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A Novel Generalized Integral Inequality Based on Free Matrices for Stability Analysis of Time-Varying Delay Systems

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ABSTRACT This paper proposes a novel generalized integral inequality based on free matrices and applies it to stability analysis of time-varying delay systems. The proposed integral inequality estimates the upper bound of the augmented quadratic term of the state and its derivative term by utilizing not only the single integral term but also the higher-order multiple integral terms. The proposed integral inequality includes several well-known integral inequalities as special cases. For the stability analysis of time-varying delay systems, a new Lyapunov-Krasovskii functional is constructed by including the double integral term with the augmented vector of the state and its derivative to utilize the proposed integral inequality when estimating the derivative of the Lyapunov-Krasovskii functional. Furthermore, to fully exploit the information on the time-varying delay, this paper divides the interval of the double integral term into two parts. Two numerical examples show that the results of the proposed method outperform those of the existing methods.

INDEX TERMS Stability analysis, time-varying delays, Lyapunov-Krasovskii functional, free matrices, generalized integral inequality.

I. INTRODUCTION

Time delays are inevitable phenomenon in the practical systems and they result in performance degradation and system instability. In recent decades, stability analysis of time delay systems has attracted considerable attention in many areas, such as communication systems, biological systems, network control systems or cyber-physical systems [1]–[5]. The main purpose of stability analysis of time delay systems is to obtain maximum delay bound that ensures the asymptotic stability for the concerned systems, and the Lyapunov-Krasovskii functional (LKF) method is an efficient way for such analysis. The LKF method consists of two essential aspects to improve conservatism: the construction of a proper LKF and the estimation of the LKF derivation.

From the aspect of the LKF construction, researches on employing more state information in the LKFs have been carried out by using the augmented vectors [6]–[8]. However, the usage of the augmented vectors is only limited to the quadratic forms of the single integral terms and constant terms in the LKFs, and the double integral terms have been only treated with a single state derivative term. Recently, the LKF which uses state and its derivative term in a double integral is proposed by using a new free-matrix-based integral inequality in [9]. The usage of additional energy from the state term and the cross-term of state and its derivative effectively improve conservatism. However, a crucial disadvantage of the new free-matrix-based integral inequality is that the higher-order multiple integral term cannot be applied, and this is one of the motivations for working on this paper.

On the other hand, from the aspect of the estimation of the LKF derivation, many integral inequalities have been proposed to provide a tighter upper bound: Jensen inequality [10], Wirtinger inequality [11], Bessel-Legendre inequality [12], free-matrix-based integral inequalities [13], and auxiliary function-based integral inequalities [14]. These integral inequalities have efficiently reduced the conservatism of constant delay systems, and Bessel-Legendre inequality ensures the tightest bound among them. However, there have been challenges for applying these inequalities to

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time-varying delay (TVD) systems due to the reciprocal convexity. To overcome these challenges, several papers related to handling the reciprocal convexity conditions have been proposed [15]–[17]. In the last several years, some papers have concentrated on developing several integral inequalities based on the Bessel-Legendre integral inequality by converting the reciprocal convexity into the convexity [18]–[20]. Especially in [20], a generalized free-matrix-based integral inequality is proposed and a less conservative stability criteria for TVD systems is derived by providing the convexity conditions. However, this inequality does not take full advantage of the free matrices, so further improvement can be done by extending the use of the free matrices.

Motivated by the above discussions, this paper proposes a novel generalized integral inequality based on free matrices (GIIBFM), and derives a stability analysis for TVD systems. The proposed GIIBFM estimates the upper bound of the augmented quadratic term of the state and its derivative term by utilizing not only the single integral term but also the higher-order multiple integral terms. The proposed GIIBFM includes several famous existing integral inequalities as special cases such as the generalized free-matrix-based integral inequality [20] and the new free-matrix-based integral inequality [9]. For the stability analysis of TVD systems, a new LKF is constructed by including the double integral term with the augmented vector of the state and its derivative to utilize the proposed GIIBFM when estimating the derivative of the LKF. Furthermore, to exploit the additional information on the TVD, this paper divides the interval of the double integral term into two parts. Finally, two numerical examples shows that the results of the proposed method outperform those of the existing methods.

Notation: The superscripts '-1' represents the inverse matrix; 'T' represents the transpose of matrix; $\mathbb{N}$ stands for non-negative integers; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{m\times n}$ stands for the set of all $n\times m$ real matrices; $0_{m\times n}$ is $m\times n$ zero matrix; 0_n is $n\times n$ zero matrix; I_n is $n\times n$ identity matrix; $\mathbf{Sym}\{X\}=X+X^T$; $Y>0$ means that Y is a symmetric and positive definite matrix; '$\star$' stands for symmetric terms in a symmetric matrix; The binomial coefficients is represented as $\binom{p}{q}=\frac{p!}{q!(p-q)!}$.

II. PROBLEM FORMULATION

A. TIME-VARYING DELAY SYSTEM

Consider the continuous-time linear system with TVD $h(t)$:

$$\begin{cases} \dot{x}(t)=Ax(t)+A_dx(t-h(t)), & t>0,\\ x(t)=\phi(t), & t\in[-\bar{h},0], \end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $A, A_d\in\mathbb{R}^{n\times n}$ are the constant matrices, and the initial condition $\phi(t)$ which is a continuous and differentiable function in $t\in[-\bar{h},0]$. The interval TVD $h(t)$ satisfying

$$0\le h(t)\le\bar{h},\quad \mu_1\le\dot{h}(t)\le\mu_2, \tag{2}$$

for constant $\bar{h}$, μ_1, and μ_2.

B. INTEGRAL INEQUALITIES

The following inequality is recently introduced to derive the stability criteria for TVD systems.

Lemma 2.1 [20]: Let $N\in\mathbb{N}$, $\zeta(t)\in\mathbb{R}^m$, and x be a continuous and differentiable function: $[a,b]\to R^n$. For symmetric matrix $R\in\mathbb{R}^{n\times n}$ and any matrix $Y\in\mathbb{R}^{(N+1)n\times m}$, the following inequality holds:

$$-\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \le (b-a)\zeta^T(t)Y^T\tilde{R}^{-1}Y\zeta(t)+\mathbf{Sym}\Big\{\zeta^T(t)Y\Gamma_N^T\zeta_N\Big\},$$

where

$$\begin{aligned}
\Gamma_N &= [\pi_N(0)\ \pi_N(1)\ \cdots\ \pi_N(N)],\\
\tilde{R} &= diag\{R, \frac{1}{3}R, \cdots, \frac{1}{2N+1}R\},\\
\zeta_N &= \begin{cases}[x^T(b)\ x^T(a)]^T, & N=0,\\ [x^T(b)\ x^T(a)\ \frac{1}{b-a}\Theta_0^T\ \cdots\ \frac{1}{b-a}\Theta_{N-1}^T]^T, & N\ge 1,\end{cases}\\
\pi_N(k) &= \begin{cases}[I_n\ \ -I_n]^T, & N=0,\\ [I_n\ (-1)^{k+1}I_n\ \theta_{Nk}^0I_n\ \cdots\ \theta_{Nk}^{N-1}I_n]^T, & N\ge 1,\end{cases}\\
\theta_{Nk}^m &= \begin{cases}(2m+1)((-1)^{k+m}-1), & m\le k,\\ 0, & m>k,\end{cases}\\
p_k(s) &= (-1)^k\sum_{i=0}^{k}\Big[(-1)^i\binom{k}{i}\binom{k+i}{i}\Big]\Big(\frac{s-a}{b-a}\Big)^i,\\
\Theta_k &= \int_a^b p_k(s)x(s)ds,\ \int_a^b p_i(s)p_j(s)ds=\begin{cases}\frac{b-a}{2i+1}, & i=j,\\ 0, & i\ne j.\end{cases}
\end{aligned}$$

Lemma 2.2 (Generalized integral inequality based on free matrices, GIIBFM): Let $\zeta(t)\in\mathbb{R}^{mn}$, $N\in\mathbb{N}$, $x(s)\in\mathbb{R}^n$ be a continuous and differentiable function in $[a,b]$. For the positive definite matrix $R\in\mathbb{R}^{2n\times 2n}$, any free-matrix $Y_{i,j}\in\mathbb{R}^{mn\times n}$ ($i=0,1,2,\cdots,N$ for $j=1$ and $i=0,1,2,\cdots,N-1$ for $j=2$), $Y_{N,2}=0_{mn\times n}$, $\eta(s)=\big[\dot{x}^T(s)\ x^T(s)\big]^T$, and $h=b-a$, the following integral inequality holds:

$$-\int_a^b \eta^T(s)R\eta(s)ds\le\zeta^T(t)\Omega_N(R,h)\zeta(t), \tag{3}$$

where

$$\Omega_N(R,h)=\sum_{k=0}^{N}\Big\{\frac{h}{2k+1}\big[Y_{k,1}\quad Y_{k,2}\big]R^{-1}\big[Y_{k,1}\quad Y_{k,2}\big]^T +\mathbf{Sym}\Big\{Y_{k,1}M_{k,1}+Y_{k,2}M_{k,2}\Big\}\Big\},$$

$$M_{k,1}\zeta(t)=\pi_N^T(k)\zeta_N,\quad M_{k,2}\zeta(t)=\Theta_k,$$

and the definitions of parameters, ζ_N, $\pi_N(k)$, and Θ_k, are same as in Lemma 2.1.

Proof 1: Before deriving this proof, the two structured matrices and the integral characteristic of $p_k(s)$ are defined such that

$$Y = \begin{bmatrix} Y_{N,1} & Y_{N,2} \\ Y_{N-1,1} & Y_{N-1,2} \\ \vdots & \vdots \\ Y_{0,1} & Y_{0,2} \end{bmatrix}, \quad P(s) = \begin{bmatrix} p_N(s) \\ \vdots \\ p_1(s) \\ p_0(s) \end{bmatrix},$$

where $p_k(s)$ is defined in Lemma 2.1. Then, Since $R > 0$, the following integral inequality is satisfied:

$$\begin{aligned} 0 &\le \int_a^b \begin{bmatrix} P(s)\zeta(t) \\ \eta(s) \end{bmatrix}^T \begin{bmatrix} YR^{-1}Y^T & Y \\ Y^T & R \end{bmatrix} \begin{bmatrix} P(s)\zeta(t) \\ \eta(s) \end{bmatrix} ds \\ &= \int_a^b \eta^T(s)R\eta(s) \\ &\quad + \zeta^T(t)\left(\int_a^b P^T(s)YR^{-1}Y^T P(s)ds\right)\zeta(t) \\ &\quad + 2\zeta^T(t)\int_a^b P^T(s)Y\eta(s)ds \\ &= \int_a^b \eta^T(s)R\eta(s) \\ &\quad + \zeta^T(t)\sum_{k=0}^{N}\Big\{\int_a^b p_k^2(s)ds\big[Y_{k,1} \quad Y_{k,2}\big]R^{-1}\big[Y_{k,1} \quad Y_{k,2}\big]^T \\ &\quad + \mathbf{Sym}\Big\{Y_{k,1}\int_a^b p_i(s)\dot{x}(s)ds + Y_{k,2}\int_a^b p_i(s)x(s)ds\Big\}\Big\}\zeta(t). \end{aligned}$$

This concludes the proof. ∎

Remark 1: Lemma 2.2 includes the existing integral inequalities. The new free-matrix-based integral inequality in [9] can be easily derived from Lemma 2.2 with $N = 1$. In addition, Lemma 2.2 with $Y_{j,2} = 0_{mn\times n}$ (for $j = 0, 1, 2, \cdots, N$) leads to Lemma 2.1, so we can conclude that Lemma 2.2 includes Lemma 2.1 as a special case.

Remark 2: A pair of free matrices $Y_{k+1,1}$ and $Y_{k,2}$ in Lemma 2.2 are related to the k-th order multiple integral term, so the structured matrix Y is constructed with $Y_{N,2} = 0_{mn\times n}$. Compare to the integral inequalities which are based on Bessel-Legendre polynomial such as [19] and Lemma 2.1, Lemma 2.2 uses not only $\pi_N^T(k)\zeta_N$ but also Θ_k by introducing the augmented vector of $x(t)$ and $\dot{x}(t)$ to fully utilize information of free matrices. When increasing the order N in Lemma 2.2, we can use the higher-order multiple integral terms, so the upper bound becomes tighter.

III. MAIN RESULT

In this section, we provide a developed stability analysis for TVD system (1). Before deriving main result, some notations are defined as follows:

$$e_k = \begin{bmatrix} 0_{n\times n(k-1)} & I_n & 0_{n\times n(9-k)} \end{bmatrix}, \ \textit{for } k = 1,2,3,\cdots,9,$$

$$\begin{aligned} \zeta(t) = \Big[&x^T(t)\ x^T(t-h(t))\ x^T(t-\bar{h})\ \dot{x}^T(t-h(t))\ \dot{x}^T(t-\bar{h}) \\ &\frac{1}{h(t)}\int_{t-h(t)}^{t} x^T(s)ds \frac{1}{\bar{h}-h(t)}\int_{t-\bar{h}}^{t-h(t)} x^T(s)ds \\ &\frac{1}{h^2(t)}\int_{-h(t)}^{0}\int_{t+r}^{t} x^T(s)dsdr \\ &\frac{1}{(\bar{h}-h(t))^2}\int_{-\bar{h}}^{-h(t)}\int_{t+r}^{t-h(t)} x^T(s)dsdr\Big]^T, \end{aligned}$$

$$\begin{aligned} \eta_1(t) = \Big[&x^T(t)\ \ x^T(t-h(t))\ \ x^T(t-\bar{h}) \ \int_{t-h(t)}^{t} x^T(s)ds \\ &\int_{t-\bar{h}}^{t-h(t)} x^T(s)ds \ \frac{1}{h(t)}\int_{t-h(t)}^{t}\int_r^t x^T(s)dsdr \\ &\frac{1}{\bar{h}-h(t)}\int_{t-\bar{h}}^{t-h(t)}\int_r^{t-h(t)} x^T(s)dsdr\Big]^T, \end{aligned}$$

$$\begin{aligned} \eta_2(t,s) = \Big[&\eta_3(s)\ \ x^T(t-h(t)) \ \int_s^t \dot{x}^T(r)dr \\ &\int_{t-\bar{h}}^{s}\dot{x}^T(r)dr \ \int_{t-h(t)}^{s}\dot{x}^T(r)dr\Big]^T, \end{aligned}$$

$$\eta_3(s) = \begin{bmatrix} \dot{x}^T(s) & x^T(s) \end{bmatrix}^T.$$

In the following Theorem 3.1, a stability analysis for TVD systems (1) is derived using Lemma 2.2 with $N = 2$.

Theorem 3.1: For a TVD $h(t)$ satisfying condition (2), the system (1) is asymptotically stable if there exist positive definite matrices $P \in \mathbb{R}^{7n\times 7n}$, $Q_k \in \mathbb{R}^{6n\times 6n}$, $R_k \in \mathbb{R}^{2n\times 2n}(k = 1, 2)$ and any matrices Y_l, $\bar{Y}_l \in \mathbb{R}^{9n\times n}(l \in \{(0,1), (0,2), (1,1), (1,2), (2,1)\})$ satisfying the following LMIs for $i = 1, 2$:

$$\Psi(0,\mu_i) < 0, \ \ \Psi(\bar{h},\mu_i) < 0, \hat{R}_1(\dot{h}(t)) > 0, \tag{4}$$

where

$$\Psi(0,\dot{h}(t)) = \begin{bmatrix} \Gamma(0,\dot{h}(t)) & \bar{h}Y_2 \\ \star & -\bar{h}\bar{R}_2 \end{bmatrix},$$

$$\Psi(h,\dot{h}(t)) = \begin{bmatrix} \Gamma(h,\dot{h}(t)) & \bar{h}Y_1 \\ \star & -\bar{h}\bar{R}_1 \end{bmatrix},$$

with

$$\begin{aligned} Y_1 &= \big[Y_{0,1}\ \ Y_{0,2}\ \mid\ Y_{1,1}\ \ Y_{1,2}\ \mid\ \ Y_{2,1}\ 0_{9n\times n}\big], \\ Y_2 &= \big[\bar{Y}_{0,1}\ \ \bar{Y}_{0,2}\ \mid\ \bar{Y}_{1,1}\ \ \ \bar{Y}_{1,2}\ \mid\ \bar{Y}_{2,1}\ \ 0_{9n\times n}\big], \\ \hat{R}_1(\dot{h}(t)) &= (1-\dot{h}(t))R_1 + \dot{h}(t)R_2, \\ \bar{R}_1 &= \begin{bmatrix} \hat{R}_1 & 0_{2n} & 0_{2n} \\ 0_{2n} & 3\hat{R}_1 & 0_{2n} \\ 0_{2n} & 0_{2n} & 5\hat{R}_1 \end{bmatrix}, \\ \bar{R}_2 &= \begin{bmatrix} R_2 & 0_{2n} & 0_{2n} \\ 0_{2n} & 3R_2 & 0_{2n} \\ 0_{2n} & 0_{2n} & 5R_2 \end{bmatrix}, \\ e_0 &= Ae_1 + A_d e_2, \\ \Pi_1(h(t)) &= [\Pi_{11}\ \ \Pi_{12}(h(t))]^T, \\ \Pi_2(\dot{h}(t)) &= [\Pi_{21}(\dot{h}(t))\Pi_{22}(\dot{h}(t))\Pi_{23}(\dot{h}(t))\ \Pi_{24}(\dot{h}(t))]^T, \\ \Pi_3 &= [e_0^T\ \ e_1^T\ \ e_2^T\ \ 0_{9n\times n}\ \ e_1^T - e_3^T\ \ e_1^T - e_2^T]^T, \\ \Pi_4 &= [e_4^T\ \ e_2^T\ \ e_2^T\ \ e_1^T - e_2^T\ \ e_2^T - e_3^T\ \ 0_{9n\times n}]^T, \\ \Pi_5 &= [e_5^T\ \ e_3^T\ \ e_2^T\ \ e_1^T - e_3^T\ \ 0_{9n\times n}\ \ e_3^T - e_2^T]^T, \\ \Pi_6(\dot{h}(t)) &= [0_{9n\times n}\ \ 0_{9n\times n}\ \ (1-\dot{h}(t))e_4^T \end{aligned}$$

$$\begin{aligned}
&e_0^T \quad -e_5^T \quad -(1-\dot{h}(t))e_4^T]^T,\\
\Pi_7(h(t)) &= [\Pi_{71}(h(t)) \;\; \Pi_{72}(h(t))]^T,\\
\Pi_8(h(t)) &= [\Pi_{81}(h(t)) \;\; \Pi_{82}(h(t)) \;\; \Pi_{83}(h(t))]^T,\\
\Pi_9 &= [e_0^T \;\; e_1^T]^T\\
\Pi_{11} &= [e_1^T \;\; e_2^T \;\; e_3^T],\\
\Pi_{12}(h(t)) &= [h(t)e_6^T \;\; (\bar{h}-h(t))e_7^T \;\; h(t)e_8^T \;\; (\bar{h}-h(t))e_9^T],\\
\Pi_{21}(\dot{h}(t)) &= [e_0^T \;\; (1-\dot{h}(t))e_4^T \;\; e_5^T],\\
\Pi_{22}(\dot{h}(t)) &= [e_1^T-(1-\dot{h}(t))e_2^T \;\; (1-\dot{h}(t))e_2^T-e_3^T],\\
\Pi_{23}(\dot{h}(t)) &= e_1^T-(1-\dot{h}(t))e_6^T-\dot{h}(t)e_8^T,\\
\Pi_{24}(\dot{h}(t)) &= (1-\dot{h}(t))e_2^T-e_7^T+\dot{h}(t)e_9^T,\\
\Pi_{71}(h(t)) &= [e_1^T-e_2^T \;\; h(t)e_6^T \;\; h(t)e_2^T \;\; h(t)(e_1^T-e_6^T)],\\
\Pi_{72}(h(t)) &= [h(t)(e_6^T-e_3^T) \;\; h(t)(e_6^T-e_2^T)],\\
\Pi_{81}(h(t)) &= [e_2^T-e_3^T \;\; (\bar{h}-h(t))e_7^T],\\
\Pi_{82}(h(t)) &= [(\bar{h}-h(t))e_2^T \;\; (\bar{h}-h(t))(e_1^T-e_7^T)],\\
\Pi_{83}(h(t)) &= [(\bar{h}-h(t))(e_7^T-e_3^T) \;\; (\bar{h}-h(t))(e_7^T-e_2^T)],\\
\Gamma(h(t),\dot{h}(t)) &= \mathbf{Sym}\Big\{\Pi_2^T(\dot{h}(t))P\Pi_1(h(t))\\
&\quad+\Pi_6^T(\dot{h}(t))Q_1\Pi_7(h(t))\\
&\quad+\Pi_6^T(\dot{h}(t))Q_2\Pi_8(h(t))\Big\}\\
&\quad+\Pi_3^TQ_1\Pi_3+(1-\dot{h}(t))\Pi_4^T(Q_2\\
&\quad-Q_1)\Pi_4-\Pi_5^TQ_2\Pi_5\\
&\quad+\Pi_9^T(h(t)R_1+(\bar{h}-h(t))R_2)\Pi_9\\
&\quad+\mathbf{Sym}\Big\{Y_{0,1}M_{0,1}+Y_{0,2}M_{0,2}+Y_{1,1}M_{1,1}\\
&\quad+Y_{1,2}M_{1,2}+Y_{2,1}M_{2,1}\\
&\quad+\bar{Y}_{0,1}\bar{M}_{0,1}+\bar{Y}_{0,2}\bar{M}_{0,2}+\bar{Y}_{1,1}\bar{M}_{1,1}\\
&\quad+\bar{Y}_{1,2}\bar{M}_{1,2}+\bar{Y}_{2,1}\bar{M}_{2,1}\Big\},\\
\Omega_2(\hat{R}_1(\dot{h}(t)),h) &= h\big[Y_{0,1}\;Y_{0,2}\big]\hat{R}_1^{-1}(\dot{h}(t))\big[Y_{0,1}\;Y_{0,2}\big]^T\\
&\quad+\frac{h}{3}\big[Y_{1,1}\;Y_{1,2}\big]\hat{R}_1^{-1}(\dot{h}(t))\big[Y_{1,1}\;Y_{1,2}\big]^T\\
&\quad+\frac{h}{5}\big[Y_{2,1}\;0_{9n\times n}\big]\hat{R}_1^{-1}(\dot{h}(t))\big[Y_{2,1}\;0_{9n\times n}\big]^T\\
&\quad+\mathbf{Sym}\Big\{Y_{0,1}M_{0,1}+Y_{0,2}M_{0,2}+Y_{1,1}M_{1,1}\\
&\quad+Y_{1,2}M_{1,2}+Y_{2,1}M_{2,1}\Big\},\\
\Omega_2(R_2,h) &= h\big[\bar{Y}_{0,1}\;\bar{Y}_{0,2}\big]R_2^{-1}\big[\bar{Y}_{0,1}\;\bar{Y}_{0,2}\big]^T\\
&\quad+\frac{h}{3}\big[\bar{Y}_{1,1}\;\bar{Y}_{1,2}\big]R_2^{-1}\big[\bar{Y}_{1,1}\;\bar{Y}_{1,2}\big]^T\\
&\quad+\frac{h}{5}\big[\bar{Y}_{2,1}\;0_{9n\times n}\big]R_2^{-1}\big[\bar{Y}_{2,1}\;0_{9n\times n}\big]^T\\
&\quad+\mathbf{Sym}\Big\{\bar{Y}_{0,1}\bar{M}_{0,1}+\bar{Y}_{0,2}\bar{M}_{0,2}+\bar{Y}_{1,1}\bar{M}_{1,1}\\
&\quad+\bar{Y}_{1,2}\bar{M}_{1,2}+\bar{Y}_{2,1}\bar{M}_{2,1}\Big\},
\end{aligned}$$

$$\begin{bmatrix} M_{01}\\ M_{02}\\ M_{11}\\ M_{12}\\ M_{21}\end{bmatrix}=\begin{bmatrix} e_1-e_2\\ h(t)e_6\\ e_1+e_2-2e_6\\ h(t)(-e_6+2e_8)\\ e_1-e_2+6e_6-12e_8\end{bmatrix},$$

$$\begin{bmatrix} \bar{M}_{01}\\ \bar{M}_{02}\\ \bar{M}_{11}\\ \bar{M}_{12}\\ \bar{M}_{21}\end{bmatrix}=\begin{bmatrix} e_2-e_3\\ (\bar{h}-h(t))e_7\\ e_2+e_3-2e_7\\ (\bar{h}-h(t))(-e_7+2e_9)\\ e_2-e_3+6e_7-12e_9\end{bmatrix}.$$

Proof 2: To evaluate the stability criterion of system with TVD (1), the LKF candidates are constructed as

$$\begin{aligned}
V(t) &= \sum_{i=1}^{3} V_i(t),\\
V_1(t) &= \eta_1^T(t)P\eta_1(t),\\
V_2(t) &= \int_{t-h(t)}^{t}\eta_2^T(t,s)Q_1\eta_2(t,s)ds\\
&\quad+\int_{t-\bar{h}}^{t-h(t)}\eta_2^T(t,s)Q_2\eta_2(t,s)ds,\\
V_3(t) &= \int_{t-h(t)}^{t}\int_{r}^{t}\eta_3^T(s)R_1\eta_3(s)dsdr\\
&\quad+\int_{t-\bar{h}}^{t-h(t)}\int_{r}^{t}\eta_3^T(s)R_2\eta_3(s)dsdr. \qquad (5)
\end{aligned}$$

The time-derivative of each LKF in (5), $V_i(t)$ where $i\in\{1,2,3\}$, can be computed as follows

$$\begin{aligned}
\dot{V}_1(t) &= \mathbf{Sym}\Big\{\Pi_2^T(\dot{h}(t))P\Pi_1(h(t))\Big\},\\
\dot{V}_2(t) &= \Pi_3^TQ_1\Pi_3+(1-\dot{h}(t))\Pi_4^T(Q_2-Q_1)\Pi_4-\Pi_5^TQ_2\Pi_5\\
&\quad+\mathbf{Sym}\Big\{\Pi_6^T(\dot{h}(t))Q_1\Pi_7(h(t))\\
&\quad+\Pi_6^T(\dot{h}(t))Q_2\Pi_8(h(t))\Big\},\\
\dot{V}_3(t) &= h(t)\Pi_9^TR_1\Pi_9+(\bar{h}-h(t))\Pi_9^TR_2\Pi_9\\
&\quad-\int_{t-h(t)}^{t}\eta_3^T(s)\hat{R}_1(\dot{h}(t))\eta_3(s)ds\\
&\quad-\int_{t-\bar{h}}^{t-h(t)}\eta_3^T(s)R_2\eta_3(s)ds. \qquad (6)
\end{aligned}$$

By applying Lemma 2.2 with $N=2$, we can obtain

$$\begin{aligned}
&-\int_{t-h(t)}^{t}\eta_3^T(s)\hat{R}_1(\dot{h}(t))\eta_3(s)ds\\
&\qquad\le\zeta^T(t)\Omega_2(\hat{R}_1(\dot{h}(t)),h(t))\zeta(t), \qquad (7)\\
&-\int_{t-\bar{h}}^{t-h(t)}\eta_3^T(s)R_2\eta_3(s)ds\\
&\qquad\le\zeta^T(t)\Omega_2(R_2,\bar{h}-h(t))\zeta(t). \qquad (8)
\end{aligned}$$

Combining (6), (7), (8) yields:

$$\begin{aligned}
\dot{V}(t) &= \dot{V}_1(t)+\dot{V}_2(t)+\dot{V}_3(t)\\
&\le\zeta^T(t)\Big\{\Gamma(h(t),\dot{h}(t))\\
&\quad+h(t)\big[Y_{0,1}\;Y_{0,2}\big]\hat{R}_1^{-1}(\dot{h}(t))\big[Y_{0,1}\;Y_{0,2}\big]^T\\
&\quad+\frac{h(t)}{3}\big[Y_{1,1}\;Y_{1,2}\big]\hat{R}_1^{-1}(\dot{h}(t))\big[Y_{1,1}\;Y_{1,2}\big]^T\\
&\quad+\frac{h(t)}{5}\big[Y_{2,1}\;0_{9n\times n}\big]\hat{R}_1^{-1}(\dot{h}(t))\big[Y_{2,1}\;0_{9n\times n}\big]^T
\end{aligned}$$

TABLE 1. The maximum admissible $\bar{h}$ for various μ.

μ	0.1	0.2	0.8	NDVs
[13]	4.788	4.060	2.615	$65n^2+11n$
[21]	4.831	4.141	2.713	$142n^2+18n$
[19]	4.908	4.199	2.735	$65n^2+8n$
[16]	4.910	4.216	2.789	$54.5n^2+6.5n$
[18] ($N=1$)	4.800	3.990	2.330	$34n^2+5n$
[18] ($N=2$)	4.930	4.220	2.660	$62.5n^2+6.5n$
[20] ($N=1$)	4.837	4.145	2.727	$77n^2+9n$
[20] ($N=2$)	4.921	4.218	2.792	$115n^2+10n$
Theorem 3.1 ($N=1$), [9]	4.838	4.154	2.736	$94.5n^2+10.5n$
Theorem 3.1 ($N=2$)	4.930	4.235	2.807	$154.5n^2+11.5n$

$$+(\bar{h}-h(t))\left[\bar{Y}_{0,1}\ \bar{Y}_{0,2}\right]R_2^{-1}\left[\bar{Y}_{0,1}\ \bar{Y}_{0,2}\right]^T + \frac{\bar{h}-h(t)}{3}\left[\bar{Y}_{1,1}\ \bar{Y}_{1,2}\right]R_2^{-1}\left[\bar{Y}_{1,1}\ \bar{Y}_{1,2}\right]^T + \frac{\bar{h}-h(t)}{5}\left[\bar{Y}_{2,1}\ 0_{9n\times n}\right]R_2^{-1}\left[\bar{Y}_{2,1}\ 0_{9n\times n}\right]^T\Bigg\}\zeta(t). \tag{9}$$

By using schur complement, it can be concluded that (9) and the LMIs in (4) are essentially equivalent for the condition (2). Therefore, if $\dot{V}(t) < 0$ which is equivalent to the LMI in (4) is satisfied, then the system (1) is asymptotically stable. This concludes the proof. ∎

Remark 3: Through the usage of the augmented vector with $x(s)$ and $\dot{x}(s)$ in the LKF, free matrices provide more freedom in deriving the stability criterion. In addition, the interval of the double integral LKF in (5) $[t-\bar{h}, t]$ is divided into $[t-\bar{h}, t-h(t)]$ and $[t-h(t), t]$ to fully exploit the information on TVD and to provide a less conservative stability criterion. It can be verified by numerical examples in the section IV. When the order of the proposed integral inequality N is increased, the higher-order multiple integral terms can be added to the LKF augmented vectors, η_1 and η_2, for constructing a less conservative stability condition.

IV. NUMERICAL EXAMPLE

In this section, the performance of the proposed method is compared with the existing methods through two numerical examples.

Examples 4.1: Consider TVD system (1) with

$$A=\begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix},\quad A_d=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix},$$

and $\mu_1=-\mu_2=\mu$. This system is a commonly-used example for verifying conservatism of many methods. In Table 1, it is shown that the highest maximum admissible upper bound is obtained by Theorem 3.1. Among the results of existing methods, the authors of this paper thought that the result of the paper [20] with $N=1$ was incorrect, so it was re-simulated and corrected from $[4.841,\ 4.154,\ 3.159,\ 2.729]$ to $[4.837,\ 4,145,\ 3.152,\ 2.727]$. According to the corrected results, Theorem 3.1 provides a less conservative stability criterion for all cases μ than other existing methods. Furthermore, with the same order of the integral inequality N,

TABLE 2. The maximum admissible $\bar{h}$ for various μ.

μ	0.1	0.2	0.8	NDVs
[21]	7.167	4.517	1.838	$142n^2+18n$
[16]	7.230	4.556	1.940	$54.5n^2+6.5n$
[20] ($N=2$)	7.309	4.670	2.072	$115n^2+10n$
Theorem 3.1 ($N=2$)	7.524	4.855	2.127	$154.5n^2+11.5n$

Theorem 3.1 shows better results compared with other generalized methods such as [18] and [20], and it is also clearly shown that Theorem 3.1 enhances a feasible region of delays by increasing the order N.

Examples 4.2: Consider TVD system (1) with

$$A=\begin{bmatrix}0 & 1.0\\ -1.0 & -2.0\end{bmatrix},\quad A_d=\begin{bmatrix}0 & 0\\ -1.0 & 1.0\end{bmatrix},$$

and $\mu_1=-\mu_2=\mu$. From the Table 2, it should be noted that the maximum admissible upper bound of Theorem 3.1 is superior than other existing methods. In addition, it can be clearly verified that Theorem 3.1 shows the dramatic improvement in performance for the same order of the integral inequality N.

V. CONCLUSION

This paper developed a novel GIIBFM to derive an improved stability analysis for TVD system. The upper bound of the augmented quadratic term of the state and its derivative term was estimated by the proposed GIIBFM by utilizing the higher-order multiple integral terms as well as the single integral term. In addition, the proposed GIIBFM included some famous integral inequalities as special cases, and made it possible to find a tighter bound than other existing generalized integral inequalities with the same order. Particularly, increasing the order of the integral inequality guaranteed a tighter upper bound and enabled the use of the higher-order multiple integral terms in deriving the stability criteria. The LKF was constructed with the augmented vectors, and we further reduced conservatism by dividing the interval of the double integral term of the LKF into two parts. The stability criteria was derived by utilizing the proposed GIIBFM at derivative of double integral LKF, and it was clearly verified that the results of the proposed method outperform those of the existing methods.

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