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Adaptive Anti-Synchronization of Julia Sets in Generalized Alternated System

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ABSTRACT In this article, the fractal behavior in generalized alternated system is discussed. Take the classical fractal set as an example, the anti-synchronization of the Julia set between two different generalized alternated systems is discussed using the adaptive control method in this article. The anti-synchronization of the Julia set between different generalized alternated systems is also accomplished by synchronizing their iterative tracks. The simulations illustrate the effectiveness and correctness of this control method.

INDEX TERMS Fractal, adaptive anti-synchronization, generalized alternated system.

I. INTRODUCTION

Fractals are everywhere. Fractal, the same as chaos, is an important branch of the nonlinear field. Fractals have important applications in many fields, such as economics, biology, environmental science, and communications [1]–[5]. With the advent of the 5G era, the application of fractals in the communication field has become the focus of scholars, where the control and synchronous control of fractal behaviors is the focus of scholars' research [6]–[10].

Danca *et al.* [11] discussed the alternated Julia set, which was obtained by the iteration $z_{n+1} = z_n^2 + c_i, i = 1, 2,$ and some properties of alternated Julia sets were given. Since then, the alternated fractal system has entered the attention of scholars. Wang and Liu [12]–[15] discussed the control and synchronization of Julia sets in several fractal alternated system, such as the classical alternating iteration system $z_{n+1} = z_n^2 + c_i$, the generalized alternating iteration system $z_{n+1} = az_n^p + c_i$, and the spatial alternating iteration system $z_{m+1,n} + az_{m,n+1} = (1+a)z_{m,n}^2 + c_i$.

In 1990, Pecora and Carrol [16] proposed a chaotic self-synchronization method, which used the drive-response method to synchronize two chaotic systems for the first time. Furthermore, some definitions of chaotic synchronization are further promoted [17]–[20], such as the complete synchronization, the coupled synchronization, generalized synchronization, and projective synchronization, and so on.

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Research on chaos synchronization has yielded fruitful results. Chaos synchronization has very attractive application prospects and potential market value in the fields of physics, chemistry, biology, mechanics, information science, and secure communication [21]–[25], especially in the application of secure communication.

Fractal has initial sensitivity and ergodicity similar to chaos. The synchronization research of fractal systems also has huge application prospects in secure communication. There are also many achievements in the synchronization research of fractal systems [26]–[30]. Wang and Liu [31] proposed an adaptive controller to realize the adaptive synchronization of Julia sets between two systems with different parameters. In practical applications, the research on the synchronization and anti-synchronization between different fractal systems is equally important for secure communication.

In this article, the definition of anti-synchronization is introduced into the Julia set of generalized alternated system. An adaptive controller is designed to realize the anti-synchronization between two different generalized alternated systems; for a system with unknown parameters, a parameter estimator is designed to complete the identification of unknown parameters.

The outline of the paper is as follows. In Sec. II, some definitions about the Julia set in generalized alternated system are given. In Sec. III, the anti-synchronization of two different Julia sets using the adaptive control method is achieved. In Sec. IV, the simulations show the feasibility of this method. Finally, in Section V, the conclusions are given.

II. FRACTAL BEHAVIOR IN GENERALIZED ALTERNATED SYSTEM - JULIA SET

In this section, some necessary knowledge about fractional theory and Julia set is reviewed briefly. More relevant contents can be found in monographs [26], [32] and references contained therein.

In this article, the following generalized alternated system is taken to discuss the anti-synchronization of Julia set,

$$z_{n+1} = az_n^p + c_i, \quad i = 1, 2 \tag{1}$$

where $a \in \mathbb{R}$, $p \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C}$ and $c_1 \neq c_2$. For the convenience of description, the symbol f is used to represent the alternated iterative system (1).

Definition 1: The filled Julia set of f denoted by K is the set of the point $z \in \mathbb{C}$ whose trajectory is limited, that is

$$K = \left\{ z : f^n(z)_{n\in\mathbb{N}} \text{ is bounded} \right\}.$$

The boundary of K is called the Julia set, which is denoted by J_f, that is $J_f = \partial K$.

In addition, some properties of Julia sets are given as follows: J_f is nonempty and bounded;

(i) J_f is fully invariant, $J_f = f(J_f) = f^{-1}(J_f)$.
(ii) $J_f = J_{f^p}$ for any positive integer p.
(iii) If ω is an attractive fixed point of f, then $J_f = \partial A(\omega)$, where $A(\omega)$ is the attractive domain of the attractive fixed pointω. It is also the same as $\omega = \infty$.

In other words, J_f is the Julia set of System (1). According to the definition of the Julia set in Logistic map, the structure of its Julia set is closely related to the iterative trajectory of System (1). Therefore, the control of the Julia set in Logistic map can be achieved by controlling its iterative trajectory.

III. ADAPTIVE ANTI-SYNCHRONIZATION OF JULIA SET IN GENERALIZED ALTERNATED SYSTEM

In order to achieve anti-synchronization of Julia sets between different generalized alternated systems, taking System (2) to be the driving system as the following form

$$z_{n+1} = \begin{cases} az_n^p + c_1, & n \text{ is even} \\ az_n^p + c_2, & n \text{ is odd} \end{cases} \tag{2}$$

The response system is as

$$w_{n+1} = \begin{cases} bw_n^q + c_3 + u_n, & n \text{ is even} \\ bw_n^q + c_4 + u_n, & n \text{ is odd.} \end{cases} \tag{3}$$

where $b \in \mathbb{R}$, $q \in \mathbb{N}$, $c_3, c_4 \in \mathbb{C}$ and $c_3 \neq c_4$, u_n is the adaptive controller to be designed. The anti-synchronous error is defined as

$$e_{n+1} = z_{n+1} + w_{n+1}. \tag{4}$$

where $e_{n+1} = e_{n+1}^r + je_{n+1}^i$.

Firstly, some definitions of synchronization of Julia set will be reviewed [16], [33].

Definition 2: For any initial value $z_0, w_0 \in \mathbb{C}$ in Systems (2) and (3), if there exists an adaptive controller u_n such that

$$\lim_{n\to\infty} \|e_n\| = \lim_{n\to\infty} \|z_n + w_n\| = 0,$$

then the trajectories of Systems (2) and (3) achieve anti-synchronization.

Case 1 For the given p and q, assuming both a and b are known, the adaptive controller is designed as follows,

$$u_n = \begin{cases} -\left(az_n^p + c_1\right) - \left(bw_n^q + c_3\right) - ke_n, & n \text{ is even} \\ -\left(az_n^p + c_2\right) - \left(bw_n^q + c_4\right) - ke_n, & n \text{ is odd,} \end{cases} \tag{5}$$

if $|k| < 1$, for any (z_0, w_0), the anti-synchronization between Julia sets of response system (2) and driving system (1) is realized.

Proof: Substitute Equations (1), (2) and (4) into Equation (3), then

$$\begin{aligned} &e_{n+1} \\ &= z_{n+1} + w_{n+1} \\ &= \begin{cases} az_n^p+c_1+bw_n^q+c_3-\left(az_n^p + c_1 + bw_n^q+c_3\right) - ke_n, \\ \quad n \text{ is even} \\ az_n^p+c_2+bw_n^q+c_4-\left(az_n^p+c_2 + bw_n^q + c_4\right) - ke_n, \\ \quad n \text{ is odd,} \end{cases} \\ &= -ke_n, \end{aligned}$$

if $|k| < 1$, there are $|e_n| \to 0$ as $n \to \infty$. According to Definition 2, the trajectories of Systems (2) and (3) achieve anti-synchronization. Therefore, the anti-synchronization of Julia sets in Systems (2) and (3) is obtained. The proof is completed.

Case 2 For the given p and q, assuming a is unknown, but b is known, for the given $d, k \in \mathbb{R}$, if $|d + k| < 1$, the adaptive controller is designed as follows,

$$u_n = \begin{cases} -\left(\hat{a}_n z_n^p + c_1\right) - \left(bw_n^q + c_3\right) - ke_n, & n \text{ is even} \\ -\left(\hat{a}_n z_n^p + c_2\right) - \left(bw_n^q + c_4\right) - ke_n, & n \text{ is odd,} \end{cases} \tag{6}$$

and the update low of $\hat{a}_n$ is

$$\hat{a}_{n+1} = \hat{a}_n + \frac{d\left(e_{n+1}z_n^p - e_n z_{n+1}^p\right)}{z_n^p z_{n+1}^p}, \tag{7}$$

for any (z_0, w_0) and $\hat{a}_0$, the anti-synchronization between Julia sets of response system (3) and driving system (2) is realized, $\hat{a}_n$ is the estimated value of the unknown parameter a.

Proof:

$$\begin{aligned} &e_{n+1} \\ &= z_{n+1} + w_{n+1} \\ &= \begin{cases} az_n^p+c_1+bw_n^q+c_3 - \left(\hat{a}_n z_n^p+c_1 + bw_n^q + c_3\right)-ke_n, \\ \quad n \text{ is even} \\ az_n^p+c_2+bw_n^q+c_4-\left(\hat{a}_n z_n^p+c_2 + bw_n^q + c_4\right) - ke_n, \\ \quad n \text{ is odd} \end{cases} \\ &= \left(a - \hat{a}_n\right) z_n^p - ke_n, \end{aligned} \tag{8}$$

where $\tilde{a}_n = a - \hat{a}_n$. From Equation (7), there are

$$\hat{a}_{n+1} - \hat{a}_n = \frac{d\left(e_{n+1}z_n^p - e_n z_{n+1}^p\right)}{z_n^p z_{n+1}^p} = \frac{de_{n+1}}{z_{n+1}^p} - \frac{de_n}{z_n^p},$$

Equivalently, there exists

$$\left(\hat{a}_{n+1} - a\right) - \left(\hat{a}_n - a\right) = \frac{de_{n+1}}{z_{n+1}^p} - \frac{de_n}{z_n^p}.$$

Since, the estimated value of $\hat{a}_n$ at step n is only related to z_n and e_n, then there are

$$\hat{a}_n - a = \frac{de_n}{z_n^p}.$$

Therefore, we have

$$\tilde{a}_n = a - \hat{a}_n = -\frac{de_n}{z_n^p}. \tag{9}$$

Equation (9) is substituted into Equation (8), there are

$$e_{n+1} = \left(a - \hat{a}_n\right)z_n^p - ke_n = -(d+k)e_n.$$

If d and k satisfy $|d+k| < 1$, then we have $|e_n| \to 0$ as $n \to \infty$. According to Definition 2, the trajectories of Systems (2) and (3) achieve anti-synchronization. And the unknown parameter $\hat{a}_n$ can be identified. The proof is completed.

Case 3 For the given p and q, assuming both a and b are unknown, for the given $d, t, k \in \mathbb{R}$, if $|d+t+k| < 1$, the adaptive controller is designed as follows,

$$u_n = \begin{cases} -\left(\hat{a}_n z_n^p + c_1\right) - \left(\hat{b}_n w_n^q + c_3\right) - ke_n, & n \text{ is even} \\ -\left(\hat{a}_n z_n^p + c_2\right) - \left(\hat{b}_n w_n^q + c_4\right) - ke_n, & n \text{ is odd,} \end{cases} \tag{10}$$

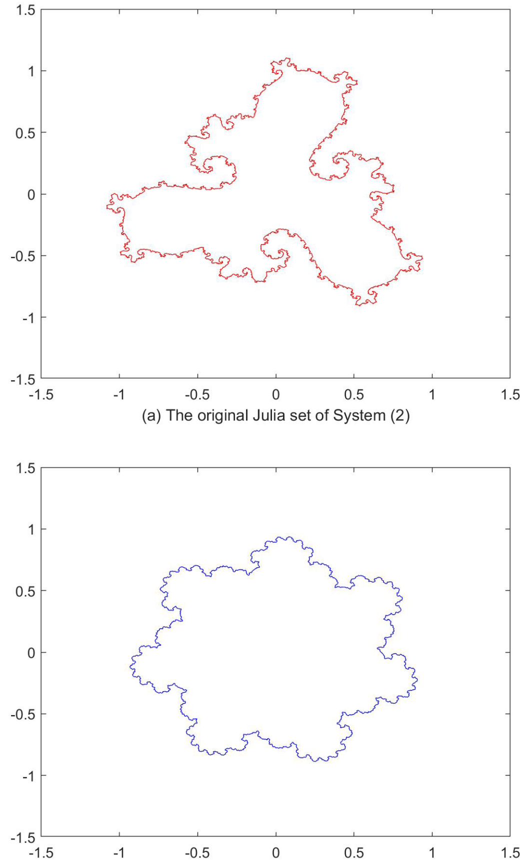
and the update lows of $\hat{a}_n$ and $\hat{b}_n$ are

$$\begin{cases} \hat{a}_{n+1} = \hat{a}_n + \dfrac{d\left(e_{n+1}z_n^p - e_n z_{n+1}^p\right)}{z_n^p z_{n+1}^p} \\ \hat{b}_{n+1} = \hat{b}_n + \dfrac{t\left(e_{n+1}w_n^q - e_n w_{n+1}^q\right)}{w_n^q w_{n+1}^q}, \end{cases} \tag{11}$$

for any (z_0, w_0) and $\left(\hat{a}_0, \hat{b}_0\right)$, the anti-synchronization between Julia sets of response system (3) and driving system (2) is realized, $\hat{a}_n$ and $\hat{b}_n$ are the estimated values of the unknown parameter a and b.

Proof:

$$\begin{aligned} &e_{n+1} \\ &= z_{n+1} + w_{n+1} \\ &= \begin{cases} az_n^p + c_1 + bw_n^q + c_3 - \left(\hat{a}_n z_n^p + c_1 + \hat{b}_n w_n^q + c_3\right) - ke_n, \\ \quad n \text{ is even} \\ az_n^p + c_2 + bw_n^q + c_4 - \left(\hat{a}_n z_n^p + c_2 + \hat{b}_n w_n^q + c_4\right) - ke_n, \\ \quad n \text{ is odd} \end{cases} \\ &= \left(a - \hat{a}_n\right)z_n^p + (b - \hat{b}_n)w_n^q - ke_n \\ &= \tilde{a}_n z_n^p + \tilde{b}_n w_n^q - ke_n, \end{aligned} \tag{12}$$


(a) The original Julia set of System (2)

(b) The original Julia set of System (3)

FIGURE 1. The original Julia sets of Systems (2) and (3).

From the update low in Equation (11), there are

$$\begin{cases} \left(\hat{a}_{n+1} - a\right) - \left(\hat{a}_n - a\right) = \dfrac{de_{n+1}}{z_{n+1}^p} - \dfrac{de_n}{z_n^p} \text{ (a)} \\ \left(\hat{b}_{n+1} - b\right) - \left(\hat{b}_n - b\right) = \dfrac{te_{n+1}}{w_{n+1}^p} - \dfrac{te_n}{w_n^p} \text{ (b).} \end{cases} \tag{13}$$

Similar to the reasoning process of Case 2, in Equation (13a), $\hat{a}_n$ at step n is only related to z_n and e_n. In Equation (13b), $\hat{b}_n$ at step n is only related to w_n and e_n. Then there are

$$\begin{cases} \tilde{a}_n = a - \hat{a}_n = -\dfrac{de_n}{z_n^p} \\ \tilde{b}_n = b - \hat{b}_n = -\dfrac{de_n}{w_n^p}. \end{cases} \tag{14}$$

Substituting Equation (14) into Equation (12), there are

$$e_{n+1} = \tilde{a}_n z_n^p + \tilde{b}_n w_z^q - ke_n = -(d+t+k)e_n.$$

If d, t and k satisfy $|d+t+k| < 1$, then we have $|e_n| \to 0$ as $n \to \infty$. According to Definition 2, the trajectories of Systems (2) and (3) achieve anti-synchronization. And the unknown parameters $\hat{a}_n$ and $\hat{b}_n$ can be identified. The proof is completed.

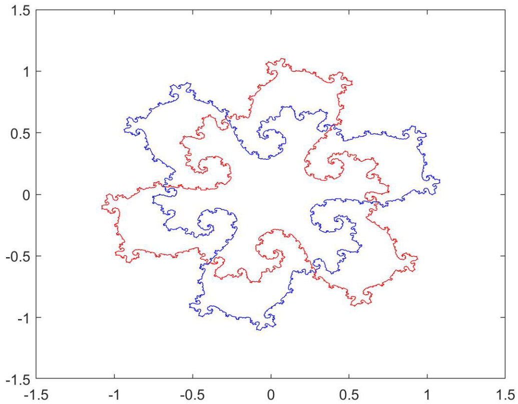


FIGURE 2. **The anti-synchronization of Julia sets between Systems (2) and (3).**

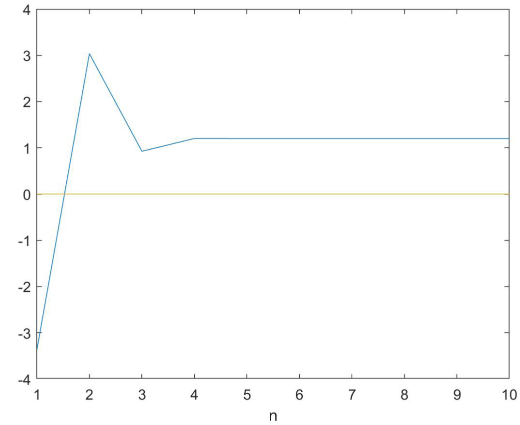


FIGURE 4. **The estimation under the adaptive controller.**

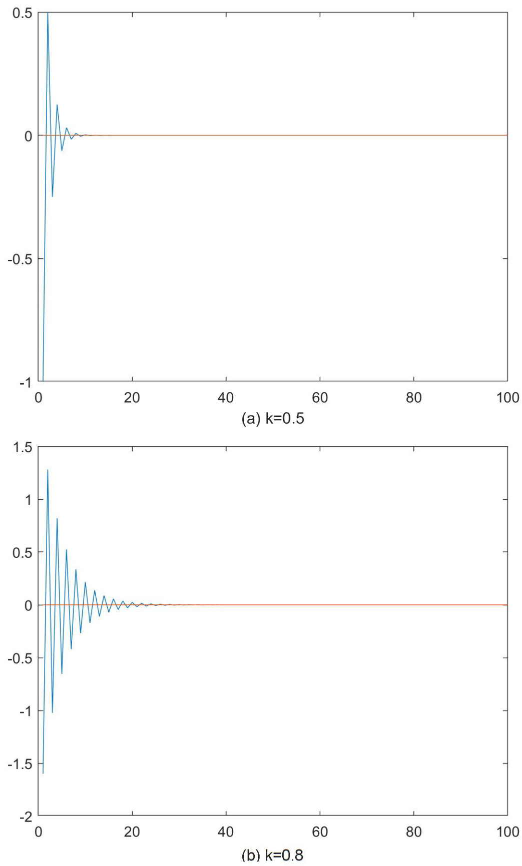


FIGURE 3. **The anti-synchronous error e_n with different k.**

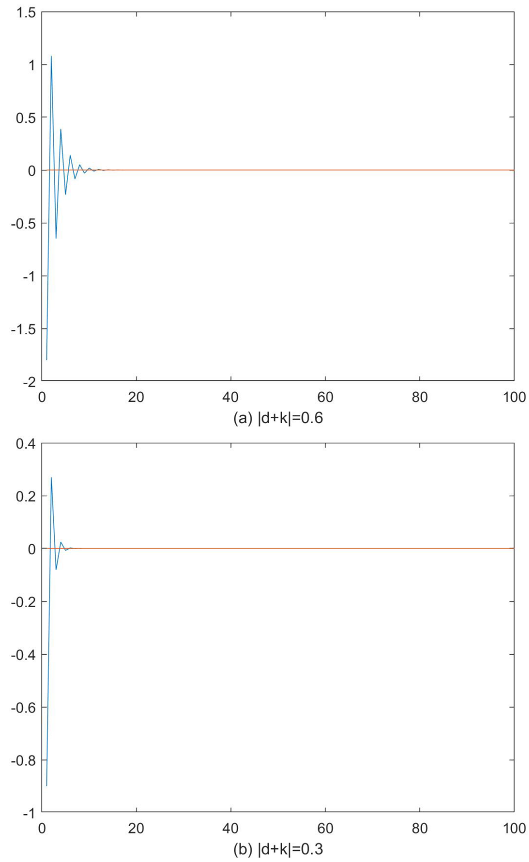


FIGURE 5. **The anti-synchronous error e_n with different $|d+k|$.**

IV. SIMULATION EXAMPLE

Some simulation examples are given in this section. In this section, taking $\boldsymbol{p} = 3, \boldsymbol{q} = 7, \boldsymbol{c}_1 = 0.5-0.5\boldsymbol{i}, \boldsymbol{c}_2 = 0.4+0.5\boldsymbol{i}, \boldsymbol{c}_3 = 0.5-0.4\boldsymbol{i}, \boldsymbol{c}_4 = 0.3+0.5\boldsymbol{i}$.

Case 1 In Systems (2) and (3), both $\boldsymbol{a}$ and $\boldsymbol{b}$ are known, and $\boldsymbol{a} = 1.2, \boldsymbol{b} = 2.3$.

Fig.1(a) illustrates the original Julia set of System (2) with $\boldsymbol{a} = 1.2, \boldsymbol{p} = 3, \boldsymbol{c}_1 = 0.5-0.5\boldsymbol{i}$ and $\boldsymbol{c}_2 = 0.4+0.5\boldsymbol{i}$. And Fig.1(b) illustrates the original Julia set of System (3) with $\boldsymbol{b} = 2.3, \boldsymbol{q} = 7, \boldsymbol{c}_3 = 0.5-0.4\boldsymbol{i}, \boldsymbol{c}_4 = 0.3+0.5\boldsymbol{i}$ and $\boldsymbol{u}_n = 0$.

The original value of anti-synchronization error $\boldsymbol{e}_n$ is taken as $\boldsymbol{e}_0 = 0.8+2.6\boldsymbol{i}$, and $\boldsymbol{k} = 0.1$. Figure 2 illustrates the anti-synchronization of Julia sets between Systems (2) and (3) with $\boldsymbol{n} = 1000$.

It can be seen from Fig.2, the Julia set of generalized alternated system (3) with the adaptive controller has shifted, which is caused by the property of Julia set. Therefore, the Julia set of response system (3) is not completely anti-symmetric to the Julia set of driving system (2), which can increase the system's anti-attack ability when used in

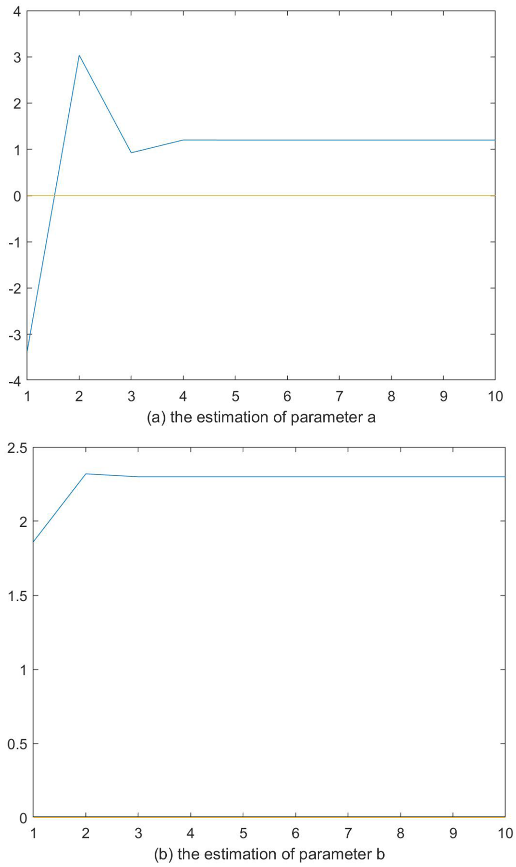


FIGURE 6. The estimation of a and b under the adaptive controller.

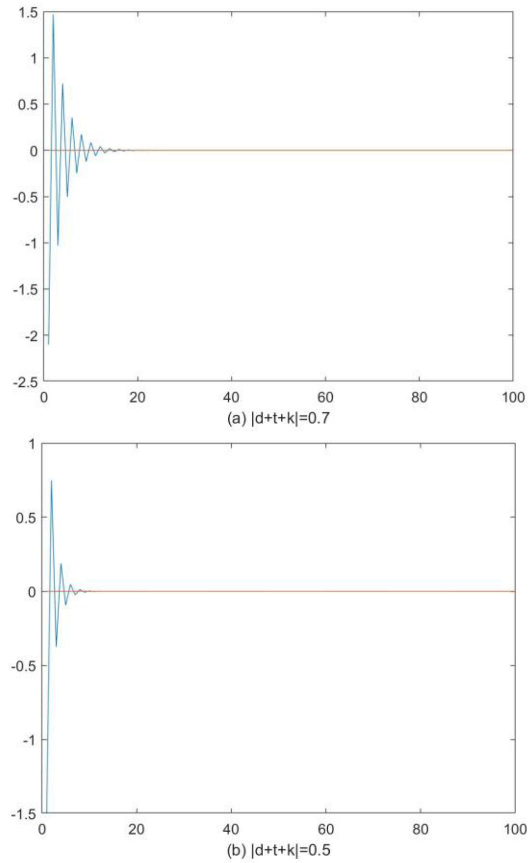


FIGURE 7. The anti-synchronous error e_n with different $|d+t+k|$.

encryption operations. Thence, the generalized alternated system is more suitable for confidential communication.

The relationship between the convergence speed of the anti-synchronous error $\boldsymbol{e_n}$ and the parameter $\boldsymbol{k}$ is shown in Figure 3 with $\boldsymbol{e}_0 = 0.8 + 2.6\boldsymbol{i}$.

Obviously, it can be seen from Fig.3, for the given $\boldsymbol{e}_0 = 0.8 + 2.6\boldsymbol{i}$, the value of parameter $\boldsymbol{k}$ determines the convergence speed of the anti-synchronous error $\boldsymbol{e_n}$. The smaller the value of k, the faster the error will converge to 0.

Case 2 In Systems (2) and (3), the parameter $\boldsymbol{b}$ is known, but parameter $\boldsymbol{a}$ is unknown, and $\boldsymbol{b} = 2.3$.

Set the initial values of Systems (2) and (3) to $z_0 = 0.2 - 0.5\boldsymbol{i}$, $\boldsymbol{w}_0 = 0.3 + 1.5\boldsymbol{i}$. The initial value of the selected parameter $\hat{\boldsymbol{a}}_0 = 5$. The initial value of anti-synchronous error $\boldsymbol{e}_0 = 3$.

Figure 4 shows the estimation of parameter $\boldsymbol{a}$. The relationship between the convergence speed of the anti-synchronous error $\boldsymbol{e_n}$ and the parameter $|\boldsymbol{d} + \boldsymbol{k}|$ is shown in Figure 5.

From Fig.5, it can be clearly seen that the value of parameter $|\boldsymbol{d} + \boldsymbol{k}|$ determines the convergence speed of the anti-synchronous error $\boldsymbol{e_n}$. The smaller the value of $|\boldsymbol{d} + \boldsymbol{k}|$, the faster the error will converge to 0.

Case 3 In Systems (2) and (3), both $\boldsymbol{a}$ and $\boldsymbol{b}$ are unknown.

Set the initial values of Systems (2) and (3) to $z_0 = 0.2 - 0.5\boldsymbol{i}$, $\boldsymbol{w}_0 = 0.3 + 1.5\boldsymbol{i}$. The initial value of the selected parameter $\hat{\boldsymbol{a}}_0 = 1$, $\hat{\boldsymbol{b}}_0 = 1$,. The initial value of anti-synchronous error $\boldsymbol{e}_0 = 3$.

Figure 6 shows the estimation of unknown parameters $\boldsymbol{a}$ and $\boldsymbol{b}$.

From Fig.7, it can be clearly seen that the value of parameter $|\boldsymbol{d} + \boldsymbol{t} + \boldsymbol{k}|$ determines the convergence speed of the anti-synchronous error $\boldsymbol{e_n}$. The smaller the value of $|\boldsymbol{d} + \boldsymbol{t} + \boldsymbol{k}|$, the faster the error will converge to 0.

V. CONCLUSION

In this article, the anti-synchronization of Julia sets between different generalized alternated systems is achieved via using the adaptive control. Moreover, we discuss the adaptive anti-synchronization between two different systems with fully unknown parameters. We designed the parameter estimator, and completed the identification of unknown parameters based on known conditions. Three examples are taken to certificate the effectiveness of the adaptive controller and the parameter estimator. These control methods and their theories are successfully applied to other aspects of fractal theory, which can help us to explain the corresponding complicated phenomena better.

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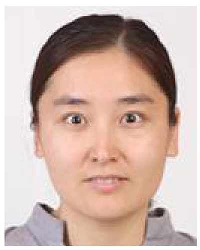

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