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A Parameter Estimation Method for Stress-Strength Model Based on Extending Markov State-Space With Variable Transition Rates

LONG WANG[1,2], TENGFEI XU[2], QINGJIE ZHANG[1], XU LUO[2], AND YUE LI[2]
[1]Aviation University of Air Force, Changchun 130022, China
[2]Laboratory of Science and Technology on Integrated Logistics Support, National University of Defense Technology, Changsha 410073, China
Corresponding author: Xu Luo (luoxu@nudt.edu.cn)

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ABSTRACT A stress-strength model usually has more than one failure mode since the component suffers at least two types of stresses, complicating the expression of the likelihood function and increasing the computational complexity of the parameter estimation for general distributions (non-exponential distributions). A phase-type distribution (also known as a PH distribution) is dense and has closure properties, which makes it suitable to reduce the computational complexity of the stress-strength model. The traditional expectation-maximization (EM) method for estimating the parameters of the PH distribution cannot be used directly when the strength changes over time since the PH distribution is a continuous-time Markov process that must satisfy the relevant properties of the infinitesimal generator in the Markov state-space. Therefore, a parameter estimation method based on extending the Markov state-space with variable transition rates for the stress-strength model is proposed. Both failure and censored samples are considered. First, the stress-strength model based on the PH distribution is briefly introduced, and the likelihood functions for different failure modes are derived. Subsequently, the principle of the method is described in detail, the derivation process of the relevant equations is provided, and the limitations of the method are discussed. The performance of the method is evaluated using two simulation cases.

INDEX TERMS Stress-strength model, parameter estimation method, PH distribution, Markov state-space, EM method.

I. INTRODUCTION

In the reliability theory, the stress-strength model is typically used to investigate a system's reliability and describe the degradation of the strength and the accumulated damage simultaneously. In essence, the model can be represented by a probability function $R = P(X < S)$ where R represents the reliability, X represents the damage, and S represents the strength [1].

Several authors have used different distributions to establish the stress-strength model, which produces a variety of the model forms and various methods to estimate the parameters. In Ref. [2], a Bayesian estimation method for stress-strength models with power Lindley components was derived, and the Markov Chain Monte-Carlo method was used for the implementation of the posterior mean method. Akgul and Senoglu [3] used a Weibull distribution to represent the stress and strength under three types of ranked set sampling. Kohansal [4] and Kohansal and Nadarajah [5] derived the point and interval estimation result of the parameters of the stress-strength model; the stress and strength were two independent Kumaraswamy random variables, and censored samples were also considered. In Ref. [6], the mean remaining strength of a parallel system consisting of strength and stress components in the stress-strength model was estimated; the components followed an exponential distribution. Liang *et al.* [7] considered a multi-component stress-strength model for censored data and, Pak *et al.* [8] developed a multi-component system with a bathtub-shaped distribution. A Bayesian estimation method was used to obtain the

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parameters of the data that were reported in terms of record values. In Ref. [9] a general form of the reliability of a stress-strength model was obtained for distributions of the stress and the strength that included non i.i.d. variables.

It is evident that a stress-strength model describes more than one failure mode since the component is affected by at least two types of stresses described by different distributions which make the form of the likelihood function quite complex in order to consider all situations and result in high computational complexity, especially for general distributions (non-exponential distributions); the convolution is particularly time-consuming.

The phase-type distribution (also known as the PH distribution) is a probability distribution cluster based on the continuous-time Markov process [10]. We assume that $\{X(t), t \geq 0\}$ is a homogeneous Markov process with continuous time and discrete states, where $X(t)$ represents the state of the random process at time t. It is assumed that the set of states in the Markov process is $\{1, 2, \ldots, k+1\}$, where $k+1$ is an absorbing state. The infinitesimal generator of the Markov process can be represented as $Q = \begin{bmatrix} D_0 & d_1 \\ 0 & 0 \end{bmatrix}$ where D_0 is a k^{th}-order transition rate matrix, and $d_1 = -D_0\vec{e}$ is a transition rate vector.

Based on the Markov process, a k^{th}-order continuous PH distribution (α, D_0) can be defined to describe the time t that is required for the transition from any initial state $\{1, 2, \ldots, k\}$ to the absorbing state $k+1$, where, $\alpha = \begin{bmatrix} \alpha_1 & \ldots & \alpha_k \end{bmatrix}$ represents a probability distribution vector of the initial phase, D_0 represents a phase-type generator, and d_1 represents an exit vector.

The cumulative density function (CDF) of the PH distribution is:

$$F(t) = P\{T \leq t\} = 1 - \alpha \exp(D_0 t)\vec{e}, \quad t > 0 \tag{1}$$

where $\vec{e}$ represents a column vector with appropriate dimensions whose elements are all 1. The probability distribution function (PDF) of the PH distribution is:

$$f(t) = \alpha e^{D_0 t} d_1, \quad t > 0 \tag{2}$$

The PH distribution is dense in the positive abscissa axis, which means it can fit any type of probability distribution if the random variable is located in the interval $(0, \infty)$. In addition, a PH distribution also has closure properties in convolution operations, which greatly simplifies the computational complexity. Three general methods for estimating the parameters of the PH distribution are the expectation-maximization (EM) method, Monte-Carlo method, and maximum likelihood method [11]; the EM method [12] has been commonly used in recent studies.

The PH distribution has been widely used for developing shock models [13]–[16] but has been rarely used for establishing stress-strength models. Montoro *et al.* [17], [18] and Montoro and Rafael [19] used the PH distribution to construct stress-strength models of components experiencing shock and continuous wear and characterize the magnitude of the damage caused by single shocks. Therefore, we attempts to introduce the PH distribution into the stress-strength model to simplify the computational complexity of the convolution operations.

The form of the likelihood function of different stress-strength model is differ since the components experience at least two types of stresses simultaneously. Thus, the model needs to be specified first. The stress-strength model proposed in this article is a single-component system that experiences continuous wear and random shocks, as shown in FIGURE 1. When the cumulative magnitude of the damage (blue line) caused by the shock sequence (green line) exceeds the strength (red line) of the component caused by wear, the component fails.

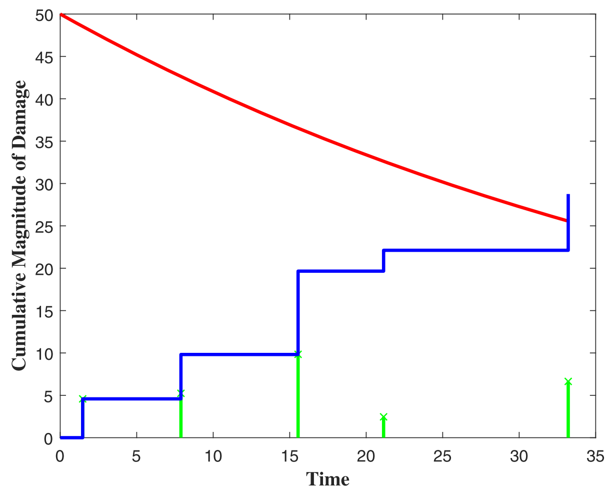

FIGURE 1. Stress-strength model.

The reliability of the stress-strength model in FIGURE 1 at time $t > 0$ can be represented as:

$$\begin{aligned} R(t) &= P(T > t) = P\left(\sum_{i=1}^{N(t)} X_i < S(t)\right) \\ &= \sum_{n=0}^{\infty} P\left(\sum_{i=1}^{n} X_i < S(t) \,|N(t) = n\right) P(N(t) = n) \end{aligned} \tag{3}$$

where $N(t)$ represents the number of shocks that the component receives, and $S(t)$ represents the strength at time t.

Since the PH distribution is obtained based on the continuous-time Markov process, the transition rate of the phases in the Markov state-space must satisfy the relevant properties of the infinitesimal generator. Therefore, the traditional EM method used for estimating the parameters of the PH distribution cannot be used directly in the stress-strength model since the strength changes over time. In response to this problem, Montoro *et al.* divided the strength of the model into different degradation stages [18], during which the strength value was constant, to avoid having to improve the traditional EM method. However, this method changes the form of the model and can't directly reflect the continuous change of the strength.

Therefore, to avoid changing the form of the model and reflect the continuous change of the strength, we propose a method that extends the Markov state-space of the original PH distribution in which the transition rate of the phases changes over time. Both censored and failure samples are considered. The remainder of this article is organized as follows: Section II briefly introduces the stress-strength model with a PH distribution and the assumptions and provides the expressions of the likelihood function for different failure modes. Section III introduces the proposed method in which the Markov state-space of the original PH distribution is extended based on the framework of the traditional EM method. The statistics in the new state-space are established, and the expressions of the model parameters for the estimation are derived. The steps and limitations of the proposed method are also introduced in Section III. Section IV describes two simulation cases to test the performance of the proposed method. Section V summarizes the method and briefly introduces follow-up works.

II. MODEL ASSUMPTIONS AND LIKELIHOOD FUNCTION

A. ASSUMPTIONS OF THE STRESS-STRENGTH MODEL

The following assumptions of the stress-strength model are made.

(1) Assume that the component is subjected to continuous wear, which is a deterministic process and causes continuous degradation of the strength, as well as a random shock sequence that causes discrete cumulative damage over time;

(2) Assume that the degradation of the strength over time is independent of the magnitude of the damage caused by the shock sequence;

(3) Assume that the arrival times of the shocks are independent of their magnitudes and the cumulative magnitude of the existing damage;

(4) Assume that the initial strength of a component is X_{th};

(5) Assume that the change in the strength over time is an exponential process, satisfying $S(t) = ab^t$, $t > 0$ where $a > 0$ and $0 < b < 1$. The function $S(t)$ is a non-increasing and differentiable function and satisfies $S(0) = a = X_{th}$;

(6) Assume that the magnitude of the damage $X_n > 0$ caused by the n-th ($n = 1, 2, \ldots$) shock is an i.i.d. random variable, and the corresponding CDF is $F(x) = P(X_n \leq x)$, $x > 0$ and the PDF is $f(x) = \frac{dF(x)}{dx}$, $x > 0$. Assume that the cumulative magnitude of the damage caused by the previous $n = 1, 2, \ldots$ shocks is $X_n^* = \sum_{i=1}^{n} X_i$. Let $X_0^* = 0$, then the corresponding CDF is $F_n(x) = P\left(X_n^* \leq x\right)$, $x > 0$ and the PDF is $f_n(x) = \frac{dF_n(x)}{dx}$, $x > 0$;

(7) Assume that the arrival time of the n-th ($n = 1, 2, \ldots$) shock is $T_n > 0$. Let $T_0 = 0$, then the corresponding CDF is $G_n(t) = P(T_n \leq t)$, $t > 0$ and the PDF is $g_n(t) = \frac{dG_n(t)}{dt}$, $t > 0$. Assume that the inter-arrival time between two adjacent shocks $\Delta T_n = T_n - T_{n-1} > 0$, $n = 1, 2, \ldots$ is an i.i.d. random variable, the corresponding CDF is $G_\Delta(t) = P(\Delta T_n \leq t)$, $t > 0$, and the PDF is $g_\Delta(t) = \frac{dG_\Delta(t)}{dt}$, $t > 0$.

Let $F_n^*(x) = P\left(X_{n-1}^* < x \leq X_n^*\right)$, $n = 1, 2, \ldots$ represents the probability that the cumulative magnitude of damage X_n^* caused by the previous $n = 1, 2, \ldots$ shocks exceeds x for the first time. Then there is:

$$F_n^*(x) = F_{n-1}(x) - F_n(x), \quad n = 1, 2, \ldots \tag{4}$$

Let $G_n^*(t) = P(N(t) = n) = P(T_n \leq t < T_{n+1})$, $n = 0, 1, \ldots$ represents the probability of the number of shocks occurring at time t. Then, there is:

$$G_n^*(t) = G_n(t) - G_{n+1}(t), \quad n = 0, 1, \ldots \tag{5}$$

According to the assumed parameters and distributions, the reliability function can be represented as:

$$R(t) = \sum_{n=0}^{\infty} F_n\left(S(t)^-\right) G_n^*(t), \quad t > 0 \tag{6}$$

where $S(t)^-$ is the left limit of $S(t)$. The reason for the limit $S(t)^-$ used here is that it is usually considered a failure state of a component when the cumulative magnitude of the damage equals the degraded strength at time t. $P(X_0 < S(t) | N(t) = 0) = 1$ constantly holds true for $t \in (0, \infty)$ in Eq. (6).

B. ASSUMPTIONS OF THE PH DISTRIBUTION PARAMETERS

The PH distribution is used to represent the above distributions as follows:

1) ASSUMPTIONS OF THE DAMAGE DISTRIBUTION PARAMETERS

Assume that the magnitude of damage X_n, $n = 1, 2, \ldots$ caused by a single shock satisfies a υ^{th}-order ($\upsilon = 1, 2, \ldots$) PH distribution (β, R_0), where $\beta = \left[\beta_1 \ \ldots \ \beta_\upsilon\right]$ represents a probability distribution vector of the initial phase, which satisfies $\sum_{i=1}^{\upsilon} \beta_i = 1$, and $R_0 = \begin{bmatrix} r_{11}^{(0)} & \ldots & r_{1\upsilon}^{(0)} \\ \ldots & \ldots & \ldots \\ r_{\upsilon 1}^{(0)} & \ldots & r_{\upsilon\upsilon}^{(0)} \end{bmatrix}$ represents a phase-type generator. Let $r_1 = \left[r_1^{(1)} \ \ldots \ r_\upsilon^{(1)}\right]^T = -R_0 \vec{e}$ represent an exit vector, then the CDF of X_n is:

$$F(x) = 1 - \beta \exp(R_0 x)\, \vec{e}, \quad x > 0 \tag{7}$$

The corresponding PDF is:

$$f(x) = \beta \exp(R_0 x)\, r_1, \quad x > 0 \tag{8}$$

The cumulative magnitude of damage X_n^* can be represented by a new PH distribution $\left(\gamma_n, Y_n^{(0)}\right)$ based on the combination of (β, R_0) [20], where $\gamma_n = \left[\beta \ 0 \ldots 0\right]_{1,\upsilon n}$ represents a probability distribution vector of the initial phase, and the subscripts respectively represent the number of rows and columns of a combined vector or matrix.

$$Y_n^{(0)} = \begin{bmatrix} R_0 & R_1 & \dots & 0 \\ 0 & R_0 & \dots & 0 \\ \dots & \dots & \dots & R_1 \\ 0 & 0 & \dots & R_0 \end{bmatrix}_{\upsilon n,\upsilon n}$$ represents a phase-type generator, where $R_1 = r_1\beta = \begin{bmatrix} r_{11}^{(1)} & \dots & r_{1\upsilon}^{(1)} \\ \dots & \dots & \dots \\ r_{\upsilon 1}^{(1)} & \dots & r_{\upsilon\upsilon}^{(1)} \end{bmatrix}$ represents an absorption-rate matrix, which is also the matrix describing the transition rate between phases when the damage cycle changes. The corresponding exit vector is $y_n^{(1)} = -Y_n^{(0)}\vec{e}$. Then the CDF of X_n^* is:

$$F_n(x) = 1 - \gamma_n \exp\left(Y_n^{(0)}x\right)\vec{e}, \quad x > 0 \tag{9}$$

and the corresponding PDF is:

$$f_n(x) = \gamma_n \exp\left(Y_n^{(0)}x\right)y_n^{(1)}, \quad x > 0 \tag{10}$$

Thus, there is:

$$F_n^*(x) = \gamma_n \exp\left(Y_n^{(0)}x\right)\vec{e}_{n,\Sigma}, \quad x > 0 \tag{11}$$

where $\vec{e}_{n,\Sigma}$ represents a column vector with appropriate dimensions in which the $((n-1)\upsilon+1)$-th to the υn-th elements are all 1, and the remaining elements are all 0.

2) ASSUMPTIONS OF THE TIME DISTRIBUTION PARAMETERS

Assume that the inter-arrival time $\Delta T_n, n = 1, 2, \dots$ satisfies a m^{th}-order $(m = 1, 2, \dots)$ PH distribution (α, D_0), where $\alpha = \begin{bmatrix} \alpha_1 & \dots & \alpha_m \end{bmatrix}$ represents a probability distribution vector of the initial phase, which also satisfies $\sum_{i=1}^{m}\alpha_i = 1$, $D_0 = \begin{bmatrix} d_{11}^{(0)} & \dots & d_{1m}^{(0)} \\ \dots & \dots & \dots \\ d_{m1}^{(0)} & \dots & d_{mm}^{(0)} \end{bmatrix}$ represents a phase-type generator, and $d_1 = \begin{bmatrix} d_1^{(1)} & \dots & d_m^{(1)} \end{bmatrix}^T = -D_0\vec{e}$ represent an exit vector.

Then the CDF of ΔT_n is:

$$G_\Delta(t) = 1 - \alpha \exp(D_0 t)\vec{e}, \quad t > 0 \tag{12}$$

and the corresponding PDF is:

$$g_\Delta(t) = \alpha \exp(D_0 t) d_1, \quad t > 0 \tag{13}$$

Similarly, the n-th $(n = 1, 2, \dots)$ arrival time T_n satisfies a PH distribution $(\mu_n, M_n^{(0)})$, which is based on the combination of (α, D_0), where $\mu_n = [\alpha\ 0 \dots 0]_{1,mn}$ represents a probability distribution vector of the initial phase.

$$M_n^{(0)} = \begin{bmatrix} D_0 & D_1 & \dots & 0 \\ 0 & D_0 & \dots & 0 \\ \dots & \dots & \dots & D_1 \\ 0 & 0 & \dots & D_0 \end{bmatrix}_{mn,mn}$$ represents a phase-type generator, and $D_1 = d_1\alpha = \begin{bmatrix} d_{11}^{(1)} & \dots & d_{1m}^{(1)} \\ \dots & \dots & \dots \\ d_{m1}^{(1)} & \dots & d_{mm}^{(1)} \end{bmatrix}$ represents an absorption-rate matrix that describes the transition rate when the shock arrives. The corresponding exit vector is $m_n^{(1)} = -M_n^{(0)}\vec{e}$. Then the CDF of T_n is:

$$G_n(t) = 1 - \mu_n \exp\left(M_n^{(0)}t\right)\vec{e}, \quad t > 0 \tag{14}$$

and the corresponding PDF is:

$$g_n(t) = \mu_n \exp\left(M_n^{(0)}t\right)m_n^{(0)}, \quad t > 0 \tag{15}$$

Thus, there is:

$$G_n^*(t) = \mu_n \exp\left(M_n^{(0)}t\right)\vec{e}_{n,\Sigma}, \quad t > 0 \tag{16}$$

C. TYPE OF SAMPLE DATA

Assume that the observed sample data are as follows:

(1) The sample size is $S > 0$, and the censored time is $t^{(c)}$.

(2) $\Delta = \{\delta_1, \delta_2, \dots, \delta_S\}$ represents the set of sample type (f-ailure mode), where $\delta_s = 0$ represents the censored sample, $\delta_s = 1$ represents the failure sample caused by the shocks, and $\delta_s = 2$ represents the failure sample caused by continuous wear.

(3) The total number of shocks of sample $s = 1, 2, \dots, S$ is $N = \{n_1, n_2, \dots, n_S\}$, which does not include the number of shocks after failure.

(4) The set of observed time is $T = \{t_1, t_2, \dots, t_S\}$ where $t_s = t^{(c)}$ represents the censored time of sample $s\ (\delta_s = 0)$, and $t_s < t^{(c)}$ represents the failure time of sample $s\ (\delta_s \neq 0)$.

D. LIKELIHOOD FUNCTION OF THE MODEL

Since the arrival time of the shock is independent of the damage it causes, the parameters can be estimated separately. Let $\theta = \{\theta_t, \theta_x\}$ represents the set of model parameters to be estimated, where $\theta_t = \{\alpha, D_0\}$ represents the set of time distribution parameters, and $\theta_x = \{b, \beta, R_0\}$ represents the set of damage-related (including strength degradation) parameters. Let $\vartheta_t^{(s)} = \{t_s, n_s, \delta_s\}$ represent the observed data related to the time parameters of sample $s = 1, 2, \dots, S$, and $\vartheta_x^{(s)} = \left\{X_{th}, \vartheta_t^{(s)}\right\}$ represent the observed data related to the damage parameters of sample $s = 1, 2, \dots, S$. The observed samples can be divided into three types, which are discussed below.

1) CENSORED SAMPLE $(\delta_s = 0)$

For the censored sample $s\ (\delta_s = 0)$, the following applies:

(1) The number of shocks at censored time $t_s = t^{(c)}$ is n_s;

(2) The cumulative magnitude of damage $X_{n_s}^*$ corresponding to the censored time $t_s = t^{(c)}$ satisfies $X_{n_s}^* < S\left(t^{(c)}\right)$. Then there is:

$$\begin{aligned} & L_s\left(\theta; \vartheta_x^{(s)}\right) \\ & = P\left(X_n^* < S\left(t^{(c)}\right) | N\left(t^{(c)}\right) = n_s\right) P\left(N\left(t^{(c)}\right) = n_s\right) \\ & = L_x^{(s)}\left(\theta_x; \vartheta_x^{(s)}\right) L_t^{(s)}\left(\theta_t; \vartheta_t^{(s)}\right) \end{aligned} \tag{17}$$

where $L_x^{(s)}\left(\theta_x; \vartheta_x^{(s)}\right)$ and $L_t^{(s)}\left(\theta_t; \vartheta_t^{(s)}\right)$ are shown in TABLE 1.

TABLE 1. Likelihood function of the damage and time ($\delta_s = 0$).

No.	Type	Likelihood Function
1	Damage	$L_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right)=F_{n_s}\left(S\left(t^{(c)}\right)\right)$ $=\int_0^{S\left(t^{(c)}\right)}\gamma_{n_s}\exp\left(Y_{n_s}^{(0)}x\right)y_{n_s}^{(1)}dx$
2	Time	$L_t^{(s)}\left(\theta_t;\vartheta_t^{(s)}\right)=G_{n_s}^*\left(t^{(c)}\right)=\mu_{n_s+1}\exp\left(M_{n_s+1}^{(0)}t^{(c)}\right)\vec{e}_{n_s+1,\Sigma}$

2) SHOCK FAILURE SAMPLE ($\delta_s = 1$)

For the failure sample s caused by shock ($\delta_s = 0$), the following applies:

(1) The failure time t_s satisfies $0 < t_s < t^{(c)}$, indicating that censoring does not occur;

(2) The n_s-th shock arrives at time t_s;

(3) The cumulative magnitude of damage $X_{n_s-1}^*$ after the (n_s-1)-th shock satisfies $X_{n_s-1}^* < S(t_s)$, and the cumulative magnitude of damage $X_{n_s}^*$ after the n_s-th shock satisfies $X_{n_s}^* \geq S(t_s)$. Then there is:

$$\begin{aligned} L_s\left(\theta;\vartheta_x^{(s)}\right) &= P\left(X_{n_s-1}^* < S(t_s) \leq X_{n_s}^* | T_{n_s}=t_s\right) P\left(T_{n_s}=t_s\right) \\ &= L_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right) L_t^{(s)}\left(\theta_t;\vartheta_t^{(s)}\right) \end{aligned} \tag{18}$$

where $L_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right)$ and $L_t^{(s)}\left(\theta_t;\vartheta_t^{(s)}\right)$ are shown in TABLE 2.

TABLE 2. Likelihood function of the damage and time ($\delta_s = 1$).

No.	Type	Likelihood Function
1	Damage	$L_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right)=F_{n_s}^*\left(S\left(t_s\right)\right)=\gamma_{n_s}\exp\left(Y_{n_s}^{(0)}S\left(t_s\right)\right)\vec{e}_{n_s,\Sigma}$
2	Time	$L_t^{(s)}\left(\theta_t;\vartheta_t^{(s)}\right)=g_{n_s}\left(t_s\right)=\mu_{n_s}\exp\left(M_{n_s}^{(0)}t_s\right)m_{n_s}^{(1)}$

3) WEAR FAILURE SAMPLE ($\delta_s = 2$)

For the failure sample s caused by wear ($\delta_s = 2$), the following applies:

(1) The failure time t_s satisfies $0 < t_s < t^{(c)}$, indicating that censoring does not occur;

(2) The strength $S(t_s)$ at time t_s is equal to the cumulati-ve magnitude of damage $X_{n_s}^*$ caused by the shocks;

(3) The number of shocks arriving at time t_s is n_s, which means that the arrival time T_{n_s} of the n_s-th shock satisfies $T_{n_s} \leq t_s$ ("=" denotes that the arrival time of the n_s-th shock is t_s, and the cumulative magnitude of damage $X_{n_s}^*$ satisfies $X_{n_s}^* = S(t_s)$), and the potential arrival time T_{n_s+1} of the (n_s+1)-th shock satisfies $T_{n_s+1} > t_{n_s}$. Then there is:

$$\begin{aligned} L_s\left(\theta|\vartheta_x^{(s)}\right) &= P\left(X_{n_s}^* = S(t_s) | N(t_s) = n_s\right) P\left(N(t_s) = n_s\right) \\ &= L_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right) L_t^{(s)}\left(\theta_t;\vartheta_t^{(s)}\right) \end{aligned} \tag{19}$$

where $L_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right)$ and $L_t^{(s)}\left(\theta_t;\vartheta_t^{(s)}\right)$ are shown in TABLE 3.

TABLE 3. Likelihood function of the damage and time ($\delta_s = 2$).

No.	Type	Likelihood Function
1	Damage	$L_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right)=\left\|S'\left(t_s\right)\right\|f_{n_s}\left(S\left(t_s\right)\right)$ $=\gamma_{n_s}\exp\left(Y_{n_s}^{(0)}S\left(t_s\right)\right)y_{n_s}^{(1)}\left\|S'\left(t_s\right)\right\|$
2	Time	$L_t^{(s)}\left(\theta_t;\vartheta_t^{(s)}\right)=G_{n_s}^*\left(t_s\right)=\mu_{n_s+1}\exp\left(M_{n_s+1}^{(0)}t_s\right)\vec{e}_{n_s+1,\Sigma}$

In TABLE 3, $\left|S'(t)\right| = -\frac{dS(t)}{dt}$. The symbol of absolute value must be added since $S(t)$ is a non-increasing and derivable function, whereas the value of the likelihood function remains non-negative.

In summary, the overall likelihood function of the sample data can be represented as:

$$\begin{aligned} L\left(\theta|X_{th}, T, N, \Delta\right) &= \prod_{s=1}^{S} L_s\left(\theta|\vartheta_x^{(s)}\right) \\ &= \prod_{s=1}^{S} L_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right) L_t^{(s)}\left(\theta_t;\vartheta_t^{(s)}\right) \end{aligned} \tag{20}$$

III. PRINCIPLE, DERIVATION AND STEPS OF THE METHOD

The damage-related parameter θ_x and the time-related parameter θ_t can be estimated separately since the arrival time of a shock is independent of the damage it causes. The parameter estimation method of θ_t is described in Refs. [20]–[22]. We only introduce the parameter estimation method of $\theta_x = \{\beta, R_0, b\}$.

A. PRINCIPLE OF THE METHOD

Compared with the likelihood functions of general PH distribution models, the likelihood function of the stress-strength model has a more complicated form that includes additional failure modes.

According to the previous assumptions, the CDF of X_n^* is $F_n(x)$. Let $x = S(t)$, there is:

$$F_n(S(t)) = 1 - \gamma_n \exp\left(Y_n^{(0)} S(t)\right)\vec{e} \tag{21}$$

The PDF with respect to time t for the n-th shocks is:

$$\begin{aligned} f^{(n)}(t) &= \frac{dF_n(S(t))}{dt} = f_n(S(t))\left|S'(t)\right| \\ &= \gamma_n \exp\left(Y_n^{(0)} S(t)\right) y_n^{(1)}\left|S'(t)\right|, \quad t > 0 \end{aligned} \tag{22}$$

Generally speaking, the phase-type generator $Y_n^{(0)}$ and exit vector $y_n^{(1)}$ with the distribution $\left(\gamma_n, Y_n^{(0)}\right)$ satisfy $y_n^{(1)} = -Y_n^{(0)}\vec{e} < \infty$, which means that the Markov state-space consisting of all phases and the absorbing state based on the distribution (β, R_0) is considered conservative. In contrast, for $f^{(n)}(t)$, the condition of the transition rate matrix in this Markov state-space cannot be met with respect to time t since $y_n^{(1)}\left|S'(t)\right| \neq -Y_n^{(0)}S(t)\vec{e}$ usually exists, especially when $y_n^{(1)}\left|S'(t)\right| > -Y_n^{(0)}S(t)\vec{e}$. Therefore, the general EM

$$
\begin{bmatrix}
\boxed{\begin{matrix} S(t)r_{11}^{(0)}\text{-}\xi & \dots & S(t)r_{1\upsilon}^{(0)} \\ \dots & \dots & \dots \\ S(t)r_{\upsilon 1}^{(0)} & \dots & S(t)r_{\upsilon\upsilon}^{(0)}\text{-}\xi \end{matrix}} & \boxed{\begin{matrix} S(t)r_{1}^{(1)}\beta_1 & \dots & S(t)r_{1}^{(1)}\beta_\upsilon \\ \dots & \dots & \dots \\ S(t)r_{\upsilon}^{(1)}\beta_1 & \dots & S(t)r_{\upsilon}^{(1)}\beta_\upsilon \end{matrix}} & \dots & 0 \\
0 & \boxed{\begin{matrix} S(t)r_{11}^{(0)}\text{-}\xi & \dots & S(t)r_{1\upsilon}^{(0)} \\ \dots & \dots & \dots \\ S(t)r_{\upsilon 1}^{(0)} & \dots & S(t)r_{\upsilon\upsilon}^{(0)}\text{-}\xi \end{matrix}} & \dots & 0 \\
\dots & \dots & \dots & \dots \\
0 & 0 & \dots & \boxed{\begin{matrix} S(t)r_{11}^{(0)}\text{-}\xi & \dots & S(t)r_{1\upsilon}^{(0)} \\ \dots & \dots & \dots \\ S(t)r_{\upsilon 1}^{(0)} & \dots & S(t)r_{\upsilon\upsilon}^{(0)}\text{-}\xi \end{matrix}}
\end{bmatrix}
$$

FIGURE 2. Phase-type generator of the extended Markov state-space.

TABLE 4. Likelihood function of the extended markov state-space.

No.	Sample Type	Likelihood Function
1	Censored sample $(\delta_s = 0)$	$\tilde{L}_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right) = \sum_{n=n_s+1}^{\infty} \gamma_n \exp\left(Y_n^{(0)} S\left(t^{(c)}\right) - \xi E\right) \vec{e}_{n,\Sigma}$
2	Shock failure samples $(\delta_s = 1)$	$\tilde{L}_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right) = \gamma_{n_s} \exp\left(Y_{n_s}^{(0)} S\left(t_s\right) - \xi E\right) \vec{e}_{n_s,\Sigma}$
3	Wear failure samples $(\delta_s = 2)$	$\tilde{L}_x^{(s)}\left(\theta_x;\vartheta_x^{(s)}\right) = \gamma_{n_s} \exp\left(Y_{n_s}^{(0)} S\left(t_s\right) - \xi E\right) y_{n_s}^{(1)} \left\lvert S'\left(t_s\right)\right\rvert$

method for estimating the parameters of the PH distribution cannot be used directly here. Thus, a method based on extending the Markov state-space with variable transition rates in response to the above situation is proposed.

According to the previous assumptions, the cumulative magnitude of the damage caused by the phase transitions from any non-absorption phase to the absorbing state satisfies the PH distribution $\left(\gamma_n, Y_n^{(0)}\right)$. We add another absorbing state to extend the original Markov state-space.

Assume that the exit vector in the extended Markov state-space from non-absorbing states to the original absorbing states is $y_n^{(1)} \left|S'(t)\right|$ and to the additional absorbing state is $y_n^{(2)}$.

The phase-type generator in the extended state-space is shown in FIGURE 2, where $\xi > 0$ is a known positive real number that is large enough to ensure that each element in exit vector $y_n^{(2)}$ is greater than or equal to 0.

Let $y_n^{(2)} = -\left(Y_n^{(0)} S(t) - \xi E\right) \vec{e} - y_n^{(1)} \left|S'(t)\right|$ to ensure that the extended Markov state-space remains conservative. Similarly to the original phase-type generator $Y_n^{(0)}$, the sub-matrix in the main diagonal boxes in FIGURE 2 represents the phase transition rate in an equivalent damage cycle. The sub-matrix in the sub-diagonal boxes represents the phase transition rate when the equivalent damage cycle changes.

Based on the extended Markov state-space, three new likelihood functions are established for different sample types of the stress-strength model in TABLE 4.

It is found that the likelihood functions in TABLE 4 are all proportional to the original likelihood function in TABLES 1, 2, and 3. Due to space limitations, only the likelihood function of the failure sample caused by wear $(\delta_s = 2)$ is proved here as follows:

Let $f^{(n)*}(t) = \gamma_n \exp\left(Y_n^{(0)} S(t) - \xi E\right) y_n^{(1)} \left|S'(t)\right|$, which can be regarded as the PDF of an equivalent cumulative magnitude of the damage in which the phase in the extended Markov state-space is assumed to transition from any non-absorbing state to the absorbing state with the phase-type generator $\left(Y_n^{(0)} S(t) - \xi E\right)$ and exit vector $y_n^{(1)} \left|S'(t)\right|$. The elements in the above matrix and vector change over time, and the equivalent cumulative magnitude is 1.

Assume that the product of a square matrix A and a square matrix B are interchangeable, namely $AB = BA$. Then, there is:

$$\exp(A + B) = \exp(A)\exp(B) \tag{23}$$

Thus, the known positive real number $\xi > 0$ is introduced to transfer the function $f^{(n)}(t)$ as follows:

$$f^{(n)}(t) = \gamma_n \exp\left(\left(Y_n^{(0)} S(t) - \xi E\right) + \xi E\right) y_n^{(1)} \left|S'(t)\right| \tag{24}$$

where E is the identity matrix with appropriate dimensions. Let $A = Y_n^{(0)} S(t) - \xi E$ and $B = \xi E$. Then, there is:

$$AB = \left(Y_n^{(0)} S(t) - \xi E\right) \xi E = \xi E \left(Y_n^{(0)} S(t) - \xi E\right) = BA \tag{25}$$

TABLE 5. Value of statistics $\left\{B_{n,i}^{(s)}\right\}$.

No.	Sample Type	Shock Number	Value of Statistics
1	Censored sample ($\delta_s = 0$)	$1 \le n \le n_s + n_s^{(h)}$	$B_{n,i}^{(s)} = \delta_{n,i}^{(s)}[0]$, satisfies $\sum_{i=1}^{\upsilon} B_{n,i}^{(s)} = 1$
		$n > n_s + n_s^{(h)}$	$B_{n,i}^{(s)} = 0$
2	Failure samples ($\delta_s \ne 0$)	$1 \le n \le n_s$	$B_{n,i}^{(s)} = \delta_{n,i}^{(s)}[0]$, satisfies $\sum_{i=1}^{\upsilon} B_{n,i}^{(s)} = 1$
		$n > n_s$	$B_{n,i}^{(s)} = 0$

The product of the square matrix A and square matrix B are interchangeable. Then there is:

$$f^{(n)}(t) = \gamma_n \exp\left(Y_n^{(0)} S(t) - \xi E\right) \exp(\xi E)\, y_n^{(1)} \left|S'(t)\right| \tag{26}$$

According to the Taylor formula:

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = E + A + \frac{A^2}{2!} + \ldots + \frac{A^m}{m!} + \ldots \tag{27}$$

it is known that:

$$\exp(\xi E) = \sum_{k=0}^{\infty} \frac{(\xi E)^k}{k!} = \sum_{k=0}^{\infty} \left(\frac{\xi^k}{k!}\right) E = \exp(\xi) E \tag{28}$$

Then, there is:

$$\begin{aligned} f^{(n)}(t) &= \gamma_n \exp\left(Y_n^{(0)} S(t) - \xi E\right) y_n^{(1)} \left|S'(t)\right| \exp(\xi) \\ &= f^{(n)^*}(t) \exp(\xi) \end{aligned} \tag{29}$$

The function $f^{(n)}(t)$ is proportional to the function $f^{(n)^*}(t)$. Thus, if the extreme value of $f^{(n)}(t_0)$ can be obtained based on the estimated result $\hat{\theta}|_{t=t_0}$, the extreme value of $f^{(n)^*}(t)$ can also be reached with parameters $\hat{\theta}|_{t=t_0}$ at the same $t = t_0$, which means that the function $f^{(n)^*}(t)$ can be used for parameter estimation. Note that the function $f^{(n)^*}(t)$ does not satisfy the basic property of the PDF, namely, $\int_0^\infty f^{(n)^*}(t)\, dx \ne 1$. However, the following derivation indicates that this does not affect the final result.

The proof of the likelihood function of the other sample types is similar. Therefore, the likelihood function based on extending the original Markov state-space with variable transition rates in TABLE 4 can be used to estimate the parameters of the stress-strength model.

B. ASSUMPTIONS OF THE STATISTICS

According to the traditional EM method, the likelihood function with complete data has to be established. First, four parameters and one function have to be defined.

(1) Let $d_{n,k}^{(s)}$ represent the phase of sample $s = 1, 2, \ldots, S$ after the k-th $(k = 0, 1, \ldots)$ phase transition in the n-th $(n = 1, 2, \ldots)$ equivalent damage cycle.

(2) Let $b_{n,k}^{(s)}$ represent the equivalent damage magnitude of sample $s = 1, 2, \ldots, S$ in phase $d_{n,k}^{(s)}$ after the k-th $(k = 0, 1, \ldots)$ phase transition in the n-th $(n = 1, 2, \ldots)$ equivalent damage cycle.

(3) Let $n_n^{(s)}$ represent the number of phase transitions of sample $s = 1, 2, \ldots, S$ in the n-th $(n = 1, 2, \ldots)$ equivalent damage cycle, including the phase transition when the cycle changes or in the absorbing state.

(4) Let $N_s^{(h)}$ represent the number of additional shocks that may cause failure if the sample $s\,(\delta_s = 0)$ is censored with shock number n_s. It can be seen that $N_s^{(h)} > 0$ is a random variable, and $n_s^{(h)}$ represents a specific sample value of $N_s^{(h)}$;

(5) Let function $\delta_{n,i}^{(s)}[k] = \begin{cases} 1 & d_{n,k}^{(s)} = i \\ 0 & d_{n,k}^{(s)} \ne i \end{cases}$ represent whether the phase of sample $s = 1, 2, \ldots, S$ after the k-th $(k = 0, 1, \ldots)$ phase transition in the n-th $(n = 1, 2, \ldots)$ equivalent damage cycle is $i = 1, 2, \ldots, \upsilon$.

The following four sets of statistics can be established based on the above assumptions.

(1) Let the set of statistics $\left\{B_{n,i}^{(s)}\right\}$ represent whether the initial phase of the sample $s = 1, 2, \ldots, S$ in the n-th $(n = 1, 2, \ldots)$ equivalent damage cycle is $i = 1, 2, \ldots, \upsilon$. The values of $\left\{B_{n,i}^{(s)}\right\}$ are shown in TABLE 5.

(2) Let the set of statistics $\left\{Z_{n,i}^{(s)}\right\}$ represents the equivalent damage magnitude of sample $s = 1, 2, \ldots, S$ in phase $i = 1, 2, \ldots, \upsilon$ in the n-th $(n = 1, 2, \ldots)$ equivalent damage cycle. The values of $\left\{Z_{n,i}^{(s)}\right\}$ are shown in TABLE 6.

(3) Let the set of statistics $\left\{M_{n,ij}^{(s)}\right\}$ represents the number of phase transitions of sample $s = 1, 2, \ldots, S$ from $i = 1, 2, \ldots, \upsilon$ to $j = 1, 2, \ldots, \upsilon\,(i \ne j)$ in the n-th equivalent damage cycle, excluding the transition when the cycle changes. The values of $\left\{M_{n,ij}^{(s)}\right\}$ are shown in TABLE 7.

(4) The set of statistics $\left\{Y_{n,i}^{(s)}\right\}$ represents whether the $n+1$-th equivalent damage cycle or the absorbing state is transferred from the phase $i = 1, 2, \ldots, \upsilon$ of sample $s = 1, 2, \ldots, S$ in the n-th $(n = 1, 2, \ldots)$ equivalent damage cycle. The values of $\left\{Y_{n,i}^{(s)}\right\}$ are shown in TABLE 8.

Let $H = \{H_1, H_2, \ldots, H_S\}$ represent the set of statistics of all samples where

$$H_s = \left\{\left\{B_{n,i}^{(s)}\right\}, \left\{Z_{n,i}^{(s)}\right\}, \left\{M_{n,ij}^{(s)}\right\}, \left\{Y_{n,i}^{(s)}\right\}\right\}$$

represents the set of statistics of sample $s = 1, 2, \ldots, S$.

TABLE 6. Value of statistics $\left\{Z_{n,i}^{(s)}\right\}$.

No.	Sample Type	Shock Number	Value of Statistics
1	Censored sample $(\delta_s = 0)$	$1 \le n \le n_s + n_s^{(h)}$	$Z_{n,i}^{(s)} = \sum_{k=0}^{n_n^{(s)}-1} \delta_{n,i}^{(s)}[k] b_{n,k}^{(s)}$
		$n = n_s + n_s^{(h)}$	$Z_{n,i}^{(s)} = \sum_{k=0}^{n_n^{(s)}} \delta_{n,i}^{(s)}[k] b_{n,k}^{(s)}$, satisfies $\sum_{n=1}^{n_s+n_s^{(h)}} \sum_{i=1}^{\upsilon} Z_{n,i}^{(s)} = 1$
		$n > n_s + n_s^{(h)}$	$Z_{n,i}^{(s)} = 0$
2	Shock failure samples $(\delta_s = 1)$	$1 \le n < n_s$	$Z_{n,i}^{(s)} = \sum_{k=0}^{n_n^{(s)}-1} \delta_{n,i}^{(s)}[k] b_{n,k}^{(s)}$
		$n = n_s$	$Z_{n,i}^{(s)} = \sum_{k=0}^{n_n^{(s)}} \delta_{n,i}^{(s)}[k] b_{n,k}^{(s)}$, satisfies $\sum_{n=1}^{n_s} \sum_{i=1}^{\upsilon} Z_{n,i}^{(s)} = 1$
		$n > n_s$	$Z_{n,i}^{(s)} = 0$
3	Wear failure sample $(\delta_s = 2)$	$1 \le n \le n_s$	$Z_{n,i}^{(s)} = \sum_{k=0}^{n_n^{(s)}-1} \delta_{n,i}^{(s)}[k] b_{n,k}^{(s)}$, satisfies $\sum_{n=1}^{n_s} \sum_{i=1}^{\upsilon} Z_{n,i}^{(s)} = 1$
		$n > n_s$	$Z_{n,i}^{(s)} = 0$

C. LIKELIHOOD FUNCTION FOR COMPLETE DATA

Based on the above four assumptions, the likelihood function for complete data can be represented as:

$$L_x(\theta_x; X_{th}, T, N, \Delta, H) = \prod_{s=1}^{S} L_x^{(s)}(\theta_x; \vartheta_x^{(s)}, H_s) = \prod_{s=1}^{S} L_x^{(s)}(\theta_x; H_s) = L_x(\theta_x; H) \quad (30)$$

where the likelihood function of the censored sample $(\delta_s = 0)$ for complete data is:

$$\begin{aligned} L_x^{(s)}(\theta_x; H_s) &= \left(\prod_{n=1}^{\infty}\prod_{i=1}^{\upsilon} \beta_i^{B_{n,i}^{(s)}}\right)\left(\prod_{n=1}^{\infty}\prod_{i=1}^{\upsilon} \exp\left(\left(S(t_s)\, r_{ii}^{(0)} - \xi\right) Z_{n,i}^{(s)}\right)\right)\cdots \\ &\quad \left(\prod_{n=1}^{\infty}\prod_{i=1}^{\upsilon}\prod_{j=1,i\neq j}^{\upsilon} \left(S(t_s)\, r_{ij}^{(0)}\right)^{M_{n,ij}^{(s)}}\right) \\ &\quad \times \left(\prod_{n=1}^{\infty}\prod_{i=1}^{\upsilon} \left(r_i^{(1)} S(t_s)\right)^{Y_{n,i}^{(s)}}\right), \end{aligned} \quad (31)$$

the likelihood function of the shock failure sample $(\delta_s = 1)$ for complete data is:

$$\begin{aligned} L_x^{(s)}(\theta_x; H_s) &= \left(\prod_{n=1}^{n_s}\prod_{i=1}^{\upsilon} \beta_i^{B_{n,i}^{(s)}}\right)\left(\prod_{n=1}^{n_s}\prod_{i=1}^{\upsilon} \exp\left(\left(S(t_s)\, r_{ii}^{(0)} - \xi\right) Z_{n,i}^{(s)}\right)\right)\cdots \\ &\quad \left(\prod_{n=1}^{n_s}\prod_{i=1}^{\upsilon}\prod_{j=1,i\neq j}^{\upsilon} \left(S(t_s)\, r_{ij}^{(0)}\right)^{M_{n,ij}^{(s)}}\right) \\ &\quad \times \left(\prod_{n=1}^{n_s-1}\prod_{i=1}^{\upsilon} \left(r_i^{(1)} S(t_s)\right)^{Y_{n,i}^{(s)}}\right), \end{aligned} \quad (32)$$

and the likelihood function of the wear failure sample $(\delta_s = 2)$ for complete data is:

$$\begin{aligned} L_x^{(s)}(\theta_x; H_s) &= \left(\prod_{n=1}^{n_s}\prod_{i=1}^{\upsilon} \beta_i^{B_{n,i}^{(s)}}\right)\left(\prod_{n=1}^{n_s}\prod_{i=1}^{\upsilon} \exp\left(\left(S(t_s)\, r_{ii}^{(0)} - \xi\right) Z_{n,i}^{(s)}\right)\right)\cdots \\ &\quad \left(\prod_{n=1}^{n_s}\prod_{i=1}^{\upsilon}\prod_{j=1,i\neq j}^{\upsilon} \left(S(t_s)\, r_{ij}^{(0)}\right)^{M_{n,ij}^{(s)}}\right) \\ &\quad \times \left(\prod_{n=1}^{n_s-1}\prod_{i=1}^{\upsilon} \left(r_i^{(1)} S(t_s)\right)^{Y_{n,i}^{(s)}}\right)\cdots \\ &\quad \left(\prod_{i=1}^{\upsilon} \left(r_i^{(1)} \left|S'(t_s)\right|\right)^{Y_{n,i}^{(s)}}\right) \end{aligned} \quad (33)$$

From function (30), the log-likelihood function can be represented as:

$$\ln L_x(\theta_x; X_{th}, T, N, \Delta, H) = \ln L_x(\theta_x; H) = \sum_{s=1}^{S} \ln L_x^{(s)}(\theta_x; H_s) \quad (34)$$

It is known from the properties of the statistics that:

$$\begin{cases} \sum_{n=1}^{\infty}\sum_{i=1}^{\upsilon} \xi Z_{n,i}^{(s)} = \xi,\ (\delta_s = 0) \\ \sum_{n=1}^{n_s}\sum_{i=1}^{\upsilon} \xi Z_{n,i}^{(s)} = \xi,\ (\delta_s \neq 0) \end{cases} \quad (35)$$

The posterior distribution of H for the overall samples can be represented as:

$$f(H|\theta_x, X_{th}, T, N, \Delta) = \frac{L_x(\theta_x; H)}{\int_H L_x(\theta_x; H)dH} \quad (36)$$

It is assumed that the damage-related parameter $\hat{\theta}_x$ has already been obtained from an iteration step of the EM

TABLE 7. Value of statistics $\left\{M_{n,ij}^{(s)}\right\}$.

No.	Sample Type	Shock Number	Value of Statistics
1	Censored sample $(\delta_s = 0)$	$1 \le n \le n_s + n_s^{(h)}$	$M_{n,ij}^{(s)} = \sum_{k=0}^{n_n^{(s)}-2} \delta_{n,i}^{(s)}[k]\delta_{n,j}^{(s)}[k+1]$
		$n = n_s + n_s^{(h)}$	$M_{n,ij}^{(s)} = \sum_{k=0}^{n_n^{(s)}-1} \delta_{n,i}^{(s)}[k]\delta_{n,j}^{(s)}[k+1]$
		$n > n_s + n_s^{(h)}$	$M_{n,ij}^{(s)} = 0$
2	Shock failure samples $(\delta_s = 1)$	$1 \le n < n_s$	$M_{n,ij}^{(s)} = \sum_{k=0}^{n_n^{(s)}-2} \delta_{n,i}^{(s)}[k]\delta_{n,j}^{(s)}[k+1]$
		$n = n_s$	$M_{n,ij}^{(s)} = \sum_{k=0}^{n_n^{(s)}-1} \delta_{n,i}^{(s)}[k]\delta_{n,j}^{(s)}[k+1]$
		$n > n_s$	$M_{n,ij}^{(s)} = 0$
3	Wear failure sample $(\delta_s = 2)$	$1 \le n \le n_s$	$M_{n,ij}^{(s)} = \sum_{k=0}^{n_n^{(s)}-2} \delta_{n,i}^{(s)}[k]\delta_{n,j}^{(s)}[k+1]$
		$n > n_s$	$M_{n,ij}^{(s)} = 0$

method. Then, combined with Eq. (35), the expectation of the log-likelihood function for the posterior distribution (36) of the statistic H can be represented as:

$$E_{\hat{\theta}_x}\left(\ln L_x\left(\theta_x; H\right)|X_{th}, T, N, \Delta\right) = \sum_{s=1}^{S} E_{\hat{\theta}_x}\left(\ln L_x^{(s)}\left(\theta_x; H_s\right)|\vartheta_x^{(s)}\right) \tag{37}$$

where the expectation of the censored sample ($\delta_s = 0$) is:

$$\begin{aligned} E_{\hat{\theta}_x}&\left(\ln L_x^{(s)}\left(\theta_x; H_s\right)|\vartheta_x^{(s)}\right) \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(B_{n,i}^{(s)}|\vartheta_x^{(s)}\right)\ln\beta_i \ldots \\ &\quad + \sum_{n=1}^{\infty}\sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)}|\vartheta_x^{(s)}\right) S\left(t_s\right) r_{ii}^{(0)} \ldots \\ &\quad + \sum_{n=1}^{\infty}\sum_{i=1}^{\upsilon}\sum_{j=1,i\neq j}^{\upsilon} E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)}|\vartheta_x^{(s)}\right)\ln S\left(t_s\right) r_{ij}^{(0)} \ldots \\ &\quad + \sum_{n=1}^{\infty}\sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_x^{(s)}\right)\ln S\left(t_s\right) r_i^{(1)} - \xi, \end{aligned} \tag{38}$$

the expectation of the shock failure sample ($\delta_s = 1$) is:

$$\begin{aligned} E_{\hat{\theta}_x}&\left(\ln L_x^{(s)}\left(\theta_x; H_s\right)|\vartheta_x^{(s)}\right) \\ &= \sum_{n=1}^{n_s}\sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(B_{n,i}^{(s)}|\vartheta_x^{(s)}\right)\ln\beta_i \ldots \\ &\quad + \sum_{n=1}^{n_s}\sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)}|\vartheta_x^{(s)}\right) S\left(t_s\right) r_{ii}^{(0)} \ldots \\ &\quad + \sum_{n=1}^{n_s}\sum_{i=1}^{\upsilon}\sum_{j=1,i\neq j}^{\upsilon} E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)}|\vartheta_x^{(s)}\right)\ln S\left(t_s\right) r_{ij}^{(0)} \ldots \\ &\quad + \sum_{n=1}^{n_s-1}\sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_x^{(s)}\right)\ln S\left(t_s\right) r_i^{(1)} - \xi, \end{aligned} \tag{39}$$

and the expectation of the wear failure sample ($\delta_s = 2$) is:

$$\begin{aligned} E_{\hat{\theta}_x}&\left(\ln L_x^{(s)}\left(\theta_x; H_s\right)|\vartheta_x^{(s)}\right) \\ &= \sum_{n=1}^{n_s}\sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(B_{n,i}^{(s)}|\vartheta_x^{(s)}\right)\ln\beta_i \ldots \\ &\quad + \sum_{n=1}^{n_s}\sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)}|\vartheta_x^{(s)}\right) S\left(t_s\right) r_{ii}^{(0)} \ldots \\ &\quad + \sum_{n=1}^{n_s}\sum_{i=1}^{\upsilon}\sum_{j=1,i\neq j}^{\upsilon} E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)}|\vartheta_x^{(s)}\right)\ln S\left(t_s\right) r_{ij}^{(0)} \ldots \\ &\quad + \sum_{n=1}^{n_s-1}\sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_x^{(s)}\right)\ln S\left(t_s\right) r_i^{(1)} \ldots. \\ &\quad + \sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(Y_{n_s,i}^{(s)}|\vartheta_x^{(s)}\right)\ln\left|S'\left(t_s\right)\right| r_i^{(1)} - \xi. \end{aligned} \tag{40}$$

D. EXPRESSIONS OF THE PARAMETERS

The parameter θ_x to be estimated satisfies the following constraints:

① The probability distribution of the initial phases satisfies $\sum_{i=1}^{\upsilon}\beta_i = 1$;

② $\sum_{j=1}^{\upsilon} r_{ij}^{(0)} + r_i^{(1)} = 0$ for any phase $i = 1, 2, \ldots, \upsilon$ in the phase-type generator;

③ the wear parameter b satisfies $0 < b < 1$.

A Lagrange multiplier $\{\eta_0, \eta_1, \ldots, \eta_\upsilon\}$ is introduced to transform the constrained optimization problem into an unconstrained optimization problem. The optimized

TABLE 8. Value of statistics $\left\{Y_{n,i}^{(s)}\right\}$.

No.	Sample Type	Shock Number	Value of Statistics
1	Censored sample $(\delta_s = 0)$	$1 \le n < n_s + n_s^{(h)}$	$Y_{n,i}^{(s)} = \delta_{n,i}^{(s)}\left[n_n^{(s)} - 1\right]$, satisfies $\sum_{n=1}^{n_s+n_s^{(h)}} \sum_{i=1}^{\upsilon} Y_{n,i}^{(s)} = n_s + n_s^{(h)} - 1$
		$n \ge n_s + n_s^{(h)}$	$Y_{n,i}^{(s)} = 0$
2	Shock failure samples $(\delta_s = 1)$	$1 \le n < n_s$	$Y_{n,i}^{(s)} = \delta_{n,i}^{(s)}\left[n_n^{(s)} - 1\right]$, satisfies $\sum_{n=1}^{n_s} \sum_{i=1}^{\upsilon} Y_{n,i}^{(s)} = n_s - 1$
		$n \ge n_s$	$Y_{n,i}^{(s)} = 0$
3	Wear failure sample $(\delta_s = 2)$	$1 \le n \le n_s$	$Y_{n,i}^{(s)} = \delta_{n,i}^{(s)}\left[n_n^{(s)} - 1\right]$, satisfies $\sum_{n=1}^{n_s} \sum_{i=1}^{\upsilon} Y_{n,i}^{(s)} = n_s$
		$n > n_s$	$Y_{n,i}^{(s)} = 0$

TABLE 9. Expressions of the parameters in (β, R_0).

No.	Expression	Condition
1	$\beta_i = \dfrac{\sum_{s=1,\delta_s=0}^{S} \sum_{n=1}^{\infty} E_{\hat{\theta}_x}\left(B_{n,i}^{(s)} \mid \vartheta_s\right) + \sum_{s=1,\delta_s\neq 0}^{S} \sum_{n=1}^{n_s} E_{\hat{\theta}_x}\left(B_{n,i}^{(s)} \mid \vartheta_s\right)}{\sum_{s=1,\delta_s=0}^{S} \sum_{n=1}^{\infty} \sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(B_{n,i}^{(s)} \mid \vartheta_s\right) + \sum_{s=1,\delta_s\neq 0}^{S} \sum_{n=1}^{n_s} \sum_{i=1}^{\upsilon} E_{\hat{\theta}_x}\left(B_{n,i}^{(s)} \mid \vartheta_s\right)}$	$i = 1, 2, \ldots, \upsilon$
2	$r_{ij}^{(0)} = \dfrac{\sum_{s=1,\delta_s=0}^{S} \sum_{n=1}^{\infty} E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)} \mid \vartheta_s\right) + \sum_{s=1,\delta_s\neq 0}^{S} \sum_{n=1}^{n_s} E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)} \mid \vartheta_s\right)}{\sum_{s=1,\delta_s=0}^{S} S(t_s) \sum_{n=1}^{\infty} E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)} \mid \vartheta_s\right) + \sum_{s=1,\delta_s\neq 0}^{S} S(t_s) \sum_{n=1}^{n_s} E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)} \mid \vartheta_s\right)}$	$i, j = 1, 2, \ldots, \upsilon; i \neq j$
3	$r_i^{(1)} = \dfrac{\sum_{s=1,\delta_s=0}^{S} \sum_{n=1}^{\infty} E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)} \mid \vartheta_s\right) + \sum_{s=1,\delta_s=1}^{S} \sum_{n=1}^{n_s-1} E_{\theta_x^{(0)}}\left(Y_{n,i}^{(s)} \mid \vartheta_s\right) + \sum_{s=1,\delta_s=2}^{S} \sum_{n=1}^{n_s} E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)} \mid \vartheta_s\right)}{\sum_{s=1,\delta_s=0}^{S} S(t_s) \sum_{n=1}^{\infty} E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)} \mid \vartheta_s\right) + \sum_{s=1,\delta_s\neq 0}^{S} S(t_s) \sum_{n=1}^{n_s} E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)} \mid \vartheta_s\right)}$	$i = 1, 2, \ldots, \upsilon$
4	$r_{ii}^{(0)} = -\left(\sum_{j=1,i\neq j}^{\upsilon} r_{ij}^{(0)} + r_i^{(1)}\right)$	$i = 1, 2, \ldots, \upsilon$

objective function is:

$$\Phi(\theta_x) = \sum_{s=1}^{S} E_{\hat{\theta}_x}\left(\ln L_x^{(s)}(\theta_x; H_s) \,|\, \vartheta_x^{(s)}\right) \ldots$$
$$+\eta_0\left(1 - \sum_{i=1}^{\upsilon}\beta_i\right) + \sum_{i=1}^{\upsilon}\eta_i\left(\sum_{j=1}^{\upsilon} r_{ij}^{(0)} + r_i^{(1)}\right) \tag{41}$$

It has no effect on the parameter estimation since the value of ξ is known. The expressions of the parameters in the distribution (β, R_0) are listed in TABLE 9.

The partial derivative of the objective function $\Phi(\theta_x)$ with respect to the parameters in $S(t_s)$ is represented in Eq. (42), as shown at the bottom of the next page.

The value of parameter b can be obtained by bringing the expressions in TABLE 9 into Eq. (42). Subsequently, the parameter values of the PH distribution (β, R_0) can be obtained by bringing b back. Note that $S(t_s)$ is not replaced by ab^{t_s} or any other specific form in the parameter estimation process. The deduction result shows that the parameter in $S(t_s)$ only exists in the unique Eq. (42), which means that the number of parameters contained in $S(t_s)$ can only be 1 if a unique estimation result has to be obtained.

It is evident that the statistics established in Section III.(B) are an infinite set of the censored samples $(\delta_s = 0)$. Therefore, an approximation method is developed by improving the method proposed in [23]. The concept is as follows:

① Determine the approximate accuracy R;

② Determine the corresponding number of shocks $n^{(T)}$ based on the current iteration value of parameter $\theta_x^{(0)}$.

To ensure that the probability of the number of shocks exceeding $n_s^{(h)} = n^{(T)} - n_s$ is less than R after the censored time for most censored samples $(\delta_s = 0)$.

E. EXPECTED STATISTICS

The expected statistics for the posterior distribution are obtained using the method in [12], [24] after making some improvements. Due to space limitations, only the results are provided. Assuming that the specific form of the parameter estimation result $\hat{\theta}_x$ after a certain number of iterations is as follows:

(1) Initial distribution vector

$$\hat{\beta} = \left[\hat{\beta}_1 \quad \ldots \quad \hat{\beta}_\upsilon\right], \quad \hat{\gamma}_n = \left[\hat{\beta} \quad 0 \ldots \quad 0\right]_{1,\upsilon n},$$

(2) Phase-type generator

$$\hat{R}_0 = \begin{bmatrix} \hat{r}_{11}^{(0)} & \ldots & \hat{r}_{1\upsilon}^{(0)} \\ \ldots & \ldots & \ldots \\ \hat{r}_{\upsilon 1}^{(0)} & \ldots & \hat{r}_{\upsilon\upsilon}^{(0)} \end{bmatrix}, \quad \hat{R}_1 = \begin{bmatrix} \hat{r}_{11}^{(1)} & \ldots & \hat{r}_{1\upsilon}^{(1)} \\ \ldots & \ldots & \ldots \\ \hat{r}_{\upsilon 1}^{(1)} & \ldots & \hat{r}_{\upsilon\upsilon}^{(1)} \end{bmatrix},$$

$$\hat{Y}_n^{(0)} \begin{bmatrix} \hat{R}_0 & \hat{R}_1 & \ldots & 0 \\ 0 & \hat{R}_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \hat{R}_1 \\ 0 & 0 & \ldots & \hat{R}_0 \end{bmatrix}_{\upsilon n,\upsilon n},$$

(3) Exit vector

$$\hat{r}_1 = -\hat{R}_0\vec{e} = \left[\hat{r}_1^{(1)} \quad \ldots \quad \hat{r}_\upsilon^{(1)}\right]^T,$$
$$\hat{y}_n^{(1)} = -\hat{Y}_n^{(0)}\vec{e} \quad \text{and } \hat{S}\left(t_s\right) = a\left(\hat{b}\right)^t$$

where $n = 1, 2, \ldots$

Let

$$\tilde{Y}_{n,s}^{(0)} = \hat{Y}_n^{(0)}\hat{S}\left(t_s\right) - \xi E, \quad \tilde{y}_{n,s}^{(1)} = \hat{y}_n^{(1)}\hat{S}(t_s),$$
$$\tilde{y}_{n,s}^{(2)} = \hat{y}_n^{(1)}\left|\hat{S}'\left(t_s\right)\right|, \quad \tilde{r}_{ij,s}^{(0)} = \hat{S}\left(t_s\right)\hat{r}_{ij}^{(0)},$$
$$\tilde{r}_{i,s}^{(1)} = \hat{S}\left(t_s\right)\hat{r}_i^{(1)} \quad \text{and } \tilde{r}_{i,s}^{(2)} = \left|\hat{S}'\left(t_s\right)\right|\hat{r}_i^{(1)}$$

where $n = 1, 2, \ldots$.

1) CENSORED SAMPLE ($\delta_s = 0$)

(1) The expected statistics $\left\{B_{n,i}^{(s)}\right\}$ are:

$$E_{\hat{\theta}_x}\left(B_{n,i}^{(s)}|\vartheta_s\right) \approx \frac{\hat{\beta}_i\vec{e}_i^{\,T}\exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right)\vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}}{\hat{\gamma}_{n_s^{(T)}}\exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right)\vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}} \tag{43}$$

where $i = 1, 2, \ldots, \upsilon$, $n = 1$, and $\vec{e}_{m,n}^{\,(\Sigma)}$ represents a column vector with appropriate dimensions whose elements from $(m-1)\upsilon + 1$ to $n\upsilon$ is 1, and the remaining elements are all 0, Eq. (47) where $i = 1, 2, \ldots, \upsilon$ and $n = 2, 3, \ldots, n_s$, and Eq. (48) where $i = 1, 2, \ldots, \upsilon$ and $n = n_s + 1, n_s + 2, \ldots, n_s^{(T)}$.

(2) The expected statistics $\left\{Z_{n,i}^{(s)}\right\}$ are Eq. (49) where $i = 1, 2, \ldots, \upsilon$, $n = 1, 2, \ldots, n_s$, and $\vec{e}_{n,i}$ represents a column vector with appropriate dimensions whose $(n-1)m+i$-th element is 1, and the remaining elements are all 0, and Eq. (50) where $i = 1, 2, \ldots, \upsilon$, and $n = n_s + 1, n_s + 2, \ldots, n_s^{(T)}$.

(3) The expected statistics $\left\{M_{n,ij}^{(s)}\right\}$ are Eq. (51) where $i, j = 1, 2, \ldots, \upsilon$ $(i \neq j)$ and $n = 1, 2, \ldots, n_s + 1$, and Eq. (52) where $i, j = 1, 2, \ldots, \upsilon$ $(i \neq j)$ and $n = n_s + 1, n_s + 2, \ldots, n_s^{(T)}$.

(4) The expected statistics $\left\{Y_{n,i}^{(s)}\right\}$ are Eq. (53) where $i = 1, 2, \ldots, \upsilon$ and $n = 1, 2, \ldots, n_s - 1$, and Eq. (54) where $i = 1, 2, \ldots, \upsilon$ and $n = n_s, n_s + 1, \ldots, n_s^{(T)} - 1$.

2) SHOCK FAILURE SAMPLE ($\delta_s = 1$)

(1) The expected statistics $\left\{B_{n,i}^{(s)}\right\}$ are:

$$E_{\hat{\theta}_x}\left(B_{n,i}^{(s)}|\vartheta_s\right) = \frac{\hat{\beta}_i\vec{e}_i^{\,T}\exp\left(\tilde{Y}_{n_s,s}^{(0)}\right)\vec{e}_{n_s,\Sigma}}{\hat{\gamma}_{n_s}\exp\left(\tilde{Y}_{n_s,s}^{(0)}\right)\vec{e}_{n_s,\Sigma}} \tag{44}$$

where $i = 1, 2, \ldots, \upsilon$, $n = 1$, and Eq. (55) where $i = 1, 2, \ldots, \upsilon$ and $n = 2, 3, \ldots, n_s$.

(2) The expected statistic $\left\{Z_{n,i}^{(s)}\right\}$ is Eq. (56) where $i = 1, 2, \ldots, \upsilon$, $n = 1, 2, \ldots, n_s$.

(3) The expected statistic $\left\{M_{n,ij}^{(s)}\right\}$ is Eq. (57) where $i, j = 1, 2, \ldots, \upsilon$ $(i \neq j)$ and $n = 2, 3, \ldots, n_s$.

(4) The expected statistic $\left\{Y_{n,i}^{(s)}\right\}$ is Eq. (58) where $i = 1, 2, \ldots, \upsilon$ and $n = 1, 2, \ldots, n_s - 1$.

$$\begin{aligned}
&\frac{\partial\Phi\left(\theta_x\right)}{\partial b} \\
&= \left(\sum_{s=1,\delta_s=0}^{S}\left(\frac{\partial S\left(t_s\right)}{\partial b}\right)\sum_{n=1}^{\infty}\sum_{i=1}^{\upsilon}E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)}|\vartheta_s\right)r_{ii}^{(0)} + \sum_{s=1,\delta_s\neq 0}^{S}\left(\frac{\partial S\left(t_s\right)}{\partial b}\right)\sum_{n=1}^{n_s}\sum_{i=1}^{\upsilon}E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)}|\vartheta_s\right)r_{ii}^{(0)}\right)\ldots \\
&\quad + \left(\sum_{s=1,\delta_s=0}^{S}\left(\frac{1}{S\left(t_s\right)}\frac{\partial S\left(t_s\right)}{\partial b}\right)\sum_{n=1}^{\infty}\sum_{i=1}^{\upsilon}\sum_{j=1,i\neq j}^{\upsilon}E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)}|\vartheta_s\right) + \sum_{s=1,\delta_s\neq 0}^{S}\left(\frac{1}{S\left(t_s\right)}\frac{\partial S\left(t_s\right)}{\partial b}\right)\sum_{n=1}^{n_s}\sum_{i=1}^{\upsilon}\sum_{j=1,i\neq j}^{\upsilon}E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)}|\vartheta_s\right)\right)\ldots \\
&\quad + \left(\sum_{s=1,\delta_s=0}^{S}\left(\frac{1}{S\left(t_s\right)}\frac{\partial S\left(t_s\right)}{\partial b}\right)\sum_{n=1}^{\infty}\sum_{i=1}^{\upsilon}E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_s\right) + \sum_{s=1,\delta_s=1}^{S}\left(\frac{1}{S\left(t_s\right)}\frac{\partial S\left(t_s\right)}{\partial b}\right)\sum_{n=1}^{n_s-1}\sum_{i=1}^{\upsilon}E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_s\right) + \ldots\right. \\
&\quad \left. + \sum_{s=1,\delta_s=2}^{S}\left(\frac{1}{S\left(t_s\right)}\frac{\partial S\left(t_s\right)}{\partial b}\right)\sum_{n=1}^{n_s-1}\sum_{i=1}^{\upsilon}E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_s\right)\right) + \left(\sum_{s=1,\delta_s=2}^{S}\left(\frac{1}{\left|S'\left(t_s\right)\right|}\frac{\partial\left|S'\left(t_s\right)\right|}{\partial b}\right)\sum_{i=1}^{\upsilon}E_{\hat{\theta}_x}\left(Y_{n_s,i}^{(s)}|\vartheta_s\right)\right) = 0
\end{aligned} \tag{42}$$

TABLE 10. Steps of the method.

Steps	Expression
1	Determine the initial value of the estimated parameter $\hat{\theta}_x\|_0$; Determine the iterative convergence threshold $\varepsilon > 0$ and determine the approximate threshold R ;
2	Calculate the value of the log-likelihood function $\ln L\left(\hat{\theta}_x \mid X_{th}, T, N, \Delta\right)$ according to $\hat{\theta}_x\|_0$;
3	Calculate the corresponding ξ using the sample $\left\{\vartheta_x^{(s)}\right\}$ and the parameter estimation values $\hat{\theta}_x\|_{n-1}$ obtained in the previous step $n-1$ to extend the original Markov state-space.
4	For the censored sample $(\delta_s = 0)$, calculate the corresponding $n^{(T)}$ using the observed sample $\left\{\vartheta_x^{(s)}\right\}$, the parameters in $\hat{\theta}_x\|_{n-1}$, and R .
5	Calculate the expected value $E_{\hat{\theta}_x}(H \mid X_{th}, T, N, \Delta)$ of the statistic H for the posterior distribution $f\left(H \mid \hat{\theta}_x, X_{th}, T, N, \Delta\right)$ using the EM method according to the parameters in $\hat{\theta}_x\|_{n-1}$; Note that for any censored sample s $(\delta_s = 0)$, the following applies: ① When $n_s < n^{(T)}$, the value of $n^{(T)}$ remains constant, and ② When $n_s \geq n^{(T)}$, $n^{(T)} = n_s + 1$ only for sample s .
6	Use the expected value $E_{\hat{\theta}_x}(H \mid X_{th}, T, N, \Delta)$ of the statistics obtained in Step ⑤ the expressions in TABLE IX and Eq. (42) to obtain the new parameter iteration values $\hat{\theta}_x\|_n$;
7	Calculate the value of log-likelihood function $\ln L\left(\hat{\theta}_x \mid X_{th}, T, N, \Delta\right)$ based on $\hat{\theta}_x\|_n$.
8	Compare the value of $\Delta^{(n)} = \left\|\left(\ln L\left(\hat{\theta}_x \mid X_{th}, T, N, \Delta\right)\Big\|_n\right) - \left(\ln L\left(\theta_x^{(n-1)} \mid X_{th}, T, N, \Delta\right)\Big\|_{n-1}\right)\right\|$ and ε ① When $\Delta^{(n)} > \varepsilon$, let $n = n+1$ and execute Steps ③~⑧ again. ② When $\Delta^{(n)} \leq \varepsilon$, end the loop and return the parameter estimation result $\hat{\theta}_x = \hat{\theta}_x\|_n$.

3) WEAR FAILURE SAMPLE $(\delta_s = 2)$

(1) The expected statistics $\left\{B_{n,i}^{(s)}\right\}$ are:

$$E_{\hat{\theta}_x}\left(B_{n,i}^{(s)} | \vartheta_s\right) = \frac{\hat{\beta}_i \vec{e}_i^{\,T} \exp\left(\tilde{Y}_{n_s,s}^{(0)}\right) \tilde{y}_{n_s,s}^{(2)}}{\hat{\gamma}_{n_s} \exp\left(\tilde{Y}_{n_s,s}^{(0)}\right) \tilde{y}_{n_s,s}^{(2)}} \tag{45}$$

where $i = 1, 2, \ldots, \upsilon$, $n = 1$, and Eq. (59) where $i = 1, 2, \ldots, \upsilon$ and $n = 2, 3, \ldots, n_s$.

(2) The expected statistic $\left\{Z_{n,i}^{(s)}\right\}$ is Eq. (60) where $i = 1, 2, \ldots, \upsilon$, $n = 1, 2, \ldots, n_s$.

(3) The expected statistic $\left\{M_{n,ij}^{(s)}\right\}$ is Eq. (61) where $i, j = 1, 2, \ldots, \upsilon$ $(i \neq j)$ and $n = 2, 3, \ldots, n_s$.

(4) The expected statistics $\left\{Y_{n,i}^{(s)}\right\}$ are Eq. (62) where $i = 1, 2, \ldots, \upsilon$ and $n = 1, 2, \ldots, n_s - 1$, and

$$E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)} | \vartheta_s\right) = \frac{\hat{\gamma}_{n_s} \exp\left(\tilde{Y}_{n_s,s}^{(0)}\right) \vec{e}_{n_s,i} \tilde{r}_{i,s}^{(2)}}{\hat{\gamma}_{n_s} \exp\left(\tilde{Y}_{n_s,s}^{(0)}\right) \tilde{y}_{n_s,s}^{(2)}} \tag{46}$$

where $i = 1, 2, \ldots, \upsilon$ and $n = n_s$. Eq. (47)–(62), as shown at the bottom of the next page.

F. STEPS OF THE METHOD

The steps of the parameter estimation method of θ_x using the model parameters and the sample data described in the previous sections are shown in TABLE 10.

IV. CASE SIMULATION AND ANALYSIS

Generally, a real data set is often used to prove the validity of a method. However, in this work, the original distribution of a real data set could not be determined, and due to the lack of an accurate control group, we could not prove the validity of the proposed method. Thus, two simulation cases are proposed.

In these cases, three non-exponential distributions (gamma distribution, Weibull distribution, and log-normal distribution) and two continuous degradation equations ($S(t) = ab^t$ and $S(t) = a\exp\left(-\frac{t}{b}\right)$) are used to establish the stress-strength model and generate the sample data.

It is unlikely to obtain the original reliability function or the PDF of the lifetime based on Eq. (3), especially since a general distribution is assumed; the censored samples $(\delta_s = 0)$ also limit the length of the time interval on the X-axis. Two steps are used to test the performance of the proposed method and balance a realistic operation and computational effort.

(1) A relatively small sample size (including censored samples $(\delta_s = 0)$) is used to estimate the parameters $\theta = \{\theta_t, \theta_x\}$;

(2) A Monte-Carlo simulation method is used, and the censored time $t^{(c)}$ is ignored to generate two large data sets of the lifetime based on the original parameters and the estimated result. The data sets are compared to test the performance of the method.

Assume that the inter-arrival time between two adjacent shocks and the magnitude of the damage caused by the single shock are i.i.d. random variables for all cases.

A. CASE 1

In this case, a gamma distribution and a Weibull distribution are used to generate the shock sequence.

1) SITUATION 1 $\left(S(t) = ab^t\right)$

(1) Model assumptions

The model assumptions are as follows:

① Assume that the sample size for the parameter estimation is $S = 500$, and the sample size for the Monte-Carlo simulation is $S_{MC} = 10000$;

② Assume that the initial strength is $X_{th} = 70$, and the censored time is $t^{(c)} = 50$;

③ Assume that the iterative convergence threshold is $\varepsilon = 0.00001$, and the approximate threshold is $R = 0.00001$;

④ Assume that the inter-arrival time satisfies a gamma distribution with shape parameter 3 and scale parameter 2;

⑤ Assume that the magnitude of the damage caused by the single shock satisfies a Weibull distribution with scale parameter 6 and shape parameter 2;

⑥ The degradation of the strength satisfies $S(t) = ab^t$ where $a = X_{th}$ and $b = 0.98$.

(2) Estimated result

In the EM method, the initial value, including the type and order of the PH distribution, should be determined first.

$$E_{\hat{\theta}_x}\left(B_{n,i}^{(s)}|\vartheta_s\right) \approx \frac{\int_0^1 \hat{\gamma}_{n-1} \exp\left(\tilde{Y}_{n-1,s}^{(0)} x\right) \tilde{y}_{n-1,s}^{(1)} \hat{\beta}_i \vec{e}_i^{\,T} \exp\left(\tilde{Y}_{n_s^{(T)}-n+1,s}^{(0)} (1-x)\right) \vec{e}_{n_s-n+2,n_s^{(T)}-n+1}^{\,(\Sigma)} dx}{\hat{\gamma}_{n_s^{(T)}} \exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right) \vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}} \tag{47}$$

$$E_{\hat{\theta}_x}\left(B_{n,i}^{(s)}|\vartheta_s\right) \approx \frac{\int_0^1 \hat{\gamma}_{n-1} \exp\left(\tilde{Y}_{n-1,s}^{(0)} x\right) \tilde{y}_{n-1,s}^{(1)} \hat{\beta}_i \vec{e}_i^{\,T} \exp\left(\tilde{Y}_{n_s^{(T)}-n+1,s}^{(0)} (1-x)\right) \vec{e}\, dx}{\hat{\gamma}_{n_s^{(T)}} \exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right) \vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}} \tag{48}$$

$$E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)}|\vartheta_s\right) \approx \frac{\int_0^1 \hat{\gamma}_{n} \exp\left(\tilde{Y}_{n,s}^{(0)} x\right) \vec{e}_{n,i} \vec{e}_i^{\,T} \exp\left(\tilde{Y}_{n_s^{(T)}-n+1,s}^{(0)} (1-x)\right) \vec{e}_{n_s-n+2,n_s^{(T)}-n+1}^{\,(\Sigma)} dx}{\hat{\gamma}_{n_s^{(T)}} \exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right) \vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}} \tag{49}$$

$$E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)}|\vartheta_s\right) \approx \frac{\int_0^1 \hat{\gamma}_{n} \exp\left(\tilde{Y}_{n,s}^{(0)} x\right) \vec{e}_{n,i} \vec{e}_i^{\,T} \exp\left(\tilde{Y}_{n_s^{(T)}-n+1,s}^{(0)} (1-x)\right) \vec{e}\, dx}{\hat{\gamma}_{n_s^{(T)}} \exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right) \vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}} \tag{50}$$

$$E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)}|\vartheta_s\right) \approx \frac{\int_0^1 \hat{\gamma}_{n} \exp\left(\tilde{Y}_{n,s}^{(0)} x\right) \vec{e}_{n,i} \tilde{r}_{ij,s}^{(0)} \vec{e}_j^{\,T} \exp\left(\tilde{Y}_{n_s^{(T)}-n+1,s}^{(0)} (1-x)\right) \vec{e}_{n_s-n+2,n_s^{(T)}-n+1}^{\,(\Sigma)} dx}{\hat{\gamma}_{n_s^{(T)}} \exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right) \vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}} \tag{51}$$

$$E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)}|\vartheta_s\right) \approx \frac{\int_0^1 \hat{\gamma}_{n} \exp\left(\tilde{Y}_{n,s}^{(0)} x\right) \vec{e}_{n,i} \tilde{r}_{ij,s}^{(0)} \vec{e}_j^{\,T} \exp\left(\tilde{Y}_{n_s^{(T)}-n+1,s}^{(0)} (1-x)\right) \vec{e}\, dx}{\hat{\gamma}_{n_s^{(T)}} \exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right) \vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}} \tag{52}$$

$$E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_s\right) \approx \frac{\int_0^1 \hat{\gamma}_{n} \exp\left(\tilde{Y}_{n,s}^{\left(0,\theta_x^{(0)}\right)} x\right) \vec{e}_{n,i} \tilde{r}_{i,s}^{(1)} \hat{\gamma}_{n_s^{(T)}-n} \exp\left(\tilde{Y}_{n_s^{(T)}-n,s}^{(0)} (1-x)\right) \vec{e}_{n_s-n+1,n_s^{(T)}-n}^{\,(\Sigma)} dx}{\hat{\gamma}_{n_s^{(T)}} \exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right) \vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}} \tag{53}$$

$$E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_s\right) \approx \frac{\int_0^1 \hat{\gamma}_{n} \exp\left(\tilde{Y}_{n,s}^{(0)} x\right) \vec{e}_{n,i} \tilde{r}_{i,s}^{(1)} \hat{\gamma}_{n_s^{(T)}-n} \exp\left(\tilde{Y}_{n_s^{(T)}-n,s}^{(0)} (1-x)\right) \vec{e}\, dx}{\hat{\gamma}_{n_s^{(T)}} \exp\left(\tilde{Y}_{n_s^{(T)},s}^{(0)}\right) \vec{e}_{n_s+1,n_s^{(T)}}^{\,(\Sigma)}} \tag{54}$$

$$E_{\hat{\theta}_x}\left(B_{n,i}^{(s)}|\vartheta_s\right) = \frac{\int_0^1 \hat{\gamma}_{n-1} \exp\left(\tilde{Y}_{n-1,s}^{(0)} x\right) \tilde{y}_{n-1,s}^{(1)} \hat{\beta}_i \vec{e}_i^{\,T} \exp\left(\tilde{Y}_{n_s-n+1,s}^{(0)} (1-x)\right) \vec{e}_{n_s-n+1,\Sigma} dx}{\hat{\gamma}_{n_s} \exp\left(\tilde{Y}_{n_s,s}^{(0)}\right) \vec{e}_{n_s,\Sigma}} \tag{55}$$

A Coxian distribution is used for fitting both cases since it can fit any distribution with an appropriate order [25].

The distribution of the inter-arrival time and the magnitude of the damage are both fitted by a 4^{th}-order Coxian distribution with the same initial values as follows:

① Initial distribution vector: $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

② Phase-type generator: $\begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.

③ Exit vector: $\begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}^T$.

Estimated result of the inter-arrival time:

① Initial distribution vector: $\hat{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

② Phase-type generator:

$$\hat{D}_0 = \begin{bmatrix} -0.5392 & 0.5392 & 0 & 0 \\ 0 & -0.5396 & 0.5377 & 0 \\ 0 & 0 & -0.5432 & 0.3074 \\ 0 & 0 & 0 & -0.5265 \end{bmatrix}.$$

③ Exit vector:

$$\hat{d}_1 = \begin{bmatrix} 0 & 0.0019 & 0.2358 & 0.5265 \end{bmatrix}^T.$$

Estimated result of the magnitude of the damage:

① Initial distribution vector: $\hat{\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

② Phase-type generator:

$$\hat{R}_0 = \begin{bmatrix} -0.7299 & 0.7299 & 0 & 0 \\ 0 & -0.6843 & 0.6512 & 0 \\ 0 & 0 & -0.6406 & 0.4721 \\ 0 & 0 & 0 & -0.5979 \end{bmatrix}.$$

③ Exit vector:

$$\hat{r}_1 = \begin{bmatrix} 0 & 0.0331 & 0.1685 & 0.5979 \end{bmatrix}^T.$$

The estimated PDF of the inter-arrival time and the magnitude of the damage based on the PH distribution and their statistical characteristics are compared with the assumed distributions in FIGURE 3 and TABLE 12, respectively.

The estimated result of b is $\hat{b} = 0.9844$. The distribution and the reliability based on the Monte-Carlo sample sets are compared; the results are shown in FIGURE 4. The statistical characteristics of the lifetime and the reliability of the samples at different time points are shown in TABLE 13 and TABLE 11, respectively.

2) SITUATION 2 $\left(S(t) = a \exp\left(-\frac{t}{b}\right)\right)$

(1) Model assumptions

The model assumptions are as follows:

① Assume that the initial strength is $X_{th} = 60$, and the censored time is $t^{(c)} = 60$;

$$E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)}|\vartheta_s\right) = \frac{\int_0^1 \hat{\gamma}_n \exp\left(\tilde{Y}_{n,s}^{(0)}x\right)\vec{e}_{n,i}\vec{e}_i^{\,T} \exp\left(\tilde{Y}_{n_s-n+1,s}^{(0)}(1-x)\right)\vec{e}_{n_s-n+1,\Sigma}dx}{\hat{\gamma}_{n_s}\exp\left(\tilde{Y}_{n_s,s}^{(0)}\right)\vec{e}_{n_s,\Sigma}} \tag{56}$$

$$E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)}|\vartheta_s\right) = \frac{\int_0^1 \hat{\gamma}_n \exp\left(\tilde{Y}_{n,s}^{(0)}x\right)\vec{e}_{n,i}\tilde{r}_{ij,s}^{(0)}\vec{e}_j^{\,T} \exp\left(\tilde{Y}_{n_s-n+1,s}^{(0)}(1-x)\right)\vec{e}_{n_s-n+1,\Sigma}dx}{\hat{\gamma}_{n_s}\exp\left(\tilde{Y}_{n_s,s}^{(0)}\right)\vec{e}_{n_s,\Sigma}} \tag{57}$$

$$E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_s\right) = \frac{\int_0^1 \hat{\gamma}_n \exp\left(\tilde{Y}_{n,s}^{(0)}x\right)\vec{e}_{n,i}\tilde{r}_{i,s}^{(1)}\hat{\gamma}_{n_s-n} \exp\left(\tilde{Y}_{n_s-n,s}^{(0)}(1-x)\right)\vec{e}_{n_s-n,\Sigma}dx}{\hat{\gamma}_{n_s}\exp\left(\tilde{Y}_{n_s,s}^{(0)}\right)\vec{e}_{n_s,\Sigma}} \tag{58}$$

$$E_{\hat{\theta}_x}\left(B_{n,i}^{(s)}|\vartheta_s\right) = \frac{\int_0^1 \hat{\gamma}_{n-1} \exp\left(\tilde{Y}_{n-1,s}^{(0)}x\right)\tilde{y}_{n-1,s}^{(1)}\hat{\beta}_i\vec{e}_i^{\,T} \exp\left(\tilde{Y}_{n_s-n+1,s}^{(0)}(1-x)\right)\tilde{y}_{n_s-n+1,s}^{(2)}dx}{\hat{\gamma}_{n_s}\exp\left(\tilde{Y}_{n_s,s}^{(0)}\right)\tilde{y}_{n_s,s}^{(2)}} \tag{59}$$

$$E_{\hat{\theta}_x}\left(Z_{n,i}^{(s)}|\vartheta_s\right) = \frac{\int_0^1 \hat{\gamma}_n \exp\left(\tilde{Y}_{n,s}^{(0)}x\right)\vec{e}_{n,i}\vec{e}_i^{\,T} \exp\left(\tilde{Y}_{n_s-n+1,s}^{(0)}(1-x)\right)\tilde{y}_{n_s-n+1,s}^{(2)}dx}{\hat{\gamma}_{n_s}\exp\left(\tilde{Y}_{n_s,s}^{(0)}\right)\tilde{y}_{n_s,s}^{(2)}} \tag{60}$$

$$E_{\hat{\theta}_x}\left(M_{n,ij}^{(s)}|\vartheta_s\right) = \frac{\int_0^1 \gamma_n \exp\left(\tilde{Y}_{n,s}^{(0)}x\right)\vec{e}_{n,i}\tilde{r}_{ij,s}^{(0)}\vec{e}_j^{\,T} \exp\left(\tilde{Y}_{n_s-n+1,s}^{(0)}(1-x)\right)\tilde{y}_{n_s-n+1,s}^{(2)}dx}{\hat{\gamma}_{n_s}\exp\left(\tilde{Y}_{n_s,s}^{(0)}\right)\tilde{y}_{n_s,s}^{(2)}} \tag{61}$$

$$E_{\hat{\theta}_x}\left(Y_{n,i}^{(s)}|\vartheta_s\right) = \frac{\int_0^1 \hat{\gamma}_n \exp\left(\tilde{Y}_{n,s}^{(0)}x\right)\vec{e}_{n,i}\tilde{r}_{i,s}^{(1)}\hat{\gamma}_{n_s-n} \exp\left(\tilde{Y}_{n_s-n,s}^{(0)}(1-x)\right)\tilde{y}_{n_s-n,s}^{(2)}dx}{\hat{\gamma}_{n_s}\exp\left(\tilde{Y}_{n_s,s}^{(0)}\right)\tilde{y}_{n_s,s}^{(2)}} \tag{62}$$

TABLE 11. Comparison of the reliability at different times (case 1, situation 1).

No.	Time	$t=15$	$t=20$	$t=25$	$t=30$	$t=35$	$t=40$	$t=45$	$t=50$ (Censored time)
1	Sample	1	0.9996	0.9883	0.9161	0.7231	0.4326	0.1964	0.0668
2	Estimation	1	0.9985	0.9870	0.9337	0.7862	0.5556	0.3225	0.1479
3	Deviation	0	-0.0011	-0.0013	0.0176	0.0631	0.1230	0.1261	0.0811

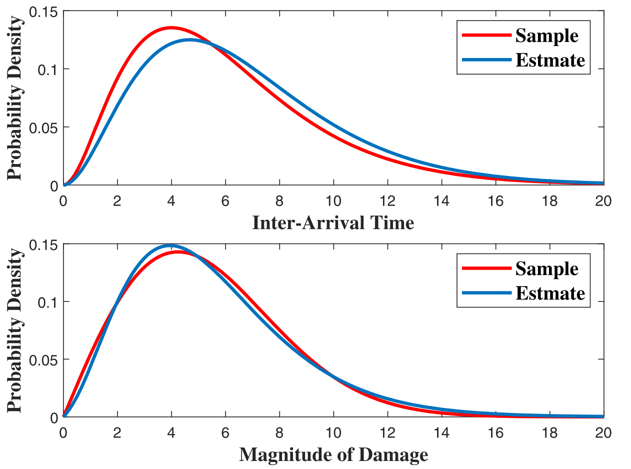

FIGURE 3. PDF of the inter-arrival time and the magnitude of the damage.

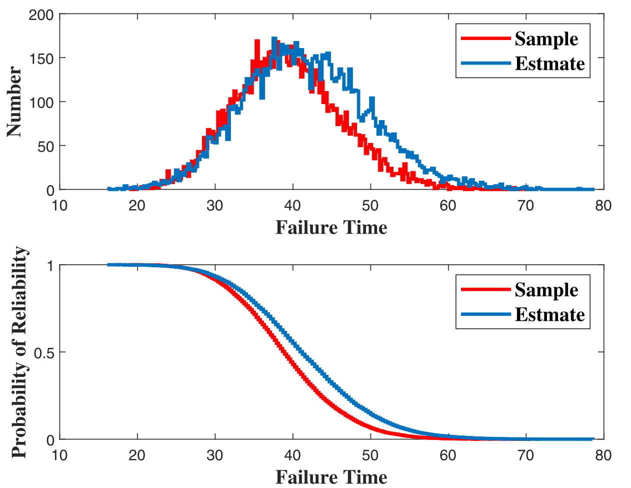

FIGURE 4. Comparison of the distribution and reliability.

② Assume that the single shock magnitude satisfies a Weibull distribution with scale parameter 3 and shape parameter 2;

③ The degradation of the strength satisfies $S(t) = a\exp\left(-\frac{t}{b}\right)$ where $a = X_{th}$ and $b = 40$;

④ The remaining parameters are the same as those in Section IV(A.1).

(2) Estimated result

The distribution of the inter-arrival time and the magnitude of the damage are both fitted by a 3$^{\text{rd}}$-order Coxian distribution with the same initial values as follows:

① Initial distribution vector: $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

TABLE 12. Statistical characteristics of the time and magnitude.

No.	Type	Statistical Characteristic	Sample	Estimate	Deviation
1	Time	Mean	6	6.6133	0.6133
		Variance	12	13.1983	1.1983
2	Magnitude	Mean	5.3174	5.4899	0.1725
		Variance	7.7257	9.1686	1.4429

TABLE 13. Statistical characteristics of the lifetime.

No.	Statistical Characteristic	Sample	Estimate	Deviation
1	Mean	39.2190	41.5381	2.3191
2	Variance	47.6619	64.4377	16.7758

② Phase-type generator: $\begin{bmatrix} -2 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

③ Exit vector: $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^T$.

Estimated result of the inter-arrival time:

① Initial distribution vector: $\hat{\alpha} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

② Phase-type generator:

$$\hat{D}_0 = \begin{bmatrix} -0.4275 & 0.4275 & 0 \\ 0 & -0.4445 & 0.3847 \\ 0 & 0 & -0.4696 \end{bmatrix}.$$

③ Exit vector: $\hat{d}_1 = \begin{bmatrix} 0 & 0.0598 & 0.4696 \end{bmatrix}^T$.

Estimated result of the magnitude of the damage:

① Initial distribution vector: $\hat{\beta} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

② Phase-type generator:

$$\hat{R}_0 = \begin{bmatrix} -1.1318 & 1.1318 & 0 \\ 0 & -1.0695 & 1.0304 \\ 0 & 0 & -1.0321 \end{bmatrix}.$$

③ Exit vector: $\hat{r}_1 = \begin{bmatrix} 0 & 0 \circ 0391 & 1.0321 \end{bmatrix}^T$.

The estimated PDF of the inter-arrival time and the magnitude of the damage based on the PH distribution and their statistical characteristics are compared with the assumed distributions in FIGURE 5 and TABLE 15, respectively.

The estimated result of b is $\hat{b} = 43.8657$. The distribution and the reliability based on the Monte-Carlo sample sets are compared; the results are shown in FIGURE 6. The statistical characteristics of the lifetime and the reliability of the samples at different time points are shown in TABLE 16 and TABLE 14, respectively.

It is observed that the estimated results based on the 4$^{\text{th}}$-order Coxian distribution are better than the results of the 3$^{\text{rd}}$-order in CASE 1. However, the computational effort is

TABLE 14. Comparison of the reliability at different times (case 1, situation 2).

No.	Time	$t = 25$	$t = 30$	$t = 35$	$t = 40$	$t = 45$	$t = 50$	$t = 55$	$t = 60$ (Censored time)
1	Sample	1	0.9986	0.9708	0.8409	0.5531	0.2540	0.0796	0.0181
2	Estimation	1	0.9988	0.9839	0.9066	0.7189	0.4477	0.2104	0.0773
3	Deviation	0	0.0002	0.0131	0.0657	0.1658	0.1937	0.1308	0.0592

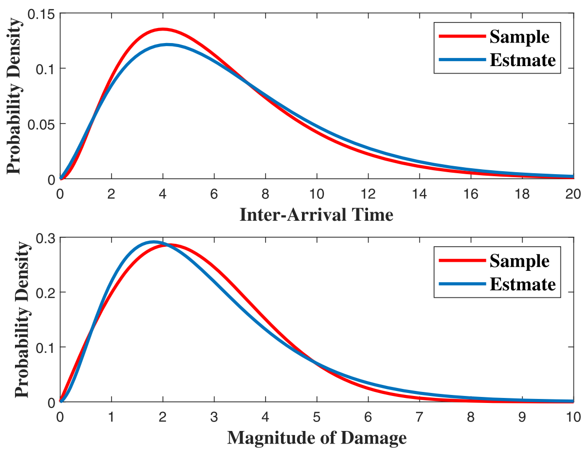

FIGURE 5. PDF of the inter-arrival time and the magnitude of the damage.

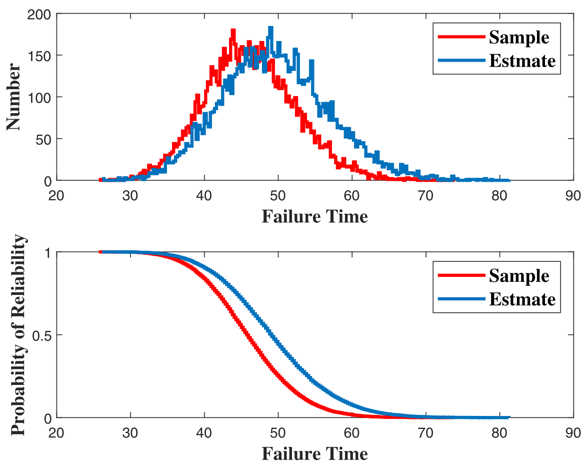

FIGURE 6. Comparison of the distribution and reliability.

higher for the 4^{th}-order than the 3^{rd}-order Coxian distribution. The failure time of the samples should be adequately represented by the chosen distribution, and the computational effort should also be acceptable when choosing the order of the distribution.

B. CASE 2

In this case, a gamma distribution and a log-normal distribution are used to generate the shock sequence.

1) SITUATION 1 $\left(S(t) = ab^t\right)$

(1) Model assumptions

The model assumptions are as follows:

① Assume that the initial strength is $X_{th} = 60$, and the censored time is $t^{(c)} = 55$;

② Assume that the inter-arrival time satisfies a gamma distribution with shape parameter 2 and scale parameter 2;

③ Assume that the single shock magnitude satisfies a log-normal distribution with a mean of 0.5 and a standard deviation of 1;

④ The degradation of the strength satisfies $S(t) = ab^t$ where $a = X_{th}$ and $b = 0.98$;

⑤ The remaining parameters are the same as those in Section IV(A.1).

(2) Estimated result

The distribution of the inter-arrival time and the magnitude of the damage are both fitted by a 4^{th}-order Coxian distribution with the same initial values as follows:

① Initial distribution vector: $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

② Phase-type generator: $\begin{bmatrix} -2 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

③ Exit vector: $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^T$.

Estimated result of the inter-arrival time:

① Initial distribution vector: $\hat{\alpha} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

② Phase-type generator:

$$\hat{D}_0 = \begin{bmatrix} -0.8748 & 0.8748 & 0 \\ 0 & -0.8384 & 0.4948 \\ 0 & 0 & -0.4579 \end{bmatrix}.$$

③ Exit vector: $\hat{d}_1 = \begin{bmatrix} 0 & 0.3436 & 0.4579 \end{bmatrix}^T$.

Estimated result of the magnitude of the damage:

① Initial distribution vector: $\hat{\beta} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

② Phase-type generator:

$$\hat{R}_0 = \begin{bmatrix} -9.1841 & 9.1841 & 0 \\ 0 & -8.8551 & 8.2471 \\ 0 & 0 & -0.4362 \end{bmatrix}.$$

TABLE 15. Statistical characteristics of the time and magnitude.

No.	Type	Statistical Characteristic	Sample	Estimate	Deviation
1	Time	Mean	6	6.4319	0.4319
		Variance	12	14.9856	2.9856
2	Magnitude	Mean	2.6587	2.7520	0.0933
		Variance	1.9314	2.5924	0.6610

TABLE 16. Statistical characteristics of the lifetime.

No.	Statistical Characteristic	Sample	Estimate	Deviation
1	Mean	46.0791	49.3767	3.2976
2	Variance	38.0595	52.4386	14.3791

TABLE 17. Statistical characteristics of the time and magnitude.

No.	Type	Statistical Characteristic	Sample	Estimate	Deviation
1	Time	Mean	4	3.6247	-9.3817
		Variance	8	6.6977	-1.3023
2	Magnitude	Mean	2.7183	2.3569	-0.3613
		Variance	12.6965	5.2555	-7.4410

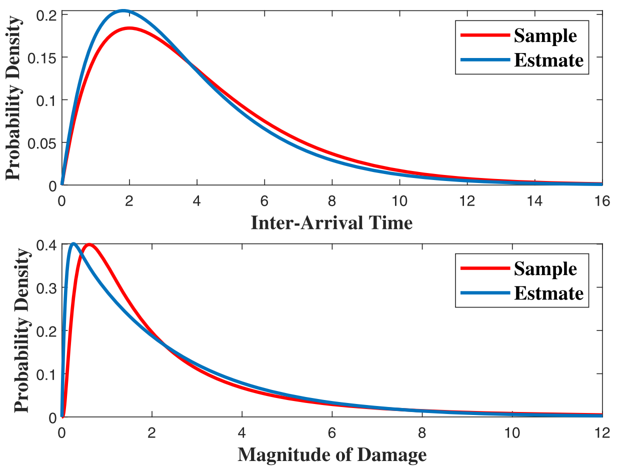

FIGURE 7. PDF of the inter-arrival time and the magnitude of the damage.

TABLE 18. Statistical characteristics of the lifetime.

No.	Statistical Characteristic	Sample	Estimate	Deviation
1	Mean	42.0364	39.7876	-2.2488
2	Variance	96.6159	59.0389	-37.5770

③ Exit vector: $\hat{r}_1 = \begin{bmatrix} 0 & 0.6080 & 0.4362 \end{bmatrix}^T$.

The estimated PDF of the inter-arrival time and the magnitude of the damage based on the PH distribution and their statistical characteristics are compared with the assumed distributions in FIGURE 7 and TABLE 17, respectively.

The estimated result of b is $\hat{b} = 0.9772$. The distribution and the reliability based on Monte-Carlo sample sets are compared; the result are shown in FIGURE 8. The statistical characteristics of the lifetime and the reliability of the samples at different time points are shown in TABLE 8 and TABLE 19, respectively.

2) SITUATION 2 $\left(S(t) = a\exp\left(-\frac{t}{b}\right)\right)$

(1) Model assumptions

The model assumptions are as follows:

① Assume that the initial strength is $X_{th} = 55$, and the censored time is $t^{(c)} = 50$;

② The degradation of the strength satisfies $S(t) = a\exp\left(-\frac{t}{b}\right)$ where $a = X_{th}$ and $b = 40$;

③ The remaining parameters are the same as those in Section IV(B.1).

(2) Estimated result

The distribution of the inter-arrival time and the magnitude of the damage are both fitted by a 4$^{\text{th}}$-order Coxian distribution with the same initial values as follows:

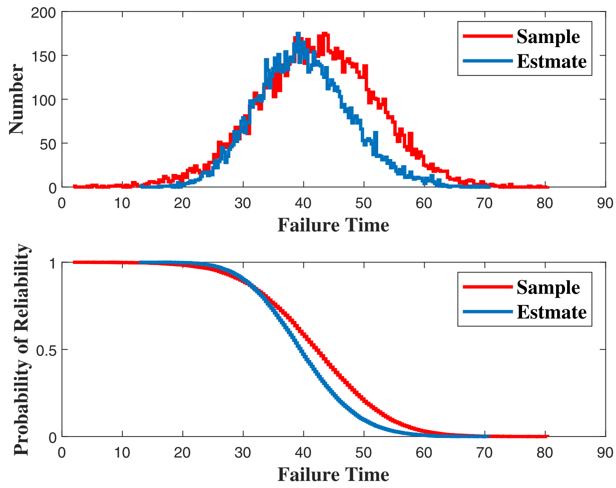

FIGURE 8. Comparison of the distribution and reliability.

① Initial distribution vector: $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

② Phase-type generator: $\begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.

③ Exit vector: $\begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}^T$.

Estimated result of the inter-arrival time:

① Initial distribution vector: $\hat{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

② Phase-type generator:

$$\hat{D}_0 = \begin{bmatrix} -0.9303 & 0.9303 & 0 & 0 \\ 0 & -0.8004 & 0.3844 & 0 \\ 0 & 0 & -0.6659 & 0.5430 \\ 0 & 0 & 0 & -0.6751 \end{bmatrix}.$$

③ Exit vector:

$$\hat{d}_1 = \begin{bmatrix} 0 & 0.4160 & 0.1229 & 0.6751 \end{bmatrix}^T.$$

Estimated result of the magnitude of the damage:

① Initial distribution vector: $\hat{\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

② Phase-type generator:

$$\hat{R}_0 = \begin{bmatrix} -2.2962 & 2.2962 & 0 & 0 \\ 0 & -2.2959 & 1.2852 & 0 \\ 0 & 0 & -0.7004 & 0.2669 \\ 0 & 0 & 0 & -0.3085 \end{bmatrix}.$$

③ Exit vector:

$$\hat{r}_1 = \begin{bmatrix} 0 & 1.0107 & 0.4335 & 0.3085 \end{bmatrix}^T.$$

The estimated PDF of the inter-arrival time and the magnitude of the damage based on the PH distribution and their statistical characteristics are compared with the assumed distributions in FIGURE 9 and TABLE 21, respectively.

The estimated result of b is $\hat{b} = 42.3342$. The distribution and the reliability based on Monte-Carlo sample sets are compared; the results are shown in FIGURE 10. The statistical characteristics of the lifetime and the reliability of the samples at different time points are shown in TABLE 22 and TABLE 20, respectively.

TABLE 19. Comparison of the reliability at different times (case 2, situation 1).

No.	Time	$t=0$	$t=10$	$t=20$	$t=30$	$t=40$	$t=50$	$t=55$ (Censored time)
1	Sample	1	0.9985	0.9846	0.8922	0.5860	0.2086	0.0887
2	Estimation	1	1	0.9979	0.9024	0.4712	0.0967	0.0302
3	Deviation	0	0.0015	0.0133	0.0102	-0.1148	-0.1119	-0.0585

TABLE 20. Comparison of the reliability at different times (case 2, situation 2).

No.	Time	$t=10$	$t=15$	$t=20$	$t=25$	$t=30$	$t=35$	$t=40$	$t=45$	$t=50$ (Censored time)
1	Sample	0.9974	0.9906	0.9701	0.9153	0.7878	0.5862	0.3447	0.1622	0.0611
2	Estimation	1	0.9996	0.9949	0.9701	0.8770	0.7077	0.4664	0.2462	0.1024
3	Deviation	0.0026	0.0090	0.0248	0.0548	0.0892	0.1215	0.1217	0.0840	0.0413

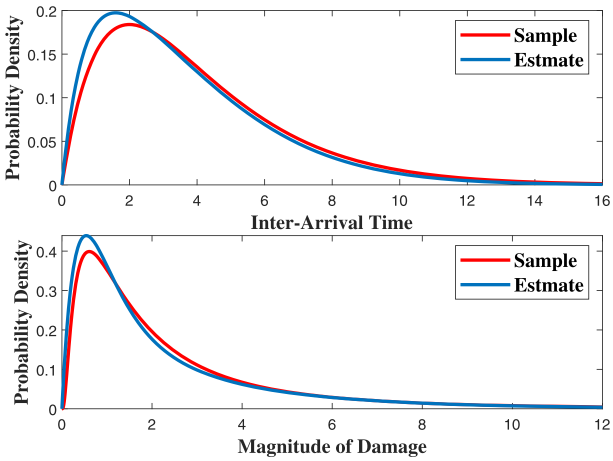

FIGURE 9. PDF of the inter-arrival time and the magnitude of the damage.

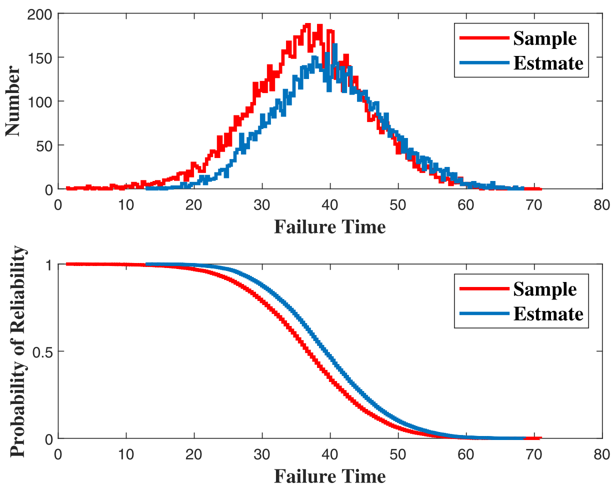

FIGURE 10. Comparison of distribution and reliability.

A log-normal distribution is usually used in the field of reliability and maintainability since its long tail is consistent with the characteristics of failure. The estimated error in CASE 2 is larger than that in CASE 1, which indicates that the selection of the initial values is not arbitrary since both cases have the same initial values. However, the estimated results in both cases are acceptable. For general distributions, the estimated error is not only affected by the number and randomness of the samples but also by the type and order of the PH distribution and its initial values. The selection of the type and order of a PH distribution has been described in [26]–[28], and the different PH distributions are described in detail in [29].

TABLE 21. Statistical characteristics of the time and magnitude.

No.	Type	Statistical Characteristic	Sample	Estimate	Deviation
1	Time	Mean	4	3.6256	-0.3744
		Variance	8	6.6500	-1.3500
2	Magnitude	Mean	2.7183	2.3617	-0.3565
		Variance	12.6965	6.8966	-5.7999

TABLE 22. Statistical characteristics of the lifetime.

No.	Statistical Characteristic	Sample	Estimate	Deviation
1	Mean	36.6788	39.5025	2.8237
2	Variance	74.7637	65.6215	-9.1421

V. CONCLUSION

In this article, a parameter estimation method based on extending Markov state-space with variable transition rates was proposed to solve the phase transition problem when using the traditional EM method to estimate the parameters of the stress-strength model with a PH distribution. The performance of the method was evaluated using two simulation cases. The results showed that the proposed method provided excellent parameter estimation results.

The following limitations of the method were observed: the degradation function of the strength can contain only one parameter, the strength degradation process is a non-random process, and the shock sequence contains only i.i.d. variables. These limitations affect the applicability of the model. Future studies will focus on solving these problems.

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REFERENCES

[1] S. Kotz, Y. Lumelskii, and M. Pensky, *The Stress-Strength Model and Its Generalizations: Theory and Applications*. Back Matter, 2003, pp. 233–253.

[2] I. Makhdoom, P. Nasiri, and A. Pak, ''Bayesian approach for the reliability parameter of power Lindley distribution,'' *Int. J. Syst. Assurance Eng. Manage.*, vol. 7, no. 3, pp. 1–15, 2016.

[3] F. G. Akgul and B. Senoglu, ''Estimation of using modifications of ranked set sampling for Weibull distribution,'' *Pakistan J. Stats Oper. Res.*, vol. 13, no. 4, pp. 931–942, 2017.

[4] A. Kohansal, ''On estimation of reliability in a multi-component stress-strength model for a kumaraswamy distribution based on progressively censored sample,'' *Stat. Papers*, vol. 4, pp. 1–40, May 2017.

[5] A. Kohansal and S. Nadarajah, ''Stress-strength parameter estimation based on type-II hybrid progressive censored samples for a kumaraswamy distribution,'' *IEEE Trans. Rel.*, vol. 68, no. 4, pp. 1296–1310, Dec. 2019.

[6] K. Fatih, ''The mean remaining strength of parallel systems in a stress-strength model based on exponential distribution,'' *Communications*, vol. 8, no. 2, pp. 1435–1451, 2019.

[7] W. Liang *et al.*, ''Reliability inference for a multi-component stress-strength model based on kumaraswamy distribution,'' *J. Comput. Appl. Math.*, vol. 376, pp. 377–427, 2020.

[8] A. Pak, N. B. Khoolenjani, and M. K. Rastogi, ''Bayesian inference on reliability in a multicomponent stress-strength bathtub-shaped model based on record values,'' *Pakistan J. Statist. Oper. Res.*, pp. 431–444, Jun. 2019.

[9] N. A. Mokhlis, E. J. Ibrahim, and D. M. Gharieb, ''Stress-strength reliability with general form distributions,'' *Commun. Statist.-Theory Methods*, vol. 46, no. 3, pp. 1230–1246, Feb. 2017.

[10] M. F. Neuts, *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*. Baltimore, MD, USA: Johns Hopkins Univ. Press, 1981.

[11] L. J. R. Esparza, ''Maximum likelihood estimation of phase-type distributions,'' Dept. Inform. Math. Model., Tech. Univ. Denmark, Lyngby, Denmark, Tech. Rep., 2011.

[12] S. Asmussen and N. M. Olsson, ''Fitting phase-type distributions via the EM algorithm,'' *Scandin. J. Statist.*, vol. 34, no. 4, pp. 419–441, 1996.

[13] S. Eryilmaz and C. Kan, ''Reliability and optimal replacement policy for an extreme shock model with a change point,'' *Rel. Eng. Syst. Saf.*, vol. 190, Oct. 2019, Art. no. 106513.

[14] M. Gong, S. Eryilmaz, and M. Xie, ''Reliability assessment of system under a generalized cumulative shock model,'' *Proc. Inst. Mech. Eng., O, J. Risk Rel.*, vol. 234, no. 1, pp. 129–137, Feb. 2020.

[15] Y. Sarada and R. Shenbagam, ''On a random lead time and threshold shock model using phase-type geometric processes,'' *Appl. Stochastic Models Bus. Ind.*, vol. 34, no. 3, pp. 407–422, May 2018.

[16] X. Zhao, K. Cai, X. Wang, and Y. Song, ''Optimal replacement policies for a shock model with a change point,'' *Comput. Ind. Eng.*, vol. 118, pp. 383–393, Apr. 2018.

[17] D. Montoro-Cazorla, P.-O. Rafael, and M. C. Segovia, ''Survival probabilities for shock and wear models governed by phase-type distributions,'' *Qual. Technol. Quant. Manage.*, vol. 4, no. 1, pp. 85–94, Jan. 2007.

[18] C. D. Montoro, P. O. Rafael, and M. C. Segovia, ''Shock and wear models under policy N using phase-type distributions,'' *Appl. Math. Model.*, vol. 33, no. 1, pp. 543–554, 2009.

[19] D. Montoro-Cazorla and R. Pérez-Ocón, ''System availability in a shock model under preventive repair and phase-type distributions,'' *Appl. Stochastic Models Bus. Ind.*, vol. 26, no. 6, pp. 689–704, Nov. 2010.

[20] M. Manoharan, H. Singh, and N. Misra, ''Preservation of phase-type distributions under Poisson shock models,'' *Adv. Appl. Probab.*, vol. 24, no. 1, pp. 223–225, Mar. 1992.

[21] M. F. Neuts and M. C. Bhattacharjee, ''Shock models with phase type survival and shock resistance,'' *Nav. Res. Logistics Quart.*, vol. 28, no. 2, pp. 213–219, Jun. 1981.

[22] J. Riascos-Ochoa, M. Sanchez-Silva, and R. Akhavan-Tabatabaei, ''Reliability analysis of shock-based deterioration using phase-type distributions,'' *Probabilistic Eng. Mech.*, vol. 38, pp. 88–101, Oct. 2014.

[23] L. Wang, Y. Li, Y. Qian, and X. Luo, ''A parameter estimation method of shock model constructed with phase-type distribution on the condition of interval data,'' *Math. Problems Eng.*, vol. 2020, pp. 1–14, Jun. 2020.

[24] M. Olsson, ''Estimation of phase-type distributions from censored data,'' *Scand. J. Statist.*, vol. 23, no. 4, pp. 443–460, 1996.

[25] J. P. Kharoufeh, C. J. Solo, and M. Y. Ulukus, ''Semi-Markov models for degradation-based reliability,'' *IIE Trans.*, vol. 42, no. 8, pp. 599–612, May 2010.

[26] T. Altiok, ''On the phase-type approximations of general distributions,'' *IIE Trans.*, vol. 17, no. 2, pp. 110–116, Jun. 1985.

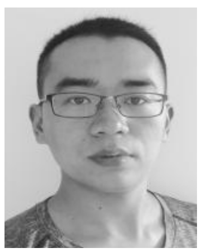
[27] M. A. Johnson, ''Selecting parameters of phase distributions: Combining nonlinear programming, heuristics, and erlang distributions,'' *ORSA J. Comput.*, vol. 5, no. 1, pp. 69–83, Feb. 1993.

[28] T. Osogami and M. Harchol-Balter, ''Closed form solutions for mapping general distributions to quasi-minimal PH distributions,'' *Perform. Eval.*, vol. 63, no. 6, pp. 524–552, Jun. 2006.

[29] J. S. William, *Probability, Markov Chains, Queues, and Simulation: The Mathematical Basis of Performance Modeling*. Princeton, NJ, USA: Princeton Univ. Press, 2009.

LONG WANG received the B.S. and M.S. degrees in mechanical engineering from the National University of Defense Technology (NUDT), Changsha, China, in 2012 and 2014, respectively, where he is currently pursuing the Ph.D. degree with the Laboratory of Science and Technology on Integrated Logistics Support.

He is currently a Lecturer with the Aviation University of Air Force. His current research interests include failure analysis, health management, and reliability testing.

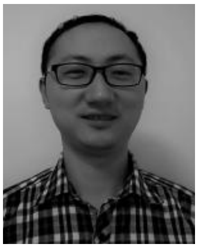


TENGFEI XU received the B.S. degree from the China University of Geosciences (CUG), Wuhan, China, in 2012, and the M.S. degree from the National University of Defense Technology (NUDT), Changsha, China, in 2014.

He is currently an Engineer with the Laboratory of Science and Technology on Integrated Logistics Support, NUDT. His current research interests include condition monitoring, fault diagnosis, and general quality characteristics assessment.

QINGJIE ZHANG received the M.S. and Ph.D. degrees in control science and engineering from the National University of Defense Technology (NUDT), Changsha, China, in 2006 and 2011, respectively.

He is currently an Associate Professor with the Aviation University of Air Force. His current research interests include optimization algorithm, system identification, and pattern recognition.



XU LUO received the M.S. and Ph.D. degrees in mechanical engineering from the National University of Defense Technology (NUDT), Changsha, China, in 2008 and 2014, respectively.

He is currently a Lecturer with the Laboratory of Science and Technology on Integrated Logistics Support, NUDT. His current research interests include maintenance engineering, maintainability design, and maintainability evaluation.

YUE LI received the M.S. and Ph.D. degrees in mechanical engineering from the National University of Defense Technology (NUDT), Changsha, China, in 1993 and 2007, respectively.

He is currently a Full Professor with the Laboratory of Science and Technology on Integrated Logistics Support, NUDT. His research interests include condition monitoring and fault diagnosis, bio-inspired self-repairing technology, and reliability testing and evaluating.