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Observer Design for Lipschitz Nonlinear Parabolic PDE Systems With Unknown Input

TENG-FEI LI
School of Information Science and Engineering, Wuhan University of Science and Technology, Wuhan 430081, China
e-mail: acelitengfei@163.com

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ABSTRACT In this article, a novel method to design the observer for a class of uncertain Lipschitz nonlinear parabolic partial differential equations (PDE) systems is investigated. First, the observer and the dynamic errors with undetermined parameters for the parabolic PDE systems subject to appropriate boundary conditions are presented. The conditions of the designed observer are involved. Then the analysis of asymptotic stability and $\mathcal{H}_\infty$ performance conditions for the observer design of uncertain nonlinear parabolic PDE systems are studied and presented in terms of matrix inequalities based on the Lyapunov stability theory. Finally, the effectiveness of the proposed method is validated by a numerical parabolic PDE system.

INDEX TERMS Parabolic PDE, $\mathcal{H}_\infty$ observer design, uncertain input, asymptotic stability.

I. INTRODUCTION

The higher requirements in reliability and safety are often put forward in the more and more complicated construction of industrial systems. Unfortunately, unexpected deviation of characteristic properties or uncertainties produced by external disturbances (such as, pulse interference, noise influence, temperature change, and so on) and internal disturbances (such as, measurement errors, device degradation, inaccurate modeling, *et al.*) which may degenerate the performance or even lead to instability of the system, always occur inherently. In order to cope with these uncertain systems which are costly task or not be able to be measured, it is paramount to estimate the state variables as they are helpful in the system analysis and synthesis. In this scenario, state estimation has been an important problem in the field of modern control theory and appealed to many researchers for a long time. The estimation of the state variables serves as a powerful tool to improve the realization about the system concerned. Thus, system state reconstruction will be seriously affected once these unexpected signals are not processed correctly. Over the last few decades, various methods were developed for the observer design of systems with uncertainties. See, for example, [1]–[6] and the references therein.

Moreover, $\mathcal{H}_\infty$ control problem is usually associated with the observer design and used to analyze the robustness of the observer for the system in the presence of unknown input. For example, Darouach *et al.* [7] investigated the $\mathcal{H}_\infty$ observer design strategy for Lipschitz nonlinear singular systems by using the parameterization of the algebraic constraints which are derived from the estimation errors. References [8]–[10] were concerned with the $\mathcal{H}_\infty$ filtering problems for fuzzy systems, discrete-time systems, and linear systems in the present of uncertainties. For more researches, one can refer to [11]–[13] and the references therein.

In science and engineering fields, such as fluid heat exchangers, thermal diffusion processes, and dissipative dynamical systems etc. [14], [15], these process models can be frequently expressed by PDE with boundary conditions. Moreover, multidimensional dynamical systems depend on space inherently, so the properties of these systems depend on spatial position as well as time variable, which is able to be frequently formulated by nonlinear parabolic PDE or evolutionary type equations. Generally, the model of parabolic PDE control system in mathematical could be modeled as:

$$\frac{\partial x(s,t)}{\partial t} = A_1(s)\frac{\partial^2 x(s,t)}{\partial s^2} + A_2(s)\frac{\partial x(s,t)}{\partial s} + f(x(s,t),s) + G(s)u(t) + D(s)w(t), \quad (1)$$

where t and s are the time and spatial position variable, $x(\cdot,t) = [x_1(s,t)\ x_2(s,t)\ \cdots\ x_n(s,t)]^T \subseteq D$ represents

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the system state, $u(t)$ denotes the control input, $\omega(t)$ is an exogenous disturbance which contains both system and measurement noise, $A_1(s)$, $A_2(s)$, $G(s)$ and $D(s)$ are known matrix functions of s, $f(x(s,t))$ is a sufficiently smooth nonlinear function and is assumed to be differentiable, D is a given domain which contains the equilibrium state of system, $\frac{\partial x(s,t)}{\partial s}$ and $\frac{\partial^2 x(s,t)}{\partial s^2}$ are respectively the first order partial derivatives and the second order partial derivatives of $x(s,t)$ with respect to s.

Parabolic PDE, a significant branch of PDE that can be found in [16]–[19] and so on, has been deeply studied by the researchers in the last few decades. These research achievements certainly promoted the progress of human research, and also provided the theoretical basis both for the applications in real-world and further research. With development of mathematical theory and the applied mathematics, many scholars devote themselves to research the observer design method for system (1). The difficulty to design the observer for parabolic PDE systems lies in the second order partial derivatives respect to the spatial position variable which is hard to cope with. Generally, the main design methods in these existing literatures can be divided into two types: by replacing the partial derivatives of parabolic PDE system to other forms which are easy to handle with by using the mathematical techniques (such as, with the aid defining a linear unbounded operator $\mathcal{A}$ satisfies $\mathcal{A}x(s,t) = A_1(s)\frac{\partial^2 x(s,t)}{\partial s^2} + A_2(s)\frac{\partial x(s,t)}{\partial s}$, a Luenberger-type PDE observer was initial developed to exponentially track the state of linear distributed parameter system in [20]; boundary control for a kind of semi-linear parabolic PDE systems in [21]; the design of robust adaptive neural observer was investigated for parabolic PDE systems which contain unknown nonlinearities and bounded disturbances by the technique of modal decomposition in [22]; for more researches, we recommend our readers to see [23], [24]), or design the observer directly (such as, based on the description of ODE (ordinary differential equation)-PDE model, adaptive observer which relies on the constraints of a first order hyperbolic that without parameter uncertainty of the PDE was presented for diffusion parabolic PDE system in [25]; [26] presented online estimates strategy for the state vector of a finite-dimensional ODE which is nonlinear with the structure of strict-feedback and the infinite-dimensional state of a linear parabolic PDE with the technic of boundary measurement sampling; Wang *et al.* [27] investigated the $\mathcal{H}_\infty$ state estimation for the T-S fuzzy model of a class of nonlinear PDE system with the technic of spatially local averaged measurements, which can guarantee the exponential stability and satisfy an $\mathcal{H}_\infty$ performance for the estimation error fuzzy PDE system; and [28] proposed an interval parabolic PDE observer with the constraint of nonnegative values of boundary and initial conditions, estimator-based $\mathcal{H}_\infty$ sampled-data fuzzy control and observer-based fuzzy fault-tolerant control can be respectively seen in [29], [30] for nonlinear parabolic PDE systems; for more researches, one can refer to [31], [32]). The above findings are significant in the field of observer design and inspire other scholars to new insight and perspective in the more in-depth studies.

Motivated and inspired by the achievements mentioned above, the purpose of this paper is to investigate the observer design method for a class of parabolic PDE systems with unknown input or disturbance. Meanwhile, we dedicate to derive the sufficient conditions of asymptotic stability for the design method. The contributions of this paper can be identified as the following parts:

1) A novel method to design the observer for a class of uncertain nonlinear parabolic PDE systems is presented. The conditions of the existence and asymptotic stability of observer is studied under $w(t) = 0$.

2) $\mathcal{H}_\infty$ observer design is investigated for the uncertain nonlinear parabolic PDE systems with $w(t) = 0$ and $\|e(s,t)\|_2 < \mu\|w(t)\|_2$ for $w(t) \neq 0$. Asymptotic stability are explored and design conditions is also derived in the form of matrix inequalities.

The remaining parts of this article is organized as: Section II describes the model of the Lipschitz nonlinear parabolic PDE system with unknown inputs, and introduces the assumptions and lemmas which are needed in this paper; Section III gives the observer design method and sufficient conditions of the existence of the observer for PDE system with unknown inputs; $\mathcal{H}_\infty$ observer design for the proposed systems with $\|e(s,t)\|_2 < \mu\|w(t)\|_2$ for $w(t) \neq 0$ is studied and sufficient design conditions are also derived in the form of matrix inequalities in Section IV; A numerical example is developed to show the merit and effectiveness of the proposed design method in Section V; Section VI concludes the paper.

Notations: Some necessary notations are needed in this paper. $\mathcal{R}$, $\mathcal{R}^+$ $\mathcal{R}^n$ and $\mathcal{R}^{m\times n}$ stand for the set of all real numbers, positive numbers, n-dimension Euclidean space, and all real matrices of dimension $m \times n$. Matrix (vector) A^T denote the transpose of the matrix (vector) A. $\mathcal{W}^{1,\bar{n}}([0,l];\mathcal{R}^n)$ is a Sobolev space of absolutely continuous vector functions $g(x) : [0,l] \to \mathcal{R}^n$ with the property of square integrable derivatives $d^{\bar{n}}g(x)/dx^{\bar{n}}$ of the order $\bar{n} \geq 1$ and the norm $\|g(\cdot)\|_{\mathcal{W}^{1,\bar{n}}} = \sqrt{\int_0^l \sum_{i=0}^{\bar{n}} \frac{dg^T(x)}{dx} \cdot \frac{dg(x)}{dx} dx}$. $\|\cdot\|_2$ means the Euclidean norm of vectors or matrices. Identity matrix is denoted by I with appropriate dimension. A symmetric matrix $A > (<, \leq) 0$ means that A is positive definite (negative definite, negative semi-definite). The symbol "$*$" is used for standing the symmetry part of a matrix, e.g.,

$$\begin{bmatrix} A + [B + C + *] & D \\ * & E \end{bmatrix} \triangleq \begin{bmatrix} A + [B + C + B^T + C^T] & D \\ D^T & E \end{bmatrix}.$$

II. PROBLEM FORMULATION AND PRELIMINARIES

In this section, we shall introduce the Assumption and Lemma which are needed, and the systems studied in this paper.

Assumption 1: Let X be a normed space and $K \subset X$ be a nonempty subset. Then an operator $f : K \to K$ is said to be

Lipschitz constraint if there exists a constant $\gamma > 0$ such that

$$\|f(u)-f(v)\|_2 \leq \gamma\|u-v\|_2, \quad \forall u,\ v \in K. \tag{2}$$

Remark 1: The value of γ in Assumption 1 is called the Lipschitz constant of f. Contractive operators are sometimes called Lipschitzian operators. If the above condition is instead satisfied for $\gamma \leq 1$, then the operator f is said to be nonexpansive.

Then, consider the Lipschitz nonlinear parabolic PDE system with unknown inputs in the form of:

$$\begin{aligned} x_t(s,t) &= \mathcal{A}x_{ss}(s,t)+f(x(s,t)) \\ &\quad +Gu(t)+Fv(t)+D_1w(t), \\ y(t) &= C\int_0^l x(s,t)ds+Ku(t)+Hv(t)+D_2w(t), \end{aligned} \tag{3}$$

where s is the spatial position variable, t is time variable, $x(s,t) \in \mathcal{D} \subseteq \mathcal{R}^n$, $x(\cdot,t) = [x_1(s,t)\ x_2(s,t)\ \cdots\ x_n(s,t)]^T$ is the state variable of the system, $\mathcal{A}, G, F, D_1, C, K, H$ and D_2 are known constant matrices, $f(x(s,t))$ is an unknown Lipschitz continuous nonlinear function with the property of sufficiently smooth and satisfies Assumption 1, $u(t)$ is the control input, and $\mathcal{D}$ is a given local domain which contains the equilibrium profile. The following Neumann boundary conditions are considered for system (3)

$$x_s(0,t) = x_s(l,t) = 0, \quad t > 0. \tag{4}$$

The following Lemma is necessary to derive our results.

Lemma 1.([33]) For a matrix $0 < R \in \mathcal{R}^{n\times n}$ and the differentiable function $z(\cdot) \in \mathcal{W}^{1,2}([0,l])$ subject to $z(0) = 0$ or $z(l) = 0$, the inequality as follows can be satisfied:

$$\int_0^l \dot{z}^T(s)R\dot{z}(s)ds \geq \frac{\pi^2}{4l^2}\int_0^l z^T(s)Rz(s)ds. \tag{5}$$

III. OBSERVER DESIGN AND STABILITY ANALYSIS

This section will present the observer design method and sufficient conditions of the existence of the observer for system (3). To estimate the state of (3), the observer is designed as:

$$\begin{aligned} z_t(s,t) &= \mathcal{B}z_{ss}(s,t)+Mf(\hat{x}(s,t))+\bar{G}u(t)+Ly(t), \\ \hat{x}(s,t) &= Tz(s,t)+Su(t)+Ey(t), \end{aligned} \tag{6}$$

with the following Dirichlet boundary conditions,

$$z(0,t) = Mx(0,t), z(l,t) = Mx(l,t) \tag{7}$$

where $z_t(s,t) \subseteq \mathcal{R}^n$, $\hat{x}(s,t)$ is the estimate of $x(s,t)$, the matrices $\mathcal{B}, M, \bar{G}, T, S, E$ are to be determined such that the error dynamics between $\hat{x}(s,t)$ and $x(s,t)$ converge to zero asymptotically. Motivated by [13], defining the error between $z(s,t)$ and $Mx(s,t)$,

$$\sigma(s,t) = z(s,t)-Mx(s,t), \tag{8}$$

and the error dynamics,

$$e(s,t) = \hat{x}(s,t)-x(s,t). \tag{9}$$

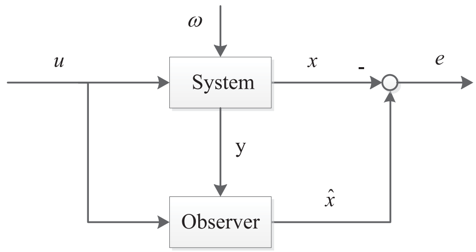

FIGURE 1. Block diagram of observer design.

Fig. 1 describes the block diagram of the observer design method in this paper.

Then

$$\begin{aligned} \sigma_t(s,t) &= z_t(s,t)-Mx_t(s,t) \\ &= \mathcal{B}z_{ss}(s,t)+Mf(\hat{x}(s,t))+\bar{G}u(t)+Ly(t) \\ &\quad -M[\mathcal{A}x_{ss}(s,t)+f(x(s,t))+Gu(t) \\ &\quad +Fv(t)+D_1v(t)] \\ &= \mathcal{B}z_{ss}(s,t)-M\mathcal{A}x_{ss}(s,t) \\ &\quad +(LD_2-MD_1)w(t) \\ &\quad +(LK+\hat{G}-MG)u(t)+(LH-MF)v(t) \\ &\quad +M[f(\hat{x}(s,t))-f(x(s,t))]+\int_0^l LCx(s,t)ds \\ &= \mathcal{B}[z_{ss}(s,t)-Mx_{ss}(s,t)]+\mathcal{B}Mx_{ss}(s,t) \\ &\quad -M\mathcal{A}x_{ss}(s,t)+\int_0^l LCx(s,t)ds \\ &\quad +M[f(\hat{x}(s,t))-f(x(s,t))] \\ &\quad +(LK+\hat{G}-MG)u(t) \\ &\quad +(LH-MF)v(t)+[LD_2-MD_1]w(t) \\ &= \mathcal{B}\sigma_{ss}(s,t)+[\mathcal{B}M-M\mathcal{A}]x_{ss}(s,t) \\ &\quad +\int_0^l LCx(s,t)ds+M[f(\hat{x}(s,t))-f(x(s,t))] \\ &\quad +(LK+\hat{G}-MG)u(t) \\ &\quad +(LH-MF)v(t)+(LD_2-MD_1)w(t), \end{aligned} \tag{10}$$

and state estimation error,

$$\begin{aligned} e(s,t) &= \hat{x}(s,t)-x(s,t) \\ &= Tz(s,t)+Su(t)-x(s,t) \\ &\quad +E[C(s)x(s,t)+Ku(t)+Hv(t)+D_2w(t)] \\ &= T[z(s,t)-Mx(s,t)]+TMx(s,t) \\ &\quad +ECx(s,t)-x(s,t)+Su(t) \\ &\quad +EKu(t)+EHv(t)+ED_2w(t) \\ &= T\sigma(s,t)+[TM+EC-I_n]x(s,t) \\ &\quad +[S+EK]u(t)+EHv(t)+ED_2w(t). \end{aligned} \tag{11}$$

If there exist matrices $\mathcal{B}, M, \bar{G}, T, S, E$ that satisfy

$$\begin{aligned} \mathcal{B}M-M\mathcal{A} &= 0, \\ LK+\bar{G}-MG &= 0, \\ LH-MF &= 0, \end{aligned}$$

$$\begin{aligned} TM + EC - I_n &= 0, \\ S + EK &= 0, \\ EH &= 0, \\ LC &= 0, \end{aligned} \tag{12}$$

then the error dynamics become,

$$\begin{aligned} \sigma_t(s,t) &= \mathcal{B}\sigma_{ss}(s,t) + M\Delta f + [LD_2 - MD_1]w(t), \\ e(s,t) &= T\sigma(s,t) + ED_2 w(t), \end{aligned} \tag{13}$$

where $\Delta f = \hat{f}(\hat{x}(s,t)) - \hat{f}(x(s,t))$.

When $w(t) = 0$, $e(s,t) = T\sigma(s,t)$. Then the asymptotic stable conditions of $e(s,t)$ are given in the following theorem.

Theorem 1: Consider the system (3) and the observer (6), for the Lipschitz constant $\gamma > 0$, if there exist proper matrixes T, M and $P > 0$ satisfy (12) and the following conditions:

$$\Gamma < I, \quad \begin{bmatrix} -\frac{\pi^2}{4l^2}[P\mathcal{B} + *] + \gamma^2 T^T T & PM \\ * & -\Gamma \end{bmatrix} < 0, \tag{14}$$

then the observer estimation error $e(t)$ is asymptotically stable to 0.

proof: Consider the Lyapunov candidate function $V(t) = \int_0^l \sigma^T(s,t)P\sigma(s,t)ds$, where P is a positive definite matrix. By taking the time derivation of $V(t)$ along $\sigma(s,t)$ in error dynamics (13), we obtain

$$\begin{aligned} \dot{V}(t) &= \int_0^l \Big[\sigma_t^T(s,t)P\sigma(s,t) + \sigma^T(s,t)P\sigma_t(s,t)\Big]ds \\ &= \int_0^l \Big[\sigma^T(s,t)P\mathcal{B}\sigma_{ss}(s,t) + \sigma^T(s,t)PM\Delta f \\ &\quad + \sigma_{ss}^T(s,t)\mathcal{B}^T P\sigma(s,t) + \Delta f^T M^T P\sigma(s,t)\Big]ds. \end{aligned} \tag{15}$$

Considering the boundary conditions (7), Lemma 1 and integrating by parts, we obtain,

$$\begin{aligned} \int_0^l \sigma^T P\mathcal{B}\sigma_{ss}ds &= \int_0^l \sigma^T P\mathcal{B}d\sigma_s \\ &= \sigma^T P\mathcal{B}\sigma_s|_{s=0}^{s=l} - \int_0^l \sigma_s^T P\mathcal{B}\sigma_s ds \\ &= -\int_0^l \sigma_s^T P\mathcal{B}\sigma_s ds \\ &\leq -\frac{\pi^2}{4l^2}\int_0^l \sigma^T P\mathcal{B}\sigma ds. \end{aligned} \tag{16}$$

Moreover, if $\Gamma \leq I$, by Assumption 1, we have

$$\begin{aligned} \Delta f^\Gamma \Delta f &\leq \Big(\hat{f}(\hat{x}(s,t)) - \hat{f}(x(s,t))\Big)^T \\ &\quad \times \Big(\hat{f}(\hat{x}(s,t)) - \hat{f}(x(s,t))\Big) \\ &\leq \gamma^2(\hat{x} - x)^T(\hat{x} - x) \\ &= \gamma^2 e^T e = \gamma^2\sigma^T T^T T\sigma. \end{aligned} \tag{17}$$

Then

$$\begin{aligned} \dot{V}(t) &\leq \int_0^l \Big[-\frac{\pi^2}{4l^2}\sigma^T[P\mathcal{B} + *]\sigma ds \\ &\quad + \sigma^T PM\Delta f + \Delta f^T M^T P\sigma\Big]ds \\ &= \int_0^l \Big[-\frac{\pi^2}{4l^2}\sigma^T[P\mathcal{B} + *]\sigma ds + \sigma^T PM\Delta f \\ &\quad + \Delta f^T M^T P\sigma + \Delta f^T\Gamma\Delta f - \Delta f^T\Gamma\Delta f\Big]ds \\ &\leq \int_0^l \Big[\sigma^T[(-\frac{\pi^2}{4l^2})[P\mathcal{B} + *] + \gamma^2 T^T T]\sigma \\ &\quad + \sigma^T PM\Delta f + \Delta f^T M^T P\sigma - \Delta f^T\Gamma\Delta f\Big]ds \\ &= \int_0^l \begin{bmatrix} \sigma^T & \Delta f^T \end{bmatrix} \Xi \begin{bmatrix} \sigma \\ \Delta f \end{bmatrix} ds, \end{aligned} \tag{18}$$

where

$$\Xi = \begin{bmatrix} -\frac{\pi^2}{4l^2}[P\mathcal{B} + *] + \gamma^2 T^T T & PM \\ * & -\Gamma \end{bmatrix}. \tag{19}$$

One can easily obtain $\dot{V}(t) < 0$ if $\Xi < 0$, which implies $e(s,t) \to 0$ as $t \to \infty$. This completes the proof. □

IV. $\mathcal{H}_\infty$ DESIGN

In this section, we investigate the stability and sufficient condition for (13) with $w(t) = 0$ and $\|e(s,t)\|_2 < \mu\|w(t)\|_2$ for $w(t) \neq 0$.

Theorem 2: Consider the observer (6) for system (3). If there exist appropriate matrices $\mathcal{B}, L, M, T, E$ and $P > 0$ satisfying the following inequalities:

$$\Gamma < I, \quad \Phi = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ * & -\Gamma & 0 \\ * & 0 & a_{33} \end{bmatrix} < 0, \tag{20}$$

where

$a_{11} = -\frac{\pi^2}{4l^2}[P\mathcal{B} + *] + (1+\gamma^2)T^T T,$

$a_{12} = PM,$

$a_{13} = P(LD_2 - MD_1) + (1+\gamma^2)T^T ED_2,$

$a_{33} = (1+\gamma^2)D_2^T E^T ED_2 - \mu^2 I.$

Then the state estimation error (13) which is produced by observer (6) asymptotically tends to 0 with $w(t) = 0$ and $\|e(s,t)\|_2 < \mu\|w(t)\|_2$ for $w(t) \neq 0$.

proof: Based on the proof of Theorem 1, let $w(t) \neq 0$, we have

$$\begin{aligned} \dot{V}(t) &= \int_0^l \Big[\sigma_t^T(s,t)P\sigma(s,t) + \sigma_t(s,t)P\sigma^T(s,t)\Big]ds \\ &= \int_0^l \Big[\sigma^T(s,t)P\mathcal{B}\sigma_{ss}(s,t) + \sigma^T(s,t)PM\Delta f \\ &\quad + \sigma_{ss}^T(s,t)\mathcal{B}^T P\sigma(s,t) + \Delta f^T M^T P\sigma(s,t) \\ &\quad + \sigma^T(s,t)P(LD_2 - MD_1)w(t) \\ &\quad + w^T(t)(LD_2 - MD_1)^T P\sigma(s,t)\Big]ds. \end{aligned} \tag{21}$$

Adding and subtracting $\int_0^l \Delta f^T\Gamma\Delta f ds - \int_0^l \Delta f^T\Gamma\Delta f ds = 0$ to the right-hand of (21), we obtain

$$\begin{aligned} \dot{V}(t) &= \int_0^l \Big[\sigma^T(s,t)P\mathcal{B}\sigma_{ss}(s,t) + \sigma^T(s,t)PM\Delta f \\ &\quad + \sigma_{ss}^T(s,t)\mathcal{B}^T P\sigma(s,t) + \Delta f^T M^T P\sigma(s,t) \end{aligned}$$

$$+\sigma^T(s,t)P(LD_2-MD_1)w(t)$$
$$+w^T(t)(LD_2-MD_1)^TP\sigma(s,t)$$
$$+\Delta f^T\Gamma\Delta f-\Delta f^T\Gamma\Delta f\Big]ds. \quad (22)$$

From (16) and (17), we arrive at

$$\dot{V}(t)\le\int_0^l\Big[\sigma^T[(-\frac{\pi^2}{4l^2})[P\mathcal{B}+*]]\sigma+\sigma^TPM\Delta f$$
$$+\Delta f^TM^TP\sigma+\sigma^TP(LD_2-MD_1)w$$
$$+w^T(LD_2-MD_1)^TP\sigma$$
$$+\gamma^2e^Te-\Delta f^T\Gamma\Delta f\Big]ds. \quad (23)$$

By (13),

$$e^Te=\sigma^TT^TT\sigma+\sigma^TT^TED_2w$$
$$+w^TD_2^TE^TT\sigma+w^TD_2^TE^TED_2w \quad (24)$$

Defining $\Upsilon=\begin{bmatrix}\sigma & \Delta f & w\end{bmatrix}^T$, then the following inequality is obtained,

$$\dot{V}(t)+\int_0^l[e^Te-\mu^2w^Tw]ds$$
$$\le\int_0^l\Big[\sigma^T[(-\frac{\pi^2}{4l^2})[P\mathcal{B}+*]+(1+\gamma^2)T^TT]\sigma$$
$$+\sigma^TPM\Delta f+\Delta f^TM^TP\sigma+\sigma^T[P(LD_2-MD_1)$$
$$+(1+\gamma^2)T^TED_2]w+w^T[(1+\gamma^2)D_2E^TT$$
$$+(LD_2-MD_1)^TP]\sigma-\Delta f^T\Gamma\Delta f\Big]ds$$
$$+w^T[(1+\gamma^2)D_2^TE^TED_2-\mu^2I]w$$
$$=\Upsilon^T\Phi\Upsilon \quad (25)$$

Then

$$\dot{V}(t)\le\int_0^l[\mu^2w^Tw-e^Te]ds, \quad (26)$$

if $\Phi<0$.

By integrating both side of (26) from 0 to ∞ respect to t yields

$$V(s,\infty)-V(s,0)=\int_0^l[\mu^2\|w(t)\|_2^2-\|e(s,t)\|_2^2]ds. \quad (27)$$

From zero initial values, we have

$$V(s,\infty)\le\int_0^l[\mu^2\|w(t)\|_2^2-\|e(s,t)\|_2^2]ds, \quad (28)$$

which results $\|e(s,t)\|_2^2<\mu^2\|w(t)\|_2^2$. This completes the proof. □

Remark 2: Note that the matrix inequalities presented in Theorem 1 and Theorem 2 can not directly solved with the LMI toolbox of MATLAB. By defining $P\mathcal{B}=\hat{P}$ and $PM=\bar{P}$, and applying Schur complement to the matrix inequalities in Theorem 1, then the parameters $\hat{P}$, $\bar{P}$ and T can be derived. Then apply the derived parameters to (12), the parameters $\mathcal{B}, M, L, \bar{G}, S$ and E can be derived.

Remark 3: It is noticed that when the Dirichlet boundary conditions (7) is changed as Neumann boundary conditions, Theorem 1 and Theorem 2 can also be derived.

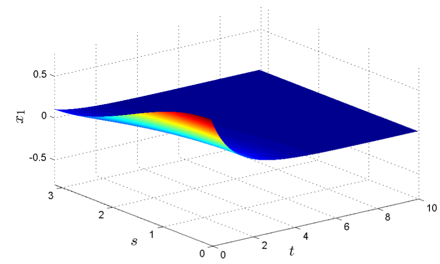

FIGURE 2. The system state x_1 when $w(t)=0$.

V. SIMULATIONS

In this section, we provide a numerical example to illustrate the effectiveness of the proposed design methods for system (3) based on Theorem 1. Consider the Lipschitz nonlinear PDE system in the form of (3) with

$$\mathcal{A}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad G=\begin{bmatrix}2\\2\end{bmatrix},\ F=\begin{bmatrix}1\\-1\end{bmatrix},$$
$$C=\begin{bmatrix}1&-1\\-1&1\end{bmatrix},\quad D_1=\begin{bmatrix}1\\1\end{bmatrix},\ D_2=\begin{bmatrix}1\\1\end{bmatrix},$$
$$K=\begin{bmatrix}1\\-1\end{bmatrix},\quad H=\begin{bmatrix}1\\1\end{bmatrix}.$$

For simulation purposes, the control input is assumed as $u(t)=-\int_0^l x(s,t)ds$, and the initial states $x_1(s,0)=0.4+0.3coss$, $x_2(s,0)=0.2+0.4coss$ with the spatial position variable $s\in[0,\pi]$. When the exogenous disturbance $w(t)$ is 0, by solving the matrices equality with the constraints of (12) and the matrices inequalities of (14), we have

$$\mathcal{B}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \bar{G}=\begin{bmatrix}1\\1\end{bmatrix},\ M=\begin{bmatrix}2&0\\0&2\end{bmatrix},$$
$$L=\begin{bmatrix}1&1\\-1&-1\end{bmatrix},\quad T=\begin{bmatrix}-\frac{1}{2}&1\\-1&\frac{3}{2}\end{bmatrix},$$
$$S=\begin{bmatrix}-2\\-2\end{bmatrix},\quad E=\begin{bmatrix}1&-1\\1&-1\end{bmatrix},$$

according to the error dynamics in (9), the evolution profiles of state $\hat{x}(s,t)$ estimated errors $e(s,t)$ are described in Figs. 2–7. It's obvious that the estimated errors $e(s,t)$ for the estimation error PDE system are asymptotically stable to 0. When the exogenous disturbance $w(t)$ is assumed to be $e^{-4-10t}sint$, by solving the matrices equalities (12) and the matrices inequality of (20), the evolution profiles of estimated error are described in Figs. 8–13. And the estimated errors $e(s,t)$ for the estimation error PDE system are asymptotically stable to 0. Moreover, the $\mathcal{H}_\infty$ performance $\|e\|_2<\mu\|w\|_2$ for $w\neq0$ is satisfied. According to the simulation results, it is easy to find that the observer design method proposed in this paper is feasible and effective to estimate the state of the uncertain Lipschitz nonlinear PDE systems.

Comparative Explanations: The presented observer design method in this paper provides an effective way in the new form for the state estimation of a class of uncertain Lipschitz nonlinear parabolic PDE systems. In contrast to the

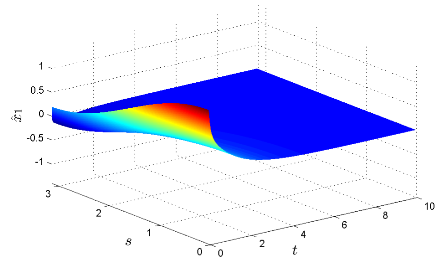

FIGURE 3. The state estimation $\hat{x}_1$ when $w(t) = 0$.

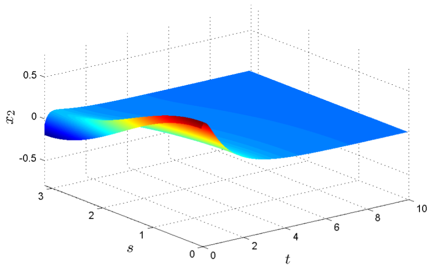

FIGURE 4. The system state x_2 when $w(t) = 0$.

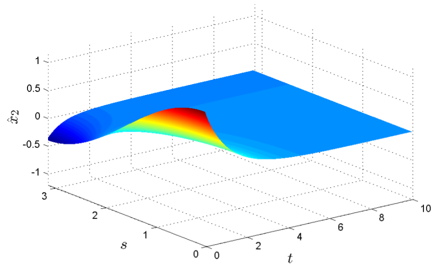

FIGURE 5. The state estimation $\hat{x}_2$ when $w(t) = 0$.

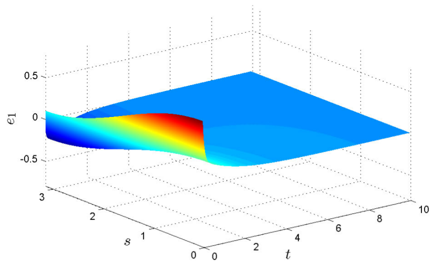

FIGURE 6. The estimated error e_1 when $w(t) = 0$.

available achievements, the main advantages of the design strategy presented could be summarized in these aspects listed as follows:

(1) Compared with the approaches of state estimation proposed in [20]–[24], we developed a novel method to design the observer for the uncertain Lipschitz nonlinear PDE systems without introducing other operators to replace the parabolic PDE system.

(2) Different from [26], [27], [29], the proposed observer design method is given without pointwise measurements and

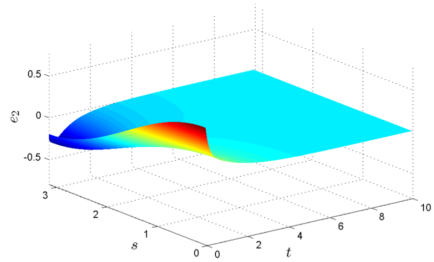

FIGURE 7. The estimated error e_2 when $w(t) = 0$.

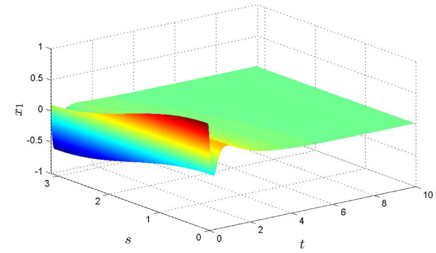

FIGURE 8. The system state x_1 when $w(t) \neq 0$.

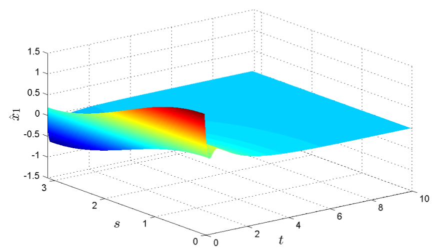

FIGURE 9. The state estimation $\hat{x}_1$ when $w(t) \neq 0$.

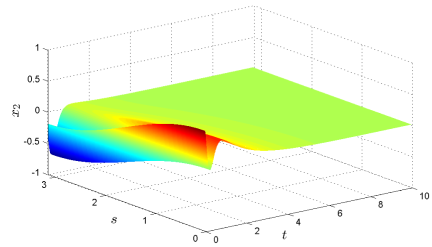

FIGURE 10. The system state x_2 when $w(t) \neq 0$.

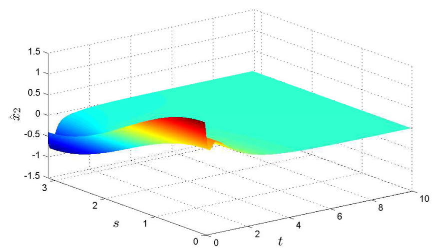

FIGURE 11. The state estimation $\hat{x}_2$ when $w(t) \neq 0$.

other constraints on system structure. Thus, the method can be applied to the system which the system information in

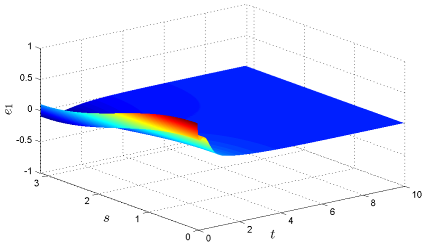

FIGURE 12. The estimated error e_1 when $w(t) \neq 0$.

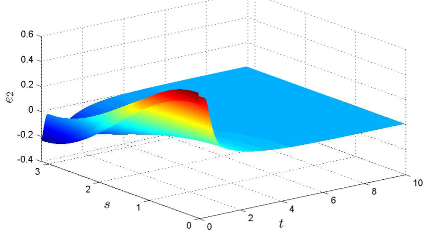

FIGURE 13. The estimated error e_2 when $w(t) \neq 0$.

the plant is difficult to be sampled or system deviation is intolerable due to sampling techniques.

VI. CONCLUSION

The $\mathcal{H}_\infty$ observer for a class of uncertain Lipschitz nonlinear parabolic PDE systems is developed in the new form in this article. The form of observer and the dynamic errors with undetermined parameters for the parabolic PDE systems subject to appropriate boundary conditions are designed under $w(t) = 0$. Sufficient conditions of the designed method with the consideration of asymptotic stability are involved. Then $\mathcal{H}_\infty$ performance of the observer is studied, and the analysis of asymptotic stability and sufficient conditions with $w(t) = 0$ and $\|e(s,t)\|_2 < \mu\|w(t)\|_2$ for $w(t) \neq 0$ are given in terms of matrix inequalities. The simulation results of a numerical example indicate the effectiveness of developed design method. However, the conditions presented in (12) are rather restrictive. In the future, we will focus our attention on the less conservative conditions for the observer design of uncertain systems, and the fuzzy observer-based filtering problems for nonlinear parabolic PDE systems by using the technic of quantization.

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TENG-FEI LI received the B.E. degree from Henan Polytechnic University, Jiaozuo, China, in 2014, and the M.S. degree from the Sichuan University of Science and Technology, Zigong, China, in 2018. He is currently pursuing the Ph.D. degree with the School of Information Science and Engineering, Wuhan University of Science and Technology, Wuhan, China. His research interests include PDE control systems, as well as fuzzy control and networked control systems.