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Adaptive Smooth Super-Twisting Sliding Mode Control of Nonlinear Systems With Unmatched Uncertainty

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ABSTRACT This article presents a novel strategy regarding the stabilization control problem for plants with unmatched uncertainties. The methodology is based on Adaptive Smooth Super Twisting Sliding Mode Control. At first, as an initial step, the plant with unmatched uncertainty is transformed into a plant with matched uncertainty. At the second step, the plant with matched uncertainty is decomposed into a unique framework containing the nominal part and some unknown terms (where these unknown terms are computed adaptively). The nominal system is stabilized by using Smooth Super Twisting Sliding Mode Control. The stabilizing controller for the plant with matched uncertainty is designed in a way; it contains some nominal control plus some compensator term. The stability of the said technique is presented impressively. The compensator controller and the adapted laws are derived in such a way that the time derivative of a Lyapunov function becomes strictly negative. The proposed method is tested on a fourth-order plant. The simulation results show the effectiveness and validity of the proposed controller.

INDEX TERMS Sliding mode control, higher-order sliding mode control, adaptive smooth super-twisting algorithm, Lyapunov function, unmatched uncertainty.

I. INTRODUCTION

Designing feedback control law for the stabilization of complex non-linear unmatched control systems has been an exciting subject for researchers in the field of control theory. Due to their wide range of applications in the ariel and underwater vehicles, such systems have gained prompt attention in the control community. In aforesaid applications, usually, actuators are responsible for moving flight control surfaces, respond quickly as compared to the engine dynamics. This fact needs to be considered in the process of control design for such systems. When faults and failures result in uncertainty in the dynamics of aircraft, such a system is categorized as an unmatched uncertainty system [1], [2]. Different techniques have been incorporated to achieve the desired performance from such systems in the presence of unmatched uncertainty, based on adaptive and Sliding Mode Control techniques (SMC) [3]–[5]. Practically speaking, precision, along with robustness, are ever demanding from the aforementioned systems. Among the said approaches, SMC fulfills the desired performance effectively.

Therefore, from the perspective of precision and robustness, the strategies based on sliding mode control have gained considerable attention from the research community of control systems [5]–[11]. SMC can be conveniently employed due to its simplicity. It is engaged in two phases, namely reaching and sliding phase, respectively. The reaching phase is considered essential in the case of conventional SMC (in some SMC formats, it may be ignored). However, in the sliding phase, the systems state trajectory is bound to slide on the sliding surface by the use of an appropriate control signal. During the sliding phase, the closed-loop system's response

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depends upon the surface parameter and remains insensitive to variations of system parameters and external disturbances [9], [10]. Contrarily during the reaching phase, the system is susceptible towards minimal disturbances, even of matched nature, resulting in deterioration in performance. Therefore, the robustness cannot be guaranteed throughout the system response [10]. Furthermore, SMC has the main obstacle in its practical application due to high-frequency oscillations (known as chattering, generated due to discontinuous control law). Chattering is undesirable for electro-mechanical systems because it invokes high-frequency dynamics, which leads to system instability [10], [11].

Various methodologies have been proposed in the literature to explain how to overcome undesirable chattering phenomenon which includes;

a) use of Higher-Order Sliding Mode Control (HOSMC) [10],
b) observer-based approach, and
c) use of saturation function instead of signum function (as discontinues control) [10], [11].

Similarly, Integral Sliding Mode Control (ISMC) is also proposed to counter the uncertainty issue during the reaching phase [9], [12]. Employment of Dynamic Sliding Mode Control (DSMC) is another solution to reduce chattering by providing new dynamics that act as compensators, and these compensators are further employed to improve system stability and performance [13]. The bottleneck for DMSC applicability is its limited application toward the non-linear system. By adopting, any of the aforementioned strategies, we have to accept some trade-offs in terms of finite-time convergence, system stability, and robustness of the systems [13], [14].

Among HOSM controllers, Second Order Sliding Mode (SOSM) controller is prevalent due to its ease and smooth implementation [14], [15]. To design SOSM, a number of algorithms have been proposed in recent years in which twisting, super twisting, sub-optimal, and drift are commonly used. Due to not requiring time derivatives of sliding variables and having insensitivity to sampling time, Super Twisting (STW) algorithm has been gained much attention from the research community, and this algorithm is used for the systems having relative degree one [11], [16]–[18]. The algorithm guarantees that system trajectories twist around the origin within finite time. In this article, STW algorithm is used in adaptive manner, in which control gains are able to adapt themselves to various uncertainties online, which increases its effectiveness.

The following are the significant contributions regarding this paper:

- A new strategy is proposed to stabilize the control system with the presence of unmatched uncertainties. The unmatched uncertainties are considered at the design phase so that the control signal efficiently controls the system under such uncertainties.
- The proposed methodology is based upon Adaptive Smooth Super-twisting (ASSTW) based SMC. In the proposed methodology, first, the plant with unmatched uncertainty is transformed into a plant with matched uncertainty. Then the said plant is decomposed into a particular structure containing a nominal part and some unknown terms (which would be computed later adaptively).
- The nominal system is stabilized, and the stabilizing controller for the plant with matched uncertainty is designed, consisting of nominal control and compensator control. The compensator controller and the adapted laws are derived based on the Lyapunov stability theory.
- A fourth-order plant is tested to confirm the validity of the proposed method.

This paper is organized in a way that the problem formulation and description are displayed in Section II. Moreover, some general control theory is also posed in Section II. Some practical examples are listed in Section III to show the effectiveness of the designed methodology. Finally, the concluding remarks are drawn in Section IV, along with some future direction, supported by the references.

II. PROBLEM FORMULATION

Considering the plant dynamics,

$$\begin{cases} \dot{x}_1 = x_2 + \theta_1 \phi(x_1) \\ \dot{x}_2 = x_3 + \theta_2 \phi(x_1) \\ \vdots \\ \dot{x}_{n-1} = x_n + \theta_{n-1} \phi(x_1) \\ \dot{x}_n = u \end{cases} \tag{1}$$

where state variables, control input, and unknown constants are represented by $x_1, x_2, \ldots, x_n$, u and, $\theta_i, i = 1, \ldots, n-1$, respectively. However, $\phi(x_1)$ is a sufficiently smooth known function with well-defined partial derivatives with respect to x_1. It can be clearly observed from equation (1) that the uncertainties do not appear with the control input, so equation (1) is referred as a plant with unmatched uncertainties. However, this unmatched uncertainty is the norm bounded.

A. PROBLEM STATEMENT

Given the desired set point $x_{des} \in \mathbb{R}^n$, construct a feedback strategy in terms of the controls $u : \mathbb{R}^n \to \mathbb{R}$ such that the desired set point x_{des} is an attractive set for (2), so that there exists an $\varepsilon > 0$, such that $x(t; 0, x_0) \to x_{des}$ as $t \to \infty$ for any initial condition $x_0 \in B(x_{des}; \varepsilon)$. Without the loss of generality, it is assumed that $x_{des} = 0$, which can be achieved by a suitable translation of the coordinate system.

The control problem is solved by transforming the system (1) with unmatched uncertainties into the system (3) with matched uncertainties through a transformation (2).

$$\begin{cases} z_1 = x_1 \\ z_2 = x_2 + \theta_1\phi(x_1) \\ z_3 = x_3 + \theta_2\phi(x_1) + \theta_1\phi^{(1)}(x_1) \\ z_4 = x_4 + \theta_3\phi(x_1) + \theta_2\phi^{(1)}(x_1) + \theta_1\phi^{(2)}(x_1) \\ \vdots \\ z_{n-1} = x_{n-1} + \theta_{n-2}\phi(x_1) + \theta_{n-3}\phi^{(1)}(x_1) \\ \qquad + \ldots + \theta_1\phi^{(n-3)}(x_1) \\ z_n = x_n + \theta_{n-1}\phi(x_1) + \theta_{n-2}\phi^{(1)}(x_1) \\ \qquad + \ldots + \theta_1\phi^{(n-2)}(x_1) \end{cases} \tag{2}$$

where $\phi^{(i)}(x_1) = \frac{d^i}{dt^i}\phi(x_1)$.

System (2) can be rewritten as $z = x + A$,
where $z = [z_1 z_2 z_3 \ldots z_n]^T$, $x = [x_1 x_2 x_3 \ldots x_n]^T$
and

$$A = \begin{bmatrix} 0 \\ \theta_1\phi \\ \theta_2\phi + \theta_1\phi^{(1)} \\ \theta_3\phi + \theta_2\phi^{(1)} + \theta_1\phi^{(2)} \\ \vdots \\ \theta_{n-2}\phi + \theta_{n-3}\phi^{(1)} + \ldots + \theta_1\phi^{(n-3)} \\ \theta_{n-1}\phi + \theta_{n-2}\phi^{(1)} + \ldots + \theta_1\phi^{(n-2)} \end{bmatrix}$$

Then

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = z_4 \\ \vdots \\ \dot{z}_{n-1} = z_n \\ \dot{z}_n = u + \theta_{n-1}\phi^{(1)}(x_1) + \theta_{n-2}\phi^{(2)}(x_1) \\ \qquad + \ldots + \theta_1\phi^{(n-1)}(x_1) \end{cases} \tag{3}$$

θ_i are unknown constants and maybe computed adaptively. Let $\hat{\theta}_i$ be an estimate of θ_i and $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ be the error in the estimation of θ_i, $i = 1, \ldots, n-1$, then, system (3) can further be expressed as (4)

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = z_4 \\ \vdots \\ \dot{z}_{n-1} = z_n \\ \dot{z}_n = u + \hat{\theta}_{n-1}\phi^{(1)}(z_1) + \hat{\theta}_{n-2}\phi^{(2)}(z_1) \\ \qquad + \ldots + \hat{\theta}_1\phi^{(n-1)}(z_1) + \tilde{\theta}_{n-1}\phi^{(1)}(z_1) \\ \qquad + \tilde{\theta}_{n-2}\phi^{(2)}(z_1) + \ldots + \tilde{\theta}_1\phi^{(n-1)}(z_1) \end{cases} \tag{4}$$

Theorem: System in (4) is asymptotically stable by choosing sliding surface $s = z_1 + \sum_{i=1}^{n-2} c_i z_{i+1} + z_n$, and $u = v_{eq} + v_s$ where

$$\begin{cases} v_{eq} = -z_2 - \sum_{i=1}^{n-2} c_i z_{i+2} - \hat{\theta}_{n-1}\phi^{(1)}(z_1) - \hat{\theta}_{n-2}\phi^{(2)}(z_1) \\ \qquad - \ldots - \hat{\theta}_1\phi^{(n-1)}(z_1) \\ v_s = -k_1 |s|^{\frac{\rho-1}{\rho}} sign(s) + w \\ \dot{w} = -k_2 |s|^{\frac{\rho-2}{\rho}} sign(s), \quad k_1, k_2 > 0, \rho \geq 2 \end{cases}$$

Consider a Lyapunov function $V = \zeta^T P \zeta + \frac{1}{2}\left(\tilde{\theta}_1^2 + \tilde{\theta}_2^2 + \ldots + \tilde{\theta}_{n-2}^2 + \tilde{\theta}_{n-1}^2\right)$, where $P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$ is the positive definite and symmetric matrix satisfying the Lyapunov equation: $A^T P + PA = -Q$, where $Q \in \mathbb{R}^{2\times 2}$ is the positive definite and symmetric matrix. $\dot{V} = -s^{-\frac{1}{\rho}}\zeta^T Q \zeta \leq 0$ if the adaptive laws are chosen as

$$\begin{cases} \dot{\hat{\theta}}_1 = -b\phi^{(n-1)}(z_1) \\ \dot{\hat{\theta}}_2 = -b\phi^{(n-2)}(z_1) \\ \vdots \\ \dot{\hat{\theta}}_{n-2} = -b\phi^{(2)}(z_1) \\ \dot{\hat{\theta}}_{n-1} = -b\phi^{(1)}(z_1) \end{cases}$$

where $b = 2y|s|^{-\frac{1}{\rho}}(p_1 s^y sign(s) + p_2 w)$ and $\dot{\hat{\theta}}_i = -\dot{\tilde{\theta}}_i$, $i = 1, \ldots, n-1$

Proof: As the sliding surface for the system (4) is defined as

$$s = z_1 + \sum_{i=1}^{n-2} c_i z_{i+1} + z_n$$

where $c_i > 0$ are chosen in such a way that s becomes Hurwitz polynomial, then

$$\begin{aligned} \dot{s} &= \dot{z}_1 + \sum_{i=1}^{n-2} c_i \dot{z}_{i+1} + \dot{z}_n \\ &= z_2 + \sum_{i=1}^{n-2} c_i z_{i+2} + u + \hat{\theta}_{n-1}\phi^{(1)}(z_1) + \hat{\theta}_{n-2}\phi^{(2)}(z_1) \\ &\quad + \ldots + \hat{\theta}_1\phi^{(n-1)}(z_1) + \tilde{\theta}_{n-1}\phi^{(1)}(z_1) \\ &\quad + \tilde{\theta}_{n-2}\phi^{(2)}(z_1) + \ldots + \tilde{\theta}_1\phi^{(n-1)}(z_1) \end{aligned} \tag{5}$$

By choosing $u = v_{eq} + v_s$, where

$$\begin{cases} v_{eq} = -z_2 - \sum_{i=1}^{n-2} c_i z_{i+2} - \hat{\theta}_{n-1}\phi^{(1)}(z_1) - \hat{\theta}_{n-2}\phi^{(2)}(z_1) \\ \qquad - \ldots - \hat{\theta}_1\phi^{(n-1)}(z_1) \\ v_s = -k_1 |s|^{\frac{\rho-1}{\rho}} sign(s) + w \\ \dot{w} = -k_2 |s|^{\frac{\rho-2}{\rho}} sign(s), \quad k_1, k_2 > 0, \rho \geq 2 \end{cases} \tag{6}$$

After employing (6) in equation (5), ones got:

$$\dot{s} = -k_1 |s|^y sign(s) + w + a \tag{7}$$

where

$$\dot{w} = -k_2 |s|^{y-\frac{1}{\rho}} sign(s), \quad k_1,\ k_2 > 0, y = \frac{\rho - 1}{\rho}$$
$$a = \tilde{\theta}_{n-1}\phi^{(1)}(z_1) + \tilde{\theta}_{n-2}\phi^{(2)}(z_1) + \ldots + \tilde{\theta}_1\phi^{(n-1)}(z_1)$$

Define $\varsigma = \begin{bmatrix} |s|^y sign(s) \\ w \end{bmatrix}$ then,

$$\dot{\varsigma} = \begin{bmatrix} y|s|^{y-1}\dot{s} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} y|s|^{\frac{-1}{\rho}}\{-k_1|s|^y sign(s) + w + a\} \\ -k_2|s|^{y-\frac{1}{\rho}} sign(s) \end{bmatrix}$$
$$= |s|^{\frac{-1}{\rho}} \begin{bmatrix} y\{-k_1|s|^y sign(s) + w + a\} \\ -k_2|s|^y sign(s) \end{bmatrix}$$
$$= |s|^{\frac{-1}{\rho}} \begin{bmatrix} -yk_1 & y \\ k_2 & 0 \end{bmatrix} \begin{bmatrix} |s|^y sign(s) \\ w + a \end{bmatrix}$$
$$= |s|^{\frac{-1}{\rho}} \begin{bmatrix} -yk_1 & y \\ k_2 & 0 \end{bmatrix} \left\{ \begin{bmatrix} |s|^y sign(s) \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ a \end{bmatrix} \right\}$$
$$= |s|^{\frac{-1}{\rho}} A\xi + |s|^{-\frac{1}{\rho}} \begin{bmatrix} ya \\ 0 \end{bmatrix}$$

where

$$A = \begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix}$$

i.e.

$$\dot{\varsigma} = |s|^{-\frac{1}{\rho}} \left(A\xi + \begin{bmatrix} ya \\ 0 \end{bmatrix} \right) \quad (8)$$

The eigenvalues of $A = \begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix}$ are the roots of the Hurwitz polynomial, i. e,

$$|\lambda I - A| = \begin{bmatrix} \lambda + yk_1 & -y \\ k_2 & \lambda \end{bmatrix} = \lambda^2 + \lambda(yk_1) + (yk_2) = 0,$$

Therefore, $A = \begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix}$ is strictly stable, then $\exists P \in \mathbb{R}^{2\times 2}$, where P is a positive definite, and symmetric matrix, satisfying the Lyapunov equation: $A^T P + PA = -Q$, where $Q \in \mathbb{R}^{2\times 2}$ is the positive definite and symmetric matrix.

Since

$$V = \zeta^T P\zeta + \frac{1}{2}\left(\tilde{\theta}_1^2 + \tilde{\theta}_2^2 + \ldots + \tilde{\theta}_{n-2}^2 + \tilde{\theta}_{n-1}^2\right) \quad (9)$$

Taking the time derivative of V and utilizing (8) yields

$$\begin{aligned}
\dot{V} &= \dot{\zeta}^T P\zeta + \zeta^T P\dot{\zeta} + \tilde{\theta}_1\dot{\tilde{\theta}}_1 + \tilde{\theta}_2\dot{\tilde{\theta}}_2 + \ldots + \tilde{\theta}_{n-2}\dot{\tilde{\theta}}_{n-2} \\
&\quad + \tilde{\theta}_{n-1}\dot{\tilde{\theta}}_{n-1} \\
&= \left\{ |s|^{-\frac{1}{\rho}} A^T\xi^T + |s|^{-\frac{1}{\rho}} \begin{bmatrix} ya & 0 \end{bmatrix} \right\} P\zeta \\
&\quad + \zeta^T P \left\{ |s|^{-\frac{1}{\rho}} A\xi + |s|^{-\frac{1}{\rho}} \begin{bmatrix} ya \\ 0 \end{bmatrix} \right\} + \tilde{\theta}_1\dot{\tilde{\theta}}_1 + \tilde{\theta}_2\dot{\tilde{\theta}}_2 \\
&\quad + \ldots + \tilde{\theta}_{n-2}\dot{\tilde{\theta}}_{n-2} + \tilde{\theta}_{n-1}\dot{\tilde{\theta}}_{n-1} \\
&= |s|^{-\frac{1}{\rho}} \left\{ \xi^T \left(A^T P + PA \right) \xi \right\} \\
&\quad + |s|^{-\frac{1}{\rho}} \left(\begin{bmatrix} ya & 0 \end{bmatrix} P\zeta + \zeta^T P \begin{bmatrix} ya \\ 0 \end{bmatrix} \right) \\
&\quad + \tilde{\theta}_1\dot{\tilde{\theta}}_1 + \tilde{\theta}_2\dot{\tilde{\theta}}_2 + \ldots + \tilde{\theta}_{n-2}\dot{\tilde{\theta}}_{n-2} + \tilde{\theta}_{n-1}\dot{\tilde{\theta}}_{n-1} \\
&= -|s|^{-\frac{1}{\rho}} \left(\xi^T Q\xi \right) + 2y|s|^{-\frac{1}{\rho}} (p_1\zeta_1 + p_2\zeta_2)a \\
&\quad + \tilde{\theta}_1\dot{\tilde{\theta}}_1 + \tilde{\theta}_2\dot{\tilde{\theta}}_2 + \ldots + \tilde{\theta}_{n-2}\dot{\tilde{\theta}}_{n-2} + \tilde{\theta}_{n-1}\dot{\tilde{\theta}}_{n-1} \\
&\quad - |s|^{-\frac{1}{\rho}} \xi^T Q\xi + b\left\{ \tilde{\theta}_{n-1}\phi^{(1)}(z_1) + \tilde{\theta}_{n-2}\phi^{(2)}(z_1) \right. \\
&\quad \left. + \ldots + \tilde{\theta}_1\phi^{(n-1)}(z_1) \right\} + \tilde{\theta}_1\dot{\tilde{\theta}}_1 + \tilde{\theta}_2\dot{\tilde{\theta}}_2 \\
&\quad + \ldots + \tilde{\theta}_{n-2}\dot{\tilde{\theta}}_{n-2} + \tilde{\theta}_{n-1}\dot{\tilde{\theta}}_{n-1} \\
&= -|s|^{-\frac{1}{\rho}} \xi^T Q\xi + \tilde{\theta}_1 \left\{ \dot{\tilde{\theta}}_1 + b\phi^{(n-1)}(z_1) \right\} \\
&\quad + \tilde{\theta}_2 \left\{ \dot{\tilde{\theta}}_2 + b\phi^{(n-2)}(z_1) \right\} + \ldots + \tilde{\theta}_{n-2} \left\{ \dot{\tilde{\theta}}_{n-2} \right. \\
&\quad \left. + b\phi^{(2)}(z_1) \right\} + \tilde{\theta}_{n-1} \left\{ \dot{\tilde{\theta}}_{n-1} + b\phi^{(1)}(z_1) \right\} \quad (10)
\end{aligned}$$

where

$$\begin{aligned}
b &= 2y|s|^{-\frac{1}{\rho}} (p_1\zeta_1 + p_2\zeta_2) \\
&= 2y|s|^{-\frac{1}{\rho}} \left(p_1 |s|^y sign(s) + p_2 w \right)
\end{aligned}$$

By using

$$\begin{cases}
\dot{\tilde{\theta}}_1 = -b\phi^{(n-1)}(z_1) \\
\dot{\tilde{\theta}}_2 = -b\phi^{(n-2)}(z_1) \\
\vdots \\
\dot{\tilde{\theta}}_{n-2} = -b\phi^{(2)}(z_1) \\
\dot{\tilde{\theta}}_{n-1} = -b\phi^{(1)}(z_1)
\end{cases}$$

where $b = 2y|s|^{-\frac{1}{\rho}}(p_1 s^y sign(s) + p_2 w)$ and $\dot{\hat{\theta}}_i = -\dot{\tilde{\theta}}_i$, $i = 1, \ldots, n-1$, in (9), one can obtain $\dot{V} = -|s|^{-\frac{1}{\rho}} \zeta^T Q\zeta \leq 0$. From this, we conclude that $\zeta \to 0$. Since $s \to 0$, therefore $z_i \to 0$, for $i = 1, \ldots, n$.

III. ILLUSTRATIVE EXAMPLES

To prove the effectiveness and stability of the proposed design methodology, a fourth-order plant is considered and illustrated in this section.

A. PLANT WITH ORDER 4, i.e., n = 4

Considering the system displayed in (1), for $n = 4$, ones have:

$$\begin{cases}
\dot{x}_1 = x_2 + \theta_1\phi(x_1) \\
\dot{x}_2 = x_3 + \theta_2\phi(x_1) \\
\dot{x}_3 = x_4 + \theta_3\phi(x_1) \\
\dot{x}_4 = u
\end{cases} \quad (11)$$

The system (11) can take the form as shown in (12), by substituting the value of $\phi(x_1) = 0.5x_1^2$

$$\begin{cases}
\dot{x}_1 = x_2 + 0.5\theta_1 x_1^2 \\
\dot{x}_2 = x_3 + 0.5\theta_2 x_1^2 \\
\dot{x}_3 = x_4 + 0.5\theta_3 x_1^2 \\
\dot{x}_4 = u
\end{cases} \quad (12)$$

State trajectories of system (12) are displayed in Fig 1, which indicates the convergence of all states at nearly sixth second.

$$\begin{cases}
z_1 = x_1 \\
z_2 = x_2 + 0.5\theta_1 x_1^2 = x_2 + 0.5\theta_1 z_1^2 \\
z_3 = x_3 + 0.5\theta_2 x_1^2 + \theta_1 x_1\dot{x}_1 = x_3 + 0.5\theta_2 z_1^2 + \theta_1 z_1 z_2 \\
z_4 = x_4 + 0.5\theta_3 x_1^2 + \theta_2 x_1\dot{x}_1 + \theta_1\left(\dot{x}_1^2 + x_1\ddot{x}_1\right) \\
= x_4 + 0.5\theta_3 z_1^2 + \theta_2 z_1 z_2 + \theta_1\left(z_2^2 + z_1 z_3\right)
\end{cases} \quad (13)$$

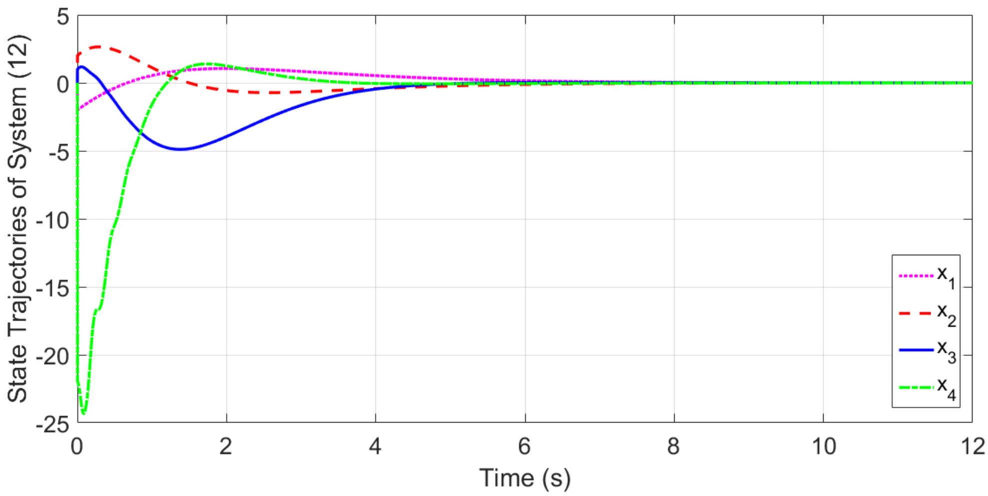

FIGURE 1. State trajectories of system (12).

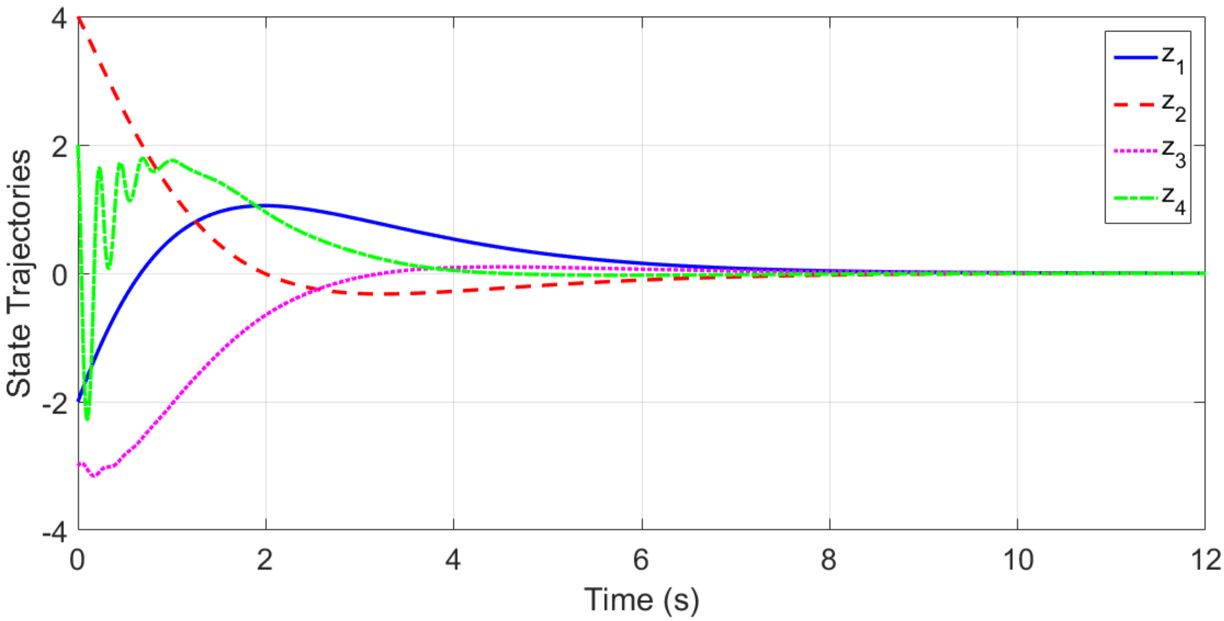

FIGURE 2. State trajectories of system (15).

Using the transformation defined in (2), the system (12) is transformed as (13). Moreover, by following the footprints defined in equation (3) and (4) at Section II, ones got equation (14) and (15) respectively

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = u + \theta_3 z_1 z_2 + \theta_2 \left(z_2^2 + z_1 z_3\right) \\ \qquad +\theta_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right) \end{cases} \tag{14}$$

State trajectories of the transformed system are displayed in Fig 2, which confirms the significant convergence of the systems states. Initial conditions $(-2, 4, -3, 2)$ are considered for Fig 2.

The system (14) can be rewritten as

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = u + \hat{\theta}_3 z_1 z_2 + \hat{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \hat{\theta}_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right) \\ +\tilde{\theta}_3 z_1 z_2 + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \tilde{\theta}_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right) \end{cases} \tag{15}$$

The sliding surface presented in Section III is considered for $n = 4$, concerning the system (15) is shown below

$$s = z_1 + 3z_2 + 3z_3 + z_4$$

then $\dot{s}$ becomes

$$\dot{s} = \dot{z}_1 + 3\dot{z}_2 + 3\dot{z}_3 + \dot{z}_4$$

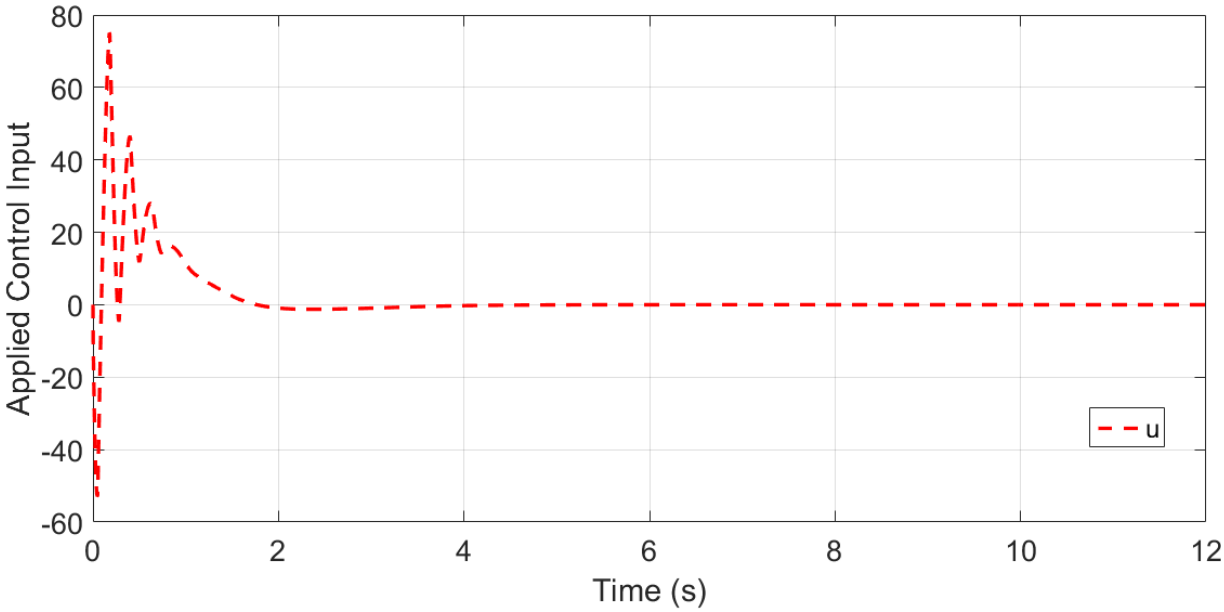

FIGURE 3. Applied control input.

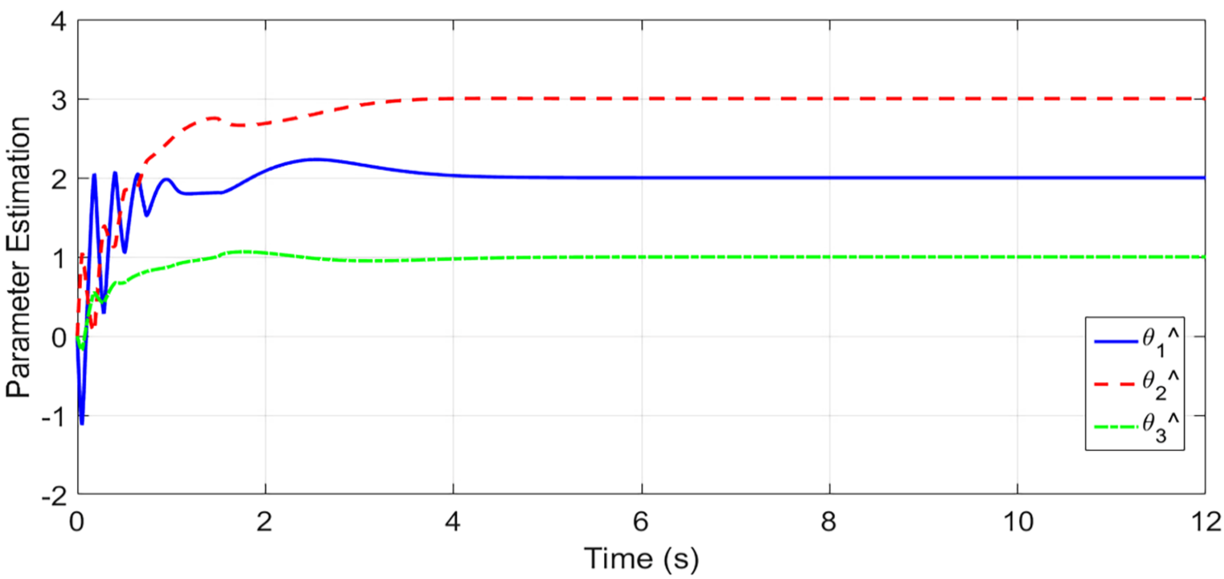

FIGURE 4. Parameter estimation.

$$
\begin{aligned}
&= z_2 + 3z_3 + 3z_4 + u + \hat{\theta}_3 z_1 z_2 + \hat{\theta}_2 \left(z_2^2 + z_1 z_3\right) \\
&\quad + \hat{\theta}_1 \left(2z_2z_3 + z_2z_3 + z_1z_4\right) + \tilde{\theta}_3 z_1 z_2 \\
&\quad + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \tilde{\theta}_1 \left(2z_2z_3 + z_2z_3 + z_1z_4\right)
\end{aligned}
$$

By choosing

$$u = v_{eq} + v_s \tag{16}$$

where

$$
\begin{aligned}
v_{eq} &= -z_2 - 3z_3 - 3z_4 - \hat{\theta}_3 z_1 z_2 + \hat{\theta}_2 \left(z_2^2 + z_1 z_3\right) \\
&\quad - \hat{\theta}_1 \left(2z_2z_3 + z_2z_3 + z_1z_4\right)
\end{aligned}
$$

$$
\begin{aligned}
v_s &= -k_1 \left|s\right|^{\frac{\rho-1}{\rho}} sign\left(s\right) + w \\
\dot{w} &= -k_2 \left|s\right|^{\frac{\rho-2}{\rho}} sign\left(s\right), \quad k_1, k_2 > 0, \rho \geq 2
\end{aligned}
$$

whereas,

$$
\begin{aligned}
\dot{s} = -k_1 \left|s\right|^{\frac{\rho-1}{\rho}} sign\left(s\right) + w + \tilde{\theta}_3 z_1 z_2 + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) \\
+ \tilde{\theta}_1 \left(2z_2z_3 + z_2z_3 + z_1z_4\right)
\end{aligned} \tag{17}
$$

Control effort is displayed in Fig. 3, indicating the system stabilized in terms of control input in the sixth second.

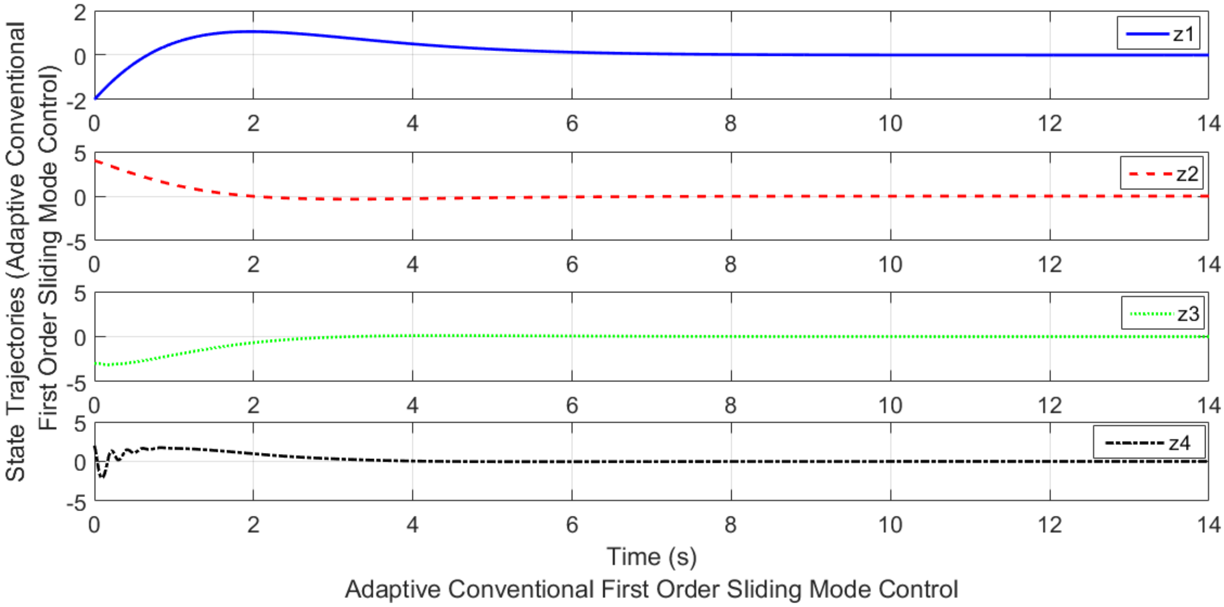

FIGURE 5. State trajectories of system (15), for adaptive conventional sliding mode control.

Define $\varsigma = \begin{bmatrix} |s|^y sign(s) \\ w \end{bmatrix}$, where $y = \frac{\rho-1}{\rho}$, then

$$\dot{\varsigma} = |s|^{-\frac{1}{\rho}} A\varsigma + |s|^{-\frac{1}{\rho}} \begin{bmatrix} y\left(\tilde{\theta}_3 z_1 z_2 + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \tilde{\theta}_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right)\right) \\ 0 \end{bmatrix}$$

where

$$A = \begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix}$$

Choose a Lyapunov function $V = \varsigma^T P \varsigma + \frac{1}{2}\left(\tilde{\theta}_1^2 + \tilde{\theta}_2^2 + \tilde{\theta}_3^2\right)$, where $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ is the positive definite and symmetric matrix satisfying the Lyapunov equation $A^T P + PA = -Q$, where $Q \in \mathbb{R}^{2\times 2}$ is the positive definite matrix.

Then

$$\dot{V} = \dot{\varsigma}^T P \varsigma + \varsigma^T P \dot{\varsigma} + \tilde{\theta}_1 \dot{\tilde{\theta}}_1 + \tilde{\theta}_2 \dot{\tilde{\theta}}_2 + \tilde{\theta}_3 \dot{\tilde{\theta}}_3$$

$$\begin{aligned} \dot{V} = &\left\{ |s|^{-\frac{1}{\rho}} \xi^T A^T + |s|^{-\frac{1}{\rho}} \left[y\left(\tilde{\theta}_3 z_1 z_2 + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \tilde{\theta}_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right)\right) \;\; 0 \right] \right\} P\varsigma \\ &+ \varsigma^T P \left\{ |s|^{-\frac{1}{\rho}} A\xi + |s|^{-\frac{1}{\rho}} \begin{bmatrix} y\left(\tilde{\theta}_3 z_1 z_2 + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \tilde{\theta}_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right)\right) \\ 0 \end{bmatrix} \right\} \\ &+ \tilde{\theta}_1 \dot{\tilde{\theta}}_1 + \tilde{\theta}_2 \dot{\tilde{\theta}}_2 + \tilde{\theta}_3 \dot{\tilde{\theta}}_3 \\ = &|s|^{-\frac{1}{\rho}} \xi^T \left(A^T P + PA\right) \xi + |s|^{-\frac{1}{\rho}} \left\{ \left[y\left(\tilde{\theta}_3 z_1 z_2 + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \tilde{\theta}_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right)\right) \;\; 0 \right] P_\varsigma + \varsigma^T P \begin{bmatrix} y\left(\tilde{\theta}_3 z_1 z_2 + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \tilde{\theta}_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right)\right) \\ 0 \end{bmatrix} \right\} \\ &+ \tilde{\theta}_1 \dot{\tilde{\theta}}_1 + \tilde{\theta}_2 \dot{\tilde{\theta}}_2 + \tilde{\theta}_3 \dot{\tilde{\theta}}_3 \\ = &- |s|^{-\frac{1}{\rho}} \varsigma^T Q \varsigma + 2y |s|^{-\frac{1}{\rho}} \left(p_1 \varsigma_1 + p_2 \varsigma_2\right) \left(\tilde{\theta}_3 z_1 z_2 + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \tilde{\theta}_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right)\right) \\ &+ \tilde{\theta}_1 \dot{\tilde{\theta}}_1 + \tilde{\theta}_2 \dot{\tilde{\theta}}_2 + \tilde{\theta}_3 \dot{\tilde{\theta}}_3 \\ = &- |s|^{-\frac{1}{\rho}} \varsigma^T Q \varsigma + b\left(\tilde{\theta}_3 z_1 z_2 + \tilde{\theta}_2 \left(z_2^2 + z_1 z_3\right) + \tilde{\theta}_1 \left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right)\right) + \tilde{\theta}_1 \dot{\tilde{\theta}}_1 + \tilde{\theta}_2 \dot{\tilde{\theta}}_2 + \tilde{\theta}_3 \dot{\tilde{\theta}}_3 \\ = &- |s|^{-\frac{1}{\rho}} \varsigma^T Q \varsigma + \tilde{\theta}_1 \left(\dot{\tilde{\theta}}_1 + b\left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right)\right) + \tilde{\theta}_2 \left(\dot{\tilde{\theta}}_2 + b\left(z_2^2 + z_1 z_3\right)\right) + \tilde{\theta}_3 \left(\dot{\tilde{\theta}}_3 + b z_1 z_2\right) \end{aligned}$$

where

$$\begin{aligned} b &= 2y |s|^{-\frac{1}{\rho}} \left(p_1 \varsigma_1 + p_2 \varsigma_2\right) \\ &= 2y |s|^{-\frac{1}{\rho}} \left(p_1 |s|^y sign(s) + p_2 w\right) \\ \dot{V} &= - |s|^{-\frac{1}{\rho}} \varsigma^T Q \varsigma \le 0 \end{aligned}$$

if adaptive laws are chosen as

$$\begin{aligned} \dot{\tilde{\theta}}_1 &= -b\left(2z_2 z_3 + z_2 z_3 + z_1 z_4\right), \\ \dot{\tilde{\theta}}_2 &= -b\left(z_2^2 + z_1 z_3\right), \\ \dot{\tilde{\theta}}_3 &= -b z_1 z_2 \end{aligned} \quad (18)$$

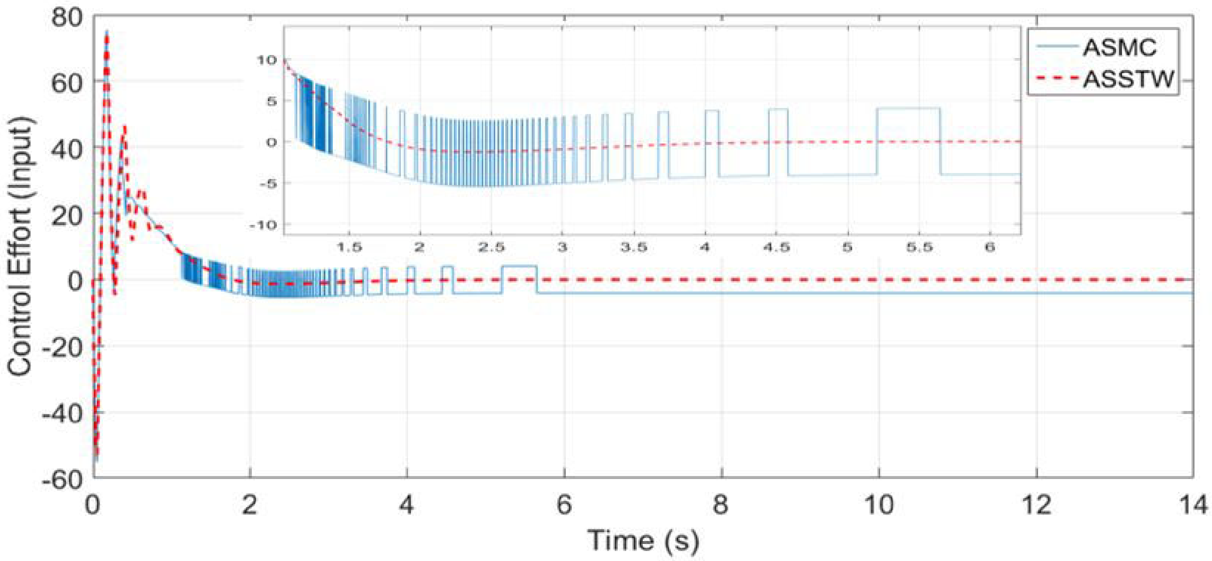

FIGURE 6. Comparison of control efforts (Adaptive SMC and ASSTW).

Parameter estimation is presented in Fig 4. It is very obvious from Fig 1 and 2 that the system states converge to zero asymptotically, and parameter estimates converge to the desired values.

To prove the effectiveness of the proposed technique, a comparison is also made with conventional adaptive first-order sliding mode control (from (5) , $u = -v_{eq} - sign\,(s)$), and the results are posed in Figs 5 and 6. Due to the adaptive phenomenon, states successfully converge to zero in finite time, which can be observed from Fig. 5. However, if we follow the comparison of control effort difference is very clear. A tremendous amount of chattering can be observed in Fig. 6. This high-frequency oscillation (known as chattering) is very dangerous for electro-mechanical systems. It may also lead to a total system failure. On the other hand, it is evident that the proposed strategy carries substantial marks considering chattering suppression.

IV. CONCLUSION

In this work, a new methodology based on Adaptive Smooth Super Twisting Sliding Mode Control (ASSTW) has been developed for plants with unmatched uncertainties. The plant with unmatched uncertainty is first converted into a plant with matched uncertainty. Further, the plant with matched uncertainty is transformed into a particular structure, consisting of two parts: 1) nominal part, 2) some unknown terms that are computed adaptively. The nominal system has been stabilized through ASSTW. The stabilizing controller for the plant with matched uncertainty has been designed, consisting of nominal control plus some compensator control. The compensator controller and the adapted laws are derived in such a way that the time derivative of a Lyapunov function becomes strictly negative. The control design procedure has been illustrated for a fourth-order plant. To prove the effectiveness of the proposed technique, a comparison is also made with conventional first-order sliding mode control. The numerical simulations verify the effectiveness and supremacy of the proposed strategy in terms of robust finite-time convergence and chattering suppression.

Considering the future perspective, the scope of this work may be extended to hardware implementation. Moreover, a fusion of the aforementioned strategy may be implemented infusion with any artificial intelligence/ machine learning algorithm like neural network (i.e., neural network-based adaptive smooth super twisting sliding mode control).

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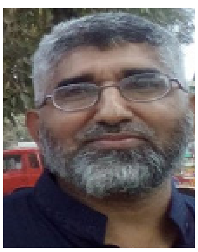

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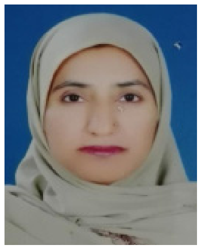

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