A Family of Derivative-Free Conjugate Gradient Methods for Constrained Nonlinear Equations and Image Restoration

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ABSTRACT In this paper, a derivative-free conjugate gradient method for solving nonlinear equations with convex constraints is proposed. The proposed method can be viewed as an extension of the three-term modified Polak-Ribiére-Polyak method (TTPRP) and the three-term Hestenes-Stiefel conjugate gradient method (TTHS) using the projection technique of Solodov and Svaiter [Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods, 1998, 355-369]. The proposed method adopts the adaptive line search scheme proposed by Ou and Li [Journal of Applied Mathematics and Computing 56.1-2 (2018): 195-216] which reduces the computational cost of the method. Under the assumption that the underlying operator is Lipschitz continuous and satisfies a weaker condition of monotonicity, the global convergence of the proposed method is established. Furthermore, the proposed method is extended to solve image restoration problem arising in compressive sensing. Numerical results are presented to demonstrate the effectiveness of the proposed method.

INDEX TERMS Unconstrained optimization, nonlinear equations, convex constrained, conjugate gradient method, projection method, compressive sensing.

I. INTRODUCTION

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a map, and Ω be a nonempty closed, convex set of $\mathbb{R}^n$. We consider the following problem: finding a vector v such that

$$\varphi(v) = 0, \quad v \in \Omega. \tag{1}$$

The problem (1) exist in a wide variety of applications that includes chemical equilibrium systems [1], economic equilibrium problems [2], power flow equations [3], non-negative matrix factorisation [4], [5], phase retrieval [6], [7], nonlinear

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compressed sensing [8], learning constrained neural networks [9] and financial forecasting problems [10]. Iterative methods such as the Newton method, fixed-point method, quasi-Newton method and the conjugate gradient method have been used to solve the unconstrained version of (1), that is, when the constraint set $\Omega = \mathbb{R}^n$.

Among the above-mentioned methods, the conjugate gradient methods (see [11] for instance) are well known and particularly effective for solving large-scale unconstrained optimization problems due to their simplicity and low storage requirement. On this note, motivated by the projection scheme in Solodov and Svaiter [12], several researchers have extended the conjugate gradient methods
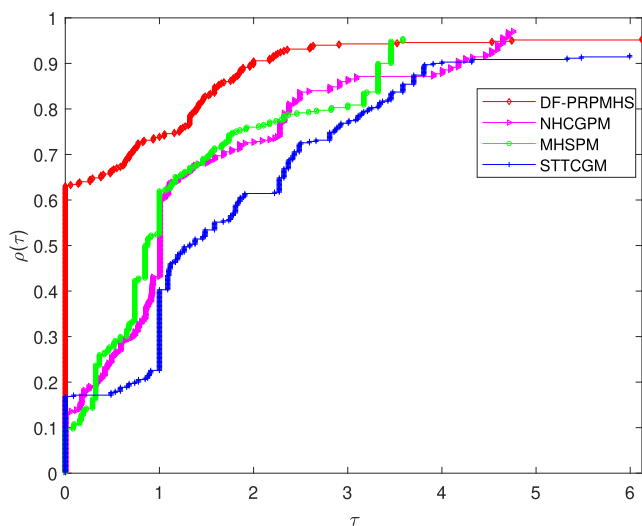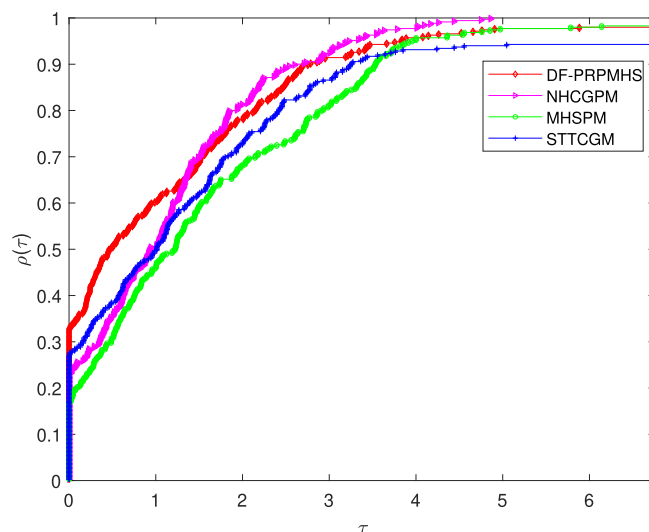
FIGURE 1. Performance profiles of iterations.

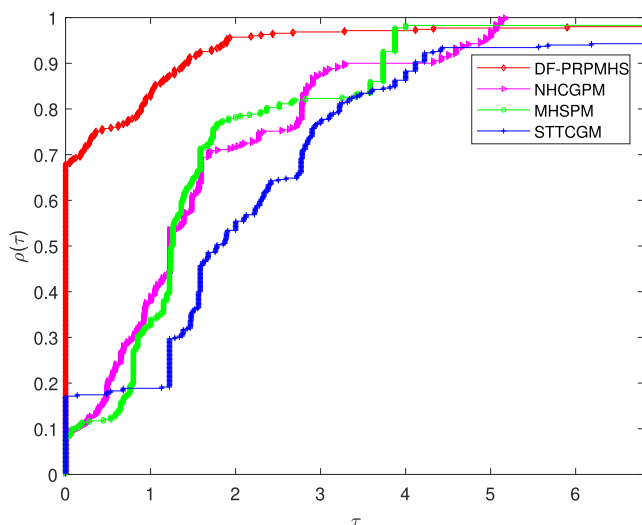

FIGURE 2. Performance profiles of function evaluations.


FIGURE 3. Performance profiles of CPU time.

for unconstrained optimization to solve large-scale nonlinear equations. Wang *et al.* [13] extended the work by Solodov and Svaiter [12], then proposed a projection-type method to solve the nonlinear equation (1) based on the inexact Newton backtracking approach. Furthermore, Ma and Wang [14] presented a modification of the extra gradient algorithm with a projection for solving constrained nonlinear monotone equations. By popularizing the idea of Zhang and Zhou [15], Yu *et al.* [16] proposed a constrained version of the spectral gradient projection algorithm for solving nonlinear monotone equations in which computing the sequence of steps does not need matrix storage as well as the solution of linear systems of equations. For recent articles see ([17]–[28]) and references therein.

In [29] and [30] Zhang, Zhou and Li proposed the three-term modified Polak-Ribiére-Polyak (TTPRP) and the three-term Hestenes-Stiefel (TTHS) conjugate gradient method.

Also, Cheng [31] proposed the three-term Hestenes-Stiefel conjugate gradient method (TTHS). The search directions of these methods are sufficiently descent and independent of the line search. They were also used to solve unconstrained optimization problems. Quite recently, Cheng, Xiao and Hu [32] proposed a derivative-free conjugate gradient method for large-scale systems of equations. The proposed method combines a derivative-free form of the three-term modified Polak-Ribiére-Polyak method (TTPRP) [29] and the two-term modified PRP method (TMPRP) proposed by Cheng [31] using a line combination. The numerical results reported indicates that they compete with the CG_DESCENT [33].

Inspired by [32], as an attempt, using the projection method [12] and the modified line search scheme proposed in [34], we effectively extend the TTPRP method and the TTHS to solve the nonlinear equation with convex constraint (1). Our proposed search direction can be viewed as an affine combination of the derivative-free version of TTPRP and TTHS method, which also satisfies the descent condition. It is worth noting that, unlike the mainstream line search, the modified line search scheme proposed in [34] has an adaptive property that gives it a suitable performance, which reduces the computation cost of the line search. Furthermore, under the assumption that the underlying operator is Lipschitz continuous and satisfies a weaker condition of monotonicity, we establish the global convergence of the proposed method. We note that, with a weaker condition of monotonicity, a larger class of functions can be considered. Numerical experiments are reported to show the robustness and effectiveness of the proposed method. Furthermore, the method is extended to solve the image restoration problem arising in compressive sensing.

The paper is organized as follows. In the next section, we recall the key methods relevant to this work. In the core Section III, the global convergence of the proposed method is
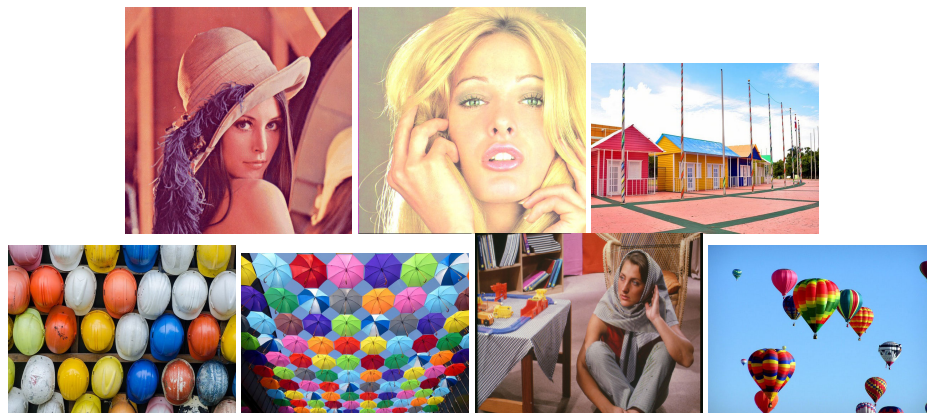
FIGURE 4. The original test images: the first row, from the left is Image 1, 2 and 3; the second row, from the left is Image 4,5,6 and 7.
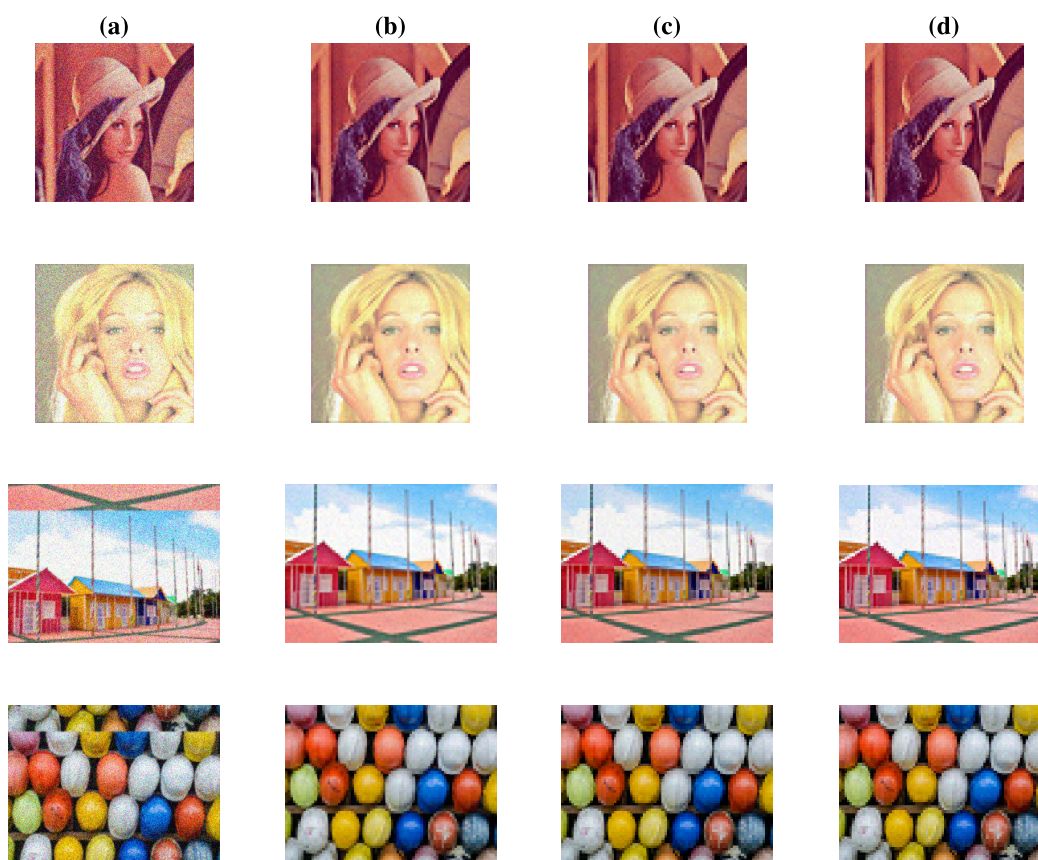
FIGURE 5. Restoration comparison of the test images. Column (a) Noisy and blurred images; Column (b) Restored images by DF-PRPMHS; Column (c) Restored images by IST and Column (d) Restored images by PSGM.

established. In Section IV we provide numerical experiments to validate the efficiency of the proposed method. Finally, in Section V, the proposed algorithm is used to solve the ℓ_1 norm regularized compressive sensing problem.

II. ALGORITHM

In this section, we begin by focusing on the conjugate gradient method designed to solve the following unconstrained

optimization problem:

$$\min\{\phi(v) : v \in \mathbb{R}^n\}, \tag{2}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a smooth nonlinear function whose gradient at point v_t is $\psi(v_t)$. The iterative formula of a nonlinear conjugate gradient method for solving (2) generate sequences of iterates recurrently by

$$v_{t+1} = v_t + \alpha_t d_t, \quad t = 0, 1, 2 \cdots, \tag{3}$$
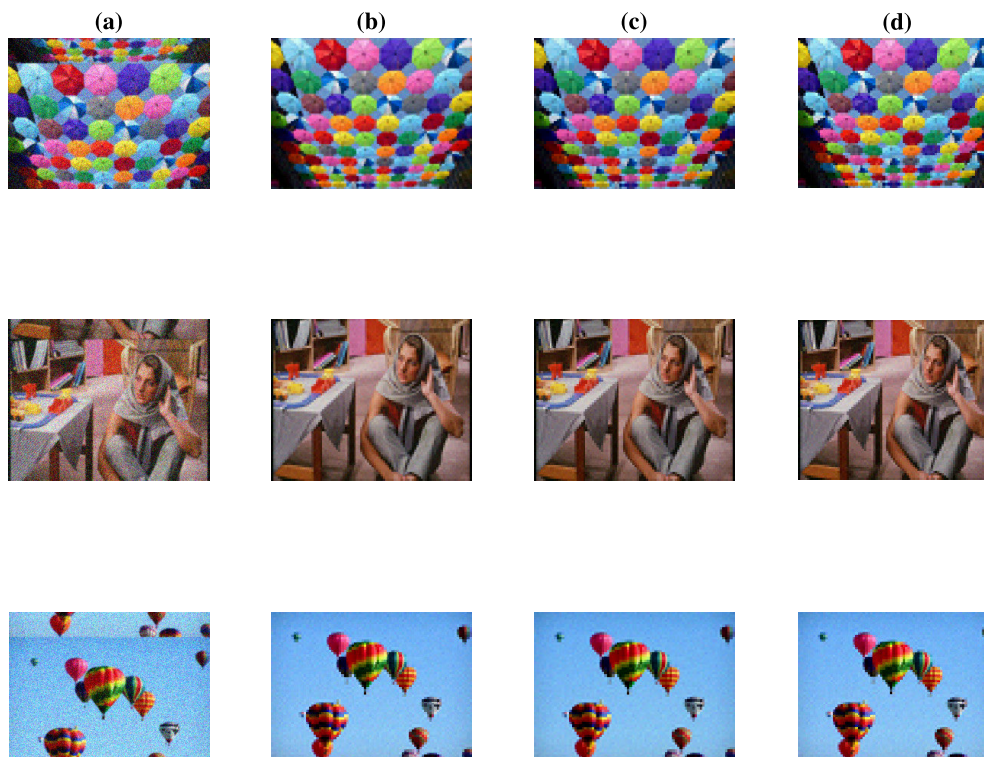
FIGURE 6. Restoration comparison of the test images. Column (a) Noisy and blurred images; Column (b) Restored images by DF-PRPMHS; Column (c) Restored images by IST and Column (d) Restored images by PSGM.

TABLE 1. Comparison result of DF-PRPMHS, IST, PSGM in terms of SNR, PSNR and SSIM.

IMAGES	DF-PRPMHS			IST			PSGM		
	SNR	PSNR	SSIM	SNR	PSNR	SSIM	SNR	PSNR	SSIM
Image 1	17.67	23.01	0.922	16.72	22.06	0.913	16.99	22.33	0.913
Image 2	21.67	23.50	0.920	20.92	22.76	0.914	21.06	22.90	0.911
Image 3	17.94	20.23	0.791	16.76	19.05	0.766	17.04	19.33	0.760
Image 4	16.42	21.52	0.779	15.55	20.66	0.751	15.55	20.66	0.749
Image 5	16.21	21.37	0.855	14.55	19.70	0.817	15.00	20.15	0.826
Image 6	14.31	20.73	0.652	13.65	20.07	0.628	13.79	20.21	0.630
Image 7	20.42	23.32	0.942	18.88	21.78	0.931	19.60	22.51	0.934

with

$$d_t = -\psi(v_t) + \beta_t d_t, \quad d_0 = -\psi(v_0), \tag{4}$$

where β_t is a suitable scalar parameter known as the conjugate gradient parameter and α_t is a positive stepsize determined by a line search.

Recently, Zhang, Zhou and Li [29] proposed a three term Polak-Ribiére-Polyak (TTPRP) conjugate gradient method with its search direction defined as follows

$$d_t = \begin{cases} -\psi(v_0) & \text{if } t=0, \\ -\psi(v_t) + \dfrac{\psi(v_t)^T y_{t-1}^*}{\|\psi(v_{t-1})\|^2} d_{t-1} - \dfrac{\psi(v_t)^T d_{t-1}}{\|\psi(v_{t-1})\|^2} y_{t-1}^* & \text{if } t>0, \end{cases} \tag{5}$$

where $y_{t-1}^* = \psi(v_t) - \psi(v_{t-1})$. Note, throughout, $\|\cdot\|$ denotes the Euclidean norm.

It is easy to see that the above search direction generated by (5) satisfy

$$\psi(v_t)^T d_t = -\|\psi(v_t)\|^2. \tag{6}$$

Thus, this implies that d_t provides a sufficient descent direction of ϕ at v_t. Zhang *et al.* [30] proposed the TTHS method. The search direction of the TTHS method has the following form

$$d_t = \begin{cases} -\psi(v_0) & \text{if } t=0, \\ -\psi(v_t) + \dfrac{\psi(v_t)^T y_{t-1}^*}{d_{t-1}^T y_{t-1}^*} d_{t-1} + \dfrac{\psi(v_t)^T d_{t-1}}{d_{t-1}^T y_{t-1}^*} y_{t-1}^* & \text{if } t\geq 1, \end{cases} \tag{7}$$

TABLE 2. Numerical results of Problem 1.

		DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
DIM	INTP	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
	v_1	1	3	0.010073	0.00E+00	22	88	0.024701	6.84E-07	9	36	0.028894	7.51E-07	11	44	0.04855	8.26E-07
	v_2	1	3	0.003513	0.00E+00	23	92	0.016499	6.12E-07	10	40	0.021791	1.70E-07	11	44	0.028804	8.12E-07
	v_3	1	3	0.003414	0.00E+00	24	96	0.019819	9.84E-07	10	40	0.013454	4.54E-07	7	28	0.013362	3.12E-07
1000	v_4	1	3	0.005125	0.00E+00	2	7	0.011458	0.00E+00	10	40	0.036765	7.47E-07	2	7	0.0057718	0
	v_5	1	3	0.003123	0.00E+00	2	7	0.003968	0.00E+00	10	40	0.015243	7.67E-07	2	7	0.012495	0
	v_6	1	3	0.00221	0.00E+00	25	100	0.018409	9.60E-07	11	44	0.029141	1.74E-07	13	52	0.011659	9.71E-07
	v_7	37	110	0.047081	7.81E-06	20	80	0.024787	7.73E-07	12	48	0.012918	4.08E-07	8	32	0.006983	2.58E-07
	v_1	1	3	0.016569	0.00E+00	23	92	0.077414	7.62E-07	10	40	0.05079	2.28E-07	14	56	0.035417	7.05E-07
	v_2	1	3	0.005611	0.00E+00	24	96	0.050714	6.81E-07	10	40	0.031394	3.60E-07	12	48	0.018983	9.38E-07
	v_3	1	3	0.004081	0.00E+00	25	100	0.041171	5.46E-07	10	40	0.05871	9.88E-07	7	28	0.026952	3.11E-07
5000	v_4	1	3	0.018386	0.00E+00	2	7	0.016275	0.00E+00		44	0.034446	2.21E-07	2	7	0.0042354	0
	v_5	1	3	0.008673	0.00E+00	2	7	0.032668	0.00E+00	11	44	0.067808	2.19E-07	2	7	0.0063592	0
	v_6	1	3	0.026969	0.00E+00	26	104	0.06676	5.32E-07	11	44	0.047885	3.88E-07	14	56	0.030277	9.08E-07
	v_7	39	116	0.086693	9.29E-06	23	92	0.052185	8.60E-07	12	48	0.042489	7.16E-07	8	32	0.020866	6.47E-07
	v_1	1	3	0.016008	0.00E+00	23	92	0.12187	5.38E-07	10	40	0.061218	3.21E-07	13	52	0.085451	6.12E-07
	v_2	1	3	0.009676	0.00E+00	24	96	0.10411	9.62E-07	10	40	0.11608	5.06E-07	12	48	0.090406	9.67E-07
	v_3	1	3	0.008518	0.00E+00	25	100	0.068306	7.71E-07	11	44	0.085508	1.94E-07	7	28	0.019895	3.11E-07
10000	v_4	1	3	0.03232	0.00E+00	2	7	0.017956	0.00E+00	11	44	0.075455	3.10E-07	2	7	0.0095236	0
	v_5	1	3	0.042558	0.00E+00	2	7	0.014646	0	11	44	0.064672	3.06E-07	2	7	0.0076368	0
	v_6	1	3	0.06953	0.00E+00	26	104	0.081514	7.50E-07	11	44	0.092137	5.49E-07	14	56	0.060192	7.72E-07
	v_7	41	122	0.31205	6.32E-06	23	92	0.10182	6.09E-07	13	52	0.11703	1.56E-07	8	32	0.028621	9.05E-07
	v_1	1	3	0.055158	0.00E+00	25	100	0.42186	6.01E-07	10	40	0.33588	7.16E-07	13	52	0.2136	8.53E-07
	v_2	1	3	0.034905	0.00E+00	25	100	0.41859	5.37E-07	11	44	0.40982	1.57E-07	14	56	0.23162	7.12E-07
	v_3	1	3	0.048703	0.00E+00	24	96	0.56779	8.61E-07	11	44	0.32748	4.32E-07	7	28	0.12214	3.11E-07
50000	v_4	1	3	0.037992	0.00E+00	2	7	0.067548	0.00E+00	11	44	0.31949	6.90E-07	2	7	0.02804	0
	v_5	1	3	0.057224	0.00E+00	2	7	0.065899	0.00E+00	11	44	0.29108	6.77E-07	2	7	0.02752	0
	v_6	1	3	0.047952	0.00E+00	26	104	0.44943	8.37E-07	12	48	0.36737	1.71E-07	15	60	0.22846	7.83E-07
	v_7	43	128	0.78007	7.18E-06	24	96	0.51976	6.82E-07	13	52	0.3576	3.20E-07	9	36	0.22834	2.02E-07
	v_1	1	3	0.19244	0.00E+00	22	88	0.84541	8.50E-07	11	44	0.72651	1.41E-07	12	48	0.37332	7.66E-07
	v_2	1	3	0.056838	0.00E+00	24	96	0.97088	7.59E-07	11	44	0.63059	2.21E-07	14	56	0.43313	7.41E-07
	v_3	1	3	0.081102	0.00E+00	26	104	0.9443	6.08E-07	11	44	0.59013	6.10E-07	7	28	0.17625	3.11E-07
100000	v_4	1	3	0.060561	0.00E+00	2	7	0.20249	0.00E+00	11	44	0.68902	9.75E-07	2	7	0.093001	0
	v_5	1	3	0.083738	0.00E+00	2	7	0.098485	0.00E+00	11	44	0.64638	9.56E-07	2	7	0.070794	0
	v_6	1	3	0.10396	0.00E+00	27	108	0.84673	5.91E-07	12	48	0.87901	2.41E-07	16	64	0.42663	5.58E-07
	v_7	45	134	1.7766	5.03E-06	23	92	0.70564	9.63E-07	13	52	0.70869	4.50E-07	9	36	0.38091	2.87E-07

It is also evident that the search direction defined by TTHS method also satisfies (6). In both work, that is [29] and [30], numerical results indicates that both methods are efficient and outperforms the CG_DESCENT method [33].

Motivated by the good practical behaviour of TTPRP and TTHS, as an attempt, we extend the TTPRP and TTHS method to solve (1). We consider the search direction d_t^* (denotes d_t determined by (5)) and d_t^{**} (denotes d_t determined by (7)), a line combination

$$d_t = (1 - \lambda_t)d_t^* + \lambda_t d_t^{**}, \qquad (8)$$

where $\{\lambda_t\}$ is a bounded sequence. The direction (8) can be rewritten as

$$d_t := \begin{cases} -\psi(v_0), & \text{if } t = 0, \\ -\psi(v_t) + (1 - \lambda_t)\left(\beta_t^{*PRP}d_{t-1} - \eta_t^* y_{t-1}^*\right) \\ \quad + \lambda_t\left(\beta_t^{*HS}d_{t-1} - \theta_t^* y_{t-1}^*\right), & \text{if } t \geq 1, \end{cases} \qquad (9)$$

where

$$\beta_t^{*PRP} := \frac{\psi(v_t)^T y_{t-1}^*}{\|\psi(v_{t-1})\|^2}, \quad \eta_t^* := \frac{\psi(v_t)^T d_{t-1}}{\|\psi(v_{t-1})\|^2},$$

$$\beta_t^{*HS} := \frac{\psi(v_t)^T y_{t-1}^*}{d_{t-1}^T y_{t-1}^*}, \quad \theta_t^* := \frac{\psi(v_t)^T d_{t-1}}{d_{t-1}^T y_{t-1}^*}.$$

We construct the search direction with the form (9) only from theoretical point of view. Observe that if we set $\lambda_t = 0$, then we get the TTPRP method, while $\lambda_t = 1$ yields the TTHS method. In the following, we focus on solving (1). Projection procedure systematic generates a sequence of iterates $\{x_t\}$ by

$$x_t = v_t + \alpha_t d_t, \quad t = 0, 1, 2 \cdots \qquad (10)$$

where α_t is the stepsize determined by a line search procedure that is later described and the search direction d_t has the following form

$$d_t := \begin{cases} -\varphi_t, & \text{if } t = 0, \\ -\varphi_t + (1 - \lambda_t)(\beta_t^{PRP}d_{t-1} - \eta_t y_{t-1}) \\ \quad + \lambda_t(\beta_t^{HS}d_{t-1} - \theta_t y_{t-1}), & \text{if } t \geq 1, \end{cases} \qquad (11)$$

where $\varphi_t = \varphi(v_t)$ for simplicity and

$$\beta_t^{PRP} := \frac{\varphi_t^T y_{t-1}}{\|\varphi_{t-1}\|^2}, \quad \eta_t := \frac{\varphi_t^T d_{t-1}}{\|\varphi_{t-1}\|^2},$$

$$\beta_t^{HS} := \frac{\varphi_t^T y_{t-1}}{d_{t-1}^T y_{t-1}}, \quad \theta_t := \frac{\varphi_t^T d_{t-1}}{d_{t-1}^T y_{t-1}}, \qquad (12)$$

with y_{t-1} defined as

$$y_{t-1} = \varphi_t - \varphi_{t-1}.$$

TABLE 3. Numerical results of Problem 2.

		DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
DIM	INTP	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
1000	v_1	5	13	0.028472	5.13E-06	9	27	0.011283	1.46E-09	8	32	0.007491	5.11E-07	12	48	0.0076633	7.89E-07
	v_2	5	13	0.006155	6.26E-06	9	27	0.005424	2.64E-08	9	36	0.013574	1.43E-07	11	44	0.0064803	8.84E-07
	v_3	5	13	0.006229	5.62E-06	11	33	0.014422	1.96E-08	9	35	0.006776	4.77E-07	12	47	0.013827	5.10E-07
	v_4	6	15	0.008135	5.94E-06	6	18	0.006724	2.92E-08	10	38	0.009377	3.06E-07	11	43	0.006318	6.20E-07
	v_5	5	13	0.008878	6.20E-06	10	30	0.00937	4.24E-08	10	38	0.009646	6.98E-07	9	35	0.0060223	1.33E-07
	v_6	7	17	0.007991	3.75E-06	12	36	0.009597	8.81E-09	11	41	0.015597	1.13E-07	11	43	0.0046367	6.12E-07
	v_7	36	103	0.036057	8.81E-06	15	53	0.014706	1.53E-09	10	39	0.005671	1.33E-07	10	39	0.014306	1.39E-07
5000	v_1	5	14	0.016744	2.18E-06	9	27	0.02296	6.45E-10	9	36	0.039752	1.09E-07	13	52	0.050436	8.50E-07
	v_2	5	14	0.025025	2.65E-06	9	27	0.022277	1.18E-08	9	36	0.024556	3.04E-07	12	48	0.063306	9.48E-07
	v_3	6	15	0.015498	2.31E-06	11	33	0.024348	8.92E-09	10	39	0.10346	1.01E-07	13	51	0.052607	5.86E-07
	v_4	6	16	0.014158	2.53E-06	6	18	0.012048	9.62E-09		38	0.026256	6.45E-07	12	47	0.039519	7.68E-07
	v_5	6	15	0.025382	3.12E-06	10	30	0.022914	1.78E-09	11	42	0.027458	1.48E-07	9	35	0.019501	2.69E-07
	v_6	7	17	0.024037	7.82E-06	12	36	0.021027	3.89E-09	11	41	0.021743	2.36E-07	12	47	0.024373	5.79E-07
	v_7	28	79	0.11372	6.49E-06	16	57	0.10587	7.63E-10	10	39	0.023205	2.74E-07	10	39	0.029167	2.94E-07
10000	v_1	5	14	0.072678	3.05E-06	9	28	0.030024	4.56E-10	9	36	0.034568	1.53E-07	14	56	0.047082	5.98E-07
	v_2	5	14	0.02103	3.72E-06	9	28	0.052354	8.37E-09	9	36	0.042186	4.27E-07	13	52	0.22773	6.67E-07
	v_3	6	15	0.047529	3.23E-06	11	34	0.034589	6.33E-09	10	39	0.080993	1.42E-07	13	51	0.045266	8.31E-07
	v_4	6	16	0.070049	3.55E-06	6	19	0.025472	6.54E-09	10	38	0.042661	9.05E-07	13	51	0.10576	5.50E-07
	v_5	6	15	0.18016	4.47E-06	10	31	0.044776	1.25E-09	11	42	0.067389	2.07E-07	9	35	0.039509	3.75E-07
	v_6	7	18	0.067249	2.20E-06	12	37	0.043637	2.75E-09	11	41	0.046018	3.31E-07	12	47	0.053134	8.00E-07
	v_7	25	70	0.19509	8.10E-06	14	55	0.063924	9.02E-07	10	39	0.049066	3.85E-07	10	39	0.043841	4.19E-07
50000	v_1	5	14	0.12845	6.77E-06	12	41	0.2147	6.36E-07	9	36	0.16193	3.41E-07	15	60	0.20056	6.66E-07
	v_2	5	14	0.11344	8.25E-06	16	57	0.25868	7.28E-07	9	36	0.16493	9.50E-07	14	56	0.2726	7.42E-07
	v_3	6	15	0.18327	7.16E-06	18	63	0.32137	5.52E-07	10	39	0.17841	3.16E-07	14	55	0.19537	9.32E-07
	v_4	6	16	0.12998	7.88E-06	13	48	0.16675	5.51E-07	11	42	0.18211	2.01E-07	14	55	0.29152	6.20E-07
	v_5	6	16	0.13091	2.02E-06	14	48	0.2804	8.64E-07	11	42	0.16367	4.60E-07	9	35	0.19281	8.30E-07
	v_6	7	18	0.12876	4.87E-06	17	58	0.41135	9.59E-07	11	41	0.18786	7.34E-07	13	51	0.20148	8.79E-07
	v_7	25	70	0.6353	2.85E-06	15	59	0.42675	5.04E-07	10	39	0.20388	8.61E-07	10	39	0.22968	9.55E-07
100000	v_1	5	14	0.20477	9.56E-06	12	41	0.55756	9.00E-07	9	36	0.30808	4.82E-07	15	60	0.451	9.42E-07
	v_2	6	16	0.26645	2.33E-06	17	61	0.67015	5.15E-07	10	40	0.3011	1.34E-07	15	60	0.40862	5.25E-07
	v_3	6	16	0.25383	2.02E-06	18	63	0.55212	7.80E-07	10	39	0.41353	4.47E-07	15	59	0.50259	6.59E-07
	v_4	7	18	0.2986	2.23E-06	13	48	0.4546	7.75E-07	11	42	0.405	2.84E-07	14	55	0.39186	8.78E-07
	v_5	6	16	0.20882	2.86E-06	15	52	0.58487	6.10E-07	11	42	0.34046	6.51E-07	10	39	0.28876	1.17E-07
	v_6	7	18	0.35029	6.88E-06	18	62	0.5441	6.78E-07	12	45	0.357	1.04E-07	14	55	0.52929	6.20E-07
	v_7	21	58	1.0712	6.21E-06	15	59	0.66119	7.12E-07	11	43	0.38562	1.21E-07	11	43	0.3747	1.32E-07

In order to ensure boundedness of the proposed direction (11), motivated by the idea of Li and Fukushima [35], we introduced the following modification. Define u_{t-1} by

$$u_{t-1} := y_{t-1} + j_{t-1} d_{t-1}, \quad j_{t-1} := 1 + \max\left\{0, -\frac{d_{t-1}^T y_{t-1}}{d_{t-1}^T d_{t-1}}\right\}. \quad (13)$$

Based on the above, we next describe our algorithm for solving the nonlinear equation (1). But first, we recall the fundamental concept and property of the projection operator. The operator of the projection P_Ω defined as a mapping from $\mathbb{R}^n$ to a nonempty closed convex set Ω, that is

$$P_\Omega[u] := \arg\min\{\|u - y\|, \ y \in \Omega\}, \quad \forall u \in \mathbb{R}^n.$$

The projection operator P_Ω has a well-known property, that is, for any $u, y \in \mathbb{R}^n$ the following nonexpansive property hold

$$\|P_\Omega[u] - P_\Omega[y]\| \le \|u - y\|. \quad (14)$$

Algorithm 1:
Input. Choose any arbitrary initial point $v_0 \in \Omega$, the positive constants: $Tol \in (0, 1)$, $\varrho \in (0, 1)$, $\zeta \in (0, 1)$, $\varsigma > 0$, $\lambda > 0$, $\tau \in (0, 2)$. Set $t := 0$.

Step 1. If $\|\varphi_t\| = 0$, stop. Otherwise go to step 2.
Step 2. Compute the search direction d_t by the following formula:

$$d_t := \begin{cases} -\varphi_t, & \text{if } t = 0, \\ -\varphi_t + (1 - \lambda_t)(\beta_t^{PRP} d_{t-1} - \eta_t y_{t-1}) \\ \quad + \lambda_t(\beta_t^{MHS} d_{t-1} - \theta_t^M y_{t-1}), & \text{if } t \ge 1, \end{cases} \quad (15)$$

where

$$\beta_t^{MHS} := \frac{\varphi_t^T y_{t-1}}{d_{t-1}^T u_{t-1}}, \quad \theta_t^M = \frac{\varphi_t^T d_{t-1}}{d_{t-1}^T u_{t-1}}$$

and β_t^{PRP}, η_t are defined as in (12).
Step 3. Determine the step-size $\alpha_t = \zeta \varrho^m$ where m is the smallest non-negative integer such that the following line search is satisfied:

$$-\varphi(v_t + \alpha_t d_t)^T d_t \ge \varsigma \alpha_t \xi_t \|d_t\|^2 \quad (16)$$

where ξ_t is defined as

$$\xi_t := \mu_t + (1 - \mu_t)\|\varphi(v_t + \alpha_t d_t)\| \quad (17)$$

with $\mu_t \in [\mu_{min}, \mu_{max}] \subseteq (0, 1]$.
Step 4. Compute

$$x_t := v_t + \alpha_t d_t. \quad (18)$$

TABLE 4. Numerical results of Problem 3.

DIM	INTP	DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
		NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
1000	v_1	4	12	0.0195	7.77E-06	15	60	0.020358	7.50E-07	8	32	0.003717	3.11E-07	11	44	0.0071504	9.73E-07
	v_2	5	14	0.007512	2.72E-06	15	60	0.01189	7.38E-07	8	32	0.005901	5.94E-07	12	48	0.0078968	9.29E-07
	v_3	4	11	0.006345	2.51E-06	16	64	0.011442	8.21E-07	9	36	0.011847	2.45E-07	11	44	0.0039543	6.21E-07
	v_4	NaN	NaN	NaN	NaN	16	64	0.017987	7.43E-07	9	36	0.008658	4.00E-07	NaN	NaN	NaN	NaN
	v_5	11	33	0.019622	6.74E-06	17	68	0.011745	9.79E-07	9	36	0.010162	5.58E-07	NaN	NaN	NaN	NaN
	v_6	11	33	0.009016	6.74E-06	17	68	0.011196	9.79E-07	10	40	0.006997	1.94E-07	NaN	NaN	NaN	NaN
	v_7	25	74	0.012704	6.16E-06	15	60	0.011265	7.70E-07	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
5000	v_1	5	14	0.014284	3.48E-06	16	64	0.041267	8.38E-07	8	32	0.011817	6.96E-07	13	52	0.01897	5.44E-07
	v_2	5	14	0.029048	6.08E-06	16	64	0.043485	8.25E-07	9	36	0.014194	1.33E-07	14	56	0.02378	5.19E-07
	v_3	4	11	0.04207	5.62E-06	17	68	0.046319	9.18E-07	9	36	0.013265	5.49E-07	12	48	0.039876	6.95E-07
	v_4	NaN	NaN	NaN	NaN	17	68	0.048831	8.30E-07		36	0.019329	8.95E-07	NaN	NaN	NaN	NaN
	v_5	12	35	0.030517	5.43E-06	19	76	0.026157	5.48E-07	10	40	0.028707	1.25E-07	NaN	NaN	NaN	NaN
	v_6	16	47	0.030287	5.59E-06	19	76	0.031615	5.48E-07	10	40	0.082786	4.34E-07	NaN	NaN	NaN	NaN
	v_7	26	77	0.21158	7.02E-06	16	64	0.061841	8.71E-07	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
10000	v_1	5	14	0.030639	4.92E-06	17	68	0.062809	5.93E-07	8	32	0.027543	9.85E-07	13	52	0.032351	7.69E-07
	v_2	5	14	0.025367	8.60E-06	17	68	0.056048	5.83E-07	9	36	0.024708	1.88E-07	14	56	0.036674	7.34E-07
	v_3	4	11	0.16734	7.95E-06	18	72	0.069021	6.49E-07	9	36	0.037891	7.76E-07	12	48	0.036616	9.82E-07
	v_4	NaN	NaN	NaN	NaN	18	72	0.068903	5.87E-07	10	40	0.033661	1.27E-07	NaN	NaN	NaN	NaN
	v_5	12	35	0.10283	7.67E-06	19	76	0.065008	7.74E-07	10	40	0.040369	1.76E-07	NaN	NaN	NaN	NaN
	v_6	16	47	0.54007	7.90E-06	19	76	0.066727	7.74E-07	10	40	0.036525	6.14E-07	NaN	NaN	NaN	NaN
	v_7	26	77	0.25869	9.93E-06	17	68	0.057077	6.18E-07	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
50000	v_1	5	15	0.10871	2.20E-06	18	72	0.34899	6.63E-07	9	36	0.10047	2.20E-06	14	56	0.14704	8.60E-07
	v_2	5	15	0.13798	3.85E-06	18	72	0.22053	6.52E-07	9	36	0.1017	4.20E-07	15	60	0.15674	8.21E-07
	v_3	4	12	0.27421	3.56E-06	19	76	0.27841	7.26E-07	10	40	0.21461	1.74E-07	14	56	0.25992	5.49E-07
	v_4	NaN	NaN	NaN	NaN	19	76	0.47431	6.56E-07	10	40	0.11346	2.83E-07	NaN	NaN	NaN	NaN
	v_5	13	38	0.16408	3.98E-06	20	80	0.3508	8.66E-07	10	40	0.13782	3.94E-07	NaN	NaN	NaN	NaN
	v_6	17	50	0.21896	6.81E-06	20	80	0.51362	8.66E-07	11	44	0.15692	1.37E-07	NaN	NaN	NaN	NaN
	v_7	27	81	0.43177	9.73E-06	18	72	0.24868	6.88E-07	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
100000	v_1	5	15	0.15733	3.11E-06	18	72	0.53097	9.37E-07	9	36	0.19232	3.11E-07	15	60	0.35619	6.08E-07
	v_2	5	15	0.15217	5.44E-06	18	72	0.66656	9.22E-07	9	36	0.24999	5.94E-07	16	64	0.32524	5.80E-07
	v_3	4	12	0.11496	5.03E-06	20	80	0.51763	5.13E-07	10	40	0.23066	2.45E-07	14	56	0.37236	7.77E-07
	v_4	NaN	NaN	NaN	NaN	19	76	0.54138	9.28E-07	10	40	0.22558	4.00E-07	NaN	NaN	NaN	NaN
	v_5	13	38	0.31724	5.63E-06	21	84	0.63587	6.12E-07	10	40	0.262	5.58E-07	NaN	NaN	NaN	NaN
	v_6	17	50	0.42843	9.63E-06	21	84	0.90363	6.12E-07	11	44	0.29401	1.94E-07	NaN	NaN	NaN	NaN
	v_7	28	83	1.0339	8.12E-06	18	72	0.57703	9.75E-07	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN

Step 5. If $x_t \in \Omega$ and $\varphi(x_t) = 0$, stop. Otherwise, compute the next iterate by

$$v_{t+1} := P_\Omega[v_t - \tau \rho_t \varphi(x_t)], \quad (19)$$

where

$$\rho_t := \frac{\varphi(x_t)^T (v_t - x_t)}{\|\varphi(x_t)\|^2} \quad (20)$$

Step 6. Finally we set $t := t + 1$ and return to step 1.

Lemma 2.1: Let the direction $\{d_t\}$ be generated by (15), it holds that for every $t \geq 0$,

$$\varphi_t^T d_t \leq -\|\varphi_t\|^2. \quad (21)$$

Proof: Remember, λ_t is a bunded sequence. For $t = 0$, (21) obviously holds. For $t \in \mathbb{N}$, we have

$$
\begin{aligned}
\varphi_t^T d_t &= -\|\varphi_t\|^2 + (1 - \lambda_t)(\beta_t^{PRP} \varphi_t^T d_{t-1} - \eta_t \varphi_t^T y_{t-1}) \\
&\quad + \lambda_t (\beta_t^{MHS} \varphi_t^T d_{t-1} - \theta_t^M \varphi_t^T y_{t-1}) \\
&= -\|\varphi_t\|^2 + \beta_t^{PRP} \varphi_t^T d_{t-1} - \eta_t \varphi_t^T y_{t-1} \\
&\quad - \lambda_t A \beta_t^{PRP} \varphi_t^T d_{t-1} + \lambda_t \eta_t \varphi_t^T y_{t-1} \\
&\quad + \lambda_t \beta_t^{MHS} \varphi_t^T d_{t-1} - \lambda_t \theta_t^M \varphi_t^T y_{t-1} \\
&\leq -\|\varphi_t\|^2 + \beta_t^{PRP} \varphi_t^T d_{t-1} - \eta_t \varphi_t^T y_{t-1} \\
&\quad - \beta_t^{PRP} \varphi_t^T d_{t-1} + \eta_t \varphi_t^T y_{t-1} \\
&\quad + \beta_t^{MHS} \varphi_t^T d_{t-1} - \theta_t^M \varphi_t^T y_{t-1} \\
&= -\|\varphi_t\|^2.
\end{aligned}
$$

∎

III. CONVERGENCE ANALYSIS

To give the convergence result, the following assumptions are required.

Assumption 1:

A1. The solution set of (1), denoted by $Sol_{\varphi,\Omega}$ is nonempty.

A2. The mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is *Lipschitz* continuous on $\mathbb{R}^n$, that is, there exists a constant $L > 0$ such that $\forall u, y \in \mathbb{R}^n$,

$$\|\varphi(u) - \varphi(y)\| \leq L\|u - y\| \quad (22)$$

A3. For any $v \in Sol_{\varphi,\Omega}$ and $w \in \mathbb{R}^n$, it holds that

$$\varphi(w)^T (w - v) \geq 0. \quad (23)$$

The Assumption A3 is obviously a weaker condition than monotonicity.

Lemma 3.1: Let the sequences $\{d_t\}$ and $\{v_t\}$ be generated by the Algorithm 1, then there always exists a step-size α_t satisfying the line search (16).

Proof: Suppose for the sake of contradiction there exist $t_0 \geq 0$ such that the line search (16) fails to hold for any nonnegative integer m, then we have

$$-\varphi(v_{t_0} + \varsigma \varrho^m d_{t_0})^T d_{t_0} < \varsigma \zeta \varrho^m \xi_{t_0} \|d_{t_0}\|^2, \quad \forall m \geq 0.$$

where ξ_{t_0} is defined by (17). It is clear that

$$\mu_{\min} \leq \xi_{t_0} \leq \max\{1, \|\varphi(v_{t_0} + \varsigma \varrho^m d_{t_0})\|\}. \quad (24)$$

TABLE 5. Numerical results of Problem 4.

DIM	INTP	DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
		NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
	v_1	1	2	0.074863	0	2	6	0.006521	0	2	6	0.005022	0	2	6	0.0039037	0
	v_2	1	2	0.003649	0	2	6	0.005736	0	2	6	0.002006	0	2	6	0.0028103	0
	v_3	1	2	0.004688	0	2	6	0.00276	0	2	6	0.004177	0	2	6	0.0030956	0
1000	v_4	1	3	0.005578	0	2	7	0.002215	0	2	7	0.003751	0	2	7	0.0021806	0
	v_5	1	3	0.004596	0	2	7	0.006439	0	2	7	0.002943	0	2	7	0.00286	0
	v_6	1	3	0.004868	0	2	7	0.001862	0	2	7	0.005061	0	2	7	0.0028707	0
	v_7	5	14	0.010955	3.59E-06	14	54	0.022683	4.63E-07	2	7	0.003093	0	2	7	0.0036899	0
	v_1	1	2	0.011777	0	2	6	0.010759	0	2	6	0.009086	0	2	6	0.0063684	0
	v_2	1	2	0.010914	0	2	6	0.009879	0	2	6	0.012934	0	2	6	0.0092309	0
	v_3	1	2	0.03201	0	2	6	0.019781	0	2	6	0.016168	0	2	6	0.0076956	0
5000	v_4	1	3	0.013617	0	2	7	0.010955	0		7	0.011516	0	2	7	0.014152	0
	v_5	1	3	0.011347	0	2	7	0.010715	0	2	7	0.010733	0	2	7	0.0079994	0
	v_6	1	3	0.009895	0	2	7	0.014082	0	2	7	0.011975	0	2	7	0.0066717	0
	v_7	5	14	0.042803	8.01E-06	14	54	0.19994	8.71E-07	2	7	0.010421	0	2	7	0.0068827	0
	v_1	1	2	0.02146	0	2	6	0.021459	0	2	6	0.013562	0	2	6	0.01488	0
	v_2	1	2	0.015462	0	2	6	0.02758	0	2	6	0.01225	0	2	6	0.024702	0
	v_3	1	2	0.01271	0	2	6	0.079126	0	2	6	0.012118	0	2	6	0.012924	0
10000	v_4	1	3	0.010691	0	2	7	0.060033	0	2	7	0.015849	0	2	7	0.015462	0
	v_5	1	3	0.011942	0	2	7	0.024961	0	2	7	0.028817	0	2	7	0.01553	0
	v_6	1	3	0.023173	0	2	7	0.020561	0	2	7	0.019245	0	2	7	0.01372	0
	v_7	6	17	0.10081	2.86E-06	15	58	0.25815	3.21E-07	2	7	0.015821	0	2	7	0.036414	0
	v_1	1	2	0.073736	0	2	6	0.11555	0	2	6	0.059885	0	2	6	0.050665	0
	v_2	1	2	0.052138	0	2	6	0.090774	0	2	6	0.05973	0	2	6	0.048581	0
	v_3	1	2	0.04503	0	2	6	0.082987	0	2	6	0.064963	0	2	6	0.058322	0
50000	v_4	1	3	0.048013	0	2	7	0.2079	0	2	7	0.077432	0	2	7	0.05689	0
	v_5	1	3	0.073447	0	2	7	0.076271	0	2	7	0.084659	0	2	7	0.083642	0
	v_6	1	3	0.083598	0	2	7	0.071039	0	2	7	0.06736	0	2	7	0.090252	0
	v_7	6	17	0.3783	6.39E-06	15	58	0.71041	7.08E-07	2	7	0.080476	0	2	7	0.066687	0
	v_1	1	2	0.13639	0	2	6	0.13429	0	2	6	0.10702	0	2	6	0.10005	0
	v_2	1	2	0.16168	0	2	6	0.13923	0	2	6	0.10869	0	2	6	0.13601	0
	v_3	1	2	0.13104	0	2	6	0.16706	0	2	6	0.12512	0	2	6	0.12575	0
100000	v_4	1	3	0.13001	0	2	7	0.15905	0	2	7	0.22447	0	2	7	0.15008	0
	v_5	1	3	0.11255	0	2	7	0.15347	0	2	7	0.13233	0	2	7	0.098322	0
	v_6	1	3	0.11251	0	2	7	0.09515	0	2	7	0.14023	0	2	7	0.11529	0
	v_7	6	17	0.71986	9.04E-06	15	58	0.95398	9.80E-07	2	7	0.15204	0	2	7	0.11711	0

By continuity of φ and setting $m \to \infty$ yields

$$-\varphi_{t_0}^T d_{t_0} \leq 0,$$

which contradicts (21). Hence, proved. ∎

Lemma 3.2: Suppose the mapping φ is Lipschitz continuous and the sequences $\{v_t\}$ and $\{x_t\}$ are generated by Algorithm 1, then

$$\alpha_t \geq \max\left\{\zeta, \frac{\varrho\|\varphi_t\|^2}{\left[L + \varsigma\left(\mu_t + (1-\mu_t)\|\varphi(v_t + \frac{\alpha_t}{\varrho}d_t\|)\right)\right]\|d_t\|^2}\right\} \tag{25}$$

Proof: As we can see, from the line search procedure (16), if $\alpha_t \neq \zeta$, then $\varrho^{-1}\alpha_t$ fail to hold for the line search procedure. That is,

$$-\varphi(v_t + \frac{\alpha_t}{\varrho}d_t)^T d_t \geq \varsigma \frac{\alpha_t}{\varrho}(\mu_t + (1-\mu_t)\|\varphi(v_t + \frac{\alpha_t}{\varrho}d_t)\|) \cdot \|d_t\|^2 \tag{26}$$

From the sufficient descent condition (21) and Lipschitz continuity, we have

$$\|\varphi_t\|^2 = -\varphi_t^T d_t$$
$$= (\varphi(v_t + \frac{\alpha_t}{\varrho}d_t) - \varphi_t)^T d_t - \varphi(v_t + \frac{\alpha_t}{\varrho}d_t)^T d_t$$
$$\leq \frac{\alpha_t}{\varrho}\left[L + \varsigma\left(\mu_t + (1-\mu_t)\|\varphi(v_t + \frac{\alpha_t}{\varrho}d_t\|)\right)\right]\|d_t\|^2$$

This yields the desired inequality (25). ∎

Lemma 3.3: Suppose that Assumption 1 hold. Let the sequences $\{v_t\}$ and $\{x_t\}$ be generated by Algorithm 1, then for any solution $v^* \in Sol_{\varphi,\Omega}$, it holds that

$$\|v_{t+1} - v^*\|^2 \leq \|v_t - v^*\|^2 - \tau(2-\tau)\frac{\varsigma^2\mu_{\min}^2\|v_t - x_t\|^4}{\|\varphi(x_t)\|^2} \tag{27}$$

Moreover, the sequence $\{v_t\}$ is bounded and

$$\sum_{t=0}^{\infty}\|v_t - x_t\|^4 < +\infty. \tag{28}$$

Proof: Using the weaker condition of monotonicity given by Assumption A3, we have

$$\varphi(x_t)^T(v_t - v^*) = \varphi(x_t)^T(v_t - x_t + x_t - v^*)$$
$$= \varphi(x_t)^T(v_t - x_t) + \varphi(x_t)^T(x_t - v^*)$$
$$\geq \varphi(x_t)^T(v_t - x_t)$$
$$\geq \varsigma\alpha_t^2\xi_t\|d_t\|^2. \tag{29}$$

Since for all t, it holds that $\xi_t \geq \mu_{min} > 0$, we have

$$\varphi(x_t)^T(v_t - x_t) \geq \varsigma\mu_{\min}\alpha_t^2\|d_t\|^2$$
$$= \varsigma\mu_{\min}\|v_t - x_t\|^2. \tag{30}$$

TABLE 6. Numerical results of Problem 5.

DIM	INTP	DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
		NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
1000	v_1	1	3	0.001995	0	15	60	0.008299	6.10E-07	8	32	0.005433	4.34E-07	11	44	0.0095265	5.14E-07
	v_2	1	3	0.00179	0	15	60	0.006156	9.56E-07	9	36	0.007017	1.16E-07	10	40	0.0047728	7.26E-07
	v_3	7	21	0.005897	8.75E-06	14	56	0.008782	7.90E-07	9	36	0.007273	3.52E-07	2	7	0.0031284	0
	v_4	1	3	0.002889	0	2	7	0.003814	0	10	40	0.008688	1.16E-07	11	44	0.0074698	7.86E-07
	v_5	1	3	0.002157	0	16	64	0.011235	9.43E-07	9	36	0.007961	5.44E-07	2	7	0.003286	0
	v_6	1	3	0.002553	0	2	7	0.002768	0	10	40	0.00375	1.16E-07	2	7	0.0019306	0
	v_7	27	81	0.016706	9.56E-06	14	56	0.008205	6.19E-07	16	64	0.012608	1.30E-07	8	32	0.0027802	1.42E-07
5000	v_1	1	3	0.015552	0	16	64	0.056854	6.82E-07	8	32	0.012332	9.70E-07	12	48	0.041417	5.75E-07
	v_2	1	3	0.009094	0	17	68	0.078564	5.34E-07	9	36	0.012981	2.60E-07	11	44	0.019252	8.12E-07
	v_3	8	23	0.10546	7.05E-06	15	60	0.19805	8.83E-07	9	36	0.013092	7.88E-07	2	7	0.0073662	0
	v_4	1	3	0.009562	0	2	7	0.009469	0		40	0.016081	2.60E-07	12	48	0.022726	8.79E-07
	v_5	1	3	0.017138	0	18	72	0.056108	5.27E-07	10	40	0.054332	1.22E-07	2	7	0.005461	0
	v_6	1	3	0.014082	0	2	7	0.020765	0	10	40	0.061402	2.60E-07	2	7	0.0048772	0
	v_7	29	87	0.047409	6.15E-06	15	60	0.02169	7.04E-07	22	88	0.085457	3.65E-07	8	32	0.0082204	4.57E-07
10000	v_1	1	3	0.027179	0	16	64	0.058381	9.65E-07	9	36	0.023934	1.37E-07	12	48	0.046548	8.13E-07
	v_2	1	3	0.0093	0	17	68	0.16688	7.55E-07	9	36	0.060715	3.68E-07	12	48	0.025783	5.74E-07
	v_3	8	23	0.033206	9.97E-06	16	64	0.054015	6.25E-07	10	40	0.02664	1.11E-07	2	7	0.0058916	0
	v_4	1	3	0.004967	0	2	7	0.014263	0	10	40	0.045663	3.68E-07	13	52	0.053098	6.21E-07
	v_5	1	3	0.00975	0	18	72	0.044269	7.45E-07	10	40	0.030735	1.72E-07	2	7	0.0051023	0
	v_6	1	3	0.007619	0	2	7	0.014522	0	10	40	0.029281	3.67E-07	2	7	0.0079739	0
	v_7	29	87	0.46085	8.69E-06	16	64	0.12164	5.01E-07	22	88	0.09543	1.15E-07	8	32	0.02006	6.23E-07
50000	v_1	1	3	0.037928	0	18	72	0.28535	5.40E-07	9	36	0.08959	3.07E-07	13	52	0.14449	9.09E-07
	v_2	1	3	0.027635	0	18	72	0.18347	8.45E-07	9	36	0.10576	8.23E-07	13	52	0.14779	6.42E-07
	v_3	9	26	0.089056	5.17E-06	17	68	0.38597	6.98E-07	10	40	0.1357	2.49E-07	2	7	0.017823	0
	v_4	1	3	0.017842	0	2	7	0.068248	0	10	40	0.18037	8.24E-07	14	56	0.24214	6.95E-07
	v_5	1	3	0.093737	0	19	76	0.25401	8.33E-07	10	40	0.12466	3.85E-07	2	7	0.034012	0
	v_6	1	3	0.060438	0	2	7	0.10326	0	10	40	0.11899	8.21E-07	2	7	0.035467	0
	v_7	31	93	0.31657	5.63E-06	17	68	0.15209	5.59E-07	30	120	0.94584	7.43E-07	9	36	0.099156	1.37E-07
100000	v_1	1	3	0.023177	0	18	72	0.44351	7.63E-07	9	36	0.15384	4.34E-07	14	56	0.3345	6.43E-07
	v_2	1	3	0.031459	0	19	76	0.50585	5.97E-07	10	40	0.21462	1.16E-07	13	52	0.22885	9.08E-07
	v_3	9	26	0.28657	7.31E-06	17	68	0.41641	9.88E-07	10	40	0.18216	3.52E-07	2	7	0.08852	0
	v_4	23	68	0.84807	7.12E-06	2	7	0.10953	0	11	44	0.21101	1.16E-07	14	56	0.28189	9.83E-07
	v_5	1	3	0.068657	0	20	80	0.66002	5.89E-07	10	40	0.24138	5.44E-07	2	7	0.047969	0
	v_6	1	3	0.049145	0	2	7	0.12864	0	11	44	0.32338	1.16E-07	2	7	0.048706	0
	v_7	31	93	0.95636	7.96E-06	17	68	0.52151	7.89E-07	34	136	2.3046	1.15E-07	9	36	0.18216	2.06E-07

Combining the property of the projection operator (14) and (30) we have

$$\|v_{t+1} - v^*\|^2$$
$$= \|P_\Omega[v_t - \tau\rho_t\varphi(x_t)] - v^*\|^2 \tag{31}$$
$$\leq \|v_t - \tau\rho_t\varphi(x_t) - v^*\|^2$$
$$= \|v_t - v^*\|^2 - 2\tau\rho_t\varphi(x_t)^T(v_t - v^*) + \|\tau\rho_t\varphi(x_t)\|^2$$
$$\leq \|v_t - v^*\|^2 - 2\tau\rho_t\varphi(x_t)^T(v_t - x_t) + \tau^2\rho_t^2\|\varphi(x_t)\|^2$$
$$= \|v_t - v^*\|^2 - \tau(2-\tau)\left(\frac{\varphi(x_t)^T(v_t - x_t)}{\|\varphi(x_t)\|}\right)^2$$
$$\leq \|v_t - v^*\|^2 - \tau(2-\tau)\frac{\varsigma^2\mu_{\min}^2\|v_t - x_t\|^4}{\|\varphi(x_t)\|^2} \tag{32}$$

We can infer from the above that the sequence $\{\|v_t - v^*\|\}$ is therefore non-increasing and convergent, therefore $\{v_t\}$ is bounded, that is there exist a positive constant say k_0 such that for all $t \geq 0$,

$$\|v_t\| \leq k_0.$$

Since φ is continuous and $\{v_t\}$ is bounded, then $\{\varphi(v_t)\}$ is bounded. That is, there exist a positive constant k_1 such that for all $t \geq 0$,

$$\|\varphi_t\| \leq k_1. \tag{33}$$

Moreover, by using the boundedness of φ_t, we can deduce that

$$\varphi(x_t)^T(v_t - x_t) = (\varphi(x_t) - \varphi_t)^T(v_t - x_t) + \varphi_t^T(v_t - x_t)$$
$$\leq \|\varphi_t\|\|v_t - x_t\|$$
$$\leq k_1\|v_t - x_t\|.$$

Thus, since from (30), it holds that

$$\varphi(x_t)^T(v_t - x_t) \geq \varsigma\mu_{\min}\|v_t - x_t\|^2,$$

we have

$$\|v_t - x_t\| \leq \frac{k_1}{\varsigma\mu_{\min}}.$$

Then, we obtain,

$$\|x_t\| \leq \frac{k_1}{\varsigma\mu_{\min}} + \|v_t\|$$

Hence the sequence $\{x_t\}$ is bounded owing to the boundedness of $\{v_t\}$. That is, there exist a positive $k_2 > 0$ such that

$$\|\varphi(x_t)\| \leq k_2, \tag{34}$$

and furthermore

$$\tau(2-\tau)\frac{\varsigma^2\mu_{\min}^2}{k_2^2}\sum_{t=0}^{\infty}\|v_t - x_t\|^4 < \|v_0 - v^*\|^2 < +\infty.$$

■

TABLE 7. Numerical results of Problem 6.

DIM	INTP	DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
		NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
1000	v_1	39	111	0.09068	7.67E-06	15	57	0.012746	8.98E-07	18	68	0.0093	1.51E-07	13	50	0.0071644	9.37E-07
	v_2	40	114	0.027908	6.71E-06	17	65	0.018818	6.83E-07	15	56	0.009489	9.45E-07	13	50	0.0040483	6.08E-07
	v_3	41	119	0.041748	8.66E-06	20	76	0.008784	8.94E-07	24	94	0.026747	1.04E-07	13	51	0.0043676	8.63E-07
	v_4	28	83	0.028243	9.19E-06	16	64	0.006333	8.77E-07	24	96	0.025848	8.07E-07	11	44	0.0069371	6.02E-07
	v_5	23	68	0.019744	7.40E-06	17	68	0.0125	5.46E-07	27	108	0.049747	4.93E-07	12	47	0.0061369	6.23E-07
	v_6	31	92	0.042536	7.80E-06	17	68	0.013243	7.29E-07	34	136	0.057405	1.77E-07	10	40	0.0086897	7.75E-07
	v_7	38	110	0.039215	7.88E-06	20	79	0.010819	6.80E-07	33	131	0.044668	6.42E-07	15	59	0.021843	4.06E-07
5000	v_1	41	117	0.26987	9.36E-06	18	68	0.031937	6.13E-07	16	60	0.036532	6.10E-07	15	58	0.025202	7.05E-07
	v_2	42	120	0.078299	7.16E-06	18	69	0.033785	8.13E-07	16	60	0.023039	1.64E-07	14	54	0.02227	8.90E-07
	v_3	43	125	0.11289	7.41E-06	20	76	0.043615	9.77E-07	33	130	0.25558	6.78E-07	12	46	0.021225	9.19E-07
	v_4	30	89	0.21801	6.09E-06	17	68	0.052279	8.55E-07		128	0.21335	1.03E-07	12	48	0.023909	5.79E-07
	v_5	24	72	0.13657	8.74E-06	19	75	0.036298	9.11E-07	36	144	0.3219	1.32E-07	15	59	0.022862	5.80E-07
	v_6	32	96	0.24971	9.63E-06	18	72	0.030302	8.71E-07	54	216	0.42061	5.92E-07	11	44	0.026281	8.48E-07
	v_7	34	96	0.34842	8.23E-06	33	131	0.070274	7.59E-07	47	187	0.47907	5.04E-07	20	79	0.046494	9.00E-07
10000	v_1	42	120	0.45048	9.91E-06	19	72	0.051244	5.65E-07	16	60	0.045622	1.34E-07	17	66	0.040692	6.82E-07
	v_2	42	120	0.51781	9.78E-06	19	73	0.051547	6.03E-07	17	64	0.096441	7.30E-07	16	62	0.038345	8.75E-07
	v_3	45	131	1.1214	4.94E-06	21	80	0.098206	7.24E-07	36	142	0.40818	1.19E-07	14	54	0.037389	5.03E-07
	v_4	30	89	0.67766	8.64E-06	18	72	0.059443	5.93E-07	35	140	0.39971	9.64E-07	12	48	0.040104	8.28E-07
	v_5	25	74	0.24024	8.45E-06	21	82	0.079211	6.36E-07	40	160	0.49824	1.29E-07	14	55	0.04331	8.61E-07
	v_6	33	98	0.35885	9.19E-06	21	84	0.072302	8.60E-07	62	248	1.0633	1.84E-07	12	48	0.033899	6.30E-07
	v_7	64	186	0.42851	8.37E-06	28	111	0.14204	7.02E-07	51	203	0.65008	4.85E-07	18	71	0.066136	8.85E-07
50000	v_1	37	105	1.0326	6.12E-06	29	112	0.565	6.20E-07	24	92	0.73202	9.49E-07	16	61	0.29376	8.88E-07
	v_2	46	132	0.74414	4.85E-06	21	80	0.28271	7.49E-07	17	64	0.26491	2.95E-07	16	61	0.18496	7.58E-07
	v_3	47	137	0.70547	5.22E-06	23	87	0.50013	9.29E-07	47	186	2.3979	6.13E-07	17	66	0.18325	5.76E-07
	v_4	29	86	0.93228	5.56E-06	19	76	0.30127	6.43E-07	48	192	2.8882	4.72E-07	14	56	0.21158	6.56E-07
	v_5	27	80	0.5081	8.16E-06	24	92	0.61032	7.60E-07	56	224	3.8834	1.85E-07	13	50	0.13432	9.96E-07
	v_6	35	104	0.72553	7.76E-06	21	84	0.29702	8.66E-07	89	356	6.9993	1.75E-07	14	56	0.25828	9.53E-07
	v_7	55	159	1.105	9.52E-06	56	223	1.0668	8.84E-07	72	287	4.8681	1.01E-07	22	87	0.41733	4.58E-07
100000	v_1	47	135	0.88397	7.49E-06	19	72	0.5357	8.40E-07	27	104	1.5436	1.13E-07	17	65	0.30951	6.78E-07
	v_2	46	132	1.9241	6.85E-06	22	84	0.64112	5.40E-07	18	68	0.52204	2.33E-07	17	65	0.33827	5.74E-07
	v_3	47	137	0.93957	7.38E-06	24	91	0.49719	6.58E-07	55	218	5.3539	7.92E-07	19	74	0.59536	5.42E-07
	v_4	29	86	0.52185	7.86E-06	19	76	0.61659	9.10E-07	54	216	6.04	8.54E-07	13	52	0.40113	6.18E-07
	v_5	28	83	1.2004	7.82E-06	25	96	0.66283	5.67E-07	66	264	8.5364	6.61E-07	14	54	0.36264	9.09E-07
	v_6	36	107	0.71922	6.82E-06	21	84	0.52058	6.24E-07	99	396	17.1225	5.30E-07	14	56	0.31088	6.89E-07
	v_7	46	132	2.108	9.33E-06	49	195	1.6031	5.89E-07	82	327	10.4083	1.31E-07	26	103	0.71176	8.78E-07

Remark 3.4: By the definition of x_t and (28), we have

$$\lim_{t\to\infty} \alpha_t \|d_t\| = \lim_{t\to\infty} \|v_t - x_t\| = 0. \qquad (35)$$

Theorem 3.5: Suppose that Assumption 1 hold, and the sequences $\{d_t\}$ and $\{v_t\}$ are generated by Algorithm 1, respectively. Then we have

$$\liminf_{t\to\infty} \|\varphi_t\| = 0. \qquad (36)$$

Furthermore, the sequence $\{v_t\}$ converges to a solution of (1).

Proof: We proof by contradiction. Suppose (36) fails to hold. That is, there exists a constant $r > 0$ such that for all $t \ge 0$, $\|\varphi_t\| \ge r$. Combining this with the sufficient descent condition implies that $\forall t \ge 0$

$$\|d_t\| \ge cr. \qquad (37)$$

Using Lipschitz continuity property on the definition of y_{t-1}, we deduce that

$$\|y_{t-1}\| \le \|\varphi_t - \varphi_{t-1}\|$$
$$= L\|v_t - v_{t-1}\|$$
$$\le L\alpha_{t-1}\|d_{t-1}\|$$

Also, from the definition of u_{t-1} in (13), it holds that
$$u_{t-1}^T d_{t-1} \ge y_{t-1}^T d_{t-1} + \|d_{t-1}\|^2 - y_t^T d_{t-1} = \|d_{t-1}\|^2. \qquad (38)$$

Since the sequences $\|\{\varphi_t\}\|$ and $\|\{\varphi(x_t)\}\|$ are bounded by (33) and (34) respectively, it follows that for all $t \ge 1$,

$$\|d_t\| \le \|\varphi_t\| + |(1-\lambda_t)|\left(|\beta_t^{PRP}|\|d_{t-1}\|\right.$$

$$\left. + |\eta_t|\|y_{t-1}\|\right) + |\lambda_t|\left(|\beta_t^{MHS}|\|d_{t-1}\| + |\theta_t^M|\|y_{t-1}\|\right)$$

$$\le \|\varphi_t\| + |(1-\lambda_t)|\left(\frac{2\|\varphi_t\|\|y_{t-1}\|}{\|\varphi_{t-1}\|^2}\|d_{t-1}\|\right)$$

$$+ |\lambda_t|\left(\frac{2\|\varphi_t\|\|y_{t-1}\|}{\|d_{t-1}\|^2}\|d_{t-1}\|\right)$$

$$\le \|\varphi_t\| + |(1-\lambda_t)|\left(\frac{2L\|\varphi_t\|\|d_{t-1}\|}{\|\varphi_{t-1}\|^2}\alpha_{t-1}\|d_{t-1}\|\right)$$

$$+ |\lambda_t|\left(2L\alpha_{t-1}\|\varphi_t\|\right)$$

$$\le (2L|\lambda_t| + 1)\|\varphi_t\|$$

$$+ |(1-\lambda_t)|\left(\frac{2L\|\varphi_t\|\|d_{t-1}\|}{\|\varphi_{t-1}\|^2}\alpha_{t-1}\|d_{t-1}\|\right)$$

$$\le (2L|\lambda_t|+1)k_1 + |(1-\lambda_t)|\left(\frac{2Lk_1\alpha_{t-1}\|d_{t-1}\|}{r^2}\|d_{t-1}\|\right)$$

Since the sequence $\{\lambda_t\}$ is a bounded sequence, we get from (35) that there exist a constant $b \in (0,1)$ and an integer n_0

TABLE 8. Numerical results of Problem 7.

DIM	INTP	DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
		NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
1000	v_1	5	15	0.17065	8.42E-06	28	112	0.037794	6.16E-07	9	36	0.012013	8.24E-07	28	112	0.023471	6.16E-07
	v_2	5	15	0.006568	8.10E-06	28	112	0.049005	5.92E-07	9	36	0.010921	7.93E-07	28	112	0.030415	5.92E-07
	v_3	5	15	0.006564	7.13E-06	28	112	0.025252	5.22E-07	9	36	0.004774	6.98E-07	28	112	0.01818	5.22E-07
	v_4	5	15	0.00743	4.88E-06	27	108	0.023047	7.14E-07	9	36	0.011578	4.78E-07	27	108	0.016171	7.14E-07
	v_5	5	15	0.007983	3.92E-06	27	108	0.028545	5.73E-07	9	36	0.008118	3.83E-07	27	108	0.032143	5.73E-07
	v_6	5	15	0.006977	2.31E-06	26	104	0.023578	6.76E-07	9	36	0.008507	2.26E-07	26	104	0.024872	6.76E-07
	v_7	5	15	0.010613	7.26E-06	28	112	0.01978	5.28E-07	9	36	0.007485	7.06E-07	28	112	0.015308	5.26E-07
5000	v_1	6	17	0.097753	3.79E-06	29	116	0.089289	6.90E-07	10	40	0.035558	1.85E-07	29	116	0.093531	6.90E-07
	v_2	6	17	0.028013	3.65E-06	29	116	0.087677	6.63E-07	10	40	0.034083	1.78E-07	29	116	0.078017	6.63E-07
	v_3	6	17	0.048763	3.21E-06	29	116	0.071294	5.84E-07	10	40	0.029418	1.57E-07	29	116	0.090887	5.84E-07
	v_4	6	17	0.036617	2.20E-06	28	112	0.11147	8.00E-07		40	0.030923	1.07E-07	28	112	0.10766	8.00E-07
	v_5	5	15	0.03253	8.82E-06	28	112	0.099611	6.42E-07	9	36	0.025486	8.61E-07	28	112	0.21386	6.42E-07
	v_6	5	15	0.019963	5.20E-06	27	108	0.06882	7.57E-07	9	36	0.02659	5.08E-07	27	108	0.067513	7.57E-07
	v_7	6	17	0.05678	3.24E-06	29	116	0.098299	5.89E-07	10	40	0.046788	1.58E-07	29	116	0.082867	5.90E-07
10000	v_1	6	17	0.12797	5.36E-06	29	116	0.16969	9.75E-07	10	40	0.048209	2.62E-07	29	116	0.24758	9.75E-07
	v_2	6	17	0.05054	5.16E-06	29	116	0.28504	9.38E-07	10	40	0.059684	2.52E-07	29	116	0.15554	9.38E-07
	v_3	6	17	0.034053	4.54E-06	29	116	0.42209	8.26E-07	10	40	0.067707	2.22E-07	29	116	0.30339	8.26E-07
	v_4	6	17	0.044923	3.11E-06	29	116	0.15585	5.66E-07	10	40	0.057455	1.52E-07	29	116	0.17212	5.66E-07
	v_5	6	17	0.032199	2.49E-06	28	112	0.26474	9.08E-07	10	40	0.046165	1.22E-07	28	112	0.17671	9.08E-07
	v_6	5	15	0.13165	7.35E-06	28	112	0.19215	5.35E-07	9	36	0.055454	7.18E-07	28	112	0.26362	5.35E-07
	v_7	6	17	0.070104	4.58E-06	29	116	0.14076	8.33E-07	10	40	0.061283	2.23E-07	29	116	0.13438	8.33E-07
50000	v_1	6	18	0.22101	2.40E-06	31	124	0.76644	5.45E-07	10	40	0.19203	5.85E-07	31	124	0.62899	5.45E-07
	v_2	6	18	0.14522	2.31E-06	31	124	0.65878	5.24E-07	10	40	0.34333	5.63E-07	31	124	0.73917	5.24E-07
	v_3	6	18	0.37249	2.03E-06	30	120	0.66796	9.24E-07	10	40	0.25464	4.96E-07	30	120	0.68746	9.24E-07
	v_4	6	17	0.2456	6.95E-06	30	120	0.61832	6.32E-07	10	40	0.2877	3.40E-07	30	120	0.68886	6.32E-07
	v_5	6	17	0.18082	5.58E-06	30	120	0.95462	5.07E-07	10	40	0.18474	2.72E-07	30	120	0.57066	5.07E-07
	v_6	6	17	0.16613	3.29E-06	29	116	0.70996	5.98E-07	10	40	0.19014	1.61E-07	29	116	0.63235	5.98E-07
	v_7	6	18	0.29392	2.05E-06	30	120	0.59246	9.32E-07	10	40	0.19029	5.00E-07	30	120	0.63632	9.32E-07
100000	v_1	6	18	0.45886	3.39E-06	31	124	1.4675	7.71E-07	10	40	0.59256	8.28E-07	31	124	1.5473	7.71E-07
	v_2	6	18	0.34433	3.26E-06	31	124	1.4599	7.42E-07	10	40	0.42673	7.96E-07	31	124	1.4846	7.42E-07
	v_3	6	18	0.57484	2.87E-06	31	124	1.4426	6.53E-07	10	40	0.49374	7.01E-07	31	124	1.5157	6.53E-07
	v_4	6	17	0.32502	9.83E-06	30	120	1.3951	8.94E-07	10	40	0.50097	4.80E-07	30	120	1.3667	8.94E-07
	v_5	6	17	0.29329	7.89E-06	30	120	1.392	7.18E-07	10	40	0.61243	3.85E-07	30	120	1.3846	7.18E-07
	v_6	6	17	0.47159	4.65E-06	29	116	1.3679	8.46E-07	10	40	0.57335	2.27E-07	29	116	1.4055	8.46E-07
	v_7	6	18	0.51657	2.90E-06	31	124	1.544	6.59E-07	10	40	0.43624	7.07E-07	31	124	1.75	6.59E-07

such that for all $t > n_0$ with $t \in \mathbb{N}$,

$$|(1 - \lambda_t)| \left(\frac{2Lk_1\alpha_{t-1}\|d_{t-1}\|}{r^2} \|d_{t-1}\| \right) < b.$$

Hence for any $t > n_0$ with $t \in \mathbb{N}$, we have

$$\|d_t\| \leq k_1 + b\|d_{t-1}\|$$
$$\leq k_1(1 + b + b^2 + \cdots + b^{t-n_0+1}) + b^{t-n_0}\|d_{n_0}\|$$
$$\leq \frac{k_1}{1-b} + \|d_{n_0}\|.$$

Setting $M = \max\left\{ \|d_1\|, \|d_2\|, \cdots, \|d_{n_0}\|, \frac{k_1}{1-b} + \|d_{n_0}\| \right\}$, we have $\|d_t\| \leq M$.

Since by the Lipschitz continuity of the mapping φ, we have established that the sequences $\{\varphi_t\}$ and $\{\varphi(x_t)\}$ are bounded, it holds that $\|\varphi(v_t + \frac{\alpha_t}{\varrho}d_t)\|$ and the sequence $\{d_t\}$ are bounded. Thus, we set

$$\Lambda = \|\varphi(v_t + \frac{\alpha_t}{\varrho}d_t)\| \leq k_3, \ k_3 > 0.$$

Now, by (25), we have that

$$\alpha_t\|d_t\| \geq \max\left\{ \varsigma, \frac{\varrho\|\varphi_t\|^2}{[L + \varsigma(\mu_t + (1 - \mu_t)\Lambda)]\|d_t\|^2} \right\} \|d_t\|$$
$$\geq \max\left\{ \varsigma cr, \frac{\varrho r^2}{[L + \varsigma(\mu_t + (1 - \mu_t)k_3)]M} \right\} > 0,$$

which contradicts (35). Thus, (36) holds. ∎

IV. NUMERICAL EXPERIMENT

We present some numerical examples in this section to visualize the behaviour of Algorithm 1. In what follows, Algorithm 1 is referred to as DF-PRPMHS. The performance of DF-PRPMHS is compared with three related algorithms of the same class, namely a modified Hestenes-Stiefel projection method (MHSPM) in [36], the New hybrid conjugate gradient projection method (NHCGPM) in [37], and a self-adaptive three-term conjugate gradient method (STTCGM) in [38]. The test problems are listed below, where the φ mapping is taken as

$$\varphi(v) = (\varphi_1(v), \varphi_2(v), \cdots, \varphi_n(v))^T,$$

where the associated initial points for these problems are

$$v_1 = (0.1, 0.1, \cdots, 0.1)^T \quad v_2 = (0.2, 0.2, \cdots, 0.2)^T$$
$$v_3 = (0.5, 0.5, \cdots, 0.5)^T, \quad v_4 = (1.2, 1.2, \cdots, 1.2)^T,$$
$$v_5 = (1.5, 1.5, \cdots 1.5)^T, \quad v_6 = (2, 2, \cdots, 2)^T,$$
$$v_7 = rand(n, 1).$$

Problem 1: This problem is the Exponential function [39] with constraint set $\Omega = \mathbb{R}^n_+$, that is,

$$\varphi_1(v) = e^{v_1} - 1,$$
$$\varphi_i(v) = e^{v_i} + v_i - 1, \quad \text{for } i = 2, 3, \ldots, n.$$

TABLE 9. Numerical results of Problem 8.

DIM	INTP	DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
		NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
1000	v_1	7	20	0.13332	5.95E-06	8	32	0.00492	2.09E-07	9	36	0.011729	2.15E-07	26	104	0.02058	7.47E-07
	v_2	7	20	0.006896	5.56E-06	8	32	0.006015	1.30E-07	9	36	0.011656	3.53E-07	25	100	0.0099579	8.96E-07
	v_3	6	17	0.006305	2.08E-06	8	32	0.007262	9.40E-08	7	28	0.00524	8.35E-07	11	44	0.0060715	7.86E-07
	v_4	8	23	0.008184	1.55E-06	8	32	0.003008	3.40E-07	9	36	0.009676	4.07E-07	28	112	0.016799	8.83E-07
	v_5	7	20	0.006947	6.03E-06	10	40	0.007067	1.66E-07	9	36	0.017397	6.65E-07	13	52	0.010727	9.77E-07
	v_6	8	22	0.006517	3.73E-06	9	35	0.003086	3.52E-07	10	39	0.009228	1.99E-07	28	111	0.021171	7.59E-07
	v_7	35	104	0.029747	7.05E-06	13	52	0.005472	2.01E-07	9	36	0.016878	9.15E-07	13	52	0.0075303	5.47E-07
5000	v_1	8	23	0.076447	2.00E-06	8	32	0.016952	4.68E-07	9	36	0.03992	4.82E-07	27	108	0.04894	8.89E-07
	v_2	8	23	0.086065	1.87E-06	8	32	0.024289	2.90E-07	9	36	0.048797	7.89E-07	27	108	0.044201	5.67E-07
	v_3	6	17	0.098133	4.66E-06	8	32	0.01589	2.10E-07	8	32	0.026769	1.95E-07	12	48	0.025773	9.35E-07
	v_4	8	23	0.11816	3.48E-06	8	32	0.016001	7.59E-07		36	0.037669	9.10E-07	30	120	0.049539	5.59E-07
	v_5	8	23	0.15192	2.03E-06	10	40	0.01497	3.72E-07	10	40	0.040952	1.55E-07	15	60	0.045844	6.18E-07
	v_6	8	22	0.061194	8.34E-06	9	35	0.021385	7.87E-07	10	39	0.05409	4.45E-07	29	115	0.050443	9.03E-07
	v_7	37	110	0.21844	6.92E-06	14	56	0.048577	2.58E-07	10	40	0.056438	2.14E-07	14	56	0.030133	6.30E-07
10000	v_1	8	23	0.073913	2.83E-06	8	32	0.024722	6.62E-07	9	36	0.060768	6.81E-07	28	112	0.073337	6.69E-07
	v_2	8	23	0.036442	2.64E-06	8	32	0.029786	4.10E-07	10	40	0.072811	1.17E-07	27	108	0.091556	8.02E-07
	v_3	6	17	0.080453	6.59E-06	8	32	0.019763	2.97E-07	8	32	0.05507	2.76E-07	13	52	0.65287	7.03E-07
	v_4	8	23	0.073725	4.91E-06	9	36	0.031559	6.86E-08	10	40	0.075413	1.34E-07	30	120	0.10341	7.90E-07
	v_5	8	23	0.16055	2.87E-06	10	40	0.026519	5.26E-07	10	40	0.069275	2.20E-07	15	60	0.058379	8.75E-07
	v_6	9	25	0.08304	1.77E-06	10	39	0.27364	7.11E-08	10	39	0.052412	6.29E-07	30	119	0.16188	6.79E-07
	v_7	37	110	0.6454	9.78E-06	14	56	0.060198	7.74E-07	10	40	0.060717	3.03E-07	14	56	0.060676	9.00E-07
50000	v_1	8	23	0.26958	6.32E-06	9	36	0.1603	9.46E-08	10	40	0.31164	1.59E-07	29	116	0.37339	7.95E-07
	v_2	8	23	0.31111	5.91E-06	8	32	0.089137	9.17E-07	10	40	0.2249	2.61E-07	28	112	0.46941	9.54E-07
	v_3	7	20	0.30503	2.22E-06	8	32	0.1482	6.65E-07	8	32	0.46068	6.17E-07	14	56	0.22248	8.37E-07
	v_4	9	26	0.27371	1.65E-06	9	36	0.13544	1.53E-07	10	40	0.30849	3.01E-07	31	124	0.67645	9.40E-07
	v_5	8	23	0.31124	6.41E-06	11	44	0.16808	7.51E-08	10	40	0.2478	4.91E-07	17	68	0.29036	5.53E-07
	v_6	9	25	0.31856	3.96E-06	10	39	0.11072	1.59E-07	11	43	0.26291	1.47E-07	31	123	0.46624	8.08E-07
	v_7	39	116	0.85695	9.58E-06	14	56	0.27275	6.20E-07	10	40	0.35015	6.76E-07	16	64	0.23432	5.69E-07
100000	v_1	8	23	0.44516	8.94E-06	9	36	0.18727	1.34E-07	10	40	0.46312	2.25E-07	30	120	0.86442	5.98E-07
	v_2	8	23	0.30357	8.35E-06	9	36	0.26301	8.28E-08	10	40	0.44504	3.69E-07	29	116	0.82362	7.18E-07
	v_3	7	20	0.20734	3.13E-06	8	32	0.16809	9.40E-07	8	32	0.52987	8.72E-07	15	60	0.38219	6.29E-07
	v_4	9	26	0.47194	2.34E-06	9	36	0.24888	2.17E-07	10	40	0.43631	4.25E-07	32	128	0.85344	7.07E-07
	v_5	8	23	0.4179	9.06E-06	11	44	0.42526	1.06E-07	10	40	0.49931	6.95E-07	17	68	0.43792	7.82E-07
	v_6	9	25	0.35162	5.60E-06	10	39	0.27171	2.25E-07	11	43	0.48504	2.08E-07	32	127	0.99269	6.08E-07
	v_7	40	119	2.3813	8.96E-06	17	68	0.60606	3.70E-07	10	40	0.50608	9.52E-07	16	64	0.46068	8.02E-07

Problem 2: Modified Logarithmic function [39] with constraint set $\Omega = \{v \in \mathbb{R}^n : \sum_{i=1}^{n} v_i \leq n, v_i > -1, i = 1, 2, \ldots, n\}$, that is,

$$\varphi_i(v) = \ln(v_i + 1) - \frac{v_i}{n}, \quad i = 2, 3, \ldots, n.$$

Problem 3: The Nonsmooth Function [40] with constraint set $\Omega = \mathbb{R}^n_+$.

$$\varphi_i(v) = 2v_i - \sin |v_i|, \quad i = 1, 2, 3, \ldots, n.$$

Problem 4 [41]: with $\Omega = \mathbb{R}^n_+$ defined by,

$$\varphi_i(v) = \min\left(\min(|v_i|, v_i^2), \max(|v_i|, v_i^3)\right) \text{ for } i = 2, 3, \ldots, n.$$

Problem 5: Strictly convex function [13], with constraint set $\Omega = \mathbb{R}^n_+$, that is,

$$\varphi_i(v) = e^{v_i} - 1, \quad i = 2, 3, \cdots, n$$

Problem 6: Strictly convex function II [13], with constraint set $\Omega = \mathbb{R}^n_+$, that is,

$$\varphi_i(v) = \frac{i}{n}e^{v_i} - 1, \quad i = 2, 3, \cdots, n$$

Problem 7: Tridiagonal Exponential function [42] with constraint set $\Omega = \mathbb{R}^n_+$, that is,

$$\varphi_1(v) = v_1 - e^{\cos(h(v_1+v_2))},$$
$$\varphi_i(v) = v_i - e^{\cos(h(v_{i-1}+v_i+v_{i+1}))}, \quad \text{for } 2 \leq i \leq n - 1,$$
$$\varphi_n(v) = v_n - e^{\cos(h(v_{n-1}+v_n))}, \quad \text{where } h = \frac{1}{n+1}$$

Problem 8: Nonsmooth function [43] with with constraint set $\Omega = \{v \in \mathbb{R}^n : \sum_{i=1}^{n} v_i \leq n, v_i \geq -1, \ 1 \leq i \leq n\}$.

$$\varphi_i(v) = v_i - \sin |v_i - 1|, \quad i = 2, 3, \cdots, n$$

Problem 9: The Trig exp function [39] with constraint set $\Omega = \mathbb{R}^n_+$, that is,

$$\varphi_1(v) = 3v_1^3 + 2v_2 - 5 + \sin(v_1 - v_2) \sin(v_1 + v_2)$$
$$\varphi_i(v) = 3v_i^3 + 2v_{i+1} - 5 + \sin(v_i - v_{i+1}) \sin(v_i + v_{i+1})$$
$$+ 4v_i - v_{i-1}e^{v_{i-1}-v_i} - 3 \quad \text{for } i = 2, 3, \ldots, n - 1$$
$$\varphi_n(v) = v_{n-1}e^{v_{n-1}-v_n} - 4v_n - 3, \quad \text{where } h = \frac{1}{n+1}.$$

Problem 10: The function $\varphi_i(v)$ with $\Omega = \mathbb{R}^n_+$ defined by,

$$\varphi_i(v) = 8^{\frac{1}{2}}v_i - 1 \quad i = 1, 2, 3, \ldots, n.$$

TABLE 10. Numerical results of Problem 9.

DIM	INTP	DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
		NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
1000	v_1	38	113	0.21301	8.95E-06	26	104	0.075244	8.52E-07	94	374	1.0058	4.18E-07	243	972	0.46322	5.00E-07
	v_2	41	122	0.15424	9.35E-06	24	96	0.083571	9.56E-07	58	232	0.54236	3.36E-07	222	888	0.43468	4.71E-07
	v_3	39	116	0.13135	6.87E-06	28	112	0.081315	3.70E-07	84	336	1.2611	3.47E-07	169	676	0.54242	3.59E-07
	v_4	31	92	0.14118	8.40E-06	5	17	0.025861	NaN	46	184	0.5303	3.88E-07	201	804	0.4889	4.55E-07
	v_5	41	122	0.16535	9.15E-06	24	96	0.099037	2.56E-07	45	180	0.47542	4.01E-07	229	916	0.48103	4.10E-07
	v_6	279	836	1.4843	9.88E-06	32	128	0.096197	9.22E-07	4	14	0.02515	NaN	255	1020	0.59168	4.57E-07
	v_7	50	149	0.20306	6.93E-06	32	128	0.093268	4.60E-07	NaN	NaN	NaN	NaN	257	1028	0.63962	9.09E-07
5000	v_1	44	131	0.34171	9.56E-06	27	108	0.398	8.44E-07	54	216	2.4776	3.52E-07	218	872	2.8958	4.47E-07
	v_2	44	131	0.73439	6.72E-06	28	112	0.33617	4.05E-07	82	328	3.879	3.53E-07	204	816	2.2671	4.71E-07
	v_3	38	113	0.87982	9.04E-06	23	92	0.36958	6.56E-07	50	200	2.1807	4.19E-07	175	700	2.028	4.14E-07
	v_4	36	107	0.82194	7.55E-06	29	116	0.34337	8.40E-07		172	2.0914	4.49E-07	208	832	2.426	4.04E-07
	v_5	58	173	2.2395	8.96E-06	24	96	0.38919	2.88E-07	56	224	2.5403	2.85E-07	243	972	2.6675	4.42E-07
	v_6	555	1664	9.2635	9.94E-06	24	96	0.36719	2.36E-07	52	203	3.2403	NaN	258	1032	3.0898	4.69E-07
	v_7	48	143	1.5296	9.93E-06	34	136	0.45922	8.53E-07	58	229	3.431	NaN	258	1032	3.1427	2.41E-07
10000	v_1	30	89	1.4309	9.39E-06	27	108	0.5776	8.80E-07	60	239	4.5898	4.02E-07	272	1088	5.5846	4.53E-07
	v_2	34	101	1.5631	8.09E-06	27	108	0.56486	4.18E-07	13	49	0.91221	NaN	260	1040	5.1899	4.53E-07
	v_3	43	128	1.4802	8.24E-06	26	104	0.59151	3.25E-07	55	220	4.8289	3.17E-07	188	752	3.2575	4.25E-07
	v_4	36	107	1.2032	7.07E-06	48	192	1.1214	1.55E-07	49	196	4.2254	4.89E-07	228	912	4.5676	3.99E-07
	v_5	57	170	2.822	9.86E-06	24	96	0.56018	2.99E-07	50	200	4.0709	3.34E-07	248	992	4.5689	4.51E-07
	v_6	696	2088	10.8374	9.99E-06	26	104	0.61383	4.71E-07	82	321	6.4065	3.33E-07	224	896	3.7615	4.63E-07
	v_7	50	149	0.67261	6.33E-06	35	140	1.427	6.70E-07	133	529	15.6434	NaN	258	1032	5.9775	3.55E-07
50000	v_1	44	131	2.3994	8.99E-06	26	104	2.7063	1.72E-07	65	259	21.6533	4.41E-07	263	1052	22.659	4.88E-07
	v_2	26	77	1.5127	9.06E-06	25	100	2.5209	7.33E-07	66	263	24.2276	3.40E-07	212	848	18.9152	4.71E-07
	v_3	24	71	1.8682	7.65E-06	26	104	2.6534	4.14E-07	71	284	24.7002	3.41E-07	212	848	19.5521	4.58E-07
	v_4	41	122	6.4642	6.59E-06	31	122	3.1164	3.11E-07	51	204	17.5241	3.76E-07	234	936	17.0699	4.18E-07
	v_5	99	296	5.8934	9.76E-06	21	84	2.1934	2.71E-07	68	271	23.6713	3.87E-07	296	1184	25.8157	4.45E-07
	v_6	NaN	NaN	NaN	NaN	25	100	2.5596	6.19E-07	88	345	29.6915	4.02E-07	216	864	16.0489	4.68E-07
	v_7	36	107	1.9702	8.79E-06	36	144	3.6538	2.33E-07	256	1019	138.9616	NaN	266	1064	25.7896	5.69E-07
100000	v_1	61	182	6.7104	8.95E-06	26	104	5.7126	1.72E-07	60	239	45.8073	3.59E-07	10	37	2.2289	NaN
	v_2	60	179	6.4546	9.48E-06	27	108	5.5797	5.98E-07	13	48	9.9641	NaN	259	1036	47.0565	5.00E-07
	v_3	24	71	3.8276	8.28E-06	25	100	5.1605	2.17E-07	63	252	45.494	4.36E-07	216	864	39.3571	4.82E-07
	v_4	39	116	4.7057	8.43E-06	28	112	6.1917	1.92E-07	55	220	40.3004	4.19E-07	242	968	45.1504	4.16E-07
	v_5	127	381	13.3068	9.99E-06	21	84	4.5056	8.26E-07	67	266	48.7264	3.67E-07	234	936	42.5959	4.81E-07
	v_6	NaN	NaN	NaN	NaN	25	100	5.614	4.28E-07	47	177	31.1865	NaN	222	888	40.2724	4.41E-07
	v_7	45	134	4.9469	8.26E-06	36	144	7.4844	3.31E-07	6	21	2.4928	NaN	268	1072	54.1131	7.32E-07

We take the parameters for DF-PRPMHS in the numerical experiment as $\zeta = 1$, $\varrho = 0.8$, $\varsigma = 10^{-4}$, $\tau = 1.2$, $Tol = 10^{-6}$, and the sequences $\lambda_t = \frac{1}{(2t+5)^2}$ and $\mu_t = \frac{1}{\exp{(t+1)^{t+1}}}$. The parameter for the algorithms compared with are chosen as given in their various papers. We considered dimensions ranging from $n = 1000, 5000, 10000, 50000, 100000$ for each test problem. The algorithms are terminated either by using the stop criterion set at $\|\varphi_t\| \leq 10^{-6}$, or if the iteration number exceeds 1000. Whenever the algorithm does not converge within 1000 iterations a failure is recorded. The performance profile developed by Dolan and Moré [44] based on the number of iterations, function evaluations and CPU computing time is used to obtain the performance profile of the methods in order to assess the detailed output of the techniques.

All the methods successfully solve all the test problems in the numerical experiments. Figures 1, 2, and 3 plot the DF-PRPMHS, NHCGPM, MHSPM, and STTCGM performance profiles in terms of number of iterations, number of function evaluations and CPU time. From the Figures 1, 2 and 3, we note that DF-PRPMHS has substantially improved over NHCGPM, MHSPM and STTCGM in terms of number of iterations, function evaluations and computing time. The methods were however very competitive for the CPU time,

with DF-PRPMHS slightly outperforming the methods compared. Based on the performance profile of the number of iterations, it is clear that DF-PRPMHS outperforms comparative methods as we can see from Figure 1 that DF-PRPMHS is a better algorithm for solving (1) since it solved about 62% of test problems with fewer iterations compared to NHCGPM, MHSPM and STTCGM which solved 15%, 10% and 18% of test problems with less number of iterations.

From Figure 2, it is easy to see that DF-PRPMHS also outperforms NHCGPM, MHSPM and STTCGM in terms of number of function evaluations, since DF-PRPMHS is a better solver for approximately 69% of test problems, while NHCGPM, MHSPM and STTCGM were better solvers for approximately 10%, 10% and 18%. Finally, regarding the CPU time, we can see the competitive nature of the time from Figure 3, but DF-PRPMHS is slightly faster than the compared methods. On the overall, it is clear that DF-PRPMHS is superior to NHCGPM, MHSPM, and STTCGM based on the performance profile metrics of Dolan and Moré, that is, number of iterations, number of function evaluations and CPU processing time.

It is worth mentioning that the compared methods (NHCGPM, MHSPM and STTCGM) made use of the

TABLE 11. Numerical results of Problem 10.

DIM	INTP	DF-PRPMHS				NHCGPM				MHSPM				STTCGM			
		NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM	NIT	NFE	CPU(s)	NM
1000	v_1	7	20	0.02701	3.31E-06	15	60	0.006603	7.75E-07	9	36	0.007072	5.78E-07	15	60	0.0042169	7.75E-07
	v_2	7	20	0.004663	2.00E-06	15	60	0.008528	4.70E-07	9	36	0.010383	3.50E-07	15	60	0.0037975	4.70E-07
	v_3	7	20	0.002814	1.91E-06	14	56	0.004029	3.60E-07	9	36	0.007861	3.34E-07	15	60	0.0061728	4.17E-07
	v_4	8	23	0.003001	1.24E-06	15	60	0.00383	6.10E-07	10	40	0.009991	2.17E-07	16	64	0.0063733	7.06E-07
	v_5	8	23	0.005718	1.68E-06	15	60	0.009696	8.26E-07	10	40	0.013116	2.94E-07	16	64	0.0085335	9.56E-07
	v_6	8	23	0.003637	2.41E-06	16	64	0.005083	3.48E-07	10	40	0.010271	4.22E-07	17	68	0.0077862	3.17E-07
	v_7	7	20	0.002673	4.22E-06	3	12	0.00254	7.77E-15	9	36	0.01071	7.57E-07	15	60	0.0044688	4.47E-07
5000	v_1	7	20	0.013418	7.40E-06	16	64	0.015715	5.08E-07	10	40	0.038331	1.45E-07	16	64	0.027345	5.08E-07
	v_2	7	20	0.014869	4.48E-06	16	64	0.013762	3.08E-07	9	36	0.028958	7.83E-07	16	64	0.02778	3.08E-07
	v_3	7	20	0.010169	4.27E-06	14	56	0.016671	8.06E-07	9	36	0.032044	7.47E-07	15	60	0.016889	9.33E-07
	v_4	8	23	0.010959	2.77E-06	16	64	0.015938	4.00E-07		40	0.030479	4.85E-07	17	68	0.023911	4.62E-07
	v_5	8	23	0.008565	3.75E-06	16	64	0.018814	5.41E-07	10	40	0.045032	6.57E-07	17	68	0.022486	6.26E-07
	v_6	8	23	0.011397	5.39E-06	16	64	0.027419	7.77E-07	10	40	0.041829	9.44E-07	17	68	0.023725	7.08E-07
	v_7	7	20	0.076228	9.44E-06	3	12	0.008202	1.69E-14	10	40	0.036245	1.83E-07	16	64	0.030109	3.00E-07
10000	v_1	8	23	0.019473	1.17E-06	16	64	0.032479	7.18E-07	10	40	0.059051	2.06E-07	16	64	0.03318	7.18E-07
	v_2	7	20	0.047894	6.34E-06	16	64	0.050172	4.35E-07	10	40	0.060166	1.24E-07	16	64	0.038713	4.35E-07
	v_3	7	20	0.033057	6.04E-06	15	60	0.036857	3.34E-07	10	40	0.07043	1.19E-07	16	64	0.040683	3.86E-07
	v_4	8	23	0.02134	3.92E-06	16	64	0.076462	5.65E-07	10	40	0.054458	6.86E-07	17	68	0.051322	6.54E-07
	v_5	8	23	0.033514	5.31E-06	16	64	0.040826	7.65E-07	10	40	0.059312	9.29E-07	17	68	0.046939	8.86E-07
	v_6	8	23	0.027504	7.62E-06	17	68	0.11946	3.22E-07	11	44	0.05542	1.50E-07	18	72	0.044261	2.93E-07
	v_7	8	23	0.039297	1.50E-06	3	12	0.009761	2.35E-14	10	40	0.063659	2.64E-07	16	64	0.046813	4.16E-07
50000	v_1	8	23	0.17924	2.62E-06	17	68	0.2339	4.70E-07	10	40	0.23996	4.60E-07	17	68	0.2083	4.70E-07
	v_2	8	23	0.11975	1.59E-06	16	64	0.127	9.73E-07	10	40	0.2181	2.78E-07	16	64	0.17722	9.73E-07
	v_3	8	23	0.080498	1.52E-06	15	60	0.2157	7.46E-07	10	40	0.21518	2.65E-07	16	64	0.13612	8.64E-07
	v_4	8	23	0.080106	8.76E-06	17	68	0.18156	3.70E-07	11	44	0.26496	1.72E-07	18	72	0.16228	4.28E-07
	v_5	9	26	0.084181	1.33E-06	17	68	0.21046	5.01E-07	11	44	0.26724	2.34E-07	18	72	0.18238	5.80E-07
	v_6	9	26	0.092134	1.91E-06	17	68	0.19207	7.20E-07	11	44	0.40948	3.35E-07	18	72	0.15541	6.56E-07
	v_7	8	23	0.12993	3.35E-06	3	12	0.034451	5.34E-15	10	40	0.21505	5.89E-07	16	64	0.18582	9.37E-07
100000	v_1	8	23	0.16004	3.71E-06	17	68	0.41272	6.65E-07	10	40	0.46267	6.50E-07	17	68	0.40316	6.65E-07
	v_2	8	23	0.20844	2.25E-06	17	68	0.32115	4.03E-07	10	40	0.58855	3.94E-07	17	68	0.3596	4.03E-07
	v_3	8	23	0.15255	2.14E-06	16	64	0.39868	3.09E-07	10	40	0.63681	3.75E-07	17	68	0.28727	3.58E-07
	v_4	9	26	0.16887	1.39E-06	17	68	0.3454	5.23E-07	11	44	0.52501	2.44E-07	18	72	0.33419	6.06E-07
	v_5	9	26	0.14231	1.88E-06	17	68	0.3376	7.09E-07	11	44	0.47679	3.30E-07	18	72	0.42389	8.21E-07
	v_6	9	26	0.15769	2.70E-06	18	72	0.29759	2.98E-07	11	44	0.6314	4.74E-07	18	72	0.28733	9.28E-07
	v_7	8	23	0.17669	4.74E-06	3	12	0.05856	7.54E-15	10	40	0.38971	8.28E-07	17	68	0.32579	3.88E-07

mainstream line search, that is

$$-\varphi(v_t + \alpha_t d_t)^T d_t \geq \varsigma \alpha_t \|d_t\|^2. \qquad (39)$$

It can be observed that the right hand side of (39) will be too large when v_t is far from the solution of the nonlinear equation with convex constraints problem (1). As such, the computing cost of the line search increases. To reduce the computation cost of the line search, the adaptive line search scheme proposed in [34] is adopted. This gives an insight of the good numerical performance of the proposed algorithm.

Table 2-11 lists the numerical results for the different test problems obtained by the different methods. In the tables, "DIM', "INTP', "NIT',' "NFE',' "CPU(s)" and "NM" represent the dimension, initial points, number of iterations, number of function values, CPU time (second) and final value of $\|\varphi_t\|$, respectively.

V. APPLICATION TO IMAGE RESTORATION PROBLEMS

Image restoration is about reconstructing or estimating uncorrupted images from noisy, blurred ones. This blurring may be caused by optical distortions, motion of objects during imaging, or turbulence in the atmosphere. Many science and engineering areas, such as aerial photography, remote sensing electron microscopy, and medical imaging, have

current or potential applications of image restoration [45]. For instance, in medical imaging, since the human visual system can be the key to decoding the elusive functions of the brain, a large amount of medical and neuroscience research is devoted to understanding the human visual system (see, for example, [46], [47]).

In this section, we focus on the first problem, i.e. the use of mathematical algorithms to perform image processing tasks. The general image restoration problem can be formulated by the inversion of the following observation model:

$$b = Av + \varpi, \qquad (40)$$

where $b \in \mathbb{R}^m$ is representing the observed data, $v \in \mathbb{R}^n$ is the unknown image, ϖ is the noise and A is a linear mapping such that $A \in \mathbb{R}^{m \times n}(m < n)$. In order to address problem (40), one of the tools usually employed is the ℓ_1-regularization. The restoration is obtained by approximating the following unconstrained optimization

$$\min_v \frac{1}{2}\|Av - b\|^2 + \Theta\|v\|_1, \qquad (41)$$

where Θ is a positive regularization parameter and $\|\cdot\|_1$ is the ℓ_1-regularization term. With the approximate equivalence between problem (1) and (41) (see [48], [49]), in what follows, we illustrate the efficiency of our algorithm in

approximating (41). Some recent methods for image restoration includes; multi-channel and multi-model based auto encoding prior for gray scale image restoration [50]; formatted learning for image restoration [51], image restoration by combined order regularization with optimal spatial adaptation [52], multi-Level encoder-decoder architectures for image restoration [53], riemannian loss for image restoration [54] and modulating image restoration with continual levels via adaptive feature modification layers [55].

In this test, the efficiency of the proposed algorithm (DF-PRPMHS) is illustrated in restoring blurred and noisy images. We considered four images of different sizes which are degraded using a Gaussian blur operator and a Gaussian noise with standard deviation 10^{-2}. The parameters selected to implement the algorithm are: $\tau = 1$; $\varsigma = 10^{-4}$; $\varrho = 0.55$; $\lambda_t = \frac{1}{(2t+5)^2}$; $\zeta = 1$; $\mu = 1$.

In addition, we compare its performance with the iterative shrinkage/thresholding (IST) [56] designed for wavelet-based image deconvolution and the PSGM method [57] to reflect the performance of the DF-PRPMHS method in restoring the blurred and noisy images. It is worth noting that the iterative procedure for all the algorithms starts using the same initial point and ends when the tolerance, $Tol < 10^{-5}$. Figure 4 shows the original images. In Figures 5 and 6, the blurred and noisy images, and the restored images by the various algorithms are presented. Images on the column labelled (a) are the blurred and noisy images, images on column (b) are the restored images by DF-PRPMHS, images on column (c) are the restored images by IST and images on column (d) are the restored images by PSGM. In Table 1, numerical result obtained from the implementation of the algorithms are presented. The performance of the methods are analysed based on the signal-to-noise ratio (SNR), [58] peak signal-to-noise ratio (PSNR), and the Structural Similarity Index (SSIM) [59] metric. From the outcome of the experiment as reported in Table 1, we can see that the restored images by DF-PRPMHS are closer to the original than those from IST and PSGM for all the test images. This is reflected by their Larger SNR, PSNR, and SSIM.

CONCLUSION

We have presented a derivative-free conjugate gradient method that combines the conjugate gradient direction of the three-term modified PRP method (TTPRP) and the three-term HS conjugate gradient method (TTHS) to solve constrained nonlinear equation with convex constraints using a projection and a modified line search procedure. Under the assumption that the underlying operator is Lipschitz continuous and satisfy a weaker monotonicity assumption, the global convergence of the proposed method is established. Another contribution from this paper is the use of our approach to solve the regularized ℓ_1-norm compressive sensing problems. The computational experiments in restoring blurred and noisy images have shown that the proposed approach is competitive with the comparative ones.

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