$	$1\le i\le j-1$
x_{2+2i}	$i+1$	i	$j-i-1$	$j-i-1$	$0\le i\le j-2$
	$i+1$	i	$j-i$	$j-i-1$	$i=j-1$

TABLE 2. Representation of edges of P_n^2.

$r(e\|R_m)$	x_1	x_2	x_{n-1}	x_n	conditions
$x_{1+2i}x_{2+2i}$	i	i	$j-1-1$	$j-i-1$	$0\le i\le j-1$
$x_{1+2i}x_{3+2i}$	0	1	$j-2$	$j-1$	$i=0$
	i	i	$j-i-2$	$j-i-1$	$1\le i\le j-2$
$x_{2+2i}x_{3+2i}$	$i+1$	i	$j-i-2$	$j-i-1$	$0\le i\le j-2$
$x_{2+2i}x_{4+2i}$	$i+1$	i	$j-i-2$	$j-i-2$	$0\le i\le j-3$
	$i+1$	i	$j-i-1$	$j-i-2$	$i=j-2$

The representation presented in Table 1 and Table 2 indicates that there are no vertices and edges of graphs having the same representations, which shows that $dim_m(P_n^2)\le 4$.

Lemma 2: When $n\equiv 0(mod2)$, $dim_m(P_n^2)\ge 4$.

Proof: In order to prove that, we assume, $dim_m(P_n^2)=3$.

(1). Let $R_m=\{x_1,x_2,x_3\}$, then we have the following,

- $r(x_{1+2i}|R_m)=r(x_{1+2i}x_{2+2i}|R_m)=r(x_{1+2i}x_{3+2i}|R_m)=(i,i,i-1)$ for $1\le i\le j-2$.
- $r(x_{2+2i}|R_m)=r(x_{2+2i}x_{3+2i}|R_m)=r(x_{2+2i}x_{4+2i}|R_m)=(i+1,i,i)$ for $1\le i\le j-2$.
- $r(x_{1+2i}|R_m)=r(x_{1+2i}x_{2+2i}|R_m)=(j-1,j-1,j-2)$ for $i=j-1$.

(2). Let $R_m=\{x_1,x_2,x_{n-1}\}$.

- $r(x_{1+2i}|R_m)=r(x_{1+2i}x_{2+2i}|R_m)=(i,i,j-i-2)$ for $1\le i\le j-1$.
- $r(x_{2+2i}|R_m))=r(x_{2+2i}x_{2+4i}||R_m))=(i+1,i,1)$ for $(i=j-2)$.
- $r(x_{2+2i}x_{3+2i}|R_m)=r(x_{2+2i}x_{4+2i}|R_m)=(i+1,i,j-i-2)$ for $0\le i\le j-3$.

(3). Let $R_m=\{x_1,x_2,x_n\}$, then we have the following;

- $r(x_{2+2i}|R_m)=r(x_{2+2i}x_{3+2i}|R_m)=(i+1,i,j-i-1)$ for $0\le i\le j-2$.
- $r(x_{1+2i}x_{2+2i}|R_m))=r(x_{1+2i}x_{3+2i}||R_m))=(i,i,j-i-1)$ for $(j-3\le i\le j-2)$.

This shows the contradiction.

Furthermore, other mixed metric generators can also be considered, which will show the same kind of contradictions, which implies that $dim_m(P_n^2)\ge 4$.

Lemma 3: If $n\equiv 1(mod2)$, and P_n^2 is the square of path graphs, we have $dim_m(P_n^2)\le 4$.

Proof: Now let us write, when $n=2j+1, j\ge 3$. Let $R_m=\{x_1,x_2,x_{n-1},x_n\}$ be the mixed resolving set for the square of a path graphs P_n^2. The distinct representation of vertices and edges with respect to R_m is shown in the tables below.

The representation in the Table 3 and Table 4 indicates that there are no vertices and edges of graphs having the same representations, which shows that $dim_m(P_n^2)\le 4$ when n is odd.

TABLE 3. Representation of vertices of P_n^2.

$r(v\|R_m)$	x_1	x_2	x_{n-1}	x_n	conditions
x_{1+2i}	0	1	j	j	$i=0$
	i	i	$j-i$	$j-i$	$1\le i\le j-1$
	i	i	$j-i+1$	$j-i$	$i=j$
x_{2+2i}	$i+1$	i	$j-i-1$	$j-i$	$0\le i\le j-1$

TABLE 4. Representation of Edges of P_n^2.

$r(e\|R_m)$	x_1	x_2	x_{n-1}	x_n	conditions
$x_{1+2i}x_{2+2i}$	i	i	$j-i-1$	$j-i$	$0\le i\le j-1$
$x_{1+2i}x_{3+2i}$	0	1	$j-1$	$j-1$	$i=0$
	i	i	$j-i-1$	$j-i-1$	$1\le i\le j-2$
	i	i	$j-i$	$j-i-1$	$i=j-1$
$x_{2+2i}x_{3+2i}$	$i+1$	i	$j-i-1$	$j-i-1$	$0\le i\le j-1$
$x_{2+2i}x_{4+2i}$	$i+1$	i	$j-i-2$	$j-i-1$	$0\le i\le j-2$

As shown in the Lemma 2, for the even case of n, there are no mixed metric generators with cardinality 3. The same kind of cases can be considered for the odd case of n so that it can be written as $dim_m(P_n^2)\ge 4$. From Lemmas 1, 2, and 3 we present the mixed metric dimension of P_n^2 which is constant for both cases of n. It does not depend upon the order of n.

Theorem 7: Let P_n^2 be the square of path graph, then we have;

$$dim_m(P_n^2)=4$$

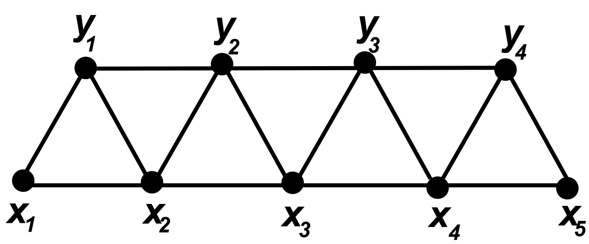

FIGURE 2. The total graph of $T(P_5)$ of path P_5.

B. TOTAL GRAPH OF PATH $T(P_n)$

The total graph of path $T(P_n)$, studied in [16], for super mean labeling. The total graph of path $T(P_5)$ is shown in the Figure 2. The total graph of path $T(P_n)$ is a graph with the vertex set $V(G)\cup E(G)$, that is let $x_1,x_2,\ldots,x_n$ be the vertices of a main path P_n, and $e_1,e_2,\ldots,e_{n-1}$ be the $n-1$ edges. The total graph of path is constructed by adding new vertices say $y_1,y_2,\ldots,y_{n-1}$ corresponding to the edges. It is also noted that $V|T(P_n)|=2n-1$, and $E|T(P_n)|=4n-5$. Mathematically the vertex and edge set can be written as,

$$V(T(P_n))=\{x_i|1\le i\le n\}\cup\{y_i|1\le i\le n-1\}$$
$$E(T(P_n))=\{x_ix_{i+1},x_iy_i,x_{i+1}y_i|1\le i\le n-1\}$$
$$\cup\{y_iy_{i+1}|1\le i\le n-2\}$$

Lemma 4: For the total graph of $T(P_n)$, mixed metric resolving set R_m must contain vertices of the main path and the path generated by the newly added vertices.

Proof: Let us assume a contradiction, that is $R_m=\{x_i|1\le i\le n\}$, contains vertices of the main path, then we have $r(x_iy_i|R_m)=r(x_i|R_m)$ $(1\le i\le n-1)$, a contradiction.

TABLE 5. Representation of vertices of $T(P_n)$.

$r(v\|R_m)$	x_1	x_n	y_1	y_{n-1}	conditions
x_{1+2i}	0	$2j-2i-1$	1	$2j-2i-1$	$i=0$
	$2i$	$2j-2i-1$	$2i$	$2j-2i-1$	$1\leq i\leq j-1$
x_{2+2i}	$2i+1$	$2j-2i-2$	$2i+1$	$2j-2i-2$	$0\leq i\leq j-2$
	$2i+1$	$2j-2i-2$	$2i+1$	$2j-2i-1$	$i=j-1$
y_{1+2i}	$2i+1$	$2j-2i-1$	$2i$	$2j-2i-2$	$0\leq i\leq j-1$
y_{2+2i}	$2i+2$	$2j-2i-2$	$2i+1$	$2j-2i-3$	$0\leq i\leq j-2$

TABLE 6. Representation of edges of $T(P_n)$.

$r(e\|R_m)$	x_1	x_n	y_1	y_{n-1}	conditions
$x_{1+2i}x_{2+2i}$	$2i$	$2j-2i-2$	$2i+1$	$2j-2i-2$	$i=0$
	$2i$	$2j-2i-2$	$2i$	$2j-2i-2$	$1\leq i\leq j-2$
	$2i$	$2j-2i-2$	$2i$	$2j-2i-1$	$i=j-1$
$x_{2+2i}x_{3+2i}$	$2i+1$	$2j-2i-3$	$2i+1$	$2j-2i-3$	$0\leq i\leq j-2$
$y_{1+2i}y_{2+2i}$	$2i+1$	$2j-2i-2$	$2i$	$2j-2i-3$	$0\leq i\leq j-2$
$y_{2+2i}y_{3+2i}$	$2i+2$	$2j-2i-3$	$2i+1$	$2j-2i-4$	$0\leq i\leq j-2$
$x_{1+2i}y_{1+2i}$	$2i$	$2j-2i-1$	$2i$	$2j-2i-2$	$0\leq i\leq j-1$
$x_{2+2i}y_{1+2i}$	$2i+1$	$2j-2i-2$	$2i$	$2j-2i-2$	$0\leq i\leq j-1$
$x_{2+2i}y_{2+2i}$	$2i+1$	$2j-2i-2$	$2i+1$	$2j-2i-3$	$0\leq i\leq j-2$
$x_{3+2i}y_{2+2i}$	$2i+2$	$2j-2i-3$	$2i+1$	$2j-2i-3$	$0\leq i\leq j-2$

Now let assume that $R_m = \{y_i|1 \leq i \leq n-1\}$, again we have $r(x_iy_i|R_m) = r(y_i|R_m)$ $(1 \leq i \leq n-1)$. So from above contradictions we conclude that for the total graph of path $T(P_n)$, R_m must contains vertices of the main path and the path generated by the newly added vertices.

Lemma 5: If $n \equiv 0(mod2)$, then for total graph of path $T(P_n)$, we have $dim_m(T(P_n)) \leq 4$.

Proof: Now we can write as, when $n = 2j$, and $j \geq 3$, the mixed metric resolving set for $T(P_n)$, is $R_m = \{x_1, x_n, y_1, y_{n-1}\}$. The distinct representation of vertices and edges with respect to R_m is shown in the tables.

The representation shown in the Table 5, and Table 6 clearly indicates that no two vertices and edges have same representation with respect to R_m thus showing that $dim_m(T(P_n)) \leq 4$.

Lemma 6: When $n \equiv 0(mod2)$, than for $T(P_n)$, we have $dim_m(T(P_n)) \geq 4$.

Proof: Now we assume that $dim_m(T(P_n)) = 3$, then the following possibilities arise;

(1). Let us assume that $R_m = \{x_1, x_n, y_1\}$.

- $r(x_{2+2i}y_{1+2i}|R_m) = r(y_{1+2i}y_{2+2i}|R_m) = (2i+1, 2j-2i-2, 2i)$ for $(0 \leq i \leq j-2)$.
- $r(x_{3+2i}y_{2+2i}|R_m) = r(y_{2+2i}y_{3+2i}|R_m) = (2i+2, 2j-2i-3, 2i+1)$ for $(0 \leq i \leq j-2)$.
- $r(x_{2+2i}|R_m) = r(x_{2+2i}y_{2+2i}|R_m) = (2i+1, 2j-2i-2, 2i+1)$ for $(0 \leq i \leq j-2)$.
- $r(x_{1+2i}|R_m) = r(x_{1+2i}y_{1+2i}|R_m) = (2i, 2j-2i-1, 2i+1)$ for $(i = 0)$.
- $r(x_{1+2i}|R_m) = r(x_{1+2i}y_{1+2i}|R_m) = (2i, 2j-2i-1, 2i)$ for $(1 \leq i \leq j-1)$.

(2). Let us assume that $R_m = \{x_1, y_1, y_{n-1}\}$.

- $r(x_{2+2i}y_{1+2i}|R_m) = r(y_{1+2i}|R_m) = (2i+1, 2i, 2j-2i-2)$ for $0 \leq i \leq j-1$.
- $r(x_{2+2i}y_{2+2i}|R_m) = r(x_{2+2i}x_{3+2i}|R_m) = (2i+1, 2i+1, 2j-2i-3)$ for $0 \leq i \leq j-2$.
- $r(x_{3+2i}y_{2+2i}|R_m) = r(y_{2+2i}|R_m) = (2i+2, 2i+1, 2j-2i-3)$ for $0 \leq i \leq j-2$.

TABLE 7. Representation of vertices of $T(P_n)$.

$r(v\|R_m)$	x_1	x_n	y_1	y_{n-1}	conditions
x_{1+2i}	0	$2j-2i$	1	$2j-2i$	$i=0$
	$2i$	$2j-2i$	$2i$	$2j-2i$	$1\leq i\leq j-1$
	$2j$	0	$2j$	1	$i=j$
x_{2+2i}	$2i+1$	$2j-2i-1$	$2i+1$	$2j-2i-1$	$0\leq i\leq j-1$
y_{1+2i}	$2i+1$	$2j-2i$	$2i$	$2j-2i-1$	$0\leq i\leq j-1$
y_{2+2i}	$2i+2$	$2j-2i-1$	$2i+1$	$2j-2i-2$	$0\leq i\leq j-1$

TABLE 8. Representation of edges of $T(P_n)$.

$r(e\|R_m)$	x_1	x_n	y_1	y_{n-1}	conditions
$x_{1+2i}x_{2+2i}$	0	$2j-2i-1$	1	$2j-2i-1$	$i=0$
	$2i$	$2j-2i-1$	$2i$	$2j-2i-1$	$1\leq i\leq j-1$
$x_{2+2i}x_{3+2i}$	$2i+1$	$2j-2i-2$	$2i+1$	$2j-2i-2$	$0\leq i\leq j-2$
	$2i+1$	0	$2i+1$	1	$i=j-1$
$y_{1+2i}y_{2+2i}$	$2i+1$	$2j-2i-1$	$2i$	$2j-2i-2$	$0\leq i\leq j-1$
$y_{2+2i}y_{3+2i}$	$2i+2$	$2j-2i-2$	$2i+1$	$2j-2i-3$	$0\leq i\leq j-2$
$x_{1+2i}y_{1+2i}$	$2i$	$2j-2i$	$2i$	$2j-2i-1$	$0\leq i\leq j-1$
$x_{2+2i}y_{1+2i}$	$2i+1$	$2j-2i-1$	$2i$	$2j-2i-1$	$0\leq i\leq j-1$
$x_{2+2i}y_{2+2i}$	$2i+1$	$2j-2i-1$	$2i+1$	$2j-2i-2$	$0\leq i\leq j-1$
$x_{3+2i}y_{2+2i}$	$2i+2$	$2j-2i-2$	$2i+1$	$2j-2i-2$	$0\leq i\leq j-1$

- $r(x_{1+2i}x_{2+2i}|R_m) = r(x_{1+2i}|R_m) = (2i, 2i, 2j-2i-1)$ for $(3 \leq i \leq j-1)$.

Thus, a contradiction. Furthermore, other mixed metric generators can be thought of, which will show the same contradictions. Thus proving that there is no mixed metric generator of cardinality 3 in the total graph of path $T(P_n)$, so $dim_m(T(P_n)) \geq 4$.

Lemma 7: If $n \equiv 1(mod2)$, then for total graph of path $T(P_n)$, we have $dim_m(T(P_n)) \leq 4$.

Proof: Now we can write as, when $n = 2j+1$, and $j \geq 3$, the mixed metric generator for $T(P_n)$, is $R_m = \{x_1, x_n, y_1, y_{n-1}\}$. The distinct representation of vertices and edges with respect to R_m is shown in the following tables.

The representation shown in the Table 7, and Table 8 clearly indicates that no two vertices and edges have same representation with respect to R_m, thus showing that $dim_m(T(P_n)) \leq 4$. In order to prove that $dim_m(T(P_n)) \geq 4$, same like Lemma 6, there exist no mixed metric generator of cardinality 3, for the $n \equiv 1(mod2)$. Thus proving that, so $dim_m(T(P_n)) \geq 4$. Form Lemmas 5, 6, and 7 the mixed metric dimension of $dim_m(T(P_n))$ is obtained in the following result. This family of graph also consist of constant mixed metric dimension.

Theorem 8: Let $T(P_n)$ be the total graph of path, then we have;

$$dim_m(T(P_n)) = 4$$

Remark 2: The mixed metric generator here can be called the edge version of a mixed metric generator as the set of vertices in R_m also form the edges for the square of path graph P_n^2, and total graph of path $T(P_n)$, respectively.

V. UNBOUNDED MIXED METRIC DIMENSION

A. MIDDLE GRAPH OF PATH $M(P_n)$

The Middle graph of path $M(P_n)$ is obtained by considering the vertices of path P_n be $y_1, y_2, \ldots, y_n$, and $e_i = y_iy_{i+1}$ $(1 \leq i \leq n-1)$, be the edges of the path then the vertex

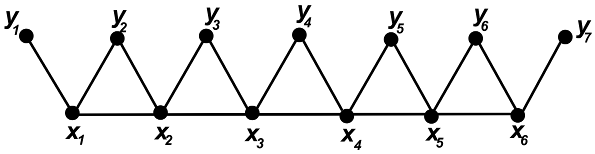

FIGURE 3. The middle graph of $M(P_7)$ of path P_7.

TABLE 9. Representation of vertices of $M(P_n)$.

$r(v\|R_1)$	y_1	y_2	y_3	y_n	conditions
x_{1+2i}	$2i+1$	$2i+1$	$2i+2$	$2j-2i-1$	$i=0$
	$2i+1$	$2i$	$2i-1$	$2j-2i-1$	$1\leq i\leq j-1$
x_{2+2i}	$2i+2$	$2i+1$	$2i+1$	$2j-2i-2$	$i=0$
	$2i+2$	$2i+1$	$2i$	$2j-2i-2$	$1\leq i\leq j-2$
y_{1+2i}	$2i$	$2i+2$	$2i+3$	$2j-2i$	$i=0$
	$2i+1$	$2i$	$2i-2$	$2j-2i$	$i=1$
	$2i+1$	$2i$	$2i-1$	$2j-2i$	$2\leq i\leq j-1$
y_{2+2i}	$2i+2$	$2i$	$2i+2$	$2j-2i-1$	$i=0$
	$2i+2$	$2i+1$	$2i$	$2j-2i-1$	$1\leq i\leq j-2$
	$2i+2$	$2i+1$	$2i$	$2j-2i-2$	$i=j-1$

set is $\{y_1, y_2, \ldots, y_n, e_1, e_2, \ldots, e_{n-1}\}$. Now by renaming the vertices e_i by x_i $(1 \leq i \leq n-1)$. The middle graph of path $M(P_n)$ is studied in [6] for the radio number. The middle graph of $T(P_7)$ of path P_7 is shown in the Figure 3. Now $deg(y_1) = deg(y_n) = 1$, $deg(x_1) = deg(x_{n-1}) = 3$, $deg(y_i) = 2(2 \leq i \leq n-1)$, and $deg(x_i) = 4(2 \leq i \leq n-2)$. It is also noted that $V(M(P_n)) = 2n-1$, and $E(M(P_n)) = 3n-4$.

Lemma 8: Let $R = \{y_1, \ldots, y_n\} \subseteq V(M(P_n))$. For some arbitrary mixed metric generator R_m of $M(P_n)$, R_m contains $\lceil \frac{2n-1}{2} \rceil$ vertices of R.

Proof: Let us assume that a contradiction, and R_m contains $\lceil \frac{2n-1}{2} \rceil - 1$ vertices. Now for instance $y_i, y_{i+1} \notin R_m$. Then $r(x_iy_i|R_m) = r(x_i|R_m)$, which is a contradiction.

We present the mixed metric dimension for the middle graph of path $M(P_n)$.

Theorem 9: For $n \geq 6$, we have;

$$dim_m(M(P_n)) = \left\lceil \frac{2n-1}{2} \right\rceil.$$

Let $R_m = \{y_1, y_2, \ldots, y_n\}$ be the mixed metric resolving set for the middle graph of path $M(P_n)$ for the cases 1,and 2.

Case 1: When n is even, $n = 2j$, and $j \geq 3$. Let us assume $R_1 = \{y_1, y_2, y_3, y_n\}$, be the mixed metric resolving set, now the representation of vertices and edges is shown with respect to R_1.

Now from the Table 9, and Table 10, we see the following contradictions.

- $r(x_{1+2i}|R_1) = r(x_{1+2i}y_{2+2i}|R_1) = 2i+1, 2i, 2i-1, 2j-2i-1$ for $(1 \leq i \leq j-2)$.
- $r(x_{1+2i}|R_1) = r(x_{1+2i}y_{1+2i}|R_1) = 2i+1, 2i, 2i-1, 2j-2i-1$ for $(2 \leq i \leq j-1)$.
- $r(x_{2+2i}|R_1) = r(x_{2+2i}y_{2+2i}|R_1) = 2i+2, 2i+1, 2i, 2j-2i-2$ for $(1 \leq i \leq j-2)$.
- $r(x_{2+2i}|R_1) = r(x_{2+2i}y_{3+2i}|R_1) = 2i+2, 2i+1, 2i, 2j-2i-2$ for $(1 \leq i \leq j-2)$.
- $r(x_{1+2i}y_{1+2i}|R_1) = r(x_{1+2i}y_{2+2i}|R_1) = 2i+1, 2i, 2i-1, 2j-2i-1$ for $(2 \leq i \leq j-2)$.

TABLE 10. Representation of edges of $M(P_n)$.

$r(e\|R_1)$	y_1	y_2	y_3	y_n	conditions
$x_{1+2i}x_{2+2i}$	$2i+1$	$2i+1$	$2i+1$	$2j-2i-2$	$i=0$
	$2i+1$	$2i$	$2i-1$	$2j-2i-2$	$1\leq i\leq j-2$
$x_{2+2i}x_{3+2i}$	$2i+2$	$2i+1$	$2i+1$	$2j-2i-3$	$i=0$
	$2i+2$	$2i+1$	$2i$	$2j-2i-3$	$1\leq i\leq j-2$
$x_{1+2i}y_{1+2i}$	$2i$	$2i+1$	$2i+2$	$2j-2i-1$	$i=0$
	$2i+1$	$2i$	$2i-2$	$2j-2i-1$	$i=1$
	$2i+1$	$2i$	$2i-1$	$2j-2i-1$	$2\leq i\leq j-1$
$x_{1+2i}y_{2+2i}$	$2i+1$	$2i$	$2i+2$	$2j-2i-1$	$i=0$
	$2i+1$	$2i$	$2i-1$	$2j-2i-1$	$1\leq i\leq j-2$
	$2i+1$	$2i$	$2i-1$	$2j-2i-2$	$i=j-1$
$x_{2+2i}y_{2+2i}$	$2i+2$	$2i$	$2i+1$	$2j-2i-2$	$i=0$
	$2i+2$	$2i+1$	$2i$	$2j-2i-2$	$1\leq i\leq j-2$
$x_{2+2i}y_{3+2i}$	$2i+2$	$2i+1$	$2i$	$2j-2i-2$	$0\leq i\leq j-2$

TABLE 11. Representation of vertices of $M(P_n)$.

$r(v\|R_1)$	y_1	y_2	y_{n-1}	y_n	conditions
x_{1+2i}	$2i+1$	$2i+1$	$2j-2i-1$	$2j-2i$	$i=0$
	$2i+1$	$2i$	$2j-2i-1$	$2j-2i$	$1\leq i\leq j-1$
x_{2+2i}	$2i+2$	$2i+1$	$2j-2i-2$	$2j-2-1$	$0\leq i\leq j-2$
	$2i+2$	$2i+1$	$2j-2i-1$	$2j-2i-1$	$i=j-1$
y_{1+2i}	$2i$	$2i+2$	$2j-2i$	$2j-2i+1$	$i=0$
	$2i+1$	$2i$	$2j-2i$	$2j-2i+1$	$1\leq i\leq j-1$
	$2i+1$	$2i$	$2j-2i+2$	$2j-2i$	$i=j$
y_{2+2i}	$2i+2$	$2i$	$2j-2i-1$	$2j-2i$	$i=0$
	$2i+2$	$2i+1$	$2j-2i-1$	$2j-2i$	$1\leq i\leq j-2$
	$2i+2$	$2i+1$	$2j-2i-2$	$2j-2i$	$i=j-1$

TABLE 12. Representation of edges of $M(P_n)$.

$r(e\|R_1)$	y_1	y_2	y_{n-1}	y_n	conditions
$x_{1+2i}x_{2+2i}$	$2i+1$	$2i+1$	$2j-2i-2$	$2j-2i-1$	$i=0$
	$2i+1$	$2i$	$2j-2i-2$	$2j-2i-1$	$1\leq i\leq j-2$
	$2i+1$	$2i$	$2j-2i-1$	$2j-2i-1$	$i=j-1$
$x_{2+2i}x_{3+2i}$	$2i+2$	$2i+1$	$2j-2i-3$	$2j-2i-2$	$0\leq i\leq j-2$
$x_{1+2i}y_{1+2i}$	$2i$	$2i+1$	$2j-2i-1$	$2j-2i$	$i=0$
	$2i+1$	$2i$	$2j-2i-1$	$2j-2i$	$1\leq i\leq j-1$
$x_{1+2i}y_{2+2i}$	$2i+1$	$2i$	$2j-2i-1$	$2j-2i$	$0\leq i\leq j-2$
	$2i+1$	$2i$	$2j-2i-2$	$2j-2i$	$i=j-1$
$x_{2+2i}y_{2+2i}$	$2i+2$	$2i$	$2j-2i-2$	$2j-2i-1$	$i=0$
	$2i+2$	$2i+1$	$2j-2i-2$	$2j-2i-1$	$1\leq i\leq j-1$
$x_{2+2i}y_{3+2i}$	$2i+2$	$2i+1$	$2j-2i-2$	$2j-2i-1$	$0\leq i\leq j-2$
	$2i+2$	$2i+1$	$2j-2i-1$	$2j-2i-2$	$i=j-1$

- $r(x_{2+2i}y_{2+2i}|R_1) = r(x_{2+2i}y_{3+2i}|R_1) = 2i+2, 2i+1, 2i, 2j-2i-2$ for $(1 \leq i \leq j-2)$.

In order to overcome these contradictions we will assume $R_1' = \{y_i|4 \leq i \leq n-1\}$, and there would be no vertices or edges having same representation between them. So combining all these factors $R_m = \{y_1, y_2, \ldots, y_{n-1}, y_n\}$ is the mixed metric generator of cardinality $\lceil \frac{2n-1}{2} \rceil$.

Case 2: When n is odd, $n = 2j+1$, and $j \geq 3$. let $R_1 = \{y_1, y_2, y_{n-1}, y_n\}$, be the mixed metric resolving set, now the representation of vertices and edges is shown with respect to mixed resolving set R_1.

Now from the Table 11, and Table 12, we have the following contradiction;

- $r(x_{2+2i}|R_1) = r(x_{2+2i}y_{3+2i}|R_1) = 2i+2, 2i+1, 2j-2i-2, 2j-2i-1$ for $(0 \leq i \leq j-2)$.
- $r(x_{2+2i}|R_1) = r(x_{2+2i}y_{2+2i}|R_1) = 2i+2, 2i+1, 2j-2i-2, 2j-2i-1$ for $(1 \leq i \leq j-2)$.
- $r(x_{1+2i}|R_1) = r(x_{1+2i}y_{2+2i}|R_1) = 2i+1, 2i, 2j-2i-1, 2j-2i$ for $(1 \leq i \leq j-2)$.

- $r(x_{1+2i}|R_1) = r(x_{1+2i}y_{1+2i}|R_1) = 2i+1, 2i, 2j-2i-1, 2j-2i$ for $(1 \le i \le j-1)$.
- $r(x_{1+2i}x_{2+2i}|R_1) = r(x_{1+2i}y_{2+2i}|R_1) = 2i+1, 2i, 2j-2i-2, 2j-2i-3$ for $(1 \le i \le j-2)$.
- $r(x_{1+2i}x_{2+2i}|R_1) = r(x_{1+2i}y_{1+2i}|R_1) = 2i+1, 2i, 2j-2i-2, 2j-2i-3$ for $(1 \le i \le j-2)$.

In order to overcome these contradictions we will assume $R'_1 = \{y_i | 3 \le i \le n-2\}$, and there would be no vertices or edges having same representation between them. So combining all these factors $R_m = \{y_1, y_2, \ldots, y_{n-1}, y_n\}$ is the mixed metric generator of cardinality $\lceil \frac{2n-1}{2} \rceil$.

B. SPLITTING GRAPH OF A PATH $S(P_n)$

Let $S(P_n)$ be the splitting graph of a path P_n. The metric dimension of splitting graph of a path $S(P_n)$ is studied in [17]. The splitting graph of path $S(P_5)$ of path P_5 is shown in the Figure 4. Now let $x_1, x_2, \ldots, x_n$ be the vertices of P_n, add new vertices $y_1, y_2, \ldots, y_n$ corresponding to the vertices $x_1, x_2, \ldots, x_n$ to form $S(P_n)$. Also $deg(y_1) = deg(y_n) = 1$, $deg(x_1) = deg(x_n) = deg(y_i) = 2(2 \le i \le n-1)$, and $deg(x_i) = 4(2 \le i \le n-1)$. For the sake of simplicity we can write as, $V_1 = \{x_1, x_2, \ldots, x_n\}$, which is the main path and $V_2 = \{y_1, y_2, \ldots, y_n\}$, the vertices adjacent to V_1. Then $V(S(P_n)) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. It is to be noted that $V|(S(P_n))| = 2n$, and $E|(S(P_n))| = 3n-3$.

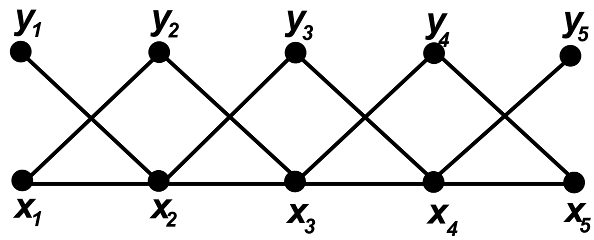

FIGURE 4. The splitting graph of path $S(P_5)$.

Lemma 9: If for any mixed metric generator R_m of $S(P_n)$, R_m contains no vertices of the main path.

Proof: Without the loss of generality, we assume a contradiction let $R_m = \{x_1, \ldots, x_n\}$. Now we can see that $r(x_i|R_m) \ne r(y_i|R_m)(1 \le i \le n) \ne r(x_ix_{i+1}|R_m)(1 \le i \le n-1)$. But for the vertices with degree 1, that is $deg(y_1) = deg(y_n) = 1$, we have $r(x_{i+1}y_i|R_m) = r(x_{i+1}|R_m)$ for $i = 1$, and also $r(x_iy_{i+1}|R_m) = r(x_i|R_m)$ for $i = n-1$. Thus R_m contains no vertices of the main path.

Next we present the mixed metric dimension for the splitting graph of a path $S(P_n)$.

Theorem 10: For $n \ge 6$, $dim_m(S(P_n))$ is,

$$dim_m(S(P_n)) = \left\lceil \frac{n}{2} \right\rceil + 1$$

Let $R_m = \{y_1, y_{2i+2} | i = 0, \ldots, \lceil \frac{n}{2} \rceil - 3, y_{n-1}, y_n\}$. Now we will prove that R_m is mixed metric resolving set for the even case of n.

Case 1: When $n = 2j$, and $j \ge 4$. Let $R_1 = \{y_1, y_2, y_{n-1}, y_n\}$. The representation of the vertices and edges with respect to R_1, is shown.

TABLE 13. Representation of vertices of $S(P_n)$.

$r(v\|R_1)$	y_1	y_2	y_{n-1}	y_n	conditions
x_{1+2i}	2	1	$2j-2i-2$	$2j-2i-1$	$0 \le i \le 1$
	$2i$	$2i-1$	$2j-2i-2$	$2j-2i-1$	$2 \le i \le j-2$
	$2i$	$2i-1$	$2j-2i$	$2j-2i-1$	$i = j-1$
x_{2+2i}	$2i+1$	2	$2j-2i-3$	$2j-2i-2$	$0 \le i \le 1$
	$2i+1$	$2i$	$2j-2i-3$	$2j-2i-2$	$2 \le i \le j-2$
	$2i+1$	$2i$	$2j-2i-1$	$2j-2i$	$i = j-1$
y_{1+2i}	$2i$	3	$2j-2i-2$	$2j-2i-1$	$0 \le i \le 1$
	$2i$	$2i-1$	$2j-2i-2$	$2j-2i-1$	$2 \le i \le j-2$
	$2i$	$2i-1$	$2j-2i-2$	$2j-2i+1$	$i = j-1$
y_{2+2i}	3	$2i$	$2j-2i-3$	$2j-2i-2$	$i = 0$
	$2i+1$	$2i$	$2j-2i-3$	$2j-2i-2$	$1 \le i \le j-3$
	$2i+1$	$2i$	3	$2j-2i-2$	$j-2 \le i \le j-1$

TABLE 14. Representation of edges of $S(P_n)$.

$r(e\|R_1)$	y_1	y_2	y_{n-1}	y_n	conditions
$x_{1+2i}x_{2+2i}$	$2i+1$	$2i+1$	$2j-2i-3$	$2j-2i-2$	$i = 0$
	$2i$	$2i-1$	$2j-2i-3$	$2j-2i-2$	$1 \le i \le j-2$
	$2i$	$2i-1$	$2j-2i-1$	$2j-2i-1$	$i = j-1$
$x_{2+2i}x_{3+2i}$	$2i+1$	$2i+1$	$2j-2i-4$	$2j-2i-3$	$i = 0$
	$2i+1$	$2i$	$2j-2i-4$	$2j-2i-3$	$1 \le i \le j-3$
	$2i+1$	$2i$	$2j-2i-3$	$2j-2i-3$	$i = j-2$
$x_{1+2i}y_{2+2i}$	$2i$	$2i$	$2j-2i-3$	$2j-2i-2$	$i = 0$
	$2i$	$2i-1$	$2j-2i-3$	$2j-2i-2$	$1 \le i \le j-3$
	$2i$	$2i-1$	2	$2j-2i-2$	$j-2 \le i \le j-1$
$x_{2+2i}y_{1+2i}$	$2i$	2	$2j-2i-3$	$2j-2i-2$	$0 \le i \le 1$
	$2i$	$2i-1$	$2j-2i-3$	$2j-2i-2$	$2 \le i \le j-2$
	$2i$	$2i-1$	$2j-2i-2$	$2j-2i$	$i = j-1$
$x_{2+2i}y_{3+2i}$	$2i+1$	$2i$	$2j-2i-4$	$2j-2i-3$	$0 \le i \le 1$
	$2i+1$	$2i$	$2j-2i-4$	$2j-2i-3$	$1 \le i \le j-3$
	$2i+1$	$2i$	$2j-2i-4$	$2j-2i-2$	$i = j-2$
$x_{3+2i}y_{2+2i}$	$2i$	$2i$	$2j-2i-4$	$2j-2i-3$	$i = 0$
	$2i+1$	$2i$	$2j-2i-4$	$2j-2i-3$	$1 \le i \le j-3$
	$2i+1$	$2i$	$2j-2i-2$	$2j-2i-3$	$i = j-2$

Now from the Table 13, and Table 14, we have the following contradictions.

- $r(x_{2+2i}|R_1) = r(y_{2+2i}|R_1) = 2i+1, 2i, 2j-2i-3, 2j-2i-2$ for $(1 \le i \le j-3)$.
- $r(x_{1+2i}|R_1) = r(y_{1+2i}|R_1) = 2i, 2i-1, 2j-2i-2, 2j-2i-1$ for $(2 \le i \le j-2)$.
- $r(x_{1+2i}x_{2+2i}|R_1) = r(x_{1+2i}y_{2+2i}|R_1) = 2i, 2i-1, 2j-2i-3, 2j-2i-2$ for $(1 \le i \le j-3)$.
- $r(x_{2+2i}x_{3+2i}|R_1) = r(x_{2+2i}y_{3+2i}|R_1),$
- $= r(x_{3+2i}y_{2+2i}|R_1) = 2i+1, 2i, 2j-2i-4, 2j-2i-3$ for $(1 \le i \le j-3)$.
- $r(x_{1+2i}x_{2+2i}|R_1) = r(x_{2+2i}y_{1+2i}|R_1) = 2i, 2i-1, 2j-2i-3, 2j-2i-2$ for $(2 \le i \le j-2)$.

In order to resolve these vertices and edges, we assume that $R'_1 = \{y_{2i+2} | i = 1, \ldots, \lceil \frac{n}{2} \rceil - 3\}$, which will present distinct representation among them. So from the above facts we can deduce that $R_m = \{y_1, y_{2i+2} | i = 0, \ldots, \lceil \frac{n}{2} \rceil - 3, y_{n-1}, y_n\}$ is the mixed metric resolving set for even case of $S(P_n)$ with cardinality $\lceil \frac{n}{2} \rceil + 1$.

Let $R_m = \{y_1, y_{2i+2} |_i = 0, \ldots, \lceil \frac{n}{2} \rceil - 2, y_n\}$ be the mixed metric bases for the odd case of n.

Case 2: To prove this we can write as, $n = 2j+1$, and $j \ge 4$. Let $R_1 = \{y_1, y_2, y_4, y_n\}$. The representation of the vertices and edges with respect to R_1, is shown.

Now from the Table 15, and Table 16, we have the following contradictions.

TABLE 15. Representation of vertices of $S(P_n)$.

$r(v\|R_1)$	y_1	y_2	y_4	y_n	conditions
x_{1+2i}	2	1	$3-2i$	$2j-2i$	$0 \le i \le 1$
	$2i$	$2i-1$	$2i-3$	$2j-2i$	$2 \le i \le j-1$
	$2i$	$2i-1$	$2i-3$	$2j-2i+2$	$i=j$
x_{2+2i}	$2i+1$	2	2	$2j-2i-1$	$0 \le i \le 1$
	$2i+1$	$2i$	$2i-2$	$2j-2i-1$	$2 \le i \le j-1$
y_{1+2i}	$2i$	3	3	$2j-2i$	$0 \le i \le 2$
	$2i$	$2i-1$	$2i-3$	$2j-2i$	$3 \le i \le j$
y_{2+2i}	3	$2i$	2	$2j-2i-1$	$i=0$
	$2i+1$	$2i$	$2i-2$	$2j-2i-1$	$1 \le i \le j-2$
	$2i+1$	$2i$	$2i-2$	$2j-2i+1$	$i=j-1$

TABLE 16. Representation of edges of $S(P_n)$.

$r(e\|R_1)$	y_1	y_2	y_4	y_n	conditions
$x_{1+2i}x_{2+2i}$	1	1	2	$2j-2i-1$	$i=0$
	$2i$	$2i-1$	1	$2j-2i-1$	$i=1$
	$2i$	$2i-1$	$2i-3$	$2j-2i-1$	$2 \le i \le j-1$
$x_{2+2i}x_{3+2i}$	1	1	1	$2j-2i-2$	$i=0$
	$2i+1$	$2i$	1	$2j-2i-2$	$i=1$
	$2i+1$	$2i$	$2i-2$	$2j-2i-2$	$2 \le i \le j-2$
	$2i+1$	$2i$	$2i-2$	$2j-2i-1$	$i=j-1$
$x_{1+2i}y_{2+2i}$	2	0	2	$2j-2i-1$	$i=0$
	$2i$	$2i-1$	0	$2j-2i-1$	$i=1$
	$2i$	$2i-1$	$2i-3$	$2j-2i-1$	$2 \le i \le j-2$
	$2i$	$2i-1$	$2i-3$	$2j-2i$	$i=j-1$
$x_{2+2i}y_{1+2i}$	$2i$	2	2	$2j-2i-1$	$0 \le i \le 1$
	$2i$	$2i-1$	$2i-2$	$2j-2i-1$	$i=2$
	$2i$	$2i-1$	$2i-3$	$2j-2i-1$	$3 \le i \le j-1$
$x_{3+2i}y_{2+2i}$	2	$2i$	1	$2j-2i-2$	$i=0$
	$2i+1$	$2i$	$2i-2$	$2j-2i-2$	$1 \le i \le j-2$
	$2i+1$	$2i$	$2i-2$	$2j-2i$	$i=j-1$
$x_{2+2i}y_{3+2i}$	$2i+1$	2	2	$2j-2i-2$	$0 \le i \le 1$
	$2i+1$	$2i$	$2i-2$	$2j-2i-2$	$2 \le i \le j-1$

- $r(x_{2+2i}|R_1) = r(y_{2+2i}|R_1) = 2i+1, 2i, 2i-2, 2j-2i-1$ for $(2 \le i \le j-2)$.
- $r(x_{1+2i}|R_1) = r(y_{1+2i}|R_1) = 2i, 2i-1, 2i-3, 2j-2i$ for $(3 \le i \le j-1)$.
- $r(x_{1+2i}x_{2+2i}|R_1) = r(x_{1+2i}y_{2+2i}|R_1) = 2i, 2i-1, 2i-3, 2j-2i-1$ for $(2 \le i \le j-3)$.
- $r(x_{2+2i}x_{3+2i}|R_1) = r(x_{2+2i}y_{3+2i}|R_1)$,
- $= r(x_{3+2i}y_{2+2i}|R_1) = 2i+1, 2i, 2i-2, 2j-2i-2$ for $(2 \le i \le j-2)$.
- $r(x_{1+2i}x_{2+2i}|R_1) = r(x_{1+2i}y_{2+2i}|R_1) = 2i, 2i-1, 2i-3, 2j-2i-1$ for $(2 \le i \le j-2)$.
- $r(x_{2+2i}x_{3+2i}|R_1) = r(x_{3+2i}y_{2+2i}|R_1) = 2i+1, 2i, 2i-2, 2j-2i-2$ for $(2 \le i \le j-2)$.

In order to resolve these vertices and edges, we assume that $R'_1 = \{y_{2i+2}|i = 2, \ldots, \lceil \frac{n}{2} \rceil - 2\}$, which will present distinct representation among them. So from the above facts we can deduce that $R_m = \{y_1, y_{2i+2}|i = 0, \ldots, \lceil \frac{n}{2} \rceil - 2, y_n\}$ is the mixed metric resolving set for even case of $S(P_n)$ with cardinality $\lceil \frac{n}{2} \rceil + 1$.

VI. METRIC DIMENSION OF $S(P_n)$

The metric dimension of splitting of path $S(P_n)$ is computed in [17], the authors proved that,

Theorem 11 [17]: $dim(S(P_n)) = \lceil \frac{n}{3} \rceil$.

In order to show $dim(S(P_n)) \le \lceil \frac{n}{3} \rceil$, the resolving set is considered as, $W = \{v_{3i-1}; i = 0, 1, \ldots, \lceil \frac{n}{3} \rceil - 1\}$. As $W \subseteq V(G)$, we believe that the resolving set should be $W = \{v_{3i+1}; i = 0, 1, \ldots, \lceil \frac{n}{3} \rceil - 1\}$. For the sake of understanding the same labeling for the vertices is shown here.

Now we consider the case for $0 \equiv (mod 3)$, we consider the splitting graph of $S(P_9)$. The resolving set is $W = \{v_1, v_4, v_7\}$. The representation of the vertices are shown graphically see Figure 5.

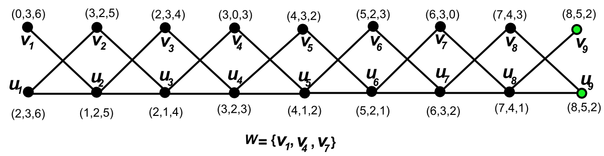

FIGURE 5. The splitting graph of $S(P_9)$.

From the Figure 5, it can be seen that $r(v_9|W) = r(u_9|W) = (8, 5, 2)$. In general it can be written as $r(v_n|W) = r(u_n|W)$. So now we give an improved result for the metric dimension of splitting of path graph $S(P_n)$.

Theorem 12: For the $n \ge 6$;

$$dim(S(P_n)) = \begin{cases} \lceil \frac{n}{3} \rceil + 1, & 0(mod 3); \\ \lceil \frac{n}{3} \rceil, & 1, 2(mod 3). \end{cases}$$

The improved resolving sets are;

$W = \{v_{3i+1}; i = 0, 1, \ldots, \lceil \frac{n}{3} \rceil - 1, v_n\}$ for $0(mod 3)$,
$W = \{v_{3i+1}; i = 0, 1, \ldots, \lceil \frac{n}{3} \rceil - 1\}$ for $1, 2(mod 3)$.

Problem 1: Compute the mixed metric dimension for the cycle related graphs.

Problem 2: The computation of edge metric dimension of path related graphs can be considered if unknown, and for which families of path related graphs $dim_e(G) = dim_m(G)$.

VII. CONCLUSION

This article deals with a newly introduced, which is known as a mixed metric dimension. The mixed metric dimension deals with both the metric and edge metric dimension of the graphs. There are several families of graphs for which metric, edge, and mixed metric dimensions are equal. In this article, we deal with some path related graphs, namely P_n^2, $T(P_n)$, $M(P_n)$, $S(P_n)$, that is square of a path, a total graph of a path, the middle graph of path and splitting of a path, respectively. We computed constant and unbounded mixed metric dimensions for these families. Further research can be thought of as finding a mixed metric dimension in some cycle related graphs. We also presented an improved result for the metric dimension of $S(P_n)$.

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