IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received August 27, 2020, accepted September 17, 2020, date of publication September 29, 2020, date of current version October 12, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3027721

Stabilization of Nonlinear Control-Affine Systems With Multiple State Constraints

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This work was support by the Ministry of Science and Technology of the Republic of China under Contract MOST 107-2221-E-019-052.

ABSTRACT This paper considers the synthesis of stabilizing controllers for nonlinear control-affine systems under multiple state constraints. A new control Lyapunov-barrier function approach is introduced for solving the considered problem. Assuming a classical control Lyapunov function, two possible methods for constructing new control Lyapunov-barrier functions are discussed. Sufficient conditions for the existence of new control Lyapunov-barrier functions are derived. With modifying the Sontag's formula, an explicit state-constrained stabilizing feedback law is presented. Finally, two numerical examples are provided to illustrate the obtained theoretical results.

INDEX TERMS Barrier function, control Lyapunov function, control Lyapunov-barrier function, nonlinear control systems, state constraint, state feedback.

I. INTRODUCTION

The design of stabilizing controllers under state constraints is a critical research topic because the state trajectories of a practical control system are not allowed to enter certain unsafe regions. In [1]–[4], the set invariance approach was developed for synthesizing state-constrained stabilizing controllers for linear/nonlinear control systems. As stated in [5], the feedback law proposed in [4] for nonlinear control systems may result in chattering as the control law is discontinuous. In [6]–[8], linear and nonlinear model predictive controllers (MPC) were designed with considering state constraints. The MPC approach has been very successful in industrial applications, but explicit guarantees of stability are not always easy [9], especially in the nonlinear case. Among different MPC formulations, Lyapunov-based MPC [10], [11] can provide explicit stability region guarantees. However, incorporating explicit state constraints in Lyapunov-based MPC is still difficult [9]. In [12]–[15], reference governors were applied for the satisfaction of state constraints. For avoiding violation of constraints, a high-level controller that generates admissible reference signals for the low-level controller is needed [16].

In the last two decades, on the basis of (control) Lyapunov functions and barrier functions, several analytical nonlinear state-constrained design methods have been proposed. In [17]–[24], barrier Lyapunov functions (BLFs) were used to design state-constrained stabilizing controllers for single-input nonlinear control systems in particular structures. The value of a BLF tends to infinity when the boundary of the constraint region is approached. Recently, the BLF approach has been applied for MIMO nonlinear control systems in particular forms [25]–[27]. Extending these results to general nonlinear control-affine systems is difficult as the backstepping technique is employed in the BLF approach. In [28], on the basis of the control Lyapunov function (CLF) (see [29]–[32]), a control Lyapunov R-function method was proposed for robust constrained stabilization of uncertain linear systems. In [33], a control barrier function (CBF) approach was developed for the control synthesis of nonlinear systems to achieve the safety objective. In [16], for nonlinear control-affine systems, a CBF and a CLF were combined by weighted average to be a smooth control Lyapunov-barrier function (CLBF) and then Sontag's formula was applied for constructing continuous controllers to ensure both safety and stability. The assumptions of a smooth CLBF and a continuous feedback law in [16] require the state constraints to those defined by unbounded sets [34]. In [34], a nonsmooth complete Lyapunov function method was introduced to synthesize discontinuous controllers for bounded state-constrained sets. In [5], [35]–[37], in combination with a CLF and a (*reciprocal* or *zeroing*) CBF, the quadratic program (QP) technique was

The associate editor coordinating the review of this manuscript and approving it for publication was Azwirman Gusrialdi.

applied for the state-constrained control design of nonlinear control systems. It has been shown in [36], [37] that the zeroing CBF (ZCBF) can provide safety, stability, and certain robustness simultaneously. For ensuring the solution of a QP is locally Lipschitz in the constrained region, a relative degree one condition must hold [5]. In this case, a closed-form expression of the solution of the QP can be obtained. In these studies, how to choose the value of the weighting factor in the QP problem to claim stability of the closed-loop system is not clear [9]. In [9], with solving a modified QP, local asymptotical stability can be guaranteed under the standard assumptions on the CLF and CBF. The controller was presented in a closed-form making it unnecessary to solve the QP online. On the other hand, barrier functions have also been applied in solving nonlinear optimal control problems subject to the trajectory, input, state, or output constraints [38]–[40]. With the introduction of optimization skills, closed-loop systems' performances can be further improved. In these approaches, conditions for the solvability of the constrained optimization problems and explicit formulas for constructing controllers are difficult to derive.

This paper introduces a new CLBF method for designing asymptotically stabilizing controllers for nonlinear control-affine systems under multiple state constraints presented in terms of state functional inequalities. Different from the CLBF in [16], the new CLBF is defined only in the safe region and its value tends to infinity as the boundary of the constrained region is approached. That the existence of new CLBFs is proven to be sufficient for the existence of state-constrained stabilizing controllers. With modifying the Sontag's formula, an explicit formula for constructing state-constrained stabilizing controllers is derived. Furthermore, through combinations of CLFs and barrier functions, two possible methods for constructing new CLBFs are discussed. More importantly, sufficient conditions for the existence of new CLBFs are derived. Under a similar concept, a barrier storage function (BSF) method has been developed in [41] for designing L_2-gain controllers for nonlinear systems with a single state constraint. The state-constrained stabilization problem considered in this paper is not a special case of the state-constrained disturbance attenuation problem in [41]. Even if considering only one state constrained condition as in [41], neither the obtained conditions for the existence of state-constrained stabilizing controllers nor the formula for constructing controllers can be deduced from those in [41] by relaxing the disturbance attenuation requirement.

The contributions of this paper are fourfold: **First**, this paper proposes a new CLBF method for designing state-constrained state feedback stabilizing controllers for nonlinear control-affine systems. Compared with related methods, the main advantage of our approach lies in the simplicity of implementing obtained controllers. The construction of state-constrained controllers is much easier than that in the classical CLBF approach [16]. Moreover, different from the QP-based methods [5], [9], and [35]–[37], in our approach state-constrained controllers can be explicitly constructed by using a modified Sontag's formula and, more importantly, asymptotical stability with the entire safe set being forward invariant and in the region of attraction can be guaranteed. **Second,** multiple general state constraints are considered. Although multiple state constraints have been also considered in [16]–[20], the constraints considered in this paper are different. In [16], unsafe regions are represented with notations of sets and therefore, explicit conditions for the existence of state-constrained controllers are difficult to derive. In [17]–[20], the considered systems must be in particular structures and the constraints are assigned to the upper and lower limits of each state. For general nonlinear control-affine systems with joint-state constraints, the methods presented in [17]–[20] are not applicable. **Third**, explicit sufficient conditions for the existence of new CLBFs are derived. Similar to the concept of Nagumo's theorem [1], the obtained conditions are expressed in terms of the positivity or negativity of functions, which only need to be verified at *some* points on the boundaries of the safe region. In [41], sufficient conditions for the existence of BSFs need to be verified in a *neighborhood* of the boundary of the safe region; and, besides the boundary, the conditions for defining the CBF [33] must hold for points in the safe region. **Fourth**, our method constitutes a constructive means of deriving a new CLBF when a traditional CLF is available and the derived conditions are satisfied. A general way of constructing ZCBFs has been proposed in [42]. This approach is complex because an inf-operation must be taken over an unbounded interval $[0, \infty)$. Reference [37] suggested an alternative approach to construct CBFs. The CBF was parameterized as a fixed-degree polynomial, and then sums-of-squares (SOS) programming [43] was used to enforce the required conditions on the CBF. In [16], a weighted average approach was proposed for constructing a classical CLBF with a CBF, denoted by B, and a CLF, denoted by V. The weighted average approach is simple, while in some proximity to the boundary of the safe region, the controller uses all inputs to make the time derivative of B satisfying $\dot{B} < 0$ ignoring the CLF part even when it is possible to satisfy $\dot{B} < 0$ and $\dot{V} < 0$ simultaneously [9]. In our approach, the construction of new CLBFs is simple and intuitive.

This paper is organized as follows: In Section II, the problem requiring a solution is formulated. Section III introduces the concept of the new CLBF. In Section IV, the construction of CLBFs is discussed, and sufficient conditions for the existence of CLBFs are derived. Two representative examples are provided in Section V. Finally, some conclusions are drawn in Section VI.

Notations: ∂S, $\bar{S}$, $\text{Int}(S)$ and S^C are the boundary, closure, interior, and complement of set S, respectively; $A \backslash B$ is the set of all elements belonging to set A but not belonging to set B; Z^+ is the set of all positive integers; $\emptyset$ denotes an empty set; $\nabla V(x) := \frac{\partial V(x)}{\partial x}$; $\text{dist}(S, x) := \min_{y \in \bar{S}} \|y - x\|$ is the minimal distance between the point x and the set $\bar{S}$; for $\epsilon > 0$, $B_D(S, \epsilon) := \{x \in D \,|\, \text{dist}(S, x) < \epsilon\}$.

For simplification, let $B_D(\epsilon) := B_D(0,\epsilon)$, $B(S,\epsilon) := B_{R^n}(S,\epsilon)$, and $B(\epsilon) := B_{R^n}(0,\epsilon)$.

II. PROBLEM FORMULATION AND PRELIMINARIES

This section formulates the state-constrained stabilization problem to be solved and introduces the CLF, the CBF, and the ZCBF.

A. PROBLEM FORMULATION

Consider the following nonlinear control system:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, and the functions $f : R^n \to R^n$ and $g : R^n \to R^{n\times m}$ are locally Lipschitz. Suppose that $f(0) = 0$.

Define

$$D_i \equiv \{x \in R^n | s_i(x) > 0\}, \quad i = 1, \ldots, N,$$

where $N \in Z^+$ and functions $s_i(\cdot)$, $i = 1, \ldots, N$, are differentiable. Assume that $\partial D_i \cap \partial D_j = \emptyset$ if $i \neq j$. Let $D \subset R^n$ be a connected region represented by

$$D \equiv \{x \in R^n | s_i(x) > 0, i = 1, \ldots, N\}. \tag{2}$$

That is,

$$D = D_1 \cap D_2 \cap \ldots \cap D_N,$$

and

$$\partial D = (\partial D_1 \cup \partial D_2 \cup \ldots \cup \partial D_N) \cap \bar{D}.$$

The region D is referred to as the safe set or the safe region.

Suppose that the origin is an interior point of D. With a little notational abuse, let $\partial D_i(\epsilon) \equiv \{x \in D \,|\, s_i(x) < \epsilon\}$ and $\partial D(\epsilon) \equiv \bigcup_{i=1}^{N} \partial D_i(\epsilon)$ for $\epsilon > 0$. In addition, suppose that there is an $\epsilon > 0$ such that $\partial D_i(\epsilon) \cap \partial D_j(\epsilon) = \emptyset$ if $i \neq j$.

The objective of this paper is to identify a *continuous* function $p : D \to R^m$ such that the state feedback controller $u = p(x)$ asymptotically stabilizes the system (1) with $x(t) \in D$ for all $t \geq 0$ if $x(0) \in D$.

Remark 1: In this paper, the initial state is required to satisfy $x(0) \in D$, otherwise, the state constraint will be violated at the beginning. The same setting can be found in many state-constrained control studies, e.g., [16], [33], and [35]. With our approach, the closed-loop state trajectory will never enter the unsafe region and therefore the feedback law can be defined only in the safe region. In some other studies, for example, the ZCBF approach [5], [36], the state trajectory is allowed to temporarily enter the unsafe region. In this case, the feedback laws must also be defined outside the safe region.

On the basis of a concept regarding regional stability introduced in [44], the following definition is given:

Definition 1: The system

$$\dot{x} = f(x) \tag{3}$$

is asymptotically stable with D being forward invariant and in the region of attraction if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each $x(0) \in B_D(\delta)$, the trajectory satisfies $x(t) \in B_D(\varepsilon)$ for all $t \geq 0$, and $\lim_{t\to\infty} x(t) \to 0$ for all $x(0) \in D$.

If there exists a differentiable positive definite function $V : D \to R$ satisfying $V(x) \to \infty$ as $\|x\| \to \infty$ or $x \to \partial D$ such that $\nabla V(x) f(x) < 0$ for all $x \in D\backslash\{0\}$, along the trajectory of the system (3) starting at $x(0) \in D$, $\dot{V}(x(t)) = \nabla V(x(t)) f(x(t)) < 0$ if $x(t) \neq 0$. This implies that $V(x(t)) < V(x(0)) < \infty$ for all $t > 0$ and therefore, the state trajectory will not pass through ∂D since $V(x) \to \infty$ as $x \to \partial D$. Then, similar to the traditional Lyapunov stability theory, we can conclude that the system (3) is asymptotically stable with D being forward invariant and in the region of attraction.

For convenience, a function $V : D \to R$ is said to be positive definite in D if $V(x) > 0$ for all $x \in D\backslash\{0\}$ and $V(0) = 0$.

B. PRELIMINARIES

This subsection simply reviews the CLF, the CBF, and the ZCBF that will be used later.

As defined in [31], a differentiable, proper, and positive definite function $V_C : R^n \to R$ is said to be a CLF of the system (1) if

$$\inf_{u\in R^m} \{\nabla V_C(x) f(x) + \nabla V_C(x)\, g(x)\, u\} < 0, \forall x \in R^n\backslash\{0\}. \tag{4}$$

Moreover, a CLF V_C satisfies the *small control property* (SCP) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each nonzero x that satisfies $\|x\| < \delta$, some u with $\|u\| < \varepsilon$ exists such that

$$\nabla V_C(x) f(x) + \nabla V_C(x)\, g(x)\, u < 0.$$

Define

$$\eta(\alpha,\beta,\mu) = -\frac{\alpha + \sqrt{\alpha^2 + \mu\,\|\beta\|^4}}{\|\beta\|^2}\beta^T,$$

where $\alpha, \mu \in R$, and $\beta \in R^m\backslash\{0\}$. Theorem 1 in [31] proves that if a CLF V_C satisfying the SCP exists, then a continuous asymptotically stabilizing controller can be constructed using Sontag's formula:

$$u = p_S(x) = \begin{cases} \eta(a_C(x), b_C(x), 1), & \text{if } b_C(x) \neq 0 \\ 0, & \text{if } b_C(x) = 0, \end{cases} \tag{5}$$

where $a_C(x) = \nabla V_C(x) f(x)$ and $b_C(x) = \nabla V_C(x) g(x)$.

Next, the definitions of the traditional CBF and the ZCBF are provided. Let $\mathcal{X}_u \subset R^n$ be the unsafe region.

Definition 2 [33]: Given a system (1) and a set of unsafe states $\mathcal{X}_u \subset R^n$. A continuously differentiable function $B : R^n \to R$ satisfying

$$B(x) > 0, \quad x \in \mathcal{X}_u, \tag{6a}$$

$$\nabla B(x) f(x) < 0, \quad \forall x \in \{x \in R^n | \nabla B(x)\, g(x)\} = 0, \tag{6b}$$

$$\left\{x \in R^n \,|\, B(x) \leq 0\right\} \neq \emptyset, \tag{6c}$$

is called a CBF of the system (1).

Let the unsafe region $\mathcal{X}_u$ be defined by a continuously differentiable function $h : \mathcal{X} \subset R^n \to R$:

$$\mathcal{X}_u = \left\{x \in R^n \mid h(x) < 0\right\}.$$

The safe region is

$$\mathcal{C} = \mathcal{X}_u^C = \left\{x \in R^n \mid h(x) \geq 0\right\}.$$

A continuous function $\alpha : (-\infty, \infty) \to (-\infty, \infty)$ is said to belong to extended class K if it is strictly increasing and $\alpha(0) = 0$ [36].

Definition 3 [36]: Let $\mathcal{C} \subset \mathcal{X} \subset R^n$ be the superlevel set of a continuously differentiable function $h : \mathcal{X} \to R$, then h is a ZCBF of the system (1) if there exists an extended class K function α such that

$$\sup_{u \in R^m} \{\nabla h(x) f(x) + \nabla h(x) g(x) u\} \geq -\alpha(h(x)), \qquad (7)$$

for all $x \in \mathcal{X}$.

III. CLBF AND STATE-CONSTRAINED STABILIZATION

In this section, on the basis of the CBFs [33], [34], the ZCBF [36], [37], and the classical CLBF [16], a new CLBF method is introduced for solving the considered constrained stabilization problem.

Definition 4: A differentiable function $V : D \to R$ is a CLBF of the system (1) if V is positive definite in D, $V(x) \to \infty$ as $\|x\| \to \infty$ or $x \to \partial D$ and,

$$\inf_{u \in R^m} \{\nabla V(x) f(x) + \nabla V(x) g(x) u\} < 0, \quad \forall x \in D \backslash \{0\}. \qquad (8)$$

Remark 2: The primary difference between traditional CLFs on sets in [45] and [46] and the new CLBFs in Definition 4 is that the new CLBFs exhibit *unbounded growth property* (i.e., $V(x) \to \infty$ as $x \to \partial D$).

Define $a(x) = \nabla V(x) f(x)$ and $b(x) = \nabla V(x) g(x)$. Condition (8) is equivalent to

$$\forall x \in D \backslash \{0\} \text{ such that } b(x) = 0 \Rightarrow a(x) < 0. \qquad (9)$$

In subsequent derivations, that the existence of CLBFs is sufficient for solving the constrained stabilization problem will be demonstrated. Nevertheless, to ensure the continuity of the obtained feedback law at the origin, a CLBF must satisfy the SCP. Similar to CLFs, a CLBF V of the system (1) satisfies the SCP if and only if (note that $0 \in D$) [47]

$$\lim_{\delta \to 0} \sup_{x \in B(\delta)} \frac{a(x)}{\|b(x)\|} \leq 0. \qquad (10)$$

The following result can be obtained.

Lemma 1: If there exists a CLBF $V : D \to R$ that satisfies the SCP for the system (1), then there exists a continuous state feedback controller $u = p(x)$ such that the closed-loop system is asymptotically stable with D being forward invariant and in the region of attraction. In this case,

$$u = p(x) = \begin{cases} \eta(a(x), b(x), \mu), & \text{if } b(x) \neq 0 \\ 0, & \text{if } b(x) = 0 \end{cases} \qquad (11)$$

with $\mu > 0$ is one such controller.

Proof: The proof is similar to that of using the traditional CLF in [31] and is therefore omitted here.

IV. CONSTRUCTIONS OF CLBFs

This section demonstrates how to construct a new CLBF using *barrier functions* for the system (1), provided that a CLF is available.

Definition 5: A differentiable function $V_{Ai} : D \to R$ is an *additive barrier function* with respect to D_i if $V_{Ai}(x) \geq 0$ for all $x \in D$, $V_{Ai}(0) = 0$, and $V_{Ai}(x) \to \infty$ as $x \to \partial D_i$. Moreover, a differentiable function $\rho_{Mi} : D \to R$ is called a *multiplicative barrier function* with respect to D_i if $\rho_{Mi}(x) > 0$ for each $x \in D \backslash \{0\}$, $\rho_{Mi}(0) \geq 0$, and $\rho_{Mi}(x) \to \infty$ as $x \to \partial D_i$.

By the definition, an additive barrier function $V_{Ai}(x)$ must vanish at the origin and be nonnegative elsewhere, and a multiplicative barrier function $\rho_{Mi}(x)$ can be nonzero at the origin and must be positive elsewhere. Consequently, $V_{Ai_1}(x) = \frac{\beta_A(x)}{s_i^q(x)}$, $V_{Ai_2}(x) = \beta_A(x) \ln \frac{1}{s_i^q(x)}$, and

$$V_{Ai_3}(x) = \begin{cases} \dfrac{\beta_A(x)(\varepsilon - s(x))^{2q}}{s_i^q(x)}, & \text{if } x \in \partial D_i(\varepsilon) \\ 0, & \text{if } x \in D \backslash \partial D_i(\varepsilon) \end{cases}$$

are all possible additive barrier functions with respect to D_i, where $q > 0$ is an integer, $\varepsilon > 0$ is a small positive real number, and $\beta_A : \bar{D} \to R$ is a differentiable function that is positive definite in D and satisfies $\inf_{x \in \delta D_i} \beta_A(x) > c$ for some $c > 0$. By contrast, $\rho_{Mi_1}(x) = \frac{\beta_M(x)}{s_i^q(x)}$ and $\rho_{Mi_2}(x) = \exp\left(\frac{\beta_M(x)}{s_i^q(x)}\right)$ are possible multiplicative barrier functions with respect to D_i, where $\beta_M : \bar{D} \to R$ is a differentiable nonnegative function that satisfies $\inf_{x \in \partial D_i} \beta_M(x) > 0$. For example, $\frac{x_1^2 + x_2^2}{3 - x_1}$ (resp. $\frac{1}{3 - x_1}$) is an additive (resp. a multiplicative) barrier function for the constrained region $\left\{x \in R^2 \mid x_1 < 3\right\}$. Notably, if $V_{Ai}(x)$ is an additive barrier function of the system (1) with respect to D_i, then $\alpha V_{Ai}(x)$ is also an additive barrier function with respect to D_i for any $\alpha > 0$. Similarly, if $\rho_{Mi}(x)$ is a multiplicative barrier function of the system (1) with respect to D_i, then $\alpha \rho_{Mi}(x)$ is also a multiplicative barrier function with respect to D_i if $\alpha > 0$.

For each $i \in \{1, 2, \ldots, N\}$, let V_{Ai} (resp., $\rho_{Mi}(x)$) be an additive (resp., multiplicative) barrier function with respect to D_i. For a CLF $V_C(x)$ of the system (1), according to (9), it is clear that the function $V(x) = V_C(x) + \sum_{i=1}^N V_{A_i}(x)$ is a CLBF if for each $x \in D \backslash \{0\}$ such that $\left(\nabla V_C(x) + \sum_{i=1}^N \nabla V_{A_i}(x)\right) g(x) = 0$, one has

$$\left(\nabla V_C(x) + \sum\nolimits_{i=1}^N \nabla V_{A_i}(x)\right) f(x) < 0. \qquad (12)$$

Similarly, the function $V(x) = V_C(x) \cdot \prod_{i=1}^N \rho_{M_i}(x)$ is a CLBF if for each $x \in D \backslash \{0\}$ such that

$\left(V_C(x)\cdot\nabla(\prod_{i=1}^{N}\rho_{M_i}(x))+\nabla V_C(x)\cdot\prod_{i=1}^{N}\rho_{M_i}(x)\right)g(x)=0$, one has

$$\left(V_C(x)\cdot\nabla(\prod_{i=1}^{N}\rho_{M_i}(x))+\nabla V_C(x)\cdot\prod_{i=1}^{N}\rho_{M_i}(x)\right) f(x)<0. \quad (13)$$

All additive and multiplicative barrier functions mentioned above can be used to construct CLBFs. The additive barrier function $V_{Ai_3}(x)$ is found convenient in discussing the existence of CLBFs. Next, with the help of $V_{Ai_3}(x)$, sufficient conditions for the existence of CLBFs for the system (1) will be proposed. For convenience, for $i=1,\ldots,N$, define the following:

$$Z_{sgi}\equiv\left\{x\in\bar{D}_i\cap\bar{D}\,|\,\nabla s_i(x)\,g(x)=0\right\},$$
$$C_{Mi}\equiv\left\{x\in Z_{sgi}\,|\,\nabla V_C(x)\,g(x)=0\right\},$$
$$C_{Li}\equiv\left\{x\in\bar{D}_i\cap\bar{D}\backslash Z_{sgi}\ |\text{there is a }\gamma>0\text{ such that }\nabla V_C(x)\,g(x)=\gamma\nabla s_i(x)g(x)\right\}.$$

Theorem 1: Suppose that $C_{Mi}\cap\partial D_i$ and $C_{Li}\cap\partial D_i$ are bounded for each $i=1,2\ldots,N$. Assume $\{0\}\notin\partial D(\epsilon)\cup D^C$ and $\partial D_i(\epsilon)\cap\partial D_j(\epsilon)=\emptyset$ if $i\neq j$ for some $\epsilon>0$. If $V_C:R^n\rightarrow R$ is a CLF for the system (1) that satisfies the SCP such that for $i=1,\ldots,N$,

a). $\nabla s_i(x)f(x)>0$ for any $x\in Z_{sgi}\cap\partial D_i$,

b). $\frac{\nabla s_i(x)f(x)}{\|\nabla s_i(x)g(x)\|}-\frac{\nabla V_C(x)f(x)}{\|\nabla V_C(x)g(x)\|}>0$ for any $x\in C_{Li}\cap\partial D_i$,

then there exists a CLBF that satisfies the SCP for the system (1).

Proof: Define

$$V(x;\epsilon)=\begin{cases}V_C(x)+\dfrac{(\epsilon-s_i(x))^2}{s_i(x)},\text{ if }x\in\partial D_i(\epsilon)\cap D,\\ \qquad i=1,\ldots,N;\\ V_C(x),\text{ if }x\in D\backslash\bigcup_{i=1}^{N}\partial D_i(\epsilon).\end{cases}\quad (14)$$

Clearly, $V(x;\epsilon)$ is differentiable and positive definite in D. We first demonstrate that for a sufficiently small ϵ, $\inf_u\nabla V(x;\epsilon)(f(x)+g(x)u)<0$ for each $x\in D\backslash\{0\}$.

Case 1: $x\in D\backslash\bigcup_{i=1}^{N}\partial D_i(\epsilon)$.

In this region, $V(x;\epsilon)=V_C(x)$. Because $V_C(x)$ is a CLF of system (1), one can see that for each nonzero $x\in D\backslash\bigcup_{i=1}^{N}\partial D_i(\epsilon)$,

$$\inf_u\nabla V(x;\epsilon)(f(x)+g(x)u)=\inf_u\nabla V_C(x)(f(x)+g(x)u)<0.$$

Case 2: $x\in\partial D_i(\epsilon)\cap D$ for $i\in\{1,2,\ldots,N\}$.

In this region,

$$\begin{aligned}\nabla V(x;\epsilon)(f(x)+g(x)u)&=\left(\nabla V_C(x)-\frac{\epsilon^2-s_i^2(x)}{s_i^2(x)}\nabla s_i(x)\right)(f(x)+g(x)u)\\&=\frac{1}{s_i^2(x)}\left(a_{V_C}(x)-a_{s_i}(x;\epsilon)\right)\\&\quad+\frac{1}{s_i^2(x)}\left(b_{V_C}(x)-b_{s_i}(x;\epsilon)\right)u,\end{aligned}\quad (15)$$

where

$$a_{V_C}(x)=s_i^2(x)\nabla V_C(x)f(x),$$
$$a_{s_i}(x;\epsilon)=\left(\epsilon^2-s_i^2(x)\right)\nabla s_i(x)f(x),$$
$$b_{V_C}(x)=s_i^2(x)\nabla V_C(x)\,g(x),$$
$$b_{s_i}(x;\epsilon)=\left(\epsilon^2-s_i^2(x)\right)\nabla s_i(x)\,g(x).$$

Let $C_{Zi}(\epsilon)\equiv\{x\in\partial D_i(\epsilon)\cap D|\nabla V(x;\epsilon)g(x)=0\}$, $i=1,2,\ldots,N$. Clearly, $C_{Zi}(\epsilon)\subset(C_{Mi}\cup C_{Li})\cap\partial D_i(\epsilon)$. For $V(x;\epsilon)$ to be a CLBF, $\nabla V(x;\epsilon)f(x)$ must be negative for each $x\in C_{Z_i}(\epsilon)$. Notice that $\partial D_i(\epsilon)$ and thus $C_{Z_i}(\epsilon)$ shrinks as ϵ becomes smaller. Under condition a), as $C_{Mi}\cap\partial D_i$ is bounded, an $\epsilon_{1i}>0$ exists such that for each ϵ satisfying $0<\epsilon<\epsilon_{1i}$, $a_{s_i}(x;\epsilon)>0$ if $x\in C_{Mi}\cap\partial D_i(\epsilon)$. In this case, for each $x\in C_{Zi}(\epsilon)\cap C_{Mi}$,

$$\nabla V(x;\epsilon)f(x)=\frac{1}{s_i^2(x)}\left(a_{V_C}(x)-a_{s_i}(x;\epsilon)\right)<0.$$

For $x\in C_{Li}$, define

$$K_i(x)=\frac{\|\nabla s_i(x)\,g(x)\|}{\|\nabla V_C(x)\,g(x)\|}.$$

Then,

$$K_i(x)\nabla V_C(x)\,g(x)=\nabla s_i(x)g(x)$$

and

$$\nabla V(x;\epsilon)\,g(x)=\frac{\nabla V_C(x)\,g(x)}{s_i^2(x)}\times\left(s_i^2(x)+K_i(x)(s_i^2(x)-\epsilon^2)\right).$$

Therefore, for $x\in C_{Li}\cap C_{Zi}(\epsilon)$, $\nabla V(x;\epsilon)\,g(x)=0$ and thus

$$s_i^2(x)=K_i(x)(\epsilon^2-s_i^2(x)).\quad (16)$$

Similarly, because $C_{Li}\cap\partial D_i$ is bounded and $\partial D_i(\epsilon)$ and $C_{Zi}(\epsilon)$ shrink as ϵ becomes smaller, according to condition b), there exists an $\epsilon_{2i}>0$ such that for each ϵ that satisfies $0<\epsilon<\epsilon_{2i}$,

$$\frac{\nabla s_i(x)f(x)}{\|\nabla s_i(x)\,g(x)\|}-\frac{\nabla V_C(x)f(x)}{\|\nabla V_C(x)\,g(x)\|}>0\text{ for }x\in C_{Zi}(\epsilon)\cap C_{Li},$$

and thus

$$K_i(x)\nabla V_C(x)f(x)-\nabla s_i(x)f(x)<0.$$

Consequently, according to (16), for each $x\in C_{Li}\cap C_{Zi}(\epsilon)$,

$$\begin{aligned}&\nabla V(x;\epsilon)f(x)\\&=\nabla V_C(x)f(x)-\frac{\epsilon^2-s_i^2(x)}{s_i^2(x)}\nabla s_i(x)f(x)\\&=\frac{\epsilon^2-s_i^2(x)}{s_i^2(x)}\left(K_i(x)\nabla V_C(x)f(x)-\nabla s_i(x)f(x)\right)\\&<0.\end{aligned}$$

The previous discussion shows that for $\epsilon<\min_i\min\{\epsilon_{1i},\epsilon_{2i}\}$, $\inf_u\nabla V(x;\epsilon)(f(x)+g(x)u)<0$ for each

$x \in D\backslash\{0\}$; therefore, $V(x;\epsilon)$ is a CLBF of the system (1). Notice that $V(x;\epsilon)$ satisfies the SCP is obvious because $V_C(x)$ satisfies the SCP and $\{0\} \in D\backslash\partial D(\epsilon)$.

With a CLBF $V: D \to R$ satisfying (8), a state-constrained stabilizing controller can be constructed by (11). However, the control signal generated by (11), with $\mu > 0$ being a constant, maybe extremely large while the state closes to the boundary of the constrained region. For example, for a point $\hat{x} \in \partial D$ satisfying $\lim_{x\to\hat{x}} \|b(x)\| \to \infty$ and $\lim_{x\to\hat{x}} \|a(x)\| = c$ for some constant $c > 0$, one has $\lim_{x\to\hat{x}} \|p(x)\| \to \infty$. Therefore, the feedback law (11) must be modified to prevent this situation.

Corollary 1: Suppose that $C_{Mi} \cap \partial D_i$ and $C_{Li} \cap \partial D_i$ are bounded for each $i = 1, 2 \ldots, N$. Assume $\{0\} \notin \partial D(\epsilon) \cup D^C$ and $\partial D_i(\epsilon) \cap \partial D_j(\epsilon) = \emptyset$ if $i \neq j$ for some $\epsilon > 0$. Let $V_C: R^n \to R$ is a CLF for system (1) that satisfies the SCP and such that conditions a) and b) of Theorem 1 hold. With $V(x;\epsilon)$ defined in (14) being a CLBF of the system (1), a state-constrained stabilizing controller can be constructed as

$$u = p(x) = \begin{cases} \eta\left(a(x), b(x), \dfrac{k(x)}{k(x) + \|b(x)\|^2}\right), & \text{if } b(x) \neq 0 \\ 0, & \text{if } b(x) = 0 \end{cases} \tag{17}$$

where

$$a(x) = \begin{cases} \left(\nabla V_C(x) - \dfrac{\epsilon^2 - s_i^2(x)}{s_i^2(x)} \nabla s_i(x)\right) f(x), \\ \qquad \text{if } x \in \partial D_i(\epsilon) \cap D \\ \nabla V_C(x) f(x), \quad \text{if} x \in D\backslash \bigcup_{i=1}^N \partial D_i(\epsilon), \end{cases}$$

$$b(x) = \begin{cases} \left(\nabla V_C(x) - \dfrac{\epsilon^2 - s_i^2(x)}{s_i^2(x)} \nabla s_i(x)\right) g(x), \\ \qquad \text{if } x \in \partial D_i(\epsilon) \cap D \\ \nabla V_C(x) g(x), \quad \text{if} x \in D\backslash \bigcup_{i=1}^N \partial D_i(\epsilon), \end{cases}$$

and $k: D \to R$ is a continuous function with $k(x) > 0$ for all $x \in D$. Moreover, for each bounded $\hat{x} \in \partial D$, with $\{x^j\}$ being a sequence of states in D that approaches $\hat{x}$,

$$\lim_{j\to\infty} \left\| p\left(x^j\right) \right\| < \infty.$$

Proof: It is not difficult to verify that controller (17) is continuous and stabilizes the system (1) under the state constraint $x(t) \in D$. The only thing that needs to be proven is $\lim_{j\to\infty} \|p(x^j)\| < \infty$.

Here that $\lim_{j\to\infty} \frac{a(x^j)}{\|b(x^j)\|} < \infty$ will first be shown. In the case of $\hat{x} \in \partial D_i \cap Z_{sgi}$ for some i, according to condition a) and noting that $\lim_{j\to\infty} s_i(x^j) = 0$ and $\lim_{j\to\infty} \nabla s_i(x^j) g(x^j) = 0$, one has

$$\begin{aligned} &\lim_{j\to\infty} \frac{a(x^j)}{\|b(x^j)\|} \\ &= \lim_{j\to\infty} \frac{s_i^2(x^j)\nabla V_C(x^j) f(x^j) - (\epsilon^2 - s_i^2(x^j)) \nabla s_i(x^j) f(x^j)}{\|s_i^2(x^j)\nabla V_C(x^j) g(x^j) - (\epsilon^2 - s_i^2(x^j)) \nabla s_i(x^j) g(x^j)\|} \\ &= \lim_{j\to\infty} \frac{-\nabla s_i(x^j) f(x^j)}{\|\nabla s_i(x^j) g(x^j)\|} < \infty. \end{aligned}$$

In the case of $\hat{x} \in \partial D_i \backslash Z_{sgi}$, for some $i \in \{1, 2, \ldots, N\}$, that $\lim_{j\to\infty} \|\nabla s_i(x^j) g(x^j)\| \neq 0$ is clear and then

$$\begin{aligned} \lim_{j\to\infty} \frac{a(x^j)}{\|b(x^j)\|} &= \lim_{j\to\infty} \frac{-\nabla s_i(x^j) f(x^j)}{\|\nabla s_i(x^j) g(x^j)\|} \\ &= \frac{-\nabla s_i(\hat{x}) f(\hat{x})}{\|\nabla s_i(\hat{x}) g(\hat{x})\|} < \infty. \end{aligned}$$

Therefore, $\lim_{j\to\infty} \frac{a(x^j)}{\|b(x^j)\|} < \infty$.

Now, that $\lim_{j\to\infty} \|p(x^j)\| < \infty$ will be proven. Since $\lim_{j\to\infty} \frac{a(x^j)}{\|b(x^j)\|} < \infty$ and $k(x^j) > 0$, one has

$$\begin{aligned} &\lim_{j\to\infty} \left\| p\left(x^j\right) \right\| \\ &= \lim_{j\to\infty} \left\{ \frac{a(x^j)}{\|b(x^j)\|} + \sqrt{\frac{a^2(x^j)}{\|b(x^j)\|^2} + \frac{k(x^j)\|b(x^j)\|^2}{k(x^j) + \|b(x^j)\|^2}} \right\} \\ &\leq \lim_{j\to\infty} \left\{ \frac{a(x^j)}{\|b(x^j)\|} + \sqrt{\frac{a^2(x^j)}{\|b(x^j)\|^2} + k(x^j)} \right\} < \infty. \end{aligned}$$

This completes the proof.

Remark 3: From Nagumo's theorem [1], one only needs to check the set invariance property by checking the conditions on the boundaries of the set. In [35], the Nagumo's theorem has been used to prove the forward invariance of $\mathcal{C}$ by the condition for defining zeroing barrier functions. Similar to the concept of Nagumo's theorem, the conditions a) and b) of Theorem 1 only need to be checked for some points on the boundaries of the constrained region D.

Notice that for function s_i satisfying condition a), it becomes a ZCBF of system (1) for the region D_i. While, condition b) is given for guaranteeing $a(x) < 0$ if $b(x) = 0$ for $x \in \partial D$. In the case that $C_{Li} \cap \partial D_i = \emptyset$, condition b) needs not to be checked. It should be noted also that conditions a) and b) are sufficient, but not necessary, for the existence of new CLBFs. If condition a) or condition b) does not hold, one might still be able to find a CLBF satisfying the condition (8). To sum up, the proposed new CLBF approach can be used in state-constrained stabilization problems with particular constrained regions that some existing methods are not applicable.

Remark 4: Deriving CLFs for general nonlinear control systems is a difficult task and is not always possible. Nevertheless, CLFs for nonlinear control systems in some particular forms (strict feedback form, canonical form, etc.) can be derived systematically. Moreover, some general nonlinear control systems can be transformed to the strict feedback form [48], [49]. Our results can be easily applied to these systems for achieving state-constrained stabilization. For a general nonlinear control system that cannot be transformed into a particular structure, how to find a CLF is still an open problem.

Remark 5: The assumption that $\partial D_i(\epsilon) \cap \partial D_j(\epsilon) = \emptyset$ is a restriction of our approach. This assumption implies that the trajectories can close to only one boundary at one time. In the case that $\partial D_i \cap \partial D_j \neq \emptyset$, for a point $\hat{x} \in \partial D_i \cap \partial D_j$, conditions a) and b) of Theorem 1 cannot guarantee $\inf_u \nabla V(x;\epsilon)(f(x)+g(x)u) < 0$ near $\hat{x}$, and therefore cannot ensure the existence of CLBFs.

Remark 6: The size of ϵ plays an essential role in the control signal near ∂D. The control signal will change rapidly near the boundary as ϵ approaches zero but still be smooth. The size of ϵ depends on the relations between individual unsafe sets and the origin. Under conditions a) and b), one can choose a small enough ϵ to satisfy $\partial D_i(\epsilon) \cap \partial D_j(\epsilon) = \emptyset$ and the conditions of CLBF, and then a state constrained controller can be constructed. While, how to explicitly determine the value of ϵ to satisfy all the conditions is a difficult problem as the considered system is nonlinear and the functions s_i, $i = 1, .., N$, are also nonlinear. For low-order systems, its value can be determined with the help of some graphing software.

V. REPRESENTATIVE EXAMPLES

To illustrate the obtained theoretical results, simulations of two examples are provided by MATLAB® in the following.

Example 1:

Consider the following nonlinear control system:

$$\begin{cases} \dot{x}_1 = 10x_1 - x_1x_2^2 + 0.5x_1^2 + 0.5x_1u_1 + 0.5u_2 \\ \dot{x}_2 = -x_2 + 0.1x_1^2 + 3x_1^2x_2 - x_1x_2u_1. \end{cases} \tag{18}$$

Define $s_1(x) = (x_1-4)^2 - (x_2-1)^2 + 1$ and $s_2(x) = -x_1 - 4x_2 + 10$. In this example, a state feedback law $u = p(x)$ will be derived to stabilize the system (18) under the state constraint:

$$x(t) \in D \equiv D_1 \cap D_2, \tag{19}$$

where $D_i = \{x \in R^2 \mid s_i(x) > 0\}, i = 1, 2.$

It is easy to verify that $V_C(x) = x_1^2 + x_2^2/2$ is a CLF that satisfies the SCP of system (18). Then,

$$\nabla V_C(x) f(x) = -x_2^2 + x_1^2\left(x_1 + x_2^2\right) + 20x_1^2 + 0.1x_1^2x_2,$$
$$\nabla V_C(x) g(x) = \left[\, x_1(x_1 - x_2^2) \;\; x_1 \,\right],$$
$$\nabla s_1(x) f(x) = 2(x_1-4)\left(-x_1x_2^2 + 0.5x_1^2 + 10x_1\right) - 2(x_2-1)(-x_2 + 0.1x_1^2 + 3x_1^2x_2),$$
$$\nabla s_1(x) g(x) = \left[\, (x_1-4)x_1 + 2(x_2-1)x_1x_2 \;\; x_1 - 4 \,\right],$$
$$\nabla s_2(x) f(x) = -\left(-x_1x_2^2 + 0.5x_1^2 + 10x_1\right) - 4\left(-x_2 + 0.1x_1^2 + 3x_1^2x_2\right),$$
$$\nabla s_2(x) g(x) = \left[\, -0.5x_1 + 4x_1x_2 \;\; -0.5 \,\right].$$

It can be demonstrated that $Z_{sg1} \cap \partial D_1 = (4, 0)$, $Z_{sg2} \cap \partial D_2 = \emptyset$, $C_{L1} \cap \partial D_1 = \emptyset$, and $C_{L2} \cap \partial D_2 = \emptyset$; additionally, $\nabla s_1(x) f(x) = 3.2 > 0$ at $(4, 0)$. That is, the conditions in Theorem 1 hold; therefore, there exists a CLBF that satisfies the SCP for the system (18). One can demonstrate that

$$V(x;\epsilon) = \begin{cases} V_C(x) + \dfrac{(\epsilon - s_i(x))^2}{s_i(x)}, & \text{if } x \in \partial D_i(\epsilon) \cap D \\ V_C(x), & \text{if } x \in D \backslash \bigcup_{i=1}^{2} \partial D_i(\epsilon) \end{cases}$$

is a CLBF of the system (18) for a sufficiently small ϵ. For example, $\epsilon = 0.8$ can verify that condition (8) holds and thus that $V(x)$ is a CLBF. A state-constrained stabilizing controller can be constructed by (17) with $N = 2$ and $k(x) = 0.1 + 0.1\|x\|$. Fig. 1 presents the state trajectories of the system (18) controlled by three different controllers. The blue curves in Fig. 1 are the state trajectories of the resultant closed-loop system (18)-(17) with $\epsilon = 0.8$ starting at several different initial states. Clearly, the closed-loop system is asymptotically stable and satisfies the state constraint (19). The red curves are the state trajectories of the closed-loop system (18)-(17) with $\epsilon = 0.1$. In this case, the trajectories can be extremely close to ∂D but will never reach D^C. Regardless of whether $\epsilon = 0.8$ or $\epsilon = 0.1$, because CLBFs are used to construct the controllers, the state constraint has not been violated. For comparison, the closed-loop trajectories of the system (18) controlled by the controller (5) are shown in Fig. 1. The green curves in Fig. 1 are the trajectories of the closed-loop system

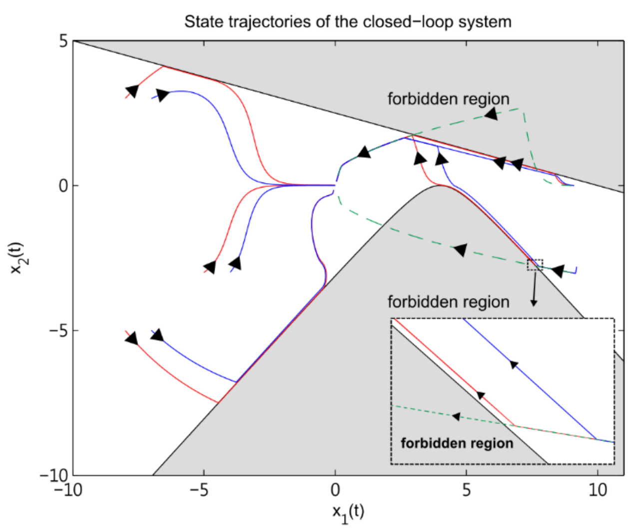

FIGURE 1. State trajectories of the closed-loop system (blue and red lines: controlled by controller (17); green lines: controlled by controller (5)).

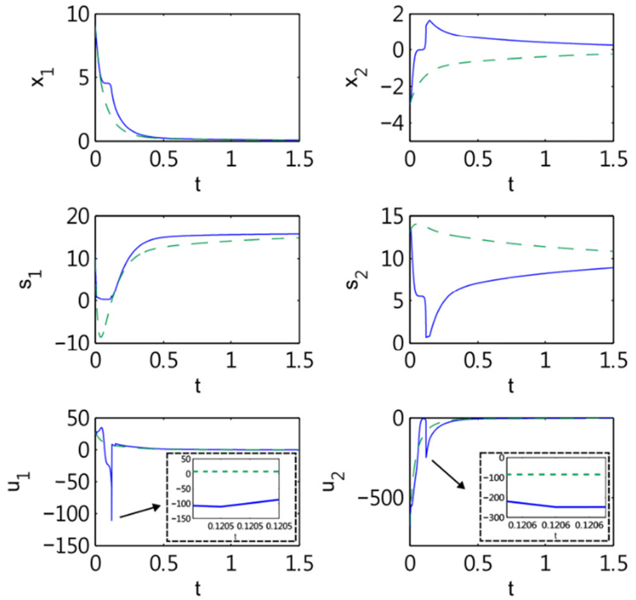

FIGURE 2. Responses of the closed-loop system with $x(0) = [9\ -3]^T$ (blue lines: controlled by controller (17); green lines: controlled by controller (5)).

(18)-(5). In this case, the state trajectories enter the unsafe region because the CLF $V_C(x)$ is used for constructing the controller but not the CLBF $V(x)$. The enlarged plot of the dashed box in Fig. 1 is to clarify the difference.

To demonstrate the responses of the closed-loop systems controlled by the controller (17) and the controller (5), Fig. 2 shows the responses of $x(t)$, $s_1(x(t))$, $s_2(x(t))$, and $u(t)$ of the closed-loop systems with the same initial condition $x(0) = \begin{bmatrix} 9 & -3 \end{bmatrix}^T$. Fig. 2 indicates that the value of $s_1(x(t))$ is always positive for the system controlled by controller (17). By contrast, the value of $s_1(x(t))$ is negative in some time interval for the system controlled by controller (5). That is, the state constraint (19) is violated in this case. In Fig. 2, some spike-like behaviors occur in the control signal. The rapid change of the control signal near the boundary of the safe region is due to the particular way of constructing CLBF in this example. The value of the barrier function is nonzero only in a neighborhood of the boundary of the constrained region and tends to infinity as the boundary is approached. Therefore, the control signal changes rapidly near the boundary as the neighborhood is small ($\epsilon = 0.8$ in this example). By the local enlarged plots in Fig. 2, one can see clearly the variations of control signals when the state trajectory approaches the boundary of the constrained region.

Example 2:

To show the practical applicability of the new CLBF approach, consider the following single-link robot arm system [50]:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{mgl}{J}\sin x_1 - \dfrac{d}{J}x_2 + \dfrac{1}{J}u, \end{cases} \tag{20}$$

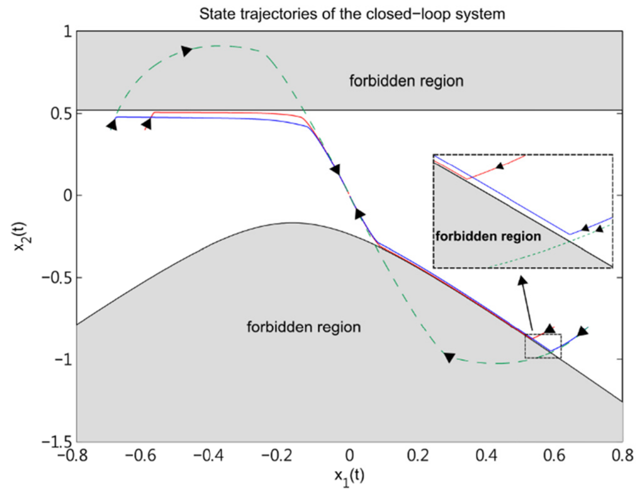

FIGURE 3. State trajectories of the closed-loop system (blue and red lines: controlled by controller (17); green lines: controlled by controller (5)).

where x_1 and x_2 denote the angle and angular velocity of the arm, $u \in R$ denotes the control input, g is the gravitational constant, and l, m, J, d denote the length, mass, inertia, and damping of the arm, respectively. The values of parameters are chosen as those in [50]: $m = 1, J = 1, d = 2$, $l = 0.5$m, and $g = 9.8\text{m/s}^2$. Define $s_1(x) = 0.52 - x_2$ and $s_2(x) = (3x_1 + 0.5)^2 - (2x_2 - 0.5)^2 + 0.7$. In this example, a state feedback law $u = p(x)$ will be found to stabilize the system (20) under the state constraint:

$$x(t) \in D \equiv D_1 \cap D_2, \tag{21}$$

where $D_i = \left\{x \in R^2 \mid s_i(x) > 0\right\}, i = 1, 2$.

That $V_C(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + 3x_1)^2$ is a CLF satisfying the SCP for system (20) is easy to verify. Therefore,

$$\begin{aligned} \nabla V_C(x) f(x) &= -3x_1^2 + 3(x_2 + 3x_1)x_2 \\ &\quad - (x_2 + 3x_1)(4.9\sin x_1 + 2x_2), \\ \nabla V_C(x) g(x) &= x_2 + 3x_1, \\ \nabla s_1(x) f(x) &= 4.9\sin x_1 + 2x_2, \\ \nabla s_1(x) g(x) &= -1, \\ \nabla s_2(x) f(x) &= 18x_1x_2 + 3x_2 \\ &\quad + (8x_2 - 2)(4.9\sin x_1 + 2x_2), \\ \nabla s_2(x) g(x) &= -8x_2 + 2. \end{aligned}$$

It can be shown that $Z_{sg1} \cap \partial D_1 = \emptyset$, $Z_{sg2} \cap \partial D_2 = \emptyset$ and

$$\begin{aligned} C_{L1} \cap \partial D_1 &= \{x \in \partial D_1 \mid x_2 + 3x_1 < 0\}, \\ C_{L2} \cap \partial D_2 &= \{x \in \partial D_2 \mid \frac{x_2 + 3x_1}{-8x_2 + 2} > 0. \end{aligned}$$

Also, it can be verified that conditions a) and b) in Theorem 1 hold and therefore, there exists a CLBF that satisfies the SCP for the system (20). In fact, the function $V(x; \epsilon)$ defined in (14) with $N = 2$ will be a CLBF for a sufficiently small ϵ, for example, $\epsilon = 0.1$. Then, a state-constrained stabilizing controller can be constructed by (17) with $k(x) = 0.1 + 0.1\|x\|$. Fig. 3 presents the state trajectories of system (20) controlled by three different controllers. The blue curves in Fig. 3 are the state trajectories of the closed-loop system (20)-(17) with $\epsilon = 0.1$, starting at several different initial states. The red curves are the state trajectories of the closed-loop system (20)-(17) with $\epsilon = 0.03$. In these two cases, the closed-loop systems are asymptotically stable and satisfy the state constraint (21). On the other hand, green curves in Fig. 3 are the trajectories of the closed-loop system (20)-(5). One can see that the state trajectories enter unsafe regions.

VI. CONCLUSION

This paper introduces a new CLBF method for designing asymptotically stabilizing state feedback controllers for nonlinear control-affine systems under multiple functional-inequality state constraints. Sufficient conditions for the existence of CLBFs were derived, and the construction of CLBFs was discussed. Possible extensions of the proposed approach include the state-constrained optimal control problem, the state-constrained robust control problem, state-constrained switched control systems, and state-constrained networked control systems.

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