

Received August 25, 2020, accepted September 17, 2020, date of publication September 24, 2020, date of current version October 7, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3026474

# A Novel Multiple Attribute Group Decision-Making Approach Based on Interval-Valued Pythagorean Fuzzy Linguistic Sets

YANG ZHOU<sup>1</sup>, YUAN XU<sup>2</sup>, WUHUAN XU<sup>2</sup>,  
JUN WANG<sup>3</sup>, AND GUANGMING YANG<sup>1</sup>

<sup>1</sup>School of Economics and Management, Zhejiang University of Science and Technology, Hangzhou 310023, China

<sup>2</sup>School of Economics and Management, Beijing Jiaotong University, Beijing 100044, China

<sup>3</sup>School of Economics and Management, Beijing University of Chemical Technology, Beijing 100029, China

Corresponding author: Jun Wang (wangjun@mail.buct.edu.cn)


This work was supported in part by the Fundamental Research Funds for the Central Universities, Zhejiang University of Science and Technology (ZUST), under Grant F701107J08, and in part by the Funds for First-Class Discipline Construction under Grant XK1802-5.

**ABSTRACT** This article investigates multi-attribute group decision-making (MAGDM) problems based on interval-valued Pythagorean fuzzy linguistic sets (IVPFLSs). The IVPFLSs are regarded as an efficient tool to describe decision makers' (DMs') evaluation information from both quantitative and qualitative aspects. However, existing IVPFLSs based MAGDM methods are still insufficient and inadequate to deal with complicated practical situations. This article aims to propose a novel MAGDM method and the main contributions of the present work are three-fold. First, we propose new operations of interval-valued Pythagorean fuzzy linguistic numbers (IVPFLNs) based on linguistic scale function. Second, we propose new aggregation operators (AOs) of IVPFLNs based on power average operator and Muirhead mean. The proposed AOs take the interrelationship among any numbers of attributes into account and eliminate the bad influence of DMs' unreasonable evaluation values on the final decision results. Third, based on the new operations and AOs of IVPFLNs, we introduce a novel approach to MAGDM and present its main steps. Finally, we discuss the effectiveness of the proposed approach and investigate their advantages through numerical examples.

**INDEX TERMS** Interval-valued Pythagorean fuzzy linguistic sets, interval-valued Pythagorean fuzzy linguistic power Muirhead mean, linguistic scale function, multiple attribute group decision-making.

## I. INTRODUCTION

Multiple attribute group decision-making (MAGDM) theories and models have gained much attention and been extensively employed in practical decision-making problems, such as supplier selection [1], [2], investment selection [3], [4], smart medical device selection [5], signal processing [6] etc. How to effectively deal with the inherent fuzziness of decision-making problems and decision makers' (DMs') evaluation information is fundamental issue before determining the optimal alternative. In order to do this, Yager [7] proposed the concept of Pythagorean fuzzy sets (PFSs), satisfying the constraint that the square sum of membership grade (MG) and non-membership grade (NMG) is equal

The associate editor coordinating the review of this manuscript and approving it for publication was Emre Koyuncu .

to or less than one. From the constraint of PFSs, we can find out that they can describe larger information space than the intuitionistic fuzzy sets (IFSs) [8]. Therefore, PFSs have been regarded as an efficient tool in portraying DMs' complicated and fuzzy evaluation information in MAGDM process and quite a few novel decision-making methods have been proposed [9]–[14]. Recently, Harish [15] proposed the interval-valued PFSs (IVPFSs), which employ interval values rather than crisp numbers to represent the MG and NMG. Evidently, compared with the traditional PFS, the IVPFSs take more information into account and can better depict DMs' judgements over alternatives in decision-making procedure. Afterwards, IVPFSs based decision-making methods have been a research focus. For example, Peng and Yang [16] investigated the basic properties of IVPFS, studied their aggregation operators (AOs), and proposed a new MAGDM

method. Harish [17] proposed a novel accuracy function for IVPFSs. Yang *et al.* [18] proposed novel operations of interval-valued Pythagorean fuzzy numbers (IVPFNs) under Frank t-norm and t-conorm and based on which the authors further proposed a set of interval-valued Pythagorean fuzzy (IVPFS) power average operators. Chen [19] proposed an IVPF compromise decision-making approach and applied it in bridge construction analysis. Wei and Mao [20] proposed an IVPF Maclaurin symmetric mean AO based decision-making method, which is powerful for its ability of capturing the interrelationship among multiple attributes. Sajjad Ali Khan *et al.* [21] proposed a hybrid IVPF decision-making method based on TOPSIS and Choquet integral. Sajjad Ali Khan and Abdullah [22] proposed an IVPF grey relational analysis method to deal with MAGDM problems with incomplete weight information. Liang *et al.* [23] introduced IVPF extended Bonferroni mean operators to proceed the heterogeneous interrelationship between aggregated IVPFNs. Garg [24] proposed some exponential operations of IVPFNs and introduced new IVPF operators. Tang *et al.* [25] proposed a series of IVPF Muirhead mean (MM) operators, which can capture the complicated interrelationship among IVPFNs flexibly. Peng and Li [26] proposed two IVPF decision making methods based on weighted distance-based approximation and multiparametric similarity measure and employed it in emergency decision making. For more recent developments of IVPFSs based decision-making methods, readers are suggested to refer [27]–[31].

More recently, Du *et al.* [32] proposed a new extension of IVPFS, called IVPF linguistic set (IVPFLS), which is a combination of IVPFS with linguistic terms set. The IVPFLS is parallel to interval-valued intuitionistic linguistic sets (IVILSs) [33] and both of them can portray both DMS' quantitative and qualitative evaluation information. But, IVPFLSs are more powerful and flexible than IVILSs as they have laxer constraint, which provides DMs more freedom to express their judgments. In [32], Du *et al.* further defined the operations of IVPF linguistic numbers (IVPFLNs), proposed their AOs and developed a new MAGDM method under IVPFLSs context. However, the decision-making method proposed by Du *et al.* [32] still has several shortcomings. First, the operations of IVPFLVs proposed by Du *et al.* [32] are not closed and they fail to handle the semantic translation requirements of different DMs. (We will discuss the drawbacks of these operations in Section 3 in detail.) Second, Du *et al.*'s [32] decision-making method cannot effectively deal with DMs' extreme evaluation values. In other words, Du *et al.*'s [32] method may produce unreasonable decision results. Third, Du *et al.*'s [32] method is based on the simple weighted average/geometric operator, which is unable to deal with the complicated interrelationship among attributes. Hence, the reliability of decision results derived by this method is weak.

Based on above analysis, this article aims to propose a new MAGDM method with IVPFL information. First, we propose novel operations of IVPFLNs based on linguistic scale function (LSF). The new IVPFL operational rules overcome the

drawbacks of existing operations of IVPFLNs. In the addition, the operations can effectively handle the semantic translation requirements of different DMs. Besides, some other notions such as comparison method and distance measure are also presented. Second, to overcome the second and third shortcomings we propose new AOs of IVPFLNs based on the power average [33] operator and Muirhead mean (MM) [34]. The PA operator was originated by Prof. Yager, and it has received much interests due to its ability of reducing the bad influence of DMs' unreasonable evaluation information [35]–[39]. Hence, in this article we firstly propose some IVPFL operators based on PA, i.e. the IVPFL power average (IVPFLPA) operator and the IVPFL power weighted average (IVPFLPWA) operator. The MM is good at capturing the interrelationship among multiple attributes and this is the reason that it has been widely employed in information aggregation process [40]–[43]. Recently, Li *et al.* [44] integrated PA with MM and proposed the power Muirhead mean (PMM) operator, which takes the advantages of both PA and MM. Hence, the PMM operator has been regarded as a promising information aggregation technique [45]–[47]. So, we further propose some AOs of IVPFLNs based on PMM, i.e. the IVPFL power Muirhead mean (IVPFLPMM) operator and the IVPFL power weighted Muirhead mean (IVPFLPWMM) operator. Finally, we present a new MAGDM method based on the proposed AOs. In the new decision-making approach, the IVPFLPWA is employed to calculate the comprehensive decision matrix and the IVPFLPWMM is used to compute the overall evaluation values of alternatives. We further provide numerical experiments to show the good performance of our proposed method.

The rest of this article is organized as follows. Section 2 reviews some existing concepts and proposes new operations of IVPFLNs. Section 3 presents new AOs of IVPFLNs and discusses their properties. Section 4 puts forward a new MAGDM method. Section 5 provides numerical examples to better illustrate the performance and advantages of our proposed method. The summary of this article and the future research issues are presented in Section 6.

## II. PRELIMINARIES

In this section, we review some basic concepts and proposes new operational laws of IVPFLNs based on LSF.

### A. THE INTERVAL-VALUED PYTHAGOREAN FUZZY LINGUISTIC SETS

*Definition 1* [32]: Let  $X$  be an ordinary fixed set and  $S$  be a continuous linguistic term set, then an interval-valued Pythagorean fuzzy linguistic set (IVPFLS)  $A$  defined on  $X$  is expressed as

$$A = \{ \langle x, [s_{\theta(x)}, (\mu_A(x), \nu_A(x))] \rangle \mid x \in X \}, \quad (1)$$

where  $\mu_A(x)$  and  $\nu_A(x)$  are two interval values in  $[0, 1]$ , denoting the MD and NMD of the element  $x \in X$  to the set  $A$ , such that  $\text{Sup}(\mu_A(x))^2 + \text{Sup}(\nu_A(x))^2 \leq 1$  for  $\forall x \in X$ . For convenience, we call  $A = \langle s_{\theta}, (\mu_A, \nu_A) \rangle$  an IVPFLN, which

can be denoted by  $\alpha = \langle s_\theta, ([a, b], [c, d]) \rangle$  for simplicity, such that  $0 \leq a \leq b \leq 1, 0 \leq c \leq d \leq 1$  and  $b^2 + d^2 \leq 1$ .

Based on the operations of IVPFNs, Du et al. [32] proposed some operational rules of IVPFLNs.

*Definition 2 [32]:* Let  $\alpha_1 = \langle s_{\theta_1}, ([a_1, b_1], [c_1, d_1]) \rangle, \alpha_2 = \langle s_{\theta_2}, ([a_2, b_2], [c_2, d_2]) \rangle$  and  $\alpha = \langle s_\theta, ([a, b], [c, d]) \rangle$  be any three IVPFLNs,  $\lambda$  be a positive real number, then

$$(1) \alpha_1 \oplus \alpha_2 =$$

$$\left\langle s_{\theta_1 + \theta_2}, \left( \left[ \left( a_1^2 + a_2^2 - a_1^2 a_2^2 \right)^{1/2}, \left( b_1^2 + b_2^2 - b_1^2 b_2^2 \right)^{1/2} \right], [c_1 c_2, b_1 b_2] \right) \right\rangle;$$

$$(2) \alpha_1 \otimes \alpha_2 = \langle s_{\theta_1 \times \theta_2}, ([a_1 a_2, b_1 b_2],$$

$$\left[ \left( c_1^2 + c_2^2 - c_1^2 c_2^2 \right)^{1/2}, \left( d_1^2 + d_2^2 - d_1^2 d_2^2 \right)^{1/2} \right] \rangle;$$

$$(3) \lambda \alpha =$$

$$\left\langle s_{\lambda \theta}, \left( \left[ \left( 1 - (1 - a^2)^\lambda \right)^{1/2}, \left( 1 - (1 - b^2)^\lambda \right)^{1/2} \right], [c^\lambda, d^\lambda] \right) \right\rangle;$$

$$(4) \alpha^\lambda =$$

$$\left\langle s_{\theta \lambda}, \left( [a^\lambda, b^\lambda], \left[ \left( 1 - (1 - c^2)^\lambda \right)^{1/2}, \left( 1 - (1 - d^2)^\lambda \right)^{1/2} \right] \right) \right\rangle.$$

To compare any two IVPFLNs, Du et al. [32] proposed a comparison method.

*Definition 3 [32]:* Let  $\alpha = \langle s_\theta, ([a, b], [c, d]) \rangle$  be a IVPFLN, then the score function  $S(\alpha)$  of  $\alpha$  is expressed as

$$S(\alpha) = s_{\theta \times (a^2 + b^2 - c^2 - d^2 + 2)} / 4, \tag{2}$$

and the accuracy function  $H(\alpha)$  of  $\alpha$  is expressed as

$$H(\alpha) = s_{\theta \times (a^2 + b^2 + c^2 + d^2)} / 2. \tag{3}$$

Let  $\alpha_1$  and  $\alpha_2$  be any two IVPFLNs, then

- (1) If  $S(\alpha_1) > S(\alpha_2)$ , then  $\alpha_1 > \alpha_2$ ;
- (2) If  $S(\alpha_1) = S(\alpha_2)$ , then,
  - If  $H(\alpha_1) = H(\alpha_2)$ , then  $\alpha_1 = \alpha_2$ ;
  - If  $H(\alpha_1) > H(\alpha_2)$ , then  $\alpha_1 > \alpha_2$ .

**B. THE POWER AVERAGE AND MUIRHEAD MEAN**

*Definition 4 [33]:* Let  $a_i (i = 1, 2, \dots, n)$  be a collection of crisp numbers, then the power average (PA) operator is expressed as

$$PA(a_1, a_2, \dots, a_n) = \frac{\bigoplus_{i=1}^n (1 + T(a_i)) a_i}{\sum_{i=1}^n (1 + T(a_i))}, \tag{4}$$

where  $T(a_i) = \sum_{j=1, i \neq j}^n Sup(a_i, a_j)$ ,  $Sup(a_i, a_j)$  is the support of  $a_i$  from  $a_j$ , satisfying the following properties

- (1)  $0 \leq Sup(a, b) \leq 1$ ;
- (2)  $Sup(a, b) = Sup(b, a)$ ;
- (3)  $Sup(a, b) \leq Sup(c, d)$ , if  $|a, b| \geq |c, d|$ .

*Definition 5 [34]:* Let  $a_i (i = 1, 2, \dots, n)$  be a collection of crisp numbers and  $P = (p_1, p_2, \dots, p_n)^T$  be a vector of parameters. If

$$MM^P(a_1, a_2, \dots, a_n) = \left( \frac{1}{n!} \sum_{\vartheta \in L_n} \prod_{j=1}^n a_{\vartheta(j)}^{p_j} \right)^{\frac{1}{\sum_{j=1}^n p_j}}, \tag{5}$$

then  $MM^P$  is the Muirhead mean (MM) operator, where  $\vartheta(j) (j = 1, 2, \dots, n)$  represents any permutation of  $(1, 2, \dots, n)$  and  $L_n$  denotes all possible permutations of  $(1, 2, \dots, n)$ .

**III. NEW OPERATIONS OF IVPFLNS BASED ON LINGUISTIC SCALE FUNCTIONS**

In this section, we aim to propose novel operations of IVPFLNs based on LSF. In order to do this, we first explain the necessity and motivations of proposing novel operations of IVPFLNs. Second, we briefly review the notion of LSFs. Further, we present new interval-valued Pythagorean fuzzy linguistic operations and discuss their properties.

**A. NECESSITY AND MOTIVATIONS**

In [32], Du et al. proposed some basic operations of IVPFLNs. However, Du et al.'s [32] operations have an obvious shortcoming. To better explain the shortcoming, we provide the following example.

*Example 1:* Let  $\alpha_1 = \langle s_3, ([0.6, 0.7], [0.3, 0.4]) \rangle$  and  $\alpha_2 = \langle s_5, ([0.4, 0.6], [0.2, 0.6]) \rangle$  be any two IVPFLNs defined on a given LTS  $S = \{s_0, s_1, \dots, s_6\}$ . According to the operations proposed by Du et al. [32], we have

- (1)  $\alpha_1 \oplus \alpha_2 = \langle s_8, ([0.68, 0.8207], [0.06, 0.24]) \rangle$ ;
- (2)  $\alpha_1 \otimes \alpha_2 = \langle s_8, ([0.24, 0.42], [0.3555, 0.68]) \rangle$ ;
- (3)  $3\alpha_1 = \langle s_9, ([0.8590, 0.9313], [0.027, 0.064]) \rangle$ ;
- (4)  $\alpha_1^3 = \langle s_{27}, ([0.216, 0.343], [0.4964, 0.6382]) \rangle$ .

From Example 1, it is easy to find out that the operations proposed by Du et al. [32] have an evident drawback, i.e. the calculation results in Example 1 have exceed the upper limit of the given LTS  $S$ . Hence, it is necessary to propose some new operational rules of IVPFLNs. Actually, similar researches can be found in recent publications. For instance, Liu et al. [48] proposed novel operations of intuitionistic uncertain linguistic variables based on LSF. Liu et al. [49] proposed operations and aggregation operators of interval-valued hesitant uncertain linguistic variables based on LSF. Therefore, we can propose new operational rules of IVPFLNs based on LSF. In order to this, we first review the concept of LSFs.

**B. THE CONCEPT OF LSF**

*Definition 6 [50]:* Let  $S = \{s_i | i = 0, 1, \dots, 2t\}$  be a linguistic term set,  $s_i \in S$  be a linguistic term and  $\tau_i \in [0, 1]$  be a real number. A linguistic scale function (LSF)  $f$  is a mapping from  $s_i$  to  $\tau_i$  ( $i = 1, 2, \dots, 2t$ ) such that

$$f : s_i \rightarrow \tau_i (i = 0, 1, 2, \dots, 2t), \tag{6}$$

where  $0 \leq \tau_0 < \tau_1 < \dots < \tau_{2t}$ . Hence,  $f$  is a strictly monotonically increasing function with regard to linguistic subscript  $i$ . Generally, there are three types of LSFs and we give a brief review in the following.

(1) The most widely used LSF is expressed as

$$f_1(s_i) = \theta_i = \frac{i}{2t} (i = 0, 1, 2, \dots, 2t), \tag{7}$$

which is a simple average calculation of the subscripts of linguistic terms.

(2) The second type of LSF is expressed as follows

$$f_2(s_i) = \theta_i = \begin{cases} \frac{\rho^t - \rho^{t-i}}{2\rho^t - 2} & (i = 0, 1, 2, \dots, t) \\ \frac{\rho^t + \rho^{i-t} - 2}{2\rho^t - 2} & (i = t + 1, t + 2, \dots, 2t), \end{cases} \tag{8}$$

(3) The third type of LSF is expressed as

$$f_3(s_i) = \theta_i = \begin{cases} \frac{t^\varepsilon - (t-i)^\varepsilon}{2t^\varepsilon} & (i = 0, 1, 2, \dots, t) \\ \frac{t^\beta + (i-t)^\beta}{2t^\beta} & (i = t + 1, t + 2, \dots, 2t), \end{cases} \tag{9}$$

For Eq. (8),  $\gamma_i \in [0, 1]$  and the absolute deviation between adjacent linguistic subscripts increase with linguistic subscripts  $i$  ( $i = 0, 1, 2, \dots, 2t$ ). We can use a subjective method to determine the value of  $\rho$ . Suppose that the weight of attribute  $A$  is much greater than attribute  $B$ , and the weight ratio is  $m$ , then  $\rho^k = m$  ( $k$  represents the scale level) and  $\rho = \sqrt[k]{m}$ . Up to now, most researchers reckon that the upper limit of the weight ratio is 9. Thus,  $\rho = \sqrt[7]{9} \approx 1.37$  can be calculated when the scale level is 7.

Especially, if  $\varepsilon = \beta = 1$ , then Eq. (9) is reduced to Eq. (7). Meanwhile, the function  $f$  can be further expanded to a continuous function such that  $f^* : \tilde{f} \rightarrow \Omega^+$  ( $\Omega^+ = \{d | d \geq 0\}, d \in R$ ), which satisfies  $f^*(s_i) = \gamma_i$ . The inverse function of  $f^*$  is depicted as  $f^{*-1}$ . Then, we can get

$$f_1^{*-1}(\gamma_i) = s_{2t * i} (i = 0, 1, 2, \dots, 2t), \tag{10}$$

$$f_2^{*-1}(\gamma_i) = \begin{cases} s_{t - \log_\rho(\rho^t - (2\rho^t - 2)\gamma_i)}, & (\gamma_i \in [0, 0.5]) \\ s_{t + \log_\rho((2\rho^t - 2)\gamma_i - \rho^t + 2)}, & (\gamma_i \in [0.5, 1.0]), \end{cases} \tag{11}$$

$$f_3^{*-1}(\gamma_i) = \begin{cases} s_{t - (t^\varepsilon - 2 \times t^\varepsilon \times \gamma_i)^{1/\varepsilon}}, & (\gamma_i \in [0, 0.5]) \\ s_{t + (2 \times t^\beta \times \gamma_i - t^\beta)^{1/\beta}}, & (\gamma_i \in [0.5, 1]) \end{cases} \tag{12}$$

**C. NEW OPERATIONS OF IVPFLNS**

*Definition 7:* Let  $\alpha_1 = \langle s_{\theta_1}, ([a_1, b_1], [c_1, d_1]) \rangle$ ,  $\alpha_2 = \langle s_{\theta_2}, ([a_2, b_2], [c_2, d_2]) \rangle$  and  $\alpha = \langle s_\theta, ([a, b], [c, d]) \rangle$  be any three IVPFLNs and  $\lambda$  be a positive real number, then

- (1)  $\alpha_1 \oplus \alpha_2 = \langle f^{*-1}(f^*(\theta_1) + f^*(\theta_2) - f^*(\theta_1)f^*(\theta_2)), \left( \left[ (a_1^2 + a_2^2 - a_1^2 a_2^2)^{1/2}, (b_1^2 + b_2^2 - b_1^2 b_2^2)^{1/2} \right], [c_1 c_2, b_1 b_2] \right) \rangle$ ;
- (2)  $\alpha_1 \otimes \alpha_2 = \langle f^{*-1}(f^*(\theta_1) \times f^*(\theta_2)), \left( [a_1 a_2, b_1 b_2], \left[ (c_1^2 + c_2^2 - c_1^2 c_2^2)^{1/2}, (d_1^2 + d_2^2 - d_1^2 d_2^2)^{1/2} \right] \right) \rangle$ ;
- (3)  $\lambda \alpha = \langle f^{*-1}(1 - (1 - f^*(\theta))^\lambda), \left( \left[ (1 - (1 - a^2)^\lambda)^{1/2}, (1 - (1 - b^2)^\lambda)^{1/2} \right], [c^\lambda, d^\lambda] \right) \rangle$ ;
- (4)  $\alpha^\lambda = \langle f^{*-1}(f^*(\theta)^\lambda), \left( [a^\lambda, b^\lambda], \left[ (1 - (1 - c^2)^\lambda)^{1/2}, (1 - (1 - d^2)^\lambda)^{1/2} \right] \right) \rangle$ .

*Example 2.* Let  $\alpha_1 = \langle s_3, ([0.6, 0.7], [0.3, 0.4]) \rangle$  and  $\alpha_2 = \langle s_5, ([0.4, 0.6], [0.2, 0.6]) \rangle$  be any two IVPFLNs defined on a given LTS  $S = \{s_0, s_1, \dots, s_6\}$ . If LSF 1 is applied in Definition 7, then we have

- (1)  $\alpha_1 \oplus \alpha_2 = \langle s_{5.5}, ([0.68, 0.8207], [0.06, 0.24]) \rangle$ ,
  - (2)  $\alpha_1 \otimes \alpha_2 = \langle s_{2.5}, ([0.24, 0.42], [0.3555, 0.68]) \rangle$ ;
  - (3)  $3\alpha_1 = \langle s_{5.25}, ([0.8590, 0.9313], [0.027, 0.064]) \rangle$ ;
  - (4)  $\alpha_1^3 = \langle s_{0.75}, ([0.216, 0.343], [0.4964, 0.6382]) \rangle$ .
- It is easy to prove that the following theorem holds.
- Theorem 1:* Let  $\alpha_1, \alpha_2$  and  $\alpha$  be any three IVPFLNs, then
- (1)  $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$ ;
  - (2)  $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$ ;
  - (3)  $\lambda (\alpha_1 \oplus \alpha_2) = \lambda \alpha_2 \oplus \lambda \alpha_1, \lambda > 0$ ;
  - (4)  $\lambda_1 \alpha \oplus \lambda_2 \alpha = (\lambda_1 + \lambda_2) \alpha, \lambda_1, \lambda_2 > 0$ ;
  - (5)  $\alpha_1^\lambda \otimes \alpha_2^\lambda = (\alpha_2 \otimes \alpha_1)^\lambda, \lambda > 0$ ;
  - (6)  $\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{\lambda_1 + \lambda_2}, \lambda_1, \lambda_2 > 0$ ;

**D. NEW COMPARISON METHOD OF IVPFLNS**

Based on LSF, we propose novel comparison method of IVPFLNs.

*Definition 8:* Let  $\alpha = \langle s_\theta, ([a, b], [c, d]) \rangle$  be an IVPFLN, then the score function of  $\alpha$  is defined as

$$S(\alpha) = \frac{1}{2} \times \left( 1 + \frac{a^2 + b^2}{2} - \frac{c^2 + d^2}{2} \right) \times f^*(\theta), \tag{13}$$

and the accuracy function of  $\alpha$  is defined as

$$H(\alpha) = \left( \frac{a^2 + b^2}{2} + \frac{c^2 + d^2}{2} \right) \times f^*(\theta), \tag{14}$$

For any two IVPFLNs  $\alpha_1 = \langle s_{\theta_1}, ([a_1, b_1], [c_1, d_1]) \rangle$  and  $\alpha_2 = \langle s_{\theta_2}, ([a_2, b_2], [c_2, d_2]) \rangle$ , then

(3) If  $S(\alpha_1) > S(\alpha_2)$ , then  $\alpha_1 > \alpha_2$ ;

(4) If  $S(\alpha_1) = S(\alpha_2)$ , then,

If  $H(\alpha_1) = H(\alpha_2)$ , then  $\alpha_1 = \alpha_2$ ;

If  $H(\alpha_1) > H(\alpha_2)$ , then  $\alpha_1 > \alpha_2$ .

*Example 3:* Let  $\alpha_1 = \langle s_3, ([0.6, 0.7], [0.4, 0.5]) \rangle$  and  $\alpha_2 = \langle s_4, ([0.6, 0.75], [0.35, 0.5]) \rangle$  be any two IVPFLNs defined on a given LST  $S = \{s_0, s_1, \dots, s_6\}$ . If we employ LSF 1, we can obtain  $S(\alpha_1) = 0.305$ ,  $H(\alpha_1) = 0.315$ ,  $S(\alpha_2) = 0.425$  and  $H(\alpha_2) = 0.4317$ . Hence,  $\alpha_1 < \alpha_2$ . If we employ LSF 2 and  $\rho = 1.37$ , then we can obtain  $S(\alpha_1) = 0.305$ ,  $H(\alpha_1) = 0.315$ ,  $S(\alpha_2) = 0.3938$  and  $H(\alpha_2) = 0.4$ . Hence,  $\alpha_1 < \alpha_2$ . If we use LSF 3 and  $\varepsilon = \beta = 1.25$ , then we get  $S(\alpha_1) = 0.305$ ,  $H(\alpha_1) = 0.315$ ,  $S(\alpha_2) = 0.3995$  and  $H(\alpha_2) = 0.4057$ . Thus,  $\alpha_1 < \alpha_2$ .

**E. DISTANCE BETWEEN TWO IVPFLNS BASED ON LSF**

Based on LSF, we propose a new concept of distance between two IVPFLNs.

*Definition 9:* Let  $\alpha_1 = \langle s_{\theta_1}, ([a_1, b_1], [c_1, d_1]) \rangle$ ,  $\alpha_2 = \langle s_{\theta_2}, ([a_2, b_2], [c_2, d_2]) \rangle$  be any two IVPFLNs, then the distance between  $\alpha_1$  and  $\alpha_2$  is expressed as

$$d(\alpha_1, \alpha_2) = \frac{1}{4} \times |f * (\theta_1) - f * (\theta_2)| \times \left( |a_1^2 - a_2^2| + |b_1^2 - b_2^2| + |c_1^2 - c_2^2| + |d_1^2 - d_2^2| \right). \quad (15)$$

*Example 4:* Let  $\alpha_1 = \langle s_3, ([0.6, 0.7], [0.3, 0.4]) \rangle$  and  $\alpha_2 = \langle s_5, ([0.4, 0.6], [0.2, 0.6]) \rangle$  be two IVPFLNs defined on a given LTS  $S = \{s_0, s_1, \dots, s_6\}$ . If we use LSF 1, then the distance between  $\alpha_1$  and  $\alpha_2$  is

$$d(\alpha_1, \alpha_2) = \frac{1}{4} \times \left| \frac{3}{6} - \frac{5}{6} \right| \times \left( |0.6^2 - 0.4^2| + |0.7^2 - 0.6^2| + |0.3^2 - 0.2^2| + |0.4^2 - 0.6^2| \right) = 0.0483.$$

**IV. NOVEL AGGREGATION OPERATORS OF IVPFLNS**

Based on PA, MM and the new operations of IVPFLNs, we propose novel AOs of IVPFLNs and discuss their properties.

**A. THE INTERVAL-VALUED PYTHAGOREAN FUZZY LINGUISTIC POWER AVERAGE OPERATOR**

*Definition 10:* Let  $\alpha_i (i = 1, 2, \dots, n)$  be a collection of IVPFLNs, then the interval-valued Pythagorean fuzzy linguistic power average (IVPFLPA) operator is expressed as

$$IVPFLPA(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{\bigoplus_{i=1}^n (1 + T(\alpha_i)) \alpha_i}{\sum_{i=1}^n (1 + T(\alpha_i))}, \quad (16)$$

where  $T(\alpha_i) = \sum_{j=1, j \neq i}^n Sup(\alpha_i, \alpha_j)$ ,  $Sup(\alpha_i, \alpha_j)$  denotes the support for  $\alpha_i$  from  $\alpha_j$ , satisfying the condition

(1)  $0 \leq Sup(\alpha_i, \alpha_j) \leq 1$ ;

(2)  $Sup(\alpha_i, \alpha_j) = Sup(\alpha_j, \alpha_i)$ ;

(3)  $Sup(\alpha, \beta) \leq Sup(\chi, \delta)$ , if  $d(\alpha, \beta) \geq d(\chi, \delta)$ .

If we assume

$$\phi_i = \frac{1 + T(\alpha_i)}{\sum_{i=1}^n (1 + T(\alpha_i))}, \quad (17)$$

then (16) can be written as

$$IVPFLPA(\alpha_1, \alpha_2, \dots, \alpha_n) = \bigoplus_{i=1}^n \phi_i \alpha_i, \quad (18)$$

where  $0 \leq \phi_i \leq 1$  and  $\sum_{i=1}^n \phi_i = 1$ .

*Theorem 2:* Let  $\alpha_i = \langle s_{\theta_i}, ([a_i, b_i], [c_i, d_i]) \rangle (i = 1, 2, \dots, n)$  be a collection of IVPFLNs, then the aggregated value by the IVPFLPA operator is an IVPFLN and

$$IVPFLPA(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle f^{*-1} \left( 1 - \left( \prod_{i=1}^n (1 - f^*(\theta_i))^{\phi_i} \right) \right), \left[ \left[ \left( 1 - \prod_{i=1}^n (1 - a_i^2)^{\phi_i} \right)^{1/2}, \left( 1 - \prod_{i=1}^n (1 - b_i^2)^{\phi_i} \right)^{1/2} \right], \left[ \prod_{i=1}^n c_i^{\phi_i}, \prod_{i=1}^n d_i^{\phi_i} \right] \right\rangle. \quad (19)$$

The proof of Theorem 1 is trivial. In addition, it is easy to prove that the IVPFLPA operator has the following properties.

*Theorem 3 (Idempotency):* Let  $\alpha_i (i = 1, 2, \dots, n)$  be a set of IVPFLNs, if  $\alpha_i = \alpha = \langle s_{\theta}, ([a, b], [c, d]) \rangle$  for any  $i$ , then

$$IVPFLPA(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha. \quad (20)$$

*Theorem 4 (Boundedness):* Let  $\alpha_i (i = 1, 2, \dots, n)$  be a collection of IVPFLNs. Let  $\alpha^- = \min(\alpha_1, \alpha_2, \dots, \alpha_n)$  and  $\alpha^+ = \max(\alpha_1, \alpha_2, \dots, \alpha_n)$ . Then

$$x \leq IVPFLPA(\alpha_1, \alpha_2, \dots, \alpha_n) \leq y. \quad (21)$$

where  $x = \bigoplus_{i=1}^n \phi_i \alpha^-$  and  $y = \bigoplus_{i=1}^n \phi_i \alpha^+$ .

**B. THE INTERVAL-VALUED PYTHAGOREAN FUZZY LINGUISTIC POWER WEIGHTED AVERAGE OPERATOR**

*Definition 11:* Let  $\alpha_i (i = 1, 2, \dots, n)$  be a collection of IVPFLNs, and  $w = (w_1, w_2, \dots, w_n)^T$  be the weight vector of  $\alpha_i (i = 1, 2, \dots, n)$ , such that  $0 \leq w_i \leq 1$  and  $\sum_{i=1}^n w_i = 1$ . The interval-valued Pythagorean fuzzy linguistic power weighted average (IVPFLPWA) operator is expressed as

$$IVPFLPWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{\bigoplus_{i=1}^n w_i (1 + T(\alpha_i)) \alpha_i}{\sum_{i=1}^n w_i (1 + T(\alpha_i))}, \quad (22)$$

where  $T(\alpha_i) = \sum_{j=1, j \neq i}^n \text{Sup}(\alpha_i, \alpha_j)$ ,  $\text{Sup}(\alpha_i, \alpha_j)$  denotes the support for  $\alpha_i$  from  $\alpha_j$ , satisfying the properties presented in Definition 10. If we assume

$$\zeta_i = \frac{w_i (1 + T(\alpha_i))}{\sum_{i=1}^n w_i (1 + T(\alpha_i))}, \quad (23)$$

then (22) can be transformed into

$$\text{IVPFLPWA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \bigoplus_{i=1}^n \zeta_i \alpha_i, \quad (24)$$

such that  $0 \leq \zeta_i \leq 1$  and  $\sum_{i=1}^n \zeta_i = 1$ .

The following theorem can be obtained based on the operations of IVPFLNs.

**Theorem 5:** Let  $\alpha_i = \langle s_{\theta_i}, ([a_i, b_i], [c_i, d_i]) \rangle (i=1, 2, \dots, n)$  be a collection of IVPFLNs, then the aggregated value by the IVPFLPWA operator is an IVPFLN and

$$\begin{aligned} & \text{IVPFLPWA}(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \left\langle f^{*-1} \left( 1 - \left( \prod_{i=1}^n (1 - f^*(\theta_i))^{\zeta_i} \right) \right), \right. \\ & \quad \left[ \left[ \left( 1 - \prod_{i=1}^n (1 - a_i^2)^{\zeta_i} \right)^{1/2}, \left( 1 - \prod_{i=1}^n (1 - b_i^2)^{\zeta_i} \right)^{1/2} \right], \right. \\ & \quad \left. \left[ \prod_{i=1}^n c_i^{\zeta_i}, \prod_{i=1}^n d_i^{\zeta_i} \right] \right] \right\rangle. \quad (25) \end{aligned}$$

In addition, it is easy to prove that the IVPFLPWA operator has the properties of boundedness.

**C. THE INTERVAL-VALUED PYTHAGOREAN FUZZY LINGUISTIC POWER MUIRHEAD MEAN OPERATOR**

**Definition 12:** Let  $\alpha_j (j=1, 2, \dots, n)$  be a collection of IVPFLNs and  $P = (p_1, p_2, \dots, p_n) \in R^n$  be vector of parameter. The interval-valued Pythagorean fuzzy linguistic power Muirhead mean (IVPFLPMM) operator is defined as

$$\begin{aligned} & \text{IVPFLPMM}^P(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \left( \frac{1}{n!} \bigoplus_{\vartheta \in L_n} \bigotimes_{j=1}^n \left( n \frac{(1 + T(\alpha_{\vartheta(j)}))}{\sum_{j=1}^n (1 + T(\alpha_j))} \alpha_{\vartheta(j)} \right)^{p_j} \right)^{\frac{1}{\sum_{j=1}^n p_j}}, \quad (26) \end{aligned}$$

where

$$T(\alpha_j) = \sum_{i=1, i \neq j}^n \text{Sup}(\alpha_i, \alpha_j), \quad (27)$$

and  $d(\alpha_i, \alpha_j)$  is the distance between  $\alpha_i$  and  $\alpha_j$ ,  $\vartheta(j) (j=1, 2, \dots, n)$  represents any permutation of  $(1, 2, \dots, n)$ ,  $L_n$  denotes all possible permutations of  $(1, 2, \dots, n)$ ,  $n$  is the balancing coefficient, and  $\text{Sup}(\alpha_i, \alpha_j)$  denotes the support for  $\alpha_i$  from  $\alpha_j$ , satisfying the properties

in Definition 10. To simplify Eq. (26), let

$$\eta_j = \frac{1 + T(\alpha_j)}{\sum_{j=1}^n (1 + T(\alpha_j))}, \quad (28)$$

then Eq. (26) can be written as

$$\begin{aligned} & \text{IVPFLPMM}^P(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \left( \frac{1}{n!} \bigoplus_{\vartheta \in L_n} \bigotimes_{j=1}^n (n \eta_{\vartheta(j)} \alpha_{\vartheta(j)})^{p_j} \right)^{\frac{1}{\sum_{j=1}^n p_j}}. \quad (29) \end{aligned}$$

where  $0 \leq \eta_j \leq 1$  and  $\sum_{j=1}^n \eta_j = 1$ .

**Theorem 6:** Let  $\alpha_j = \langle s_{\theta_j}, ([a_j, b_j], [c_j, d_j]) \rangle (j=1, 2, \dots, n)$  be a collection of IVPFLNs, then the aggregated value by the IVPFLPMM operator is still an IVPFLN and (30), as shown at the bottom of the next page.

*Proof:* According to Definition 7, we can get

$$\begin{aligned} & n \eta_{\vartheta(j)} \alpha_{\vartheta(j)} \\ &= \left\langle f^{*-1} \left( 1 - (1 - f^*(\theta_{\vartheta(j)}))^{n \eta_{\vartheta(j)}} \right), \right. \\ & \quad \left[ \left[ \left( 1 - (1 - a_{\vartheta(j)}^2)^{n \eta_{\vartheta(j)}} \right)^{1/2}, \left( 1 - (1 - b_{\vartheta(j)}^2)^{n \eta_{\vartheta(j)}} \right)^{1/2} \right], \right. \\ & \quad \left. \left[ c_{\vartheta(j)}^{n \eta_{\vartheta(j)}}, d_{\vartheta(j)}^{n \eta_{\vartheta(j)}} \right] \right] \right\rangle, \end{aligned}$$

and

$$\begin{aligned} & (n \eta_{\vartheta(j)} \alpha_{\vartheta(j)})^{p_j} \\ &= \left\langle f^{*-1} \left( (1 - (1 - f^*(\theta_{\vartheta(j)}))^{n \eta_{\vartheta(j)}})^{p_j} \right), \right. \\ & \quad \left[ \left[ \left( 1 - (1 - a_{\vartheta(j)}^2)^{n \eta_{\vartheta(j)}} \right)^{\frac{1}{2} p_j}, \left( 1 - (1 - b_{\vartheta(j)}^2)^{n \eta_{\vartheta(j)}} \right)^{\frac{1}{2} p_j} \right], \right. \\ & \quad \left[ \left( 1 - (1 - (c_{\vartheta(j)}^{n \eta_{\vartheta(j)}})^2)^{p_j} \right)^{\frac{1}{2}}, \right. \\ & \quad \left. \left. \left( 1 - (1 - (d_{\vartheta(j)}^{n \eta_{\vartheta(j)}})^2)^{p_j} \right)^{\frac{1}{2}} \right] \right] \right] \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} & \bigotimes_{j=1}^n (n \eta_{\vartheta(j)} \alpha_{\vartheta(j)})^{p_j} \\ &= \left\langle f^{*-1} \left( \prod_{j=1}^n (1 - (1 - f^*(\theta_{\vartheta(j)}))^{n \eta_{\vartheta(j)}})^{p_j} \right), \right. \\ & \quad \left[ \left[ \prod_{j=1}^n \left( 1 - (1 - a_{\vartheta(j)}^2)^{n \eta_{\vartheta(j)}} \right)^{\frac{p_j}{2}}, \right. \right. \\ & \quad \left. \left. \prod_{j=1}^n \left( 1 - (1 - b_{\vartheta(j)}^2)^{n \eta_{\vartheta(j)}} \right)^{\frac{p_j}{2}} \right] \right] \right\rangle, \end{aligned}$$

$$\left( \left[ \left( 1 - \prod_{j=1}^n \left( 1 - (c_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2 \right)^{p_j} \right)^{\frac{1}{2}} \right. \right. \\ \left. \left. \left( 1 - \prod_{j=1}^n \left( 1 - (d_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2 \right)^{p_j} \right)^{\frac{1}{2}} \right] \right) \right\}.$$

and

$$\bigoplus_{\vartheta \in L_n} \bigotimes_{j=1}^n (n\eta_{\vartheta(j)} \alpha_{\vartheta(j)})^{p_j} \\ = \left\langle f *^{-1} \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - f * (\theta_{\vartheta(j)}))^{n\eta_{\vartheta(j)}})^{p_j} \right) \right) \right. \\ \left( \left[ \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - a_{\vartheta(j)}^2)^{n\eta_{\vartheta(j)}})^{p_j} \right) \right)^{1/2} \right. \right. \\ \left. \left. \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - b_{\vartheta(j)}^2)^{n\eta_{\vartheta(j)}})^{p_j} \right) \right)^{1/2} \right] \right. \\ \left( \left[ \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n \left( 1 - (c_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2 \right)^{p_j} \right)^{1/2} \right. \right. \\ \left. \left. \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n \left( 1 - (d_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2 \right)^{p_j} \right)^{1/2} \right] \right) \right\}.$$

Then,

$$\frac{1}{n!} \bigoplus_{\vartheta \in L_n} \bigotimes_{j=1}^n (n\eta_{\vartheta(j)} \alpha_{\vartheta(j)})^{p_j} \\ = \left\langle f *^{-1} \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - f * (\theta_{\vartheta(j)}))^{n\eta_{\vartheta(j)}})^{p_j} \right) \right)^{\frac{1}{n!}} \right. \\ \left( \left[ \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - a_{\vartheta(j)}^2)^{n\eta_{\vartheta(j)}})^{p_j} \right) \right) \right)^{\frac{1}{n!}} \right]^{1/2} \right. \\ \left. \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - b_{\vartheta(j)}^2)^{n\eta_{\vartheta(j)}})^{p_j} \right) \right) \right)^{\frac{1}{n!}} \right]^{1/2} \right. \\ \left( \left[ \prod_{\vartheta \in T_n} \left( 1 - \prod_{j=1}^n \left( 1 - (c_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2 \right)^{p_j} \right)^{1/2} \right]^{\frac{1}{n!}} \right. \\ \left. \left[ \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n \left( 1 - (d_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2 \right)^{p_j} \right)^{1/2} \right)^{\frac{1}{n!}} \right] \right\}.$$

Finally,  $\left( \frac{1}{n!} \bigoplus_{\vartheta \in L_n} \bigotimes_{j=1}^n (n\eta_{\vartheta(j)} \alpha_{\vartheta(j)})^{p_j} \right)^{\frac{1}{\sum_{j=1}^n p_j}}$ , as shown at the bottom of the next page.

$$IVPFLPMM^P(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle f *^{-1} \left( \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - f * (\theta_{\vartheta(j)}))^{n\eta_{\vartheta(j)}})^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right. \\ \left( \left[ \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - a_{\vartheta(j)}^2)^{n\eta_{\vartheta(j)}})^{p_j} \right) \right)^{\frac{1}{n!}} \right]^{\frac{1}{2 \sum_{j=1}^n p_j}} \right. \\ \left. \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - b_{\vartheta(j)}^2)^{n\eta_{\vartheta(j)}})^{p_j} \right) \right)^{\frac{1}{n!}} \right]^{\frac{1}{2 \sum_{j=1}^n p_j}} \right. \\ \left[ \left( 1 - \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n \left( 1 - (c_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2 \right)^{p_j} \right) \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right]^{1/2} \\ \left. \left( 1 - \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n \left( 1 - (d_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2 \right)^{p_j} \right) \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right]^{1/2} \right\}. \tag{30}$$

**Theorem 7 (Idempotency):** Let  $\alpha_j$  ( $j = 1, 2, \dots, n$ ) be a set of IVPFLNs, if  $\alpha_j = \alpha = \langle s_\theta, ([a, b], [c, d]) \rangle$  of all  $j$ , then

$$IVPFLPMM^P(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha. \quad (31)$$

*Proof:* As  $\alpha_j = \alpha = \langle s_\theta, ([a, b], [c, d]) \rangle$  ( $j = 1, 2, \dots, n$ ), then  $Sup(\alpha_i, \alpha_j) = 1$  for  $i, j = 1, 2, \dots, n$  ( $i \neq j$ ) is obtained. Thus,  $\eta_j = 1/n$  ( $j = 1, 2, \dots, n$ ) holds for all  $j$ . According to Theorem 5, we have

**Theorem 8 (Boundedness):** Let  $\alpha_j = \langle s_{\theta_j}, ([a_j, b_j], [c_j, d_j]) \rangle$  ( $j = 1, 2, \dots, n$ ) be a collection of IVPFLNs, then

$$\alpha^- \leq IVPFLPMM^P(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha^+, \quad (32)$$

where

$$\alpha^- = \left\langle s_{\min(\theta_j)}, \left( \left[ \min_{j=1}^n(a_j), \min_{j=1}^n(b_j) \right], \left[ \max_{j=1}^n(c_j), \max_{j=1}^n(d_j) \right] \right) \right\rangle$$

and

$$\alpha^+ = \left\langle s_{\max(\theta_j)}, \left( \left[ \max_{j=1}^n(a_j), \max_{j=1}^n(b_j) \right], \left[ \min_{j=1}^n(c_j), \min_{j=1}^n(d_j) \right] \right) \right\rangle.$$

*Proof:* As the LSF  $f$  is a strictly monotonically increasing function, then we have

In addition

Hence, we have  $\alpha^- \leq IVPFLPMM^P(\alpha_1, \alpha_2, \dots, \alpha_n)$ . Similarly, we can get  $IVPFLPMM^P(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha^+$ . Therefore  $\alpha^- \leq IVPFLPMM^P(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha^+$ .

In the following, we investigate some special cases of the proposed IVPFLPMM operator with respect to the parameter vector  $P$ .

**Case 1.** If  $P = (1, 0, \dots, 0)$ , then the IVPFLPMM operator reduces to the IVPFLPA operator, i.e.

$$\begin{aligned} & IVPFLPMM^{(1,0,\dots,0)}(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \left\langle f *^{-1} \left( 1 - \prod_{j=1}^n (1 - f * (\theta_j))^{\eta_j} \right), \right. \\ & \quad \left. \left( \left[ \left( 1 - \prod_{j=1}^n (1 - a_j^2)^{\eta_j} \right)^{1/2}, \left( 1 - \prod_{j=1}^n (1 - b_j^2)^{\eta_j} \right)^{1/2} \right], \right. \right. \\ & \quad \left. \left. \left[ \prod_{j=1}^n c_j^{\eta_j}, \prod_{j=1}^n d_j^{\eta_j} \right] \right) \right\rangle. \\ &= \bigoplus_{j=1}^n \eta_j \alpha_j = IVPFLPA(\alpha_1, \alpha_2, \dots, \alpha_n). \end{aligned} \quad (33)$$

In this case, if  $Sup(\alpha_i, \alpha_j) = t$  ( $t > 0$ ) for  $i, j = 1, 2, \dots, n$  ( $i \neq j$ ), then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic average

$$\begin{aligned} & \left( \frac{1}{n!} \bigoplus_{\vartheta \in L_n} \bigotimes_{j=1}^n (n \eta_{\vartheta(j)} \alpha_{\vartheta(j)})^{p_j} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \\ &= \left\langle f *^{-1} \left( \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - f * (\theta_{\vartheta(j)}))^{\eta_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right), \right. \\ & \quad \left( \left[ \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - a_{\vartheta(j)}^2)^{\eta_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{2 \sum_{j=1}^n p_j}}, \right. \right. \\ & \quad \left. \left. \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - b_{\vartheta(j)}^2)^{\eta_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{2 \sum_{j=1}^n p_j}} \right], \right. \\ & \quad \left. \left[ \left( 1 - \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (c_{\vartheta(j)}^{\eta_{\vartheta(j)}})^2)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right)^{1/2}, \right. \right. \\ & \quad \left. \left. \left( 1 - \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (d_{\vartheta(j)}^{\eta_{\vartheta(j)}})^2)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right)^{1/2} \right] \right) \right\rangle. \end{aligned}$$



(IVPFLA) operator, i.e.

$$\begin{aligned}
 & IVPFLPMM^{(1,0,0,\dots,0)}(\alpha_1, \alpha_2, \dots, \alpha_n) \\
 &= \left\langle f *^{-1} \left( 1 - \prod_{j=1}^n (1 - f * (\theta_j))^{1/n} \right), \right. \\
 & \quad \left( \left[ \left( 1 - \prod_{j=1}^n (1 - a_j^2)^{1/n} \right)^{1/2}, \right. \right. \\
 & \quad \left. \left. \left( 1 - \prod_{j=1}^n (1 - b_j^2)^{1/n} \right)^{1/2} \right] \right), \\
 & \quad \left. \left( 1 - \prod_{j=1}^n (1 - b_j^2)^{1/n} \right)^{1/2} \right\rangle,
 \end{aligned}$$

$$\begin{aligned}
 & \left. \left( \prod_{j=1}^n c_j^{1/n}, \prod_{j=1}^n d_j^{1/n} \right) \right\rangle. \\
 &= \frac{1}{n} \oplus_{j=1}^n \alpha_j = IVPFLA(\alpha_1, \alpha_2, \dots, \alpha_n). \quad (34)
 \end{aligned}$$

**Case 2.** If  $P = (1, 1, 0, 0, \dots, 0)$ , then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic power Bonferroni mean (IVPFLPBM) operator, i.e.

In this case, if  $Sup(\alpha_i, \alpha_j) = t (t > 0)$  for  $i, j = 1, 2, \dots, n (i \neq j)$ , then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic Bonferroni mean (IVPFLBM) operator, i.e.

$$\begin{aligned}
 & \left( \frac{1}{n!} \oplus_{\vartheta \in L_n} \otimes_{j=1}^n (n\eta_{\vartheta(j)} \alpha_{\vartheta(j)})^{p_j} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \\
 &= \left\langle f *^{-1} \left( \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - f * (\theta))^{\frac{1}{n}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right), \right. \\
 & \quad \left( \left[ \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - a^2)^{\frac{1}{n}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{2 \sum_{j=1}^n p_j}}, \right. \right. \\
 & \quad \left. \left. \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - b^2)^{\frac{1}{n}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{2 \sum_{j=1}^n p_j}} \right] \right), \\
 & \quad \left[ \left( 1 - \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (c)^2)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right)^{1/2}, \right. \\
 & \quad \left. \left. \left( 1 - \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (d)^2)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right)^{1/2} \right] \right\rangle. \\
 &= \langle \theta, ([a, b], [c, d]) \rangle = \alpha.
 \end{aligned}$$

$$\begin{aligned}
 & f *^{-1} \left( \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - f * (\theta_{\vartheta(j)}))^{n\eta_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right) \\
 & \left( \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - f * (\min_{j=1}^n (\theta_{\vartheta(j)})))^{n\eta_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right)^{1/2} = \min_{j=1}^n (\theta_j).
 \end{aligned}$$

Case 3: If  $P = \overbrace{(1, 1, \dots, 1)}^k \overbrace{(0, 0, \dots, 0)}^{n-k}$ , then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic power Maclaurin symmetric mean (IVPFLPMSM) operator, i.e.

In this case, In this case, if  $Sup(\alpha_i, \alpha_j) = t (t > 0)$  for  $i, j = 1, 2, \dots, n (i \neq j)$ , then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic Maclaurin symmetric mean (IVPFLMSM) operator, i.e.

Case 4: If  $P = (1, 1, \dots, 1)$  or  $P = (1/n, 1/n, \dots, 1/n)$ , then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic power geometric (IVPFLPG) operator, i.e.

$$\begin{aligned}
 & IVPFLPMM^{(1,1,\dots,1) \text{ or } (1/n,1/n,\dots,1/n)}(\alpha_1, \alpha_2, \dots, \alpha_n) \\
 &= \bigotimes_{j=1}^n (n\eta_j \alpha_j)^{1/n} \\
 &= \left\langle f *^{-1} \left( \prod_{j=1}^n (1 - (1 - f * (\theta_j))^{n\eta_j})^{1/n} \right), \right. \\
 &\quad \left( \left[ \left( \prod_{j=1}^n (1 - (1 - a_j^2)^{n\eta_j}) \right)^{1/2n}, \right. \right. \\
 &\quad \left. \left. \left( \prod_{j=1}^n (1 - (1 - b_j^2)^{n\eta_j}) \right)^{1/2n} \right] \right. \\
 &\quad \left. \left[ \left( 1 - \prod_{j=1}^n (1 - (c_j^{n\eta_j})^2)^{\frac{1}{n}} \right)^{1/2}, \right. \right. \\
 &\quad \left. \left. \left( 1 - \prod_{j=1}^n (1 - (d_j^{n\eta_j})^2)^{\frac{1}{n}} \right)^{1/2} \right] \right] \right\rangle. \tag{39}
 \end{aligned}$$

In this case, if  $Sup(\alpha_i, \alpha_j) = t (t > 0)$  for  $i, j = 1, 2, \dots, n (i \neq j)$ , then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic geometric (IVPFLG) operator, i.e.

$$\begin{aligned}
 & IVPFLPMM^{(1,1,\dots,1) \text{ or } (1/n,1/n,\dots,1/n)}(\alpha_1, \alpha_2, \dots, \alpha_n) \\
 &= \left\langle f *^{-1} \left( \prod_{j=1}^n (f * (\theta_j))^{1/n} \right), \left( \left[ \prod_{j=1}^n a_j^{1/n}, \prod_{j=1}^n b_j^{1/n} \right], \right. \\
 &\quad \left[ \left( 1 - \prod_{j=1}^n (1 - c_j^2)^{1/n} \right)^{1/2}, \right. \\
 &\quad \left. \left. \left( 1 - \prod_{j=1}^n (1 - d_j^2)^{1/n} \right)^{1/2} \right] \right] \right\rangle \\
 &= \bigotimes_{j=1}^n \alpha_j^{1/n} = IVPFLG(\alpha_1, \alpha_2, \dots, \alpha_n). \tag{40}
 \end{aligned}$$

**D. THE INTERVAL-VALUED PYTHAGOREAN FUZZY LINGUISTIC POWER WEIGHTED MUIRHEAD MEAN OPERATOR**

Definition 13: Let  $\alpha_j (j = 1, 2, \dots, n)$  be a collection of IVPFLNs and  $P = (p_1, p_2, \dots, p_n) \in R^n$  be vector of parameter. Let  $w = (w_1, w_2, \dots, w_n)^T$  be the weight vector, such that  $\sum_{j=1}^n w_j = 1$  and  $0 \leq w_j \leq 1$ . The interval-valued Pythagorean fuzzy linguistic power weighted Muirhead mean (IVPFLPWMM) operator is defined as

$$\begin{aligned}
 & IVPFLPWMM^P(\alpha_1, \alpha_2, \dots, \alpha_n) \\
 &= \left( \frac{1}{n!} \bigoplus_{\vartheta \in L_n} \bigotimes_{j=1}^n \left( n \frac{w_{\vartheta(j)} (1 + T(\alpha_{\vartheta(j)}))}{\sum_{j=1}^n w_j (1 + T(\alpha_j))} \alpha_{\vartheta(j)} \right)^{p_j} \right)^{\frac{1}{\sum_{j=1}^n p_j}}, \tag{41}
 \end{aligned}$$

$$\begin{aligned}
 & \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - a_{\vartheta(j)}^2)^{n\eta_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \geq \min_{j=1}^n (a_j), \\
 & \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - b_{\vartheta(j)}^2)^{n\eta_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \geq \min_{j=1}^n (b_j), \\
 & \left( 1 - \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (c_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \leq \max_{j=1}^n (c_j) \\
 & \left( 1 - \left( 1 - \left( \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (d_{\vartheta(j)}^{n\eta_{\vartheta(j)}})^2)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \leq \max_{j=1}^n (d_j).
 \end{aligned}$$

where  $T(\alpha_j) = \sum_{i=1, i \neq j}^n Sup(\alpha_i, \alpha_j)$ ,  $Sup(\alpha_i, \alpha_j) = 1 - d(\alpha_i, \alpha_j)$ , and  $d(\alpha_i, \alpha_j)$  is the distance between  $\alpha_i$  and  $\alpha_j$ ,  $\vartheta(j)$  ( $j = 1, 2, \dots, n$ ) represents any permutation of  $(1, 2, \dots, n)$ ,  $L_n$  denotes all possible permutations of  $(1, 2, \dots, n)$ ,  $n$  is the balancing coefficient, and  $Sup(\alpha_i, \alpha_j)$  denotes the support for  $\alpha_i$  from  $\alpha_j$ , satisfying the properties in Definition 10. To simplify Eq. (41), let

$$\gamma_j = \frac{w_j(1 + T(\alpha_j))}{\sum_{j=1}^n w_j(1 + T(\alpha_j))}, \quad (42)$$

then Eq. (41) can be written as

$$\begin{aligned} & IVPFLPWMM^P(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \left( \frac{1}{n!} \bigoplus_{\vartheta \in L_n} \bigotimes_{j=1}^n (n\gamma_{\vartheta(j)}\alpha_{\vartheta(j)})^{p_j} \right)^{\frac{1}{\sum_{j=1}^n p_j}}. \end{aligned} \quad (43)$$

where  $0 \leq \eta_j \leq 1$  and  $\sum_{j=1}^n \eta_j = 1$ .

**Theorem 9:** Let  $\alpha_j = \langle s_{\theta_j}, ([a_j, b_j], [c_j, d_j]) \rangle$  ( $j = 1, 2, \dots, n$ ) be a collection of IVPFLNs, then the aggregated value by the IVPFLPWMM operator is still an IVPFLN and

The poof of Theorem 9 is similar to that of Theorem 2.

### V. A NEW MAGDM METHOD UNDER IVPFLS

Based on the proposed AOs of IVPFLNs, we propose a new MAGDM method to deal with decision-making problems under IVPFLSs. Let's consider a MAGDM problem with IVPFL information. There are  $m$  feasible alternatives  $\{A_1, A_2, \dots, A_m\}$  that to be evaluated under  $n$  attributes, i.e.  $\{C_1, C_2, \dots, C_n\}$ . The weight vector of attributes is  $w = (w_1, w_2, \dots, w_n)^T$ , such that  $\sum_{i=1}^n w_i = 1$  and  $0 \leq w_i \leq 1$ . A group of DMs are required to evaluate the performance of all the possible alternatives. Let  $\{D_1, D_2, \dots, D_t\}$  be the DM set, with the weight vector being  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)^T$ , such that  $0 \leq \lambda_e \leq 1$  and  $\sum_{e=1}^t \lambda_e = 1$ . For attribute  $C_j$  ( $j = 1, 2, \dots, n$ ) for alter-

$$\begin{aligned} & IVPFLPMM^{(1,1,0,\dots,0)}(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \left\langle f *^{-1} \left( \left( \prod_{\substack{i,j=1 \\ i \neq j}}^n (1 - (1 - (1 - f * (\theta_i))^{nm_i}) (1 - (1 - f * (\theta_j))^{nm_j}))^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right) \right. \\ & \left( \left[ \left( \prod_{\substack{i,j=1 \\ i \neq j}}^n (1 - (1 - (1 - a_i^2)^{nm_i}) (1 - (1 - a_j^2)^{nm_j}))^{\frac{1}{n(n-1)}} \right)^{\frac{1}{4}} \right. \right. \\ & \left. \left( \prod_{\substack{i,j=1 \\ i \neq j}}^n (1 - (1 - (1 - b_i^2)^{nm_i}) (1 - (1 - b_j^2)^{nm_j}))^{\frac{1}{n(n-1)}} \right)^{\frac{1}{4}} \right] \\ & \left. \left( \prod_{\substack{i,j=1 \\ i \neq j}}^n (1 - (1 - (1 - (c_i^{nm_i})^2 + (c_j^{nm_j})^2 - (c_i^{nm_i})^2 (c_j^{nm_j})^2))^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right)^{1/2} \right. \\ & \left. \left( \prod_{\substack{i,j=1 \\ i \neq j}}^n (1 - (1 - (1 - (d_i^{nm_i})^2 + (d_j^{nm_j})^2 - (d_i^{nm_i})^2 (d_j^{nm_j})^2))^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right) = \left( \frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^n (n\eta_i\alpha_i \otimes n\eta_j\alpha_j) \right)^{\frac{1}{2}} \\ &= IVPFLPBM^{1,1}(\alpha_1, \alpha_2, \dots, \alpha_n). \end{aligned} \quad (35)$$

native  $A_i$  ( $i = 1, 2, \dots, m$ ), DM  $D_e$  ( $e = 1, 2, \dots, t$ ) utilizes  $\alpha_{ij}^e = \left\langle s_{\theta_{ij}^e}, \left( \left[ a_{ij}^e, b_{ij}^e \right], \left[ c_{ij}^e, d_{ij}^e \right] \right) \right\rangle$  to express his/her evaluation value, where  $\alpha_{ij}^e$  is an IVPFLN defined on the linguistic set  $S$ . Hence,  $t$  interval-valued Pythagorean fuzzy decision matrices are obtained. In the following, based on the proposed AOs, we give the main steps of determining the rank of feasible alternatives.

*Step 1.* Generally, there are two types of attributes, i.e., benefit type and cost type. Hence, the original decision matrices should be normalized. If  $C_j$  is benefit type, then the original decision matrices do not need to be normalized. If  $C_j$  is cost type, then the original decision matrices should be changed according to the following formula

$$\alpha_{ij}^e = \left\langle s_{\theta_{ij}^e}, \left( \left[ c_{ij}^e, d_{ij}^e \right], \left[ a_{ij}^e, b_{ij}^e \right] \right) \right\rangle, \quad (45)$$

*Step 2.* Compute  $Sup(\alpha_{ij}^k, \alpha_{ij}^d)$  ( $k, d = 1, 2, \dots, t; k \neq d$ ) according to the following equation

$$Sup(\alpha_{ij}^k, \alpha_{ij}^d) = 1 - d(\alpha_{ij}^k, \alpha_{ij}^d), \quad (46)$$

where  $d(\alpha_{ij}^k, \alpha_{ij}^d)$  is the distance between the two IVPFLNs  $\alpha_{ij}^k$  and  $\alpha_{ij}^d$ . Definition 8 illustrates how to calculate the distance between any two IVPFLNs.

*Step 3.* Calculate the overall supports  $T(\alpha_{ij}^k)$  by

$$T(\alpha_{ij}^k) = \sum_{g=1, g \neq k}^t Sup(\alpha_{ij}^k, \alpha_{ij}^g). \quad (47)$$

*Step 4.* For DM  $D_k$ , compute the power weight associated with the IVPFLN  $\alpha_{ij}^k$  by

$$\delta_{ij}^k = \frac{\lambda_k (1 + T(\alpha_{ij}^k))}{\sum_{k=1}^t \lambda_k (1 + T(\alpha_{ij}^k))}. \quad (48)$$

*Step 5.* Use the IVPFLPWA operator to determine the comprehensive decision matrix

$$\alpha_{ij} = IVPFLPWA(\alpha_{ij}^1, \alpha_{ij}^2, \dots, \alpha_{ij}^t). \quad (49)$$

*Step 6.* Calculate the support between the two IVPFLNs  $\alpha_{il}$  and  $\alpha_{if}$  by

$$Sup(\alpha_{il}, \alpha_{if}) = 1 - d(\alpha_{il}, \alpha_{if}), \quad (50)$$

$$\begin{aligned} & IVPFLPMM^{(1,1,0,\dots,0)}(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \left\langle f *^{-1} \left( \left( 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^n (1 - f * (\theta_i) f * (\theta_j))^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right), \right. \\ & \left. \left( \left[ \left( \left( 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^n (1 - a_i^2 a_j^2)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}}, \left( 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^n (1 - b_i^2 b_j^2)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right], \right. \right. \\ & \left. \left[ \left( 1 - \left( 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^n (c_i^2 + c_j^2 - c_i^2 c_j^2)^{\frac{1}{n(n+1)}} \right)^{\frac{1}{2}} \right)^{1/2}, \right. \right. \\ & \left. \left. \left( 1 - \left( 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^n (d_i^2 + d_j^2 - d_i^2 d_j^2)^{\frac{1}{n(n+1)}} \right)^{\frac{1}{2}} \right)^{1/2} \right] \right] \right) \right\rangle \\ &= \left( \frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^n (\alpha_i \otimes \alpha_j) \right)^{\frac{1}{2}} = IVPFLBM^{1,1}(\alpha_1, \alpha_2, \dots, \alpha_n). \quad (36) \end{aligned}$$

wherein  $l, f = 1, 2, \dots, n; l \neq j$

Step 7. Compute the overall  $T(\alpha_{ij})$  by

$$T(\alpha_{ij}) = \sum_{j=1; j \neq i}^n Sup(\alpha_{ij}, \alpha_{is}). \quad (51)$$

Step 8. Compute the power weight associated with the IVPFLN  $\alpha_{ij}$  by

$$\gamma_{ij} = \frac{w_j (1 + T(\alpha_{ij}))}{\sum_{j=1}^n w_j (1 + T(\alpha_{ij}))}. \quad (52)$$

Step 9. For each alternative, use the proposed IVPFLP-WMM to compute the overall evaluation values, i.e.

$$\alpha_i = IVPFLPWMM^Y(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in}), \quad (53)$$

and a series of overall evaluation values  $\alpha_i (i = 1, 2, \dots, m)$  are obtained.

Step 10. Compute the scores of  $\alpha_i (i = 1, 2, \dots, m)$  according to Definition 8.

Step 11. Rank the corresponding alternatives and select the optimal one.

## VI. NUMERICAL EXAMPLES

Example 5 (Revised from [33]): Suppose that there are four command and control systems  $\{A_1, A_2, A_3, A_4\}$  that to be evaluated under three attributes, i.e. system availability ( $G_1$ ), information accuracy ( $G_2$ ) and picture completeness ( $G_3$ ). The weight vector of the attributes is  $\lambda = (0.3727, 0.3500, 0.2773)^T$ . Let  $S = \{s_0, s_1, \dots, s_6\}$  be a linguistic term set and three decision experts  $D_1, D_2$ , and  $D_3$  utilize IVPFLNs defined on  $S$  to express their decision ideas. Hence, three IVPFL decision matrices  $R_1, R_2$  and  $R_3$  are obtained, which are shown in Tables 1-3. The weight vector of DMs is  $\lambda = (0.3, 0.4, 0.3)^T$ .

### A. THE DECISION-MAKING PROCESS

We use the MAGDM method introduced in Section V to resolve Example 5, and the particular methods procedure is presented as follows.

Step 1: It is clearly that all attributes are benefit type, there is no need to normalize the original decision matrix.

Step 2: Calculate the  $Sup(\alpha_{ij}^k, \alpha_{ij}^d)$  according to Eq. (46) (Suppose LFS1 is utilized as the specified LSF in the calculation process). For convenience, we utilize the symbol  $S_d^k$  to represent the support between  $\alpha_{ij}^k$  and  $\alpha_{ij}^d (i = 1, 2, 3, 4, 5; j = 1, 2, 3; k, d = 1, 2, 3; k \neq d)$ .

$$\begin{aligned} & IVPFLPMM^{\overbrace{(1, 1, \dots, 1)}^k, \overbrace{(0, 0, \dots, 0)}^{n-k}}(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \left\langle f *^{-1} \left( \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k (1 - (1 - f * (\theta_{ij}))^{nm_{ij}}) \right)^{\frac{1}{c_n^k}} \right)^{\frac{1}{k}} \right), \right. \\ & \left. \left( \left[ \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k (1 - (1 - a_{ij}^2)^{nm_{ij}}) \right)^{1/c_n^k} \right)^{1/2k}, \right. \right. \right. \\ & \left. \left. \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k (1 - (1 - b_{ij}^2)^{nm_{ij}}) \right)^{1/c_n^k} \right)^{1/2k} \right], \right. \\ & \left. \left[ \left( 1 - \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k (1 - (c_{ij}^{nm_{ij}})^2) \right)^{1/c_n^k} \right)^{1/k} \right)^{1/2}, \right. \right. \\ & \left. \left. \left( 1 - \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k (1 - (d_{ij}^{nm_{ij}})^2) \right)^{1/c_n^k} \right)^{1/k} \right)^{1/2} \right] \right] \right) \right. \\ & \left. \left( \frac{1}{C_n^k} \bigoplus_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \bigotimes_{j=1}^k (nm_{ij} \alpha_{ij}) \right)^{1/k} \right) \\ &= IVPFLPMSM^{(k)}(\alpha_1, \alpha_2, \dots, \alpha_n). \quad (37) \end{aligned}$$

Hence, we obtain the following results

$$T \left( \alpha_{ij}^k \right) (i = 1, 2, 3, 4, 5; j = 1, 2, 3; k = 1, 2, 3)$$

$$\begin{aligned}
 S_2^1 = S_1^2 &= \begin{bmatrix} 0.9842 & 0.9858 & 0.9863 \\ 0.9588 & 1.0000 & 0.9875 \\ 0.9225 & 0.9963 & 1.0000 \\ 0.9525 & 0.9408 & 0.9842 \\ 1.0000 & 0.9483 & 0.9800 \end{bmatrix} & T^1 &= \begin{bmatrix} 1.9392 & 1.9858 & 1.9579 \\ 1.9179 & 1.9929 & 1.9625 \\ 1.8908 & 1.9746 & 2.0000 \\ 1.9254 & 1.9408 & 1.9842 \\ 1.9733 & 1.9317 & 1.9550 \end{bmatrix} \\
 S_3^1 = S_1^3 &= \begin{bmatrix} 0.9550 & 1.0000 & 0.9717 \\ 0.9592 & 0.9929 & 0.9750 \\ 0.9683 & 0.9783 & 1.0000 \\ 0.9729 & 1.0000 & 1.0000 \\ 0.9733 & 0.9833 & 0.9750 \end{bmatrix} & T^2 &= \begin{bmatrix} 1.9692 & 1.9792 & 1.9750 \\ 1.9346 & 1.9733 & 1.9729 \\ 1.9154 & 1.9867 & 2.0000 \\ 1.9467 & 1.9308 & 1.9750 \\ 1.9896 & 1.9392 & 1.9700 \end{bmatrix} \\
 S_3^2 = S_2^3 &= \begin{bmatrix} 0.9850 & 0.9933 & 0.9888 \\ 0.9758 & 0.9733 & 0.9854 \\ 0.9929 & 0.9904 & 1.0000 \\ 0.9942 & 0.9900 & 0.9908 \\ 0.9896 & 0.9908 & 0.9900 \end{bmatrix} & T^3 &= \begin{bmatrix} 1.9400 & 1.9933 & 1.9604 \\ 1.9350 & 1.9663 & 1.9604 \\ 1.9613 & 1.9688 & 2.0000 \\ 1.9671 & 1.9900 & 1.9908 \\ 1.9629 & 1.9742 & 1.9650 \end{bmatrix}
 \end{aligned}$$

Step 3: Calculate  $T \left( \alpha_{ij}^k \right)$  according to Eq. (47). For convenience, we use the symbol  $T^k$  to represent the values

Step 4: For DM  $D_k$ , calculate his/her power weight associated with the IVPFLV  $\alpha_{ij}^k$  on the basis of his/her weight  $\gamma_k$  according to Eq. (49). For convenience, we use the symbol  $\delta^k$  to represent the values  $\delta_{ij}^k$  ( $i, j = 1, 2, 3, 4; k = 1, 2, 3$ ).

$$\begin{aligned}
 & IVPFLPM \overbrace{(1, 1, \dots, 1, 0, 0, \dots, 0)}^k \overbrace{(\alpha_1, \alpha_2, \dots, \alpha_n)}^{n-k} \\
 &= \left\langle f *^{-1} \left( \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k f * (\theta_{i_j}) \right)^{1/C_n^k} \right)^{1/k} \right), \right. \\
 & \quad \left( \left[ \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k a_{i_j}^2 \right)^{1/C_n^k} \right)^{1/2k}, \right. \right. \\
 & \quad \left. \left. \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k b_{i_j}^2 \right)^{1/C_n^k} \right)^{1/2k} \right] \right), \\
 & \quad \left[ \left( 1 - \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k (1 - c_{i_j}^2) \right)^{1/C_n^k} \right)^{1/k} \right)^{1/2}, \right. \\
 & \quad \left. \left. \left( 1 - \left( 1 - \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \left( 1 - \prod_{j=1}^k (1 - d_{i_j}^2) \right)^{1/C_n^k} \right)^{1/k} \right)^{1/2} \right] \right) \right\rangle. \\
 &= \left( \frac{1}{C_n^k} \bigoplus_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \bigotimes_{j=1}^k \alpha_{i_j} \right)^{1/k} \\
 &= IVPFLMSM^{(k)} (\alpha_1, \alpha_2, \dots, \alpha_n).
 \end{aligned} \tag{38}$$

TABLE 1. The interval-valued Pythagorean fuzzy linguistic decision matrix  $R_1$  of Example 1 provided by  $D_1$ .

	$G_1$	$G_2$	$G_3$
$A_1$	$\langle s_2, ([0.6, 0.7], [0.2, 0.3]) \rangle$	$\langle s_3, ([0.6, 0.7], [0.1, 0.3]) \rangle$	$\langle s_5, ([0.6, 0.7], [0.1, 0.3]) \rangle$
$A_2$	$\langle s_4, ([0.9, 1.0], [0, 0]) \rangle$	$\langle s_3, ([0.5, 0.6], [0.2, 0.3]) \rangle$	$\langle s_4, ([0.6, 0.8], [0.1, 0.2]) \rangle$
$A_3$	$\langle s_2, ([0.5, 0.6], [0.3, 0.4]) \rangle$	$\langle s_6, ([0.6, 0.7], [0, 0.2]) \rangle$	$\langle s_3, ([0.7, 0.8], [0, 0.1]) \rangle$
$A_4$	$\langle s_4, ([0.7, 0.9], [0, 0.1]) \rangle$	$\langle s_5, ([0.7, 0.8], [0, 0.1]) \rangle$	$\langle s_2, ([0.5, 0.6], [0, 0.3]) \rangle$
$A_5$	$\langle s_4, ([0.8, 0.9], [0, 0.1]) \rangle$	$\langle s_1, ([0.6, 0.8], [0, 0.1, 0.2]) \rangle$	$\langle s_3, ([0.8, 0.9], [0, 0.1]) \rangle$

Therefore, we can obtain the following results.

$$\delta^1 = \begin{bmatrix} 0.2988 & 0.3000 & 0.2992 \\ 0.2988 & 0.3016 & 0.2996 \\ 0.2968 & 0.2997 & 0.3000 \\ 0.2979 & 0.2989 & 0.3002 \\ 0.2997 & 0.2984 & 0.2991 \end{bmatrix}$$

$$\delta^2 = \begin{bmatrix} 0.4024 & 0.3992 & 0.4013 \\ 0.4007 & 0.3995 & 0.4009 \\ 0.3991 & 0.4012 & 0.4000 \\ 0.4000 & 0.3972 & 0.3990 \\ 0.4017 & 0.3989 & 0.4008 \end{bmatrix}$$

$$\delta^3 = \begin{bmatrix} 0.2988 & 0.3008 & 0.2995 \\ 0.3005 & 0.2989 & 0.2994 \\ 0.3041 & 0.2991 & 0.3000 \\ 0.3021 & 0.3039 & 0.3008 \\ 0.2986 & 0.3027 & 0.3001 \end{bmatrix}$$

Step 5. Utilize the IVPFLPWA operator to aggregate the individual decision matrices into a collective one, which is listed in Table 4.

Step 6. For Table 4, calculate the support between  $\alpha_{il}$  and  $\alpha_{if}$ , that is,  $Sup(\alpha_{il}, \alpha_{if})$  according to Eq. (50). For convenience, we utilize the symbol  $S^{lf}$  to represent the value  $Sup(\alpha_{il}, \alpha_{if})$  ( $i, = 1, 2, 3, 4, 5; l, f = 1, 2, 3; l \neq f$ ). Hence, we can obtain the following results

$$S^{12} = S^{21} = (0.9547, 0.9747, 0.9853, 0.9946),$$

$$S^{13} = S^{31} = (0.9782, 0.9550, 0.9988, 0.9928),$$

$$S^{23} = S^{32} = (0.9975, 0.9983, 0.9863, 0.9899).$$

Step 7: Calculate the support  $T(\alpha_{ij})$  according to Eq. (51). Similarly, we use the symbol  $T_{ij}$  to denote the value  $T(\alpha_{ij})$

$$IVPFLPMM^P(\alpha_1, \alpha_2, \dots, \alpha_n)$$

$$= \left\langle f *^{-1} \left( \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - f * (\theta_{\vartheta(j)}))^{n\gamma_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right), \right.$$

$$\left( \left[ \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - a_{\vartheta(j)}^2)^{n\gamma_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{2 \sum_{j=1}^n p_j}}, \right.$$

$$\left. \left. \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (1 - b_{\vartheta(j)}^2)^{n\gamma_{\vartheta(j)}})^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{2 \sum_{j=1}^n p_j}} \right], \right.$$

$$\left[ \left( 1 - \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (c_{\vartheta(j)}^{n\gamma_{\vartheta(j)}})^2)^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right)^{1/2}, \right.$$

$$\left. \left. \left( 1 - \left( 1 - \prod_{\vartheta \in L_n} \left( 1 - \prod_{j=1}^n (1 - (d_{\vartheta(j)}^{n\gamma_{\vartheta(j)}})^2)^{p_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right)^{1/2} \right] \right\rangle. \tag{44}$$

TABLE 2. The interval-valued Pythagorean fuzzy linguistic decision matrix  $R_2$  of Example 1 provided by  $D_2$ .

	$G_1$	$G_2$	$G_3$
$A_1$	$\langle s_4, ([0.5, 0.7], [0.1, 0.2]) \rangle$	$\langle s_2, ([0.7, 0.8], [0, 0.2]) \rangle$	$\langle s_4, ([0.7, 0.8], [0.1, 0.2]) \rangle$
$A_2$	$\langle s_5, ([0.7, 0.8], [0, 0.2]) \rangle$	$\langle s_3, ([0.7, 0.9], [0, 0.1]) \rangle$	$\langle s_2, ([0.6, 0.7], [0.1, 0.2]) \rangle$
$A_3$	$\langle s_4, ([0.7, 0.9], [0, 0.1]) \rangle$	$\langle s_5, ([0.6, 0.7], [0.2, 0.3]) \rangle$	$\langle s_3, ([0.6, 0.7], [0.1, 0.3]) \rangle$
$A_4$	$\langle s_2, ([0.6, 0.7], [0.2, 0.3]) \rangle$	$\langle s_3, ([0.5, 0.6], [0.2, 0.4]) \rangle$	$\langle s_4, ([0.4, 0.6], [0.2, 0.4]) \rangle$
$A_5$	$\langle s_4, ([0.4, 0.7], [0.1, 0.3]) \rangle$	$\langle s_5, ([0.5, 0.7], [0.1, 0.3]) \rangle$	$\langle s_2, ([0.6, 0.8], [0, 0.2]) \rangle$

TABLE 3. The interval-valued Pythagorean fuzzy linguistic decision matrix  $R_3$  of Example 1 provided by  $D_3$ .

	$G_1$	$G_2$	$G_3$
$A_1$	$\langle s_6, ([0.5, 0.6], [0.1, 0.3]) \rangle$	$\langle s_3, ([0.6, 0.8], [0, 0.1]) \rangle$	$\langle s_3, ([0.5, 0.8], [0.1, 0.1]) \rangle$
$A_2$	$\langle s_3, ([0.6, 0.7], [0.1, 0.2]) \rangle$	$\langle s_4, ([0.5, 0.7], [0, 0.3]) \rangle$	$\langle s_1, ([0.6, 0.9], [0.1, 0.1]) \rangle$
$A_3$	$\langle s_3, ([0.7, 0.8], [0, 0.1]) \rangle$	$\langle s_4, ([0.4, 0.7], [0.1, 0.3]) \rangle$	$\langle s_3, ([0.5, 0.6], [0, 0.3]) \rangle$
$A_4$	$\langle s_3, ([0.5, 0.7], [0.1, 0.3]) \rangle$	$\langle s_5, ([0.4, 0.6], [0.1, 0.4]) \rangle$	$\langle s_2, ([0.4, 0.5], [0.2, 0.4]) \rangle$
$A_5$	$\langle s_5, ([0.6, 0.7], [0.1, 0.2]) \rangle$	$\langle s_3, ([0.6, 0.7], [0.1, 0.3]) \rangle$	$\langle s_4, ([0.5, 0.8], [0.1, 0.2]) \rangle$

TABLE 4. The comprehensive evaluation decision matrix of Example 5.

	$C_1$	$C_2$	$C_3$
$A_1$	$\langle s_6, ([0.5336, 0.6739], [0.1230, 0.2548]) \rangle$	$\langle s_6, ([0.5336, 0.6739], [0.1230, 0.2548]) \rangle$	$\langle s_6, ([0.5336, 0.6739], [0.1230, 0.2548]) \rangle$
$A_2$	$\langle s_{4.2899}, ([0.7705, 1], [0, 0]) \rangle$	$\langle s_{3.3424}, ([0.5976, 0.7949], [0, 0.1934]) \rangle$	$\langle s_{2.5256}, ([0.6, 0.8113], [0.1, 0.1625]) \rangle$
$A_3$	$\langle s_{3.2208}, ([0.6543, 0.818], [0, 0.1509]) \rangle$	$\langle s_6, ([0.553, 0.7], [0, 0.2657]) \rangle$	$\langle s_3, ([0.6107, 0.7129], [0, 0.2158]) \rangle$
$A_4$	$\langle s_{3.0170}, ([0.6103, 0.7874], [0, 0.2163]) \rangle$	$\langle s_{4.4529}, ([0.5552, 0.6791], [0, 0.2643]) \rangle$	$\langle s_{2.9665}, ([0.4337, 0.5733], [0.1624, 0.3669]) \rangle$
$A_5$	$\langle s_{4.3739}, ([0.6318, 0.7878], [0, 0.1912]) \rangle$	$\langle s_{3.7457}, ([0.5641, 0.7351], [0.1, 0.2658]) \rangle$	$\langle s_{3.0191}, ([0.6595, 0.8382], [0, 0.1626]) \rangle$

for simplicity, and we can obtain the following matrix

$$T = \begin{bmatrix} 3.8276 & 3.9303 & 3.9501 \\ 3.8738 & 3.9576 & 3.9757 \\ 3.9233 & 3.9300 & 3.9890 \end{bmatrix}$$

Step 8: Calculate the power weight  $\gamma_{ij}$  associated with the IVPFLV  $\alpha_{if}$  according to Eq. (52), and we have

$$\gamma = \begin{bmatrix} 0.3694 & 0.3720 & 0.3712 \\ 0.3503 & 0.3513 & 0.3504 \\ 0.2803 & 0.2768 & 0.2784 \end{bmatrix}$$

Step 9: For alternative  $A_i$  ( $i = 1, 2, 3, 4, 5$ ), utilize the IVPFLPMM operator to calculate the overall evaluation  $\alpha_i$  ( $i = 1, 2, 3, 4, 5$ ). Without the loss of generality, let  $H = (1, 1, 1)$  and the overall evaluation values are shown as follows

$$\begin{aligned} \alpha_1 &= \langle s_{3.9562}, ([0.5967, 0.7387], [0.1010, 0.2119]) \rangle \\ \alpha_2 &= \langle s_{3.2478}, ([0.6468, 0.8531], [0.0852, 0.1634]) \rangle \\ \alpha_3 &= \langle s_{3.8006}, ([0.6022, 0.7373], [0.0000, 0.2272]) \rangle \\ \alpha_4 &= \langle s_{3.3774}, ([0.5241, 0.6682], [0.1282, 0.3110]) \rangle \end{aligned}$$

$$\alpha_5 = \langle s_{3.6080}, ([0.6155, 0.7853], [0.0515, 0.2127]) \rangle$$

Step 10: Calculate the score values  $S(\alpha_i)$  ( $i = 1, 2, 3, 4$ ) and we can get

$$\begin{aligned} S(\alpha_1) &= 0.4693 \\ S(\alpha_2) &= 0.4211 \\ S(\alpha_3) &= 0.4520 \\ S(\alpha_4) &= 0.3670 \\ S(\alpha_5) &= 0.4431 \end{aligned}$$

Step 11: According to the score values  $S(\alpha_i)$  ( $i = 1, 2, 3, 4, 5$ ), the ranking order of the alternatives can be determined, that is,  $A_1 > A_3 > A_5 > A_2 > A_4$ . Therefore,  $A_3$  is the best alternative.

### B. THE INFLUENCE OF THE PARAMETERS ON THE RESULTS

In this section, we try to investigate the influence of the parameter vector  $P$  and LSF  $f$  on the decision results.



**TABLE 5.** The ranking results with different parameter vectors  $P$  in the IVPFLWMM operator.

Parameters	Score functions $S(\alpha_i)(i=1,2,3,4,5)$	Ranking orders
$P = (1,0,0)$	$S(\alpha_1) = 0.7176$ $S(\alpha_2) = 0.5141$ $S(\alpha_3) = 0.7253$ $S(\alpha_4) = 0.4110$ $S(\alpha_5) = 0.4743$	$A_3 \succ A_1 \succ A_2 \succ A_5 \succ A_4$
$P = (1,1,0)$	$S(\alpha_1) = 0.4919$ , $S(\alpha_2) = 0.4408$ , $S(\alpha_3) = 0.4747$ , $S(\alpha_4) = 0.3821$ , $S(\alpha_5) = 0.4542$	$A_1 \succ A_3 \succ A_5 \succ A_2 \succ A_4$
$P = (1,1,1)$	$S(\alpha_1) = 0.4693$ , $S(\alpha_2) = 0.4211$ , $S(\alpha_3) = 0.4520$ , $S(\alpha_4) = 0.3670$ , $S(\alpha_5) = 0.4431$	$A_1 \succ A_3 \succ A_5 \succ A_2 \succ A_4$

**TABLE 6.** Score functions and ranking orders by different LSFs when  $H = (1,1,1)$ .

Parameters	Score functions $S(\alpha_i)(i=1,2,3,4,5)$	Ranking orders
Our method based on LSF1 ( $t = 3$ )	$S(\alpha_1) = 0.4693$ , $S(\alpha_2) = 0.4211$ , $S(\alpha_3) = 0.4520$ , $S(\alpha_4) = 0.3670$ , $S(\alpha_5) = 0.4431$	$A_1 \succ A_3 \succ A_5 \succ A_2 \succ A_4$
Our method based on LSF2 ( $t=3, \rho = 1.37$ )	$S(\alpha_1) = 0.4632$ , $S(\alpha_2) = 0.4096$ $S(\alpha_3) = 0.4480$ , $S(\alpha_4) = 0.3569$ , $S(\alpha_5) = 0.4230$	$A_1 \succ A_3 \succ A_5 \succ A_2 \succ A_4$
Our method based on LSF3 ( $t = 3, \epsilon = \beta = 0.5$ )	$S(\alpha_1) = 0.5268$ , $S(\alpha_2) = 0.4923$ $S(\alpha_3) = 0.5432$ , $S(\alpha_4) = 0.4530$ , $S(\alpha_5) = 0.5400$	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$

1) THE INFLUENCE OF THE PARAMETER VECTOR  $P$

We assign different values in  $P$  in the IVPFLWMM operator and present the decision results in Table 5 (LSF 1 is used in the calculation process).

As we can see from Table 5, when  $P = (1, 0, 0)$  the ranking order of alternatives is  $A_3 \succ A_1 \succ A_2 \succ A_5 \succ A_4$ , and the optimal alternative is  $A_3$ . When  $P = (1, 1, 0)$  and  $P = (1, 1, 1)$ , the ranking order of alternatives is  $A_1 \succ A_3 \succ A_5 \succ A_2 \succ A_4$ , and the best alternative is  $A_1$ . This is because when  $P = (1, 0, 0)$ , our proposed decision-making method does not consider the interrelationship among attributes. In other word, when  $P = (1, 0, 0)$ , it is assumed that all the attributes are independent. When  $P = (1, 1, 0)$  and  $P = (1, 1, 1)$ , our proposed method takes the interrelationship among attributes. In this Example 5, there exists evident interrelationship between attributes. Let  $N_P$  denotes the number of related attributes ( $N_P = 1, 2, 3$ ). We notice that with the increase of  $N_P$ , the score values of alternatives will decrease, which illustrates the flexibility of our proposed decision-making method. In actually decision situations, DMs can select the proper parameter  $P$  according to practical needs.

2) THE IMPACT OF THE LSF ON THE RESULTS

Then, we investigate how the LSF  $f$  affects the decision results. We take different LSFs in the operations of IVPFLNs and presented the decision results in Table 6. As seen from Table 6, different score values of the overall evaluation values are obtained by using different LSF in the calculation process, which further leads to different ranking orders of all the feasible alternatives. In real decision-making situations, DMs and select a proper LSF according to actual needs.

C. VALIDITY OF OUR PROPOSED METHOD

In this subsection, we attempt to prove the effectiveness of our proposed method through comparison method. First, we compare our method with some other methods based on IVPFLSs. Then, as IVILS is a special case of IVPFLS we then compare our method with those based on IVILSs to further explain the effectiveness of our method.

1) COMPARED WITH DECISION-MAKING METHOD BASED ON IVPFLS

We compare our proposed method based on the IVPFLWMM operator with Du et al.'s method based on interval-valued Pythagorean fuzzy linguistic weighted average (IVPFLWA) operator. The two methods are used to solve the following decision-making problem and we compare their final results.

*Example 6 (Adopted from Du et al. [32]):* The government wants to evaluate the performance of five departments during the rescue work after the earthquake occurred. Let  $A_i (i = 1, 2, \dots, 5)$  be the five departments and DMs evaluate their emergency response capabilities under five attributes, i.e. the emergency forecasting capability ( $G_1$ ), the emergency process capability ( $G_2$ ), the after-disaster loss evaluation capability ( $G_3$ ), the emergency support capability ( $G_4$ ), and the after-disaster reconstruction capability ( $G_5$ ). Let  $w = (0.15, 0.28, 0.18, 0.25, 0.14)^T$  be the weight vector of the attributes. Assume  $S = (s_0, s_1, \dots, s_6)$  to be a predefined linguistic term set and the DMs employ IVPFLNs defined on  $S$  to express their evaluation values. The decision matrix is presented in Table 7. We use our proposed method and Du et al.'s [32] method to solve Example 6 and present in the decision results in Table 8.

TABLE 7. The interval-valued Pythagorean fuzzy linguistic decision matrix of Example 6.

	$A_1$	$A_2$	$A_3$	$A_4$
$G_1$	$\langle s_3, ([0.6, 0.7], [0.4, 0.5]) \rangle$	$\langle s_4, ([0.6, 0.75], [0.35, 0.5]) \rangle$	$\langle s_5, ([0.7, 0.8], [0.45, 0.5]) \rangle$	$\langle s_4, ([0.65, 0.78], [0.4, 0.55]) \rangle$
$G_2$	$\langle s_3, ([0.55, 0.7], [0.5, 0.6]) \rangle$	$\langle s_4, ([0.55, 0.75], [0.5, 0.65]) \rangle$	$\langle s_{4.5}, ([0.7, 0.75], [0.35, 0.45]) \rangle$	$\langle s_5, ([0.6, 0.75], [0.4, 0.5]) \rangle$
$G_3$	$\langle s_{3.5}, ([0.5, 0.7], [0.55, 0.63]) \rangle$	$\langle s_4, ([0.65, 0.7], [0.45, 0.6]) \rangle$	$\langle s_5, ([0.7, 0.75], [0.4, 0.5]) \rangle$	$\langle s_{4.3}, ([0.65, 0.72], [0.42, 0.55]) \rangle$
$G_4$	$\langle s_3, ([0.7, 0.75], [0.4, 0.5]) \rangle$	$\langle s_4, ([0.6, 0.7], [0.5, 0.6]) \rangle$	$\langle s_{3.5}, ([0.65, 0.75], [0.45, 0.55]) \rangle$	$\langle s_{2.6}, ([0.7, 0.8], [0.4, 0.5]) \rangle$
$G_5$	$\langle s_{2.8}, ([0.6, 0.75], [0.4, 0.5]) \rangle$	$\langle s_{2.8}, ([0.6, 0.75], [0.4, 0.5]) \rangle$	$\langle s_4, ([0.6, 0.7], [0.4, 0.5]) \rangle$	$\langle s_{3.5}, ([0.75, 0.8], [0.3, 0.5]) \rangle$

TABLE 8. The decision results of Example 6 by different methods.

Methods	Score functions $S(\alpha_i) (i=1,2,3,4)$	Ranking orders
Du et al.'s [32] method	$S(\alpha_1) = 1.9769, S(\alpha_2) = 2.2098, S(\alpha_3) = 2.7942, S(\alpha_4) = 2.4756$	$A_3 \succ A_4 \succ A_2 \succ A_1$
The proposed method when $P = (1, 1, 1, 1, 1)$	$S(\alpha_1) = 0.3323, S(\alpha_2) = 0.3424, S(\alpha_3) = 0.4436, S(\alpha_4) = 0.3846$	$A_3 \succ A_4 \succ A_2 \succ A_1$

TABLE 9. The ranking results with different methods of Example 5.

Methods	Score functions $S(\alpha_i) (i=1,2,3,4,5)$	Ranking orders
Dong and Wan's [51] method	$S(\alpha_1) = 2.7478, S(\alpha_2) = 3.0466, S(\alpha_3) = 2.9457,$ $S(\alpha_4) = 2.4734, S(\alpha_5) = 2.8129$	$A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
The proposed method when $P = (1, 1, 1, 1, 1)$	$S(\alpha_1) = 0.4211, S(\alpha_2) = 0.4693, S(\alpha_3) = 0.4520,$ $S(\alpha_4) = 0.3670, S(\alpha_5) = 0.4431$	$A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$

As we can see from Table 8, although the score values of alternatives produced by the two different methods are different, the ranking orders of alternatives are the same and the best alternative is  $A_3$ , which illustrates the validity of our proposed method.

2) COMPARED WITH DECISION-MAKING METHODS UNDER IIVLS

We continue to compare our proposed method with that proposed by Dong and Wan's [33] based on the interval-valued intuitionistic linguistic weighted arithmetic average (IIVLWAA) operator. We employ our propose method and Dong and Wang's method to solve Example 5 and present the decision results in Table 9. As we can see from Table 9, the ranking order produced by our proposed method is the same as that obtained by Dong and Wan's [33] decision-making method, and that illustrates the validity of our proposed method.

D. ADVANTAGES OF OUR METHOD

In this section, we detailly investigate the advantages and superiorities of our proposed method though numerical examples.

1) IT EFFECTIVELY DEALS WITH DMS' UNREASONABLE EVALUATION VALUES

In some practical decision-making problems, some DMS may provide unreasonable decision-making values. The reasons

are usually two-fold. First, due to the intricacy of MAGDM problems, DMS can hardly get all the information of all the feasible alternatives in a limited time and they may provide unduly high or low evaluation values. Second, as DMS usually have different background and priori knowledge some of them maybe prejudiced against some alternatives and they probably provide tendentious evaluation values. Evidently, the biased evaluation information offered by prejudiced DMS have negative influence on the final decision results. If such bad impact is not eliminated or reduces, unreasonable decision outcomes may be gained. It is noted that our method is based on the PA operator. The PA operator was originated to deal with possible unreasonable evaluation values and hence, our proposed method can also effectively handle the unduly high or low evaluation values.

2) IT CONSIDERS THE COMPLICATED INTERRELATIONSHIP BETWEEN ATTRIBUTES

As we can see from Table 8, when  $P = (1, 0, 0, 0, 0)$ , then our proposed method produces the same ranking order as Du et al.'s [32] method. This is because Du et al.'s [32] method is based on the simply weighted average operator, which does not consider the interrelationship between attributes. When  $P = (1, 0, 0, 0, 0)$ , our proposed method does not take the interrelationship between the attributes, which is the same as the weighted average operator. In addition, when  $P = (1, 1, 0, 0, 0)$ , our method takes the interrelationship between any two attributes into consideration.

When  $P = (1, 1, 1, 1, 1)$ , the interrelationship among multiple attribute values is taken into consideration. Similarly, as seen from Table 8 our proposed method produces the same ranking order as Du *et al.*'s [32] method when  $P = (1, 0, 0)$ . In this case, both our proposed method and Dong and Wang's [33] method assume that the attributes are independent and there is no interrelationship between attributes. When  $P = (1, 1, 0)$ , our method reflects the interrelationship among any two attributes. When  $P = (1, 1, 1)$ , the interrelationship among all the attributes is taken into account. In most real decision-making problems, the attributes are usually correlated and there exists complicated interrelationship among attributes. To determine the optimal alternative, the interrelationship among attributes should be taken into consideration. If there indeed no interrelationship between attributes, we can assign  $P = (s, 0, 0, \dots, 0)$  ( $s > 0$ ) in our method. Hence, our method is more powerful and flexible than those proposed by Du *et al.* [32] and Dong and Wan [33].

### 3) THE PROPOSED OPERATIONS OF IVPFLNs ARE MORE REASONABLE AND FLEXIBLE

In addition, the operations in this article are more flexible. First, they consider satisfy the constraint of LTS. Second, they consider DMs' attitude toward to optimism and pessimism. Hence, they can more suitable and powerful to deal with practical MAGDM problems. However, it should be noted that the operations proposed by Du *et al.* [32] and Dong and Wan [33] are based on simple operators and they have shortcomings in dealing with practical MAGDM problems.

## VII. CONCLUSION

This article investigates MAGDM method wherein DMs' evaluation information is expressed as IVPFLNs. The main contributions of this article include three aspects. First, we proposed new operational laws of IVPFLNs based on LSF. The new operations are more reasonable and can depict DMs' attitude towards optimism and pessimism. Second, we proposed novel AOs of IVPFLNs based on the PA and PMM operators. These operators can more effectively deal with DMs' biased evaluation values and take the interrelationship among attributes into consideration. Third, we presented a new MAGDM method. In the new decision-making method, the IVPFLPWA operator is used to compute the collective decision matrix and the IVPFLPWMM operator is employed to calculate the overall evaluation value of each alternative. Finally, we showed the validity of the proposed method. We also conducted comparison analysis to illustrate the advantages and superiorities of our method.

In the future, we will continue our research from two aspects. First, we shall study more applications of our decision-making methods in more practical MAGDM problems. Second, we will study more MAGDM methods based on IVPFL information to provide DMs more options of manners to select the best alternative.

## REFERENCES

- [1] R. Davoudabadi, S. M. Mousavi, V. Mohagheghi, and B. Vahdani, "Resilient supplier selection through introducing a new interval-valued intuitionistic fuzzy evaluation and decision-making framework," *Arabian J. Sci. Eng.*, vol. 44, pp. 1–10, May 2019, doi: [10.1007/s13369-019-03891-x](https://doi.org/10.1007/s13369-019-03891-x).
- [2] F. Jia, Y. Liu, and X. Wang, "An extended MABAC method for multi-criteria group decision making based on intuitionistic fuzzy rough numbers," *Expert Syst. Appl.*, vol. 127, pp. 241–255, Aug. 2019, doi: [10.1016/j.eswa.2019.03.016](https://doi.org/10.1016/j.eswa.2019.03.016).
- [3] W. Zhou and Z. Xu, "Extreme intuitionistic fuzzy weighted aggregation operators and their applications in optimism and pessimism decision-making processes," *J. Intell. Fuzzy Syst.*, vol. 32, no. 1, pp. 1129–1138, Jan. 2017, doi: [10.3233/JIFS-16516](https://doi.org/10.3233/JIFS-16516).
- [4] Z. Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 6, pp. 1179–1187, Dec. 2007, doi: [10.1109/TFUZZ.2006.890678](https://doi.org/10.1109/TFUZZ.2006.890678).
- [5] G. Büyükköçkan and F. Göçer, "Smart medical device selection based on intuitionistic fuzzy choquet integral," *Soft Comput.*, vol. 23, no. 20, pp. 10085–10103, Oct. 2019, doi: [10.1007/s00500-018-3563-5](https://doi.org/10.1007/s00500-018-3563-5).
- [6] H. Garg and G. Kaur, "TOPSIS based on nonlinear-programming methodology for solving decision-making problems under cubic intuitionistic fuzzy set environment," *Comput. Appl. Math.*, vol. 38, no. 3, p. 114, Sep. 2019, doi: [10.1007/s40314-019-0869-6](https://doi.org/10.1007/s40314-019-0869-6).
- [7] R. R. Yager, "Pythagorean membership grades in multicriteria decision making," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 958–965, Aug. 2014, doi: [10.1109/TFUZZ.2013.2278989](https://doi.org/10.1109/TFUZZ.2013.2278989).
- [8] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 87–96, Aug. 1986.
- [9] T.-Y. Chen, "A mixed-choice-strategy-based consensus ranking method for multiple criteria decision analysis involving pythagorean fuzzy information," *IEEE Access*, vol. 6, pp. 79174–79199, 2018, doi: [10.1109/ACCESS.2018.2884895](https://doi.org/10.1109/ACCESS.2018.2884895).
- [10] Y. Xing, R. Zhang, J. Wang, and X. Zhu, "Some new pythagorean fuzzy choquet-frank aggregation operators for multi-attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 11, pp. 2189–2215, Nov. 2018, doi: [10.1002/int.22025](https://doi.org/10.1002/int.22025).
- [11] W. Yang and Y. Pang, "New pythagorean fuzzy interaction maclaurin symmetric mean operators and their application in multiple attribute decision making," *IEEE Access*, vol. 6, pp. 39241–39260, 2018, doi: [10.1109/ACCESS.2018.2856270](https://doi.org/10.1109/ACCESS.2018.2856270).
- [12] R. Zhang, J. Wang, X. Zhu, M. Xia, and M. Yu, "Some generalized pythagorean fuzzy bonferroni mean aggregation operators with their application to multiattribute group decision-making," *Complexity*, vol. 2017, Aug. 2017, Art. no. 5937376, doi: [10.1155/2017/5937376](https://doi.org/10.1155/2017/5937376).
- [13] T.-Y. Chen, "Novel generalized distance measure of pythagorean fuzzy sets and a compromise approach for multiple criteria decision analysis under uncertainty," *IEEE Access*, vol. 7, pp. 58168–58185, 2019, doi: [10.1109/ACCESS.2019.2914703](https://doi.org/10.1109/ACCESS.2019.2914703).
- [14] X. Zhu, K. Bai, J. Wang, R. Zhang, and Y. Xing, "Pythagorean fuzzy interaction power partitioned bonferroni means with applications to multi-attribute group decision making," *J. Intell. Fuzzy Syst.*, vol. 36, no. 4, pp. 3423–3438, Apr. 2019, doi: [10.3233/JIFS-181171](https://doi.org/10.3233/JIFS-181171).
- [15] H. Garg, "A novel accuracy function under interval-valued pythagorean fuzzy environment for solving multicriteria decision making problem," *J. Intell. Fuzzy Syst.*, vol. 31, no. 1, pp. 529–540, Jun. 2016, doi: [10.3233/JIFS-162165](https://doi.org/10.3233/JIFS-162165).
- [16] X. Peng and Y. Yang, "Fundamental properties of interval-valued pythagorean fuzzy aggregation operators," *Int. J. Intell. Syst.*, vol. 31, no. 5, pp. 444–487, May 2016.
- [17] H. Garg, "A novel improved accuracy function for interval valued pythagorean fuzzy sets and its applications in the decision-making process," *Int. J. Intell. Syst.*, vol. 32, no. 12, pp. 1247–1260, Dec. 2017, doi: [10.1002/int.21898](https://doi.org/10.1002/int.21898).
- [18] Y. Yang, Z.-S. Chen, Y.-H. Chen, and K.-S. Chin, "Interval-valued pythagorean fuzzy frank power aggregation operators based on an isomorphic Frank dual triple," *Int. J. Comput. Intell. Syst.*, vol. 11, no. 1, pp. 1091–1110, May 2018, doi: [10.2991/ijcis.11.1.83](https://doi.org/10.2991/ijcis.11.1.83).
- [19] T.-Y. Chen, "An interval-valued pythagorean fuzzy compromise approach with correlation-based closeness indices for multiple-criteria decision analysis of bridge construction methods," *Complexity*, vol. 2018, Nov. 2018, Art. no. 6463039, doi: [10.1155/2018/6463039](https://doi.org/10.1155/2018/6463039).

- [20] G. Wei and M. Lu, "Pythagorean fuzzy maclaurin symmetric mean operators in multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 5, pp. 1043–1070, May 2018, doi: [10.1002/int.21911](https://doi.org/10.1002/int.21911).
- [21] M. S. A. Khan, S. Abdullah, M. Y. Ali, I. Hussain, and M. Farooq, "Extension of TOPSIS method base on choquet integral under interval-valued pythagorean fuzzy environment," *J. Intell. Fuzzy Syst.*, vol. 34, no. 1, pp. 267–282, Jan. 2018, doi: [10.3233/JIFS-171164](https://doi.org/10.3233/JIFS-171164).
- [22] M. S. A. Khan and S. Abdullah, "Interval-valued pythagorean fuzzy GRA method for multiple-attribute decision making with incomplete weight information," *Int. J. Intell. Syst.*, vol. 33, no. 8, pp. 1689–1716, Aug. 2018, doi: [10.1002/int.21992](https://doi.org/10.1002/int.21992).
- [23] D. Liang, A. P. Darko, and Z. Xu, "Interval-valued pythagorean fuzzy extended bonferroni mean for dealing with heterogenous relationship among attributes," *Int. J. Intell. Syst.*, vol. 33, no. 7, pp. 1381–1411, Jul. 2018, doi: [10.1002/int.21973](https://doi.org/10.1002/int.21973).
- [24] H. Garg, "New exponential operational laws and their aggregation operators for interval-valued pythagorean fuzzy multicriteria decision-making," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 653–683, Mar. 2018, doi: [10.1002/int.21966](https://doi.org/10.1002/int.21966).
- [25] X. Tang, G. Wei, and H. Gao, "Models for multiple attribute decision making with interval-valued pythagorean fuzzy muirhead mean operators and their application to green suppliers selection," *Informatica*, vol. 30, no. 1, pp. 153–186, Mar. 2019, doi: [10.15388/Informatica.2018.202](https://doi.org/10.15388/Informatica.2018.202).
- [26] X. Peng and W. Li, "Algorithms for interval-valued pythagorean fuzzy sets in emergency decision making based on multiparametric similarity measures and WDBA," *IEEE Access*, vol. 7, pp. 7419–7441, 2019, doi: [10.1109/ACCESS.2018.2890097](https://doi.org/10.1109/ACCESS.2018.2890097).
- [27] K. Rahman and S. Abdullah, "Some induced generalized interval-valued pythagorean fuzzy einstein geometric aggregation operators and their application to group decision-making," *Comput. Appl. Math.*, vol. 38, no. 3, p. 139, Jun. 2019, doi: [10.1007/s40314-019-0868-7](https://doi.org/10.1007/s40314-019-0868-7).
- [28] N. Jan, M. Aslam, K. Ullah, T. Mahmood, and J. Wang, "An approach towards decision making and shortest path problems using the concepts of interval-valued pythagorean fuzzy information," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2403–2428, Oct. 2019, doi: [10.1002/int.22154](https://doi.org/10.1002/int.22154).
- [29] E. Haktamir and C. Kahraman, "A novel interval-valued pythagorean fuzzy QFD method and its application to solar photovoltaic technology development," *Comput. Ind. Eng.*, vol. 132, pp. 361–372, Jun. 2019, doi: [10.1016/j.cie.2019.04.022](https://doi.org/10.1016/j.cie.2019.04.022).
- [30] C. Yu, Y. Shao, K. Wang, and L. Zhang, "A group decision making sustainable supplier selection approach using extended TOPSIS under interval-valued pythagorean fuzzy environment," *Expert Syst. Appl.*, vol. 121, pp. 1–17, May 2019, doi: [10.1016/j.eswa.2018.12.010](https://doi.org/10.1016/j.eswa.2018.12.010).
- [31] A. Biswas and B. Sarker, "Interval-valued pythagorean fuzzy TODIM approach through point operator-based similarity measures for multicriteria group decision making," *Kybernetes*, vol. 48, no. 3, pp. 496–519, Mar. 2019, doi: [10.1108/K-12-2017-0490](https://doi.org/10.1108/K-12-2017-0490).
- [32] Y. Du, F. Hou, W. Zafar, Q. Yu, and Y. Zhai, "A novel method for multiattribute decision making with interval-valued pythagorean fuzzy linguistic information," *Int. J. Intell. Syst.*, vol. 32, no. 10, pp. 1085–1112, Oct. 2017, doi: [10.1002/int.21881](https://doi.org/10.1002/int.21881).
- [33] J. Dong and S.-P. Wan, "Arithmetic aggregation operators for interval-valued intuitionistic linguistic variables and application to multi-attribute group decision making," *Iranian J. Fuzzy Syst.*, vol. 13, no. 1, pp. 1–23, Feb. 2016, doi: [10.22111/IJFS.2016.2284](https://doi.org/10.22111/IJFS.2016.2284).
- [34] R. F. Muirhead, "Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters," *Proc. Edinburgh Math. Soc.*, vol. 21, pp. 144–162, Feb. 1902, doi: [10.1017/S001309150003460X](https://doi.org/10.1017/S001309150003460X).
- [35] P. Liu, Y. Li, and F. Teng, "Bidirectional projection method for probabilistic linguistic multi-criteria group decision-making based on power average operator," *Int. J. Intell. Syst.*, vol. 21, no. 8, pp. 1–14, Sep. 2019, doi: [10.1007/s40815-019-00705-y](https://doi.org/10.1007/s40815-019-00705-y).
- [36] P. Liu, S.-M. Chen, and P. Wang, "Multiple-attribute group decision-making based on q-rung orthopair fuzzy power maclaurin symmetric mean operators," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 50, no. 10, pp. 3741–3756, Oct. 2020, doi: [10.1109/TSMC.2018.2852948](https://doi.org/10.1109/TSMC.2018.2852948).
- [37] H. Garg and R. Arora, "Generalized intuitionistic fuzzy soft power aggregation operator based on t-norm and their application in multicriteria decision-making," *Int. J. Intell. Syst.*, vol. 34, no. 2, pp. 215–246, Feb. 2019, doi: [10.1002/int.22048](https://doi.org/10.1002/int.22048).
- [38] H. Liu and L. Jiang, "A multidistance based on generalized OWA operator and its application in power average operator," *J. Intell. Fuzzy Syst.*, vol. 36, no. 4, pp. 3087–3094, Apr. 2019, doi: [10.3233/JIFS-172140](https://doi.org/10.3233/JIFS-172140).
- [39] Y. Zhong, H. Gao, X. Guo, Y. Qin, M. Huang, and X. Luo, "Dombi power partitioned heronian mean operators of q-rung orthopair fuzzy numbers for multiple attribute group decision making," *PLoS ONE*, vol. 14, no. 10, Oct. 2019, Art. no. e0222007, doi: [10.1371/journal.pone.0222007](https://doi.org/10.1371/journal.pone.0222007).
- [40] Y. Xu, X. Shang, J. Wang, H. Zhao, R. Zhang, and K. Bai, "Some interval-valued q-rung dual hesitant fuzzy muirhead mean operators with their application to multi-attribute decision-making," *IEEE Access*, vol. 7, pp. 54724–54745, 2019, doi: [10.1109/ACCESS.2019.2912814](https://doi.org/10.1109/ACCESS.2019.2912814).
- [41] J. Wang, R. Zhang, X. Zhu, Z. Zhou, X. Shang, and W. Li, "Some q-rung orthopair fuzzy muirhead means with their application to multi-attribute group decision making," *J. Intell. Fuzzy Syst.*, vol. 36, no. 2, pp. 1599–1614, Mar. 2019, doi: [10.3233/jifs-18607](https://doi.org/10.3233/jifs-18607).
- [42] Y. Xu, X. Shang, J. Wang, R. Zhang, W. Li, and Y. Xing, "A method to multi-attribute decision making with picture fuzzy information based on muirhead mean," *J. Intell. Fuzzy Syst.*, vol. 36, no. 4, pp. 3833–3849, Apr. 2019, doi: [10.3233/JIFS-172130](https://doi.org/10.3233/JIFS-172130).
- [43] J. Wang, H. Gao, and G. Wei, "Some 2-tuple linguistic neutrosophic number muirhead mean operators and their applications to multiple attribute decision making," *J. Exp. Theor. Artif. Intell.*, vol. 31, no. 3, pp. 409–439, May 2019.
- [44] L. Li, R. Zhang, J. Wang, X. Zhu, and Y. Xing, "Pythagorean fuzzy power muirhead mean operators with their application to multi-attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 35, no. 2, pp. 2035–2050, Aug. 2018, doi: [10.3233/JIFS-171907](https://doi.org/10.3233/JIFS-171907).
- [45] W. Xu, X. Shang, J. Wang, and W. Li, "A novel approach to multi-attribute group decision-making based on interval-valued intuitionistic fuzzy power muirhead mean," *Symmetry*, vol. 11, no. 3, p. 441, Mar. 2019, doi: [10.3390/sym11030441](https://doi.org/10.3390/sym11030441).
- [46] P. Liu, Q. Khan, T. Mahmood, and N. Hassan, "T-spherical fuzzy power muirhead mean operator based on novel operational laws and their application in multi-attribute group decision making," *IEEE Access*, vol. 7, pp. 22613–22632, 2019, doi: [10.1109/ACCESS.2019.2896107](https://doi.org/10.1109/ACCESS.2019.2896107).
- [47] Q. Khan, N. Hassan, and T. Mahmood, "Neutrosophic cubic power muirhead mean operators with uncertain data for multi-attribute decision-making," *Symmetry*, vol. 10, no. 10, p. 444, Sep. 2018, doi: [10.3390/sym10100444](https://doi.org/10.3390/sym10100444).
- [48] Z. Liu, H. Xu, X. Zhao, P. Liu, and J. Li, "Multi-attribute group decision making based on intuitionistic uncertain linguistic hamy mean operators with linguistic scale functions and its application to health-care waste treatment technology selection," *IEEE Access*, vol. 7, pp. 20–46, 2019, doi: [10.1109/ACCESS.2018.2882508](https://doi.org/10.1109/ACCESS.2018.2882508).
- [49] Z. Liu, P. Liu, and X. Liang, "Multiple attribute group decision-making method based on generalized interval-valued hesitant uncertain linguistic power aggregation operators and linguistic-scale functions," *Int. J. Fuzzy Syst.*, vol. 20, no. 6, pp. 1995–2015, Aug. 2018, doi: [10.1007/s40815-018-0465-8](https://doi.org/10.1007/s40815-018-0465-8).
- [50] J. Wang, J. Wu, J. Wang, H. Zhang, and X. Chen, "Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems," *Inf. Sci.*, vol. 288, pp. 5–72, Dec. 2014.



**YANG ZHOU** received the Ph.D. degree from Beijing Jiaotong University, Beijing, China, in 2019. He is currently a Lecturer with School of Economics and Management, Zhejiang University of Science and Technology, Hangzhou, Zhejiang, China. His research interests include machine learning, big data analysis, and operational research.



**YUAN XU** received the B.S. degree from Hebei University, Baoding, Hebei, China, in 2017, and the M.S. degree from Beijing Jiaotong University, Beijing, China, in 2020, where she is currently pursuing the Ph.D. degree with the School of Economic and Management. Her current research interests include group decision making, aggregations operators, health informatics, and data-driven healthcare management.



**JUN WANG** received the B.S. degree from Hebei University, in 2013, and the Ph.D. degree from Beijing Jiaotong University, China, in 2019. He is currently a Lecturer with the School of Economics and Management, Beijing University of Chemical Technology, Beijing, China. His research interests include aggregation operators, multiple attribute decision making, fuzzy logics, operational research, and big data.



**WUHUAN XU** received the B.S. degree from Hebei University, Baoding, Hebei, China, in 2018. She is currently pursuing the master's degree with the School of Economic and Management, Beijing Jiaotong University, Beijing, China. Her current research interests include multiple attributes decision making, clinical text mining, and data-driven healthcare management.



**GUANGMING YANG** received the Ph.D. degree from the Huazhong University of Science and Technology, Wuhan, China, in 2012. He is currently a Lecturer with the School of Economics and Management, Zhejiang University of Science and Technology, Hangzhou, Zhejiang, China. His research interests include big data analysis, electronic commerce, and operational research.

...