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The Pairwise-Markov Bernoulli Filter

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ABSTRACT The Bernoulli filter is a general, Bayes-optimal solution for tracking a single disappearing and reappearing target, using a sensor whose observations are corrupted by missed detections and a general, known clutter process. Like virtually all target-tracking algorithms it presumes restrictive independence assumptions, namely a hidden Markov model (HMM) structure on the sensor and target. That is, the current state of the target depends only on its previous state, and the measurement collected from it depends only on its current state. Pieczynski's pairwise Markov model (PMM) relaxes these restrictions. In it, the current target state can additionally depend on the previous measurement; and the current measurement can additionally depend on the previous measurement and previous target state. In this paper we show how to correctly generalize the PMM to the multitarget (MPMM) case; and use the MPMM to derive a "PMM Bernoulli filter" that obeys PMM rather than restrictive HMM sensor/target statistics.

INDEX TERMS Target tracking, random finite set, finite-set statistics, recursive Bayes filter, Bernoulli filter, hidden Markov model, pairwise Markov model.

I. INTRODUCTION

The Bernoulli filter was independently and contemporaneously devised by Vo [1] and Mahler [2, Sec. 14.7]. It is a general and Bayes-optimal solution for tracking a single disappearing and reappearing target, using a sensor whose observations are corrupted by missed detections and a general, known clutter process. It propagates a *probability hypothesis density* (PHD) D via time-update and measurement-update steps $D(\mathbf{x}_{k-1}|Z_{1:k-1}) \rightarrow D(\mathbf{x}_k|Z_{1:k-1})$ and $D(\mathbf{x}_k|Z_{1:k-1}) \rightarrow D(\mathbf{x}_k|Z_{1:k})$, where $Z_{1:k} : Z_1,\ldots, Z_k$ is the time-sequence of collected measurement-sets. See Section VI-A for more detail.

Like virtually all target-tracking algorithms, the Bernoulli filter presumes restrictive independence assumptions, namely a *hidden Markov model* (HMM) structure on the sensor and target. That is, at time t_k the target's state $\mathbf{x}_k$ depends only on its previous state $\mathbf{x}_{k-1}$ with Markov transition density $f(\mathbf{x}_k|\mathbf{x}_{k-1})$; and the measurement $\mathbf{y}_k$ that the sensor collects from it depends only on $\mathbf{x}_k$ with measurement density $f(\mathbf{y}_k|\mathbf{x}_k)$. Pieczynski's *pairwise Markov model* (PMM) [3]–[7] relaxes these restrictions.

A. THE PAIRWISE MARKOV MODEL (PMM)

The PMM generalizes the HMM by treating the target and sensor as a joint dynamical system with joint state $(\mathbf{x}_k, \mathbf{y}_k)$, which is governed by a Markov transition density

$$f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \quad (1)$$

where the factorization on the right is due to Bayes' rule. In the PMM, the current target state can additionally depend on the previous measurement (as described by $f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ (i.e., the target can be non-Markovian); and in that the current measurement can additionally depend on the previous measurement and the previous target state as described by $f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ (and thus measurement noise can be colored or correlated with plant noise [7, p. 4487]). See Section III for more detail.

Pieczynski and Desbouvries [6] have described practical Kalman filter-based implementations of PMMs to single-target tracking. Petetin and Desbouvries [7] proposed a PMM generalization of the probability hypothesis density (PHD) filter of [2, Sec. 16.3]; described concrete practical applications and implementations; and demonstrated that their PMM-PHD filter has better tracking performance than the classical HMM-PHD filter under non-HMM conditions. This work has been extended to nonlinear models [8].

B. THE MULTITARGET PMM (MPMM)

Let X_k be the state-set of a multitarget system at time t_k and Y_k the multitarget measurement-set generated by both targets and clutter. In [9] Mahler generalized the PMM to

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the multitarget PMM (MPMM); and also proposed a concrete formula—see (51,53) below—for the MPMM transition density $f(X_k, Y_k|X_{k-1}, Y_{k-1})$, based on the "standard" multitarget Markov density $f(X_k|X_{k-1})$ [2, Eq. 14.273], [10, Eq. 5.94]; and the "standard" multitarget measurement density $f(Y_k|X_k)$ [2, Eq. 14.290], [10, Eq. 5.104].

Remark 1: The MPMM transition model (51,53) turns out to be erroneous—see Section V-C. It will be replaced by the corrected, theoretically rigorous model in (60,61).

We shall see that the evolution $f(X_{k-1}, Y_{k-1}|Z_{1:k-2}) \to f(X_k, Y_k|Z_{1:k-1})$ of an MPMM is described in terms of "MPMM densities" $f(X_k, Y_k|Z_{1:k-1})$, which describe not only X_k and Y_k but also the statistical correlation between them (Section IV). In this paper we will we consider the evolution of "Bernoulli MPMM's" (X_k, Y_k)—i.e., those such that $|X_k| \leq 1$ for all $k \geq 1$ (where $|X|$ denotes the number of elements in X). In such an MPMM the following dynamical transitions are possible: $(\{\mathbf{x}_{k-1}\}, Y_{k-1}) \to (\{\mathbf{x}_k\}, Y_k)$ (target survives); $(\{\mathbf{x}_{k-1}\}, Y_{k-1}) \to (\emptyset, Y_k)$ (target disappears); or $(\emptyset, Y_{k-1}) \to (\{\mathbf{x}_k\}, Y_k)$ (target appears or reappears).

Since $f(X_k, Y_k|Z_{1:k-1}) = 0$ identically if $|X_k| > 1$, the state of a Bernoulli MPMM at time t_k is completely described by $f(\emptyset, Y_k|Z_{1:k-1})$ and $f(\{\mathbf{x}_k\}, Y_k|Z_{1:k-1})$; and its evolution from time t_{k-1} to time t_k is described by the update $f(\emptyset, Y_{k-1}|Z_{1:k-2})$, $f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Z_{1:k-2}) \to f(\emptyset, Y_k|Z_{1:k-1}), f(\{\mathbf{x}_k\}, Y_k|Z_{1:k-1})$. The ultimate result is a Bayes-optimal "PMM Bernoulli filter" in which the sensor can have correlated-noise statistics and the target can have non-Markovian dynamics.

C. SUMMARY OF MAIN RESULTS

These are as follows:

1. The corrected MPMM transition model, (60,61).
2. Evolution models for the "elementary" MPMM pairs (X_k, Y_k)—i.e., those with $|X_k|, |Y_k| \leq 1$ (Sections V-D through V-H).
3. The "Bernoulli MPMM filter," which recursively propagates Bernoulli MPMM densities $f(X_k, Y_k|Z_{1:k-1})$ with $|X_k| \leq 1$ (Section VI-D).
4. The "PMM Bernoulli filter," which, like the usual HMM Bernoulli filter, recursively propagates PHD's $D(\mathbf{x}_k|Z_{1:k})$ (see (10)).

The PMM Bernoullii filter can be summarized as follows. Let us be given: (i) $\kappa_k(Y_k)$ (the multi-object probability density function of the clutter process); (ii) $p_S(\mathbf{x}_{k-1})$ (the probability that the target will not disappear at time t_{k-1}); (iii) q_k^B (the probability that the target will reappear at time t_{k-} after having disappeared); (iv) $s_k^B(\mathbf{x}_k)$ (the target's spatial density after reappearance); (v) $p_D(\mathbf{x}_k)$ (the target's probability of detection); (vi) $f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ (the PMM transition density); (vii) $f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ (the marginal of $f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$; (viii) $M_{\mathbf{x}_k}(\mathbf{x}_{k-1}) = f(\mathbf{x}_k|\mathbf{x}_{k-1})$ (Markov density associated with transition $(\{\mathbf{x}_{k-1}\}\}, \emptyset) \to (\{\mathbf{x}_k\}, \emptyset)$, see (76); and (ix) $L_{\mathbf{y}_k}(\mathbf{x}_k) = f(\mathbf{y}_k|\mathbf{x}_k)$ (measurement density associated with $(\emptyset, \emptyset) \to (\{\mathbf{x}_k\},\{\mathbf{y}_k\})$, see (89). Define

$$\ell_{Y_k}(\mathbf{x}_k) = 1 - p_D(\mathbf{x}_k) + p_D(\mathbf{x}_k) \sum_{\mathbf{y}_k \in Y_k} L_{\mathbf{y}_k}(\mathbf{x}_k) \cdot \frac{\kappa_k(Y_k - \{\mathbf{y}_k\})}{\kappa_k(Y_k)} \tag{2}$$

where by convention the summation vanishes if $Y_k = \emptyset$. Also, if $f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ is the marginal of $f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$,

$$\begin{aligned}\ell_{Y_k,Y_{k-1}}(\mathbf{x}_k|\mathbf{x}_{k-1}) &= \frac{1-p_D(\mathbf{x}_k)}{|Y_{k-1}|} \sum_{\mathbf{y}_{k-1} \in Y_{k-1}} f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ &+ \frac{p_D(\mathbf{x}_k)}{|Y_{k-1}|} \sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \sum_{\mathbf{y}_k \in Y_k} f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ &\cdot \frac{\kappa_k(Y_k - \{\mathbf{y}_k\})}{\kappa_k(\mathbf{y}_k)}\end{aligned} \tag{3}$$

if $Y_{k-1} \neq \emptyset$ (and where by convention the second summation vanishes if $Y_k = \emptyset$); whereas if $Y_{k-1} = \emptyset$,

$$\ell_{Y_k,\emptyset}(\mathbf{x}_k|\mathbf{x}_{k-1}) = \ell_{Y_k}(\mathbf{x}_k) \cdot f(\mathbf{x}_k|\mathbf{x}_{k-1}) = \ell_{Y_k}(\mathbf{x}_k) \cdot M_{\mathbf{x}_k}(\mathbf{x}_{k-1}). \tag{4}$$

Abbreviate

$$D_{k|k-1}(\mathbf{x}_k) = D(\mathbf{x}_k|Z_{1:k-1}), D_{k|k}(\mathbf{x}_k) = D(\mathbf{x}_k|Z_{1:k}) \tag{5}$$

$$\ell_{Y_k,Y_{k-1},\mathbf{x}_k}(\mathbf{x}_{k-1}) = \ell_{Y_k,Y_{k-1}}(\mathbf{x}_k|\mathbf{x}_{k-1}). \tag{6}$$

Define

$$\begin{aligned}\tilde{\ell}_{Y_k,Y_{k-1}}(\mathbf{x}_{k-1}) &= \frac{1}{|Y_{k-1}|} \sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \int (1-p_D(\mathbf{x}_k)) \cdot f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) d\mathbf{x}_k \\ &+ \frac{1}{|Y_{k-1}|} \sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \sum_{\mathbf{y}_k \in Y_k} \frac{\kappa_k(Y_k - \{\mathbf{y}_k\})}{\kappa_k(Y_k)} \\ &\times \int p_D(\mathbf{x}_k) f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) d\mathbf{x}_k\end{aligned} \tag{7}$$

if $Y_{k-1} \neq \emptyset$ whereas if $Y_{k-1} = \emptyset$ (and employing the notation defined in (47)),

$$\tilde{\ell}_{Y_k,\emptyset}(\mathbf{x}_{k-1}) = \int \ell_{Y_k}(\mathbf{x}_k) \cdot M_{\mathbf{x}_k}(\mathbf{x}_{k-1}) d\mathbf{x}_k = M_{\ell_{Y_k}}(\mathbf{x}_{k-1}). \tag{8}$$

Also, if $f(\mathbf{x})$ is a density function and $0 \leq h(\mathbf{x}) \leq 1$ a unitless function, define

$$f[h] = \int h(\mathbf{x}) \cdot f(\mathbf{x}) d\mathbf{x}. \tag{9}$$

Given this, the *PMM Bernoulli filter* is given by the following single-step recursive update, (10), as shown at the bottom of the next page. This equation is derived in Appendix B.

If the PMM is actually an HMM, then

$$f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = f(\mathbf{y}_k|\mathbf{x}_k) \cdot f(\mathbf{x}_k|\mathbf{x}_{k-1}), \tag{11}$$

(see (36)) from which it follows that

$$\ell_{Y_k,Y_{k-1}}(\mathbf{x}_k|\mathbf{x}_{k-1}) = \ell_{Y_k}(\mathbf{x}_k) \cdot f(\mathbf{x}_k|\mathbf{x}_{k-1}) \tag{12}$$

in which case, as will be shown in (200), (10) reduces to the single-step HMM Bernoulli filter as given in (111).

D. ORGANIZATION OF THE PAPER

The remainder of the paper is organized as follows: A brief summary of the mathematical theory required to understand the paper (Section II); the PMM (Section III); the MPMM (Section IV); the corrected MPMM transition density (Section V); and the Bernoulli MPMM filter (Section VI). Conclusions can be found in Section VII, and the Bernoulli MPMM and PMM Bernoulli filters are derived in Appendices A and B, respectively. The following notation will be employed hereafter: $A := B$ means "A is defined to be B"; and $A \mathrel{!=} B$ means "A is an abbreviation of B."

II. OVERVIEW OF FINITE-SET STATISTICS (FISST)

This section summarizes the theory necessary to understand the remainder of the paper. Greater detail can be found in books [2], [10]–[12], tutorials [13]–[15], and a short survey of advances c. 2015 [16]. Also, systematic investigations of FISST vs. "point processes" can be found in [17], [18] and of FISST vs. measurement-to-track approaches in [19].

Significant recent advances can be found in [20], [21]. Specifically, [20] describes an implementation of the generalized labeled multi-Bernoulli (GLMB) filter that is capable of simultaneously tracking over a million 2D targets in significant clutter in real time using off-the-shelf computing equipment, as well as a theoretically rigorous, large-scale track quality measure, "OSPA$^{(2)}$"; and [21] describes a multiscan extension of the GLMB filter.

The section is organized as follows: random finite sets (Section II-A); multitarget calculus (Section II-B); Bernoulli RFSs (Section II-C); and the multitarget recursive Bayes filter (Section II-D).

A. RANDOM FINITE SETS (RFSs)

Let $\mathfrak{I}$ be a single-target state-space with $\mathbf{x} \in \mathfrak{I}$, and let $\mathfrak{R}$ be the sensor measurement space with $\mathbf{z} \in \mathfrak{R}$. Then the state of a multitarget system is represented as a finite subset $X = \{\mathbf{x}_1,\ldots,\mathbf{x}_n\} \subseteq \mathfrak{I}$ with $X = \emptyset$ for $n = 0$. The number of elements in X is denoted as $|X|$. In a Bayesian approach, unknown states are random variables. Thus an unknown multitarget state is a random finite set (RFS) $\Xi \subseteq \mathfrak{I}$.

B. MULTITARGET CALCULUS

A *multitarget density function* is a function $f(X) \geq 0$ of the finite-set variable $X \subseteq \mathfrak{I}$ such that the unit of measurement of $f(X)$ is $\iota_{\mathfrak{I}}^{-|X|}$, where $\iota_{\mathfrak{I}}$ is the unit of measurement of $\mathfrak{I}$. The *set integral* of $f(X)$ is

$$\int f(X)\delta X = f(\emptyset) + \sum_{n\geq 1} \int f_n(\mathbf{x}_1,\ldots,\mathbf{x}_n) d\mathbf{x}_1 \cdots d\mathbf{x}_n \quad (13)$$

where $f_n(\mathbf{x}_1,\ldots,\mathbf{x}_n) := f(\{\mathbf{x}_1,\ldots,\mathbf{x}_n\})/n!$ for distinct $\mathbf{x}_1,\ldots,\mathbf{x}_n$ and $f_n(\mathbf{x}_1,\ldots,\mathbf{x}_n) := 0$ otherwise. Every random finite state-set Ξ has a multitarget probability distribution $f_\Xi(X)$ with $\int f_\Xi(X)\delta X = 1$.

An *MPMM density function* is a function $f(X,Y) \geq 0$ of the finite-set variables $X \subseteq \mathfrak{I}$, $Y \subseteq \mathfrak{R}$ such that the unit of measurement of $f(X,Y)$ is $\iota_{\mathfrak{I}}^{-|X|}\iota_{\mathfrak{R}}^{-|Y|}$ where $\iota_{\mathfrak{R}}$ is the unit of measurement of $\mathfrak{R}$. An MPMM density function is a *joint probability density* if $\int f(X,Y)\delta X\delta Y = 1$. If $\Xi \subseteq \mathfrak{I}$ and $\Sigma \subseteq \mathfrak{R}$ are RFSs then Ξ, Σ have an MPMM probability density $f_{\Xi,\Sigma}(X,Y)$ that describes the statistical correlation between them.

The *probability generating functional* (p.g.fl.) of Ξ is, for unitless "test functions" $0 \leq h(\mathbf{x}) \leq 1$,

$$G_\Xi[h] := \int h^X \cdot f_\Xi(X)\delta X \quad (14)$$

where $h^X = 1$ if $X = \emptyset$ and $h^X = \prod_{\mathbf{x}\in X} h(\mathbf{x})$ otherwise. The simplest nontrivial p.g.fl.'s are

$$s[h] = \int h(\mathbf{x}) \cdot s(\mathbf{x}) d\mathbf{x} \quad (15)$$

where $s(\mathbf{x}) \geq 0$ is a probability density function on $\mathfrak{I}$. If $0 \leq g(\mathbf{z}) \leq 1$ for $\mathbf{z} \in \mathfrak{R}$ then the joint p.g.fl. of Ξ, Σ is

$$G_{\Xi,\Sigma}[h,g] := \int h^X \cdot g^Y \cdot f_{\Xi,\Sigma}(X,Y)\delta X\delta Y. \quad (16)$$

The *intuitive* definition of the *Volterra functional derivative* of $G_\Xi[h]$ is:

$$\frac{\delta G_\Xi}{\delta \mathbf{x}}[h] := \lim_{\varepsilon\to 0^+} \frac{G_\Xi[h + \varepsilon \cdot \delta_{\mathbf{x}}] - G_\Xi[h]}{\varepsilon} \quad (17)$$

where $\delta_{\mathbf{x}}(\mathbf{y})$ is the Dirac delta function concentrated at $\mathbf{x}$. (For a rigorous definition see [14].) If $X = \{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ with $|X| = n$ then the iterated functional derivative is

$$\frac{\delta G_\Xi}{\delta X}[h] := \frac{\delta^n G_\Xi}{\delta \mathbf{x}_1 \cdots \delta \mathbf{x}_n}[h] := \frac{\delta}{\delta \mathbf{x}_n} \frac{\delta^{n-1} G_\Xi}{\delta \mathbf{x}_1 \cdots \delta \mathbf{x}_{n-1}}[h] \quad (18)$$

if $|X| \geq 1$ and $= G_\Xi[h]$ if otherwise. There is an extensive "toolbox" of "turn-the-crank" rules for set integrals and functional derivatives [2, pp. 383-389], [10, pp. 69-80].

The joint functional derivatives of $G_{\Xi,\Sigma}[h,g]$ are:

$$\frac{\delta G_{\Xi,\Sigma}}{\delta X \bullet \delta Y}[h] := \frac{\delta}{\delta X}\frac{\delta}{\delta Y} G_{\Xi,\Sigma}[h,g] \quad (19)$$

where the "$\bullet$" notation indicates that $\delta/\delta X$ is taken with respect to the variable h and $\delta/\delta Y$ with respect to the variable g. When $Y = \emptyset$ or $X = \emptyset$ we have:

$$\frac{\delta G_{\Xi,\Sigma}}{\delta X \bullet}[h,g] := \frac{\delta}{\delta X} G_{\Xi,\Sigma}[h,g],$$
$$\frac{\delta G_{\Xi,\Sigma}}{\bullet \delta Y}[h,g] := \frac{\delta}{\delta Y} G_{\Xi,\Sigma}[h,g]. \quad (20)$$

$$D_{k|k}(\mathbf{x}_k) = \frac{\ell_{Z_k}(\mathbf{x}_k) \cdot q_k^B s_k^B(\mathbf{x}_k) \cdot (1 - D_{k-1|k-1}[1]) + D_{k-1|k-1}[p_S \ell_{Z_k,Z_{k-1},\mathbf{x}_k}]}{\left(\begin{array}{l}(1 - q_k^B s_k^B + q_k^B s_k^B[\ell_{Z_k}]) \cdot (1 - D_{k-1|k-1}[1]) \\ + D_{k-1|k-1}[1 - p_S + p_S \tilde{\ell}_{Z_k,Z_{k-1}}]\end{array}\right)}. \quad (10)$$

The p.g.fl. and distibution of an RFS are related by:

$$f_\Xi(X) = \frac{\delta G_\Xi}{\delta X}[0]. \tag{21}$$

Likewise, the bivariate p.g.fl. and bivariate multitarget distibution of RFSs Ξ, Σ are related by:

$$f_{\Xi,\Sigma}(X, Y) = \frac{\delta G_{\Xi,\Sigma}}{\delta X^\bullet \delta Y}[0, 0]. \tag{22}$$

C. THE BERNOULLI RFS

An RFS of special importance for this paper, the *Bernoulli RFS*, is most easily described using its p.g.fl.: $G_\Xi[h] = 1 - q + q \cdot s[h]$ where $0 \le q \le 1$ and the probability density $s(\mathbf{x})$ are, respectively, the existence probability and spatial distribution of a single target.

D. MULTITARGET RECURSIVE BAYES FILTER

Given a time-sequence $Z_{1:k}$:$Z_1, \ldots, Z_k$ of collected measurement-sets from a sensor, this is:

$$\ldots \rightarrow f(X_{k-1}|Z_{1:k-1}) \rightarrow f(X_k|Z_{1:k-1}) \rightarrow f(X_k|Z_{1:k}) \rightarrow \ldots$$

where

$$f(X_k|Z_{1:k-1}) = \int f(X_k|X_{k-1}, Z_{1:k-1}) \cdot f(X_{k-1}|Z_{1:k-1})\delta X_{k-1} \tag{23}$$

$$f(X_k|Z_{1:k}) \propto f(Z_k|X_k, Z_{1:k-1}) \cdot f(X_k|Z_{1:k-1}); \tag{24}$$

and where $f(X_k|X_{k-1}, Z_{1:k-1})$ is the multitarget state-transition density and $f(Z_k|X_k, Z_{1:k-1})$ is the sensor multitarget measurement density. It is assumed that $f(X_k|X_{k-1}, Z_{1:k-1}) = f(X_k|X_{k-1})$ (Markov assumption) and $f(Z_k|X_k, Z_{1:k-1}) = f(Z_k|X_k)$.

In this paper we will be concerned with $f(X_k|X_{k-1})$ and $f(Z_k|X_k)$ for only the "standard" multitarget motion and measurement models, respectively—see Section V-A.

III. THE PAIRWISE MARKOV MODEL (PMM)

The section is organized as follows: single-target recursive Bayes filter (Section III-A); the PMM (Section III-B); and single-target tracking using PMMs (Section III-C).

A. SINGLE-TARGET RECURSIVE BAYES FILTER

The PMM concept is most easily explained via the single-target recursive Bayes filter. Let $\mathbf{x} \in \mathfrak{I}$ denote a single-target state and $\mathbf{z} \in \mathfrak{R}$ a single-target measurement. In the Bayesian approach, the unknown state at time t_k is a random variable $\mathbf{X}_{k|k} \in \mathfrak{I}$ and the measurement process at time t_k is a random variable $\mathbf{Z}_k \in \mathfrak{R}$. Let us be given:

1. the distribution $f(\mathbf{x}_0)$ of the initial state $\mathbf{X}_{0|0}$;
2. the sequence $\mathbf{z}_{1:k}$: $\mathbf{z}_1, \ldots, \mathbf{z}_k$ of measurements collected from the target at times $t_1, \ldots, t_k$;
3. the transition density $f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{z}_{1:k-1})$, which describes the evolution of $\mathbf{X}_{k-1|k-1}$ at time t_{k-1} to $\mathbf{X}_{k|k-1}$ at time t_k; and
4. the measurement density $f(\mathbf{z}_k|\mathbf{x}_k, \mathbf{z}_{1:k-1})$ at time t_k, which characterizes the statistics of $\mathbf{Z}_k$ if $\mathbf{x}_k$ is a realization of $\mathbf{X}_{k|k-1}$.

Given this, the recursive Bayes filter is defined by the time- and measurement-update equations

$$f(\mathbf{x}_k|\mathbf{z}_{1:k-1}) = \int f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{z}_{1:k-1}) \cdot f(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1}) d\mathbf{x}_{k-1} \tag{25}$$

$$f(\mathbf{x}_k|\mathbf{z}_{1:k}) = \frac{f(\mathbf{z}_k|\mathbf{x}_k, \mathbf{z}_{1:k-1}) \cdot f(\mathbf{x}_k|\mathbf{z}_{1:k-1})}{f(\mathbf{z}_k|\mathbf{z}_{1:k-1})} \tag{26}$$

$$f(\mathbf{z}_k|\mathbf{z}_{1:k-1}) = \int f(\mathbf{z}_k|\mathbf{x}_k, \mathbf{z}_{1:k-1}) \cdot f(\mathbf{x}_k|\mathbf{z}_{1:k-1}) d\mathbf{x}_{k-1}. \tag{27}$$

It is usually assumed that $f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{z}_{1:k-1}) = f(\mathbf{x}_k|\mathbf{x}_{k-1})$ and $f(\mathbf{z}_k|\mathbf{x}_k, \mathbf{z}_{1:k-1}) = f(\mathbf{z}_k|\mathbf{x}_k)$.

B. THE PAIRWISE MARKOV MODEL

Now let the state space be the Cartesian product $\mathfrak{I} \times \mathfrak{R}$ rather than $\mathfrak{I}$. In this case, the unknown quantity at time t_k is the joint state of the joint target-measurement system, and is represented as the random pair $(\mathbf{X}_{k|k}, \mathbf{Y}_k) \in \mathfrak{I} \times \mathfrak{R}$. What is unknown is not only $\mathbf{X}_{k|k}$ and $\mathbf{Y}_k$ but also their statistical correlation, as described by the posterior PMM density $f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{z}_{1:k-1})$. Let us be given

1. a PMM transition density

$$f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \tag{28}$$

that describes the evolution of the PMM system;

2. the measurement density $f(\mathbf{z}_k|\mathbf{x}_k, \mathbf{y}_k) = \delta_{\mathbf{y}_k}(\mathbf{z}_k)$ of the PMM system, in which case it follows that the measurement equation is $\mathbf{z}_k = \eta(\mathbf{x}_k, \mathbf{y}_k)$ with measurement function $\eta(\mathbf{x}, \mathbf{y}) := \mathbf{y}$—i.e., if the joint system has state $(\mathbf{x}, \mathbf{y})$ then $\mathbf{y}$ is the only measurement that can be collected from it.

Given this, $f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{z}_{1:k-1})$ can be recursively derived from $f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}|\mathbf{z}_{1:k-2})$ as (29)–(31), shown at the bottom of the next page. Here, (29) is the Bayes' filter time-update step for the PMM; (30) incorporates the Bayes' filter measurement-update step for the PMM; and (31) follows from the fact that $f(\mathbf{z}_k|\mathbf{x}_k, \mathbf{y}_k) = \delta_{\mathbf{y}_k}(\mathbf{z}_k)$. The initial PMM distribution for the recursion is $f(\mathbf{x}_1, \mathbf{y}_1) = f(\mathbf{y}_1|\mathbf{x}_1) \cdot f(\mathbf{x}_1)$ where $f(\mathbf{x}_1)$ is an initial target distribution and $f(\mathbf{y}_1|\mathbf{x}_1)$ is an initial measurement density.

Remark 2: Since $\mathbf{Z}_k = \mathbf{Y}_k$, in the PMM literature the distinction between $\mathbf{z}_j$(a collected measurement) and $\mathbf{y}_j$(a realization of the unknown random variable $\mathbf{Y}_j$) is notationally suppressed:

$$f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{y}_{1:k-1}) = \frac{\int f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}|\mathbf{y}_{1:k-2}) d\mathbf{x}_{k-1}}{\int f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}|\mathbf{y}_{1:k-2}) d\mathbf{x}_{k-1}}. \tag{32}$$

The estimated measurement density at time t_k can be determined from $f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{z}_{1:k-1})$ via

$$f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{y}_{1:k-1}) = \frac{f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{y}_{1:k-1})}{f(\mathbf{x}_k|\mathbf{y}_{1:k-1})} = \frac{f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{y}_{1:k-1})}{\int f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{y}_{1:k-1})d\mathbf{y}_k}. \tag{33}$$

Likewise, if $f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ is the marginal of $f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ then the estimated Markov density at time t_k is

$$f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{1:k-2}) = \frac{\int f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}|\mathbf{y}_{1:k-2})d\mathbf{y}_{k-1}}{f(\mathbf{x}_{k-1}|\mathbf{y}_{1:k-2})}. \tag{34}$$

A PMM reduces to an HMM if, for $k > 1$,

$$f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = f(\mathbf{y}_k|\mathbf{x}_k), f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = f(\mathbf{x}_k|\mathbf{x}_{k-1}) \tag{35}$$

in which case (28) reduces to:

$$f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = f(\mathbf{x}_k|\mathbf{x}_{k-1}) \cdot f(\mathbf{y}_k|\mathbf{x}_k). \tag{36}$$

Thus PMMs significantly weaken HMM's to encompass non-Markov targets and correlated sensor noise [7, p. 4487].

C. SINGLE-TARGET TRACKING USING PMMs

The single-target posterior distribution $f(\mathbf{x}_k|\mathbf{y}_{1:k})$ can be recursively propagated as follows [7, Eq. 12]:

$$f(\mathbf{x}_k|\mathbf{y}_{1:k}) = \frac{\int f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}|\mathbf{y}_{1:k-1})d\mathbf{x}_{k-1}}{\int f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}|\mathbf{y}_{1:k-1})d\mathbf{x}_{k-1}d\mathbf{x}_k}. \tag{37}$$

Note that $f(\mathbf{x}_k|\mathbf{y}_{1:k})$ is related to $f(\mathbf{x}_k, \mathbf{y}_k|y_{1:k-1})$ as follows:

$$f(\mathbf{x}_k|\mathbf{y}_{1:k}) = \frac{f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{y}_{1:k-1})}{f(\mathbf{y}_k|\mathbf{y}_{1:k-1})} = \frac{f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{y}_{1:k-1})}{\int f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{y}_{1:k-1})d\mathbf{x}_k}, \tag{38}$$

so that

$$\hat{\mathbf{x}}_k = \arg\sup_{\mathbf{x}_k \in \mathfrak{I}} f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{y}_{1:k-1}) \tag{39}$$

is the MAP estimate of the target state given $\mathbf{y}_{1:k}$. The predicted target distribution is

$$f(\mathbf{x}_k|\mathbf{y}_{1:k-1}) = \int f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{y}_{1:k-1})d\mathbf{y}_k. \tag{40}$$

IV. MULTITARGET PMM (MPMM)

This is a direct generalization of the single-target PMM. Let $Fin(\mathfrak{I})$ denote the set of multitarget states (i.e., the finite subsets X of $\mathfrak{I}$) and let $Fin(\mathfrak{R})$ denote the set of multitarget measurements (i.e., the finite subsets Z of $\mathfrak{R}$). Then the unknown multitarget state at time t_k is an RFS $\Xi_{k|k} \subseteq \mathfrak{I}$ and the multitarget measurement process is an RFS $\Sigma_k \subseteq \mathfrak{R}$.

Now let the state space be $Fin(\mathfrak{I}) \times Fin(\mathfrak{R})$. Then the unknown state at time t_k is that of the joint multitarget, multimeasurement system, as represented by the random pair $(\Xi_{k|k}, \Sigma_k) \in Fin(\mathfrak{I}) \times Fin(\mathfrak{R})$. What is unknown is not only $\Xi_{k|k}$ and Σ_k but also their statistical correlation, as described by the posterior MPMM density $f(X_k, Y_k|Y_{1:k-1})$. Let us be given:

1. an MPMM transition density

$$f(X_k, Y_k|X_{k-1}, Y_{k-1}) = f(X_k|X_{k-1}, Y_{k-1}) \cdot f(Y_k|X_k, X_{k-1}, Y_{k-1}) \tag{41}$$

   describing the evolution of the MPMM system;
2. the MPMM measurement density $f(Z_k|X_k, Y_k) = \delta_{Y_k}(Z_k)$ of the MPMM system.

Then as with the single-target case, the recursions for the MPMM density $f(X_k, Y_k|Y_{1:k-1})$ and the multitarget posterior $f(X_k|Y_{1:k})$ are, respectively,

$$f(X_k, Y_k|Y_{1:k-1}) = \frac{\int f(X_k, Y_k|X_{k-1}, Y_{k-1}) f(X_{k-1}, Y_{k-1}|Y_{1:k-2})\delta X_{k-1}}{\int f(X_{k-1}, Y_{k-1}|Y_{1:k-2})\delta X_{k-1}} \tag{42}$$

$$f(X_k|Y_{1:k}) = \frac{\int f(X_k, Y_k|X_{k-1}, Y_{k-1}) \cdot f(X_{k-1}|Y_{1:k-1})\delta X_{k-1}}{\int f(X_k, Y_k|X_{k-1}, Y_{k-1}) \cdot f(X_{k-1}|Y_{1:k-1})\delta X_{k-1}\delta X_k}. \tag{43}$$

From (42) we see that the p.g.fl. of $f(X_k, Y_k|Y_{1:k-1})$ is

$$G[h_k, g_k|Y_{1:k-1}] = \int h_k^{X_k} \cdot g_k^{Y_k} \cdot f(X_k, Y_k|Y_{1:k-1})\delta X_k \delta Y_k \tag{44}$$

$$= \frac{\int G[h_k, g_k|X_{k-1}, Y_{k-1}] \cdot f(X_{k-1}, Y_{k-1}|Y_{1:k-2})\delta X_{k-1}}{\int f(X_{k-1}, Y_{k-1}|Y_{1:k-2})\delta X_{k-1}} \tag{45}$$

$$f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{z}_{1:k-1}) = \int f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}|\mathbf{z}_{1:k-1})d\mathbf{x}_{k-1}d\mathbf{y}_{k-1} \tag{29}$$

$$= \frac{\int \begin{pmatrix} f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{z}_{k-1}|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}|\mathbf{z}_{1:k-2}) \end{pmatrix} d\mathbf{x}_{k-1}d\mathbf{y}_{k-1}}{\int f(\mathbf{z}_{k-1}|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}|\mathbf{z}_{1:k-2})d\mathbf{x}_{k-1}d\mathbf{y}_{k-1}} \tag{30}$$

$$= \frac{\int f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{z}_{k-1}) \cdot f(\mathbf{x}_{k-1}, \mathbf{z}_{k-1}|\mathbf{z}_{1:k-2})d\mathbf{x}_{k-1}}{\int f(\mathbf{x}_{k-1}, \mathbf{z}_{k-1}|\mathbf{z}_{1:k-2})d\mathbf{x}_{k-1}}. \tag{31}$$

where the p.g.fl. of $f(X_k, Y_k|X_{k-1}, Y_{k-1})$ is

$$G[h_k, g_k|X_{k-1}, Y_{k-1}] = \int h_k^{X_k} g_k^{Y_k} \cdot f(X_k, Y_k|X_{k-1}, Y_{k-1})\delta X_k \delta Y_k. \quad (46)$$

Thus $f(X_k, Y_k|X_{k-1}, Y_{k-1})$ can be specified by providing a formula for $G[h_k, g_k|X_{k-1}, Y_{k-1}]$ as in the next section.

V. MPMM TRANSITION DENSITIES

The section is organized as follows: the "standard" multitarget motion and measurement models (Section V-A); the original MPMM transition model (Section V-B); the corrected MPMM transition model (Section V-C); and the evolution of the MPMM pair $(\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\})$ according to this model (Section V-D). The remaining subsections address extensions of this basic evolution model: the general evolution of $(\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\})$ (Section V-E); the evolution of $(\{\mathbf{x}_{k-1}\}, \emptyset)$ (Section V-F); the evolution of $(\emptyset, \emptyset)$ (Section V-G); and the evolution of $(\emptyset,\{\mathbf{y}_{k-1}\})$ (Section V-H).

A. THE "STANDARD" MULTITARGET MOTION AND MEASUREMENT MODELS

What is $f(X_k, Y_k|X_{k-1}, Y_{k-1})$? This question was addressed in [9] by endeavoring to infer the form of its p.g.fl. $G[h_k, g_k|X_{k-1}, Y_{k-1}]$ from the p.g.fl.'s $G[h_k|X_{k-1}]$ and $G[g_k|X_k]$ of, respectively, the "standard" multitarget motion and measurement models.

The "standard" multitarget motion model [2, Eq. 14.273], [10, Eq. 5.94], presumes that: (a) individual target motions are statistically independent; (b) the probability that a target with state $\mathbf{x}_{k-1}$ at time t_{k-1} will survive to time t_k is $p_S(\mathbf{x}_{k-1})$; (c) if so, then $f(\mathbf{x}_k|\mathbf{x}_{k-1})$ is the probability (density) that it will transition to a target with state $\mathbf{x}_k$; and (d) $f^B(X_k)$ is the multitarget density of newly-appearing targets, with corresponding p.g.fl. $G^B[h_k]$. This motion model is used to construct the "standard" multitarget Markov density $f(X_k|X_{k-1})$, with corresponding p.g.fl.

$$G[h_k|X_{k-1}] = G^B[h_k] \cdot (1 - p_S + p_S M_{h_k})^{X_{k-1}},$$
$$M_{h_k}(\mathbf{x}_{k-1}) := \int h_k(\mathbf{x}_k) \cdot f(\mathbf{x}_k|\mathbf{x}_{k-1}) d\mathbf{x}_k. \quad (47)$$

The "standard" multitarget measurement model [2, Eq. 14.290], [10, Eq. 5.104] presumes that: (e) all measurements are generated independently of each other; (f) the probability that a target with state $\mathbf{x}_k$ at time t_k will generate a measurement is $p_D(\mathbf{x}_k)$; (g) if so, then $f(\mathbf{y}_k|\mathbf{x}_k)$ is the probability (density) that the measurement is $\mathbf{y}_k$; and (h) $f^\kappa(Z_k)$ is the multi-object density of the clutter process, with corresponding p.g.fl. $G^\kappa[g_k]$. This model is used to construct the standard multitarget measurement density $f(Y_k|X_k)$, with corresponding p.g.fl.

$$G[g_k|X_k] = G^\kappa[g_k] \cdot (1 - p_D + p_D L_{g_k})^{X_k},$$
$$L_{g_k}(\mathbf{x}_k) := \int g_k(\mathbf{y}_k) \cdot f(\mathbf{y}_k|\mathbf{x}_k) d\mathbf{y}_k. \quad (48)$$

B. THE ORIGINAL MPMM TRANSITION MODEL

The multitarget analog of (36) is

$$f(X_k, Y_k|X_{k-1}, Y_{k-1}) = f(X_k|X_{k-1}) \cdot f(Y_k|X_k). \quad (49)$$

Given this, it was shown in [9, Sec. 2.4] via substitution that the p.g.fl. of $f(X_k, Y_k|X_{k-1}, Y_{k-1})$ is

$$G[h_k, g_k|X_{k-1}, Y_{k-1}] = \int h_k^{X_k} g_k^{Y_k} \cdot f(X_k, Y_k|X_{k-1}, Y_{k-1})\delta X_k \delta Y_k \quad (50)$$
$$= G^\kappa[g_k] \cdot G^B[h_k(1 - p_D + p_D L_{g_k})] \cdot G^E[h_k, g_k|X_{k-1}] \quad (51)$$

where $G^\kappa[g_k]$ characterizes the clutter process; $G^B[h_k]$ characterizes the target-appearance process; and where the evolution model is

$$G^E[h_k, g_k|X_{k-1}] = (1 - p_S + p_S M_{h_k(1-p_D+p_D L_{g_k})})^{X_{k-1}}. \quad (52)$$

From this, in [9, Sec. 2.4] it was proposed that if $X_{k-1} \neq \emptyset$ and $Y_{k-1} \neq \emptyset$ then a plausible generalization of $G^E[h_k, g_k|X_{k-1}]$ to $G^E[h_k, g_k|X_{k-1}, Y_{k-1}]$ is

$$G^E[h_k, g_k|X_{k-1}, Y_{k-1}] = \prod_{(\mathbf{x}_{k-1},\mathbf{y}_{k-1})\in X_{k-1}\times Y_{k-1}} G^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}] \quad (53)$$

where the evolution of the PMM pair $(\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\})$ is described by the p.g.fl.

$$G^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}] = \ddot{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \quad (54)$$
$$:= \int \begin{pmatrix} (1 - p_S(\mathbf{x}_{k-1}) + p_S(\mathbf{x}_{k-1}) \cdot h_k(\mathbf{x}_k) \\ \cdot\{1 - p_D(\mathbf{x}_k) + p_D(\mathbf{x}_k) \cdot g_k(\mathbf{y}_k)\}) \end{pmatrix} \times f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) d\mathbf{x}_k d\mathbf{y}_k. \quad (55)$$

Equation (53) thus presumes that the elementary pairs $(\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\})$ evolve independently of each other.

C. THE CORRECTED MPMM TRANSITION MODEL

In retrospect, (53) cannot be correct for at least two reasons. First, when the underlying PMM is an HMM—i.e., when $f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = f(\mathbf{x}_k|\mathbf{x}_{k-1}) \cdot f(\mathbf{y}_k|\mathrm{x}_k)$—then

$$\ddot{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = M_{1-p_S+p_S h_k(1-p_D+p_D L_{g_k})}(\mathbf{x}_{k-1}). \quad (56)$$

Thus from (52-54) we see that

$$G^E[h_k, g_k|X_{k-1}, Y_{k-1}] = \left((1 - p_S + p_S M_{1-p_D+p_D L_{g_k}})^{X_{k-1}}\right)^{|Y_{k-1}|}$$
$$= G^E[h_k, g_k|X_{k-1}]^{|Y_{k-1}|} \quad (57)$$

rather than, as should be the case, $G^E[h_k, g_k|X_{k-1}, Y_{k-1}] = G^E[h_k, g_k|X_{k-1}]$.[1]

Second, suppose that the scenario contains at most a single target obscured by clutter. Then the multi-object system will

[1] The MPMM-CPHD filter proposed in [9] therefore cannot be correct.

always be described by an MPMM pair (X_k, Y_k) with $|X_k| \leq 1$. In this case the evolution of the system must consist of transitions $\ldots \to (X_{k-1}, Y_{k-1}) \to (X_k, Y_k) \to \ldots$ where $|X_{k-1}|, |X_k| \leq 1$. Such an evolution is impossible if it is given as (53) since

$$G^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, Y_{k-1}] = \prod_{\mathbf{y}_{k-1}\in Y_{k-1}} G^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}] \tag{58}$$

describes a system that has as many as $|Y_{k-1}|$ targets.

Accordingly, a corrected model is required, as follows. If $|Y_{k-1}| > 0$ or $|Y_{k-1}| = 0$ define, respectively,

$$\begin{aligned} &\bar{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, Y_{k-1}) \\ &:= \frac{1}{|Y_{k-1}|}\sum_{\mathbf{y}_{k-1}\in Y_{k-1}} \ddot{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ &\quad \bar{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \emptyset) \\ &:= M_{1-p_S+p_S h_k(1-p_D+p_D L_{g_k})}(\mathbf{x}_{k-1}). \end{aligned} \tag{59}$$

Then replace (53) with

$$\begin{aligned} &\tilde{G}^E[h_k, g_k|X_{k-1}, Y_{k-1}] \\ &:= \prod_{\mathbf{x}_{k-1}\in X_{k-1}} \bar{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, Y_{k-1}). \end{aligned} \tag{60}$$

In this case

$$\begin{aligned} &G[h_k, g_k|X_{k-1}, Y_{k-1}] \\ &= G^\kappa[g_k]\cdot G^B[h_k(1-p_D+p_D L_{g_k})]\cdot \tilde{G}^E[h_k, g_k|X_{k-1}, Y_{k-1}] \end{aligned} \tag{61}$$

does properly reduce to (51). Thus the incorrect (58) is replaced with

$$\begin{aligned} &\tilde{G}^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, Y_{k-1}] \\ &\quad = \bar{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, Y_{k-1}). \end{aligned} \tag{62}$$

Equation (62) has a physical interpretation. Suppose that $p_S = 1$ (the target does not disappear) and $p_D = 1$ (the target is detected) and $|Y_{k-1}| > 0$. Then (62) reduces to

$$\begin{aligned} &\tilde{G}^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, Y_{k-1}] \\ &= \frac{1}{|Y_{k-1}|}\sum_{\mathbf{y}_{k-1}\in Y_{k-1}} \int h_k(\mathbf{x}_k)\cdot g_k(\mathbf{y}_k) \\ &\quad \cdot f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) d\mathbf{x}_k d\mathbf{y}_k. \end{aligned} \tag{63}$$

The corresponding distribution is: $f^E(X_k, Y_k|\{\mathbf{x}_{k-1}\}, Y_{k-1}) = 0$ unless $|X_k| = |Y_k| = 1$, in which case

$$\begin{aligned} &f^E(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, Y_{k-1}) \\ &\quad = \frac{1}{|Y_{k-1}|}\sum_{\mathbf{y}_{k-1}\in Y_{k-1}} f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}). \end{aligned} \tag{64}$$

That is, the transition from $(\{\mathbf{x}_{k-1}\}, Y_{k-1})$ to $(\{\mathbf{x}_k\},\{\mathbf{y}_k\})$ is the average transition from $(\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\})$ to $(\{\mathbf{x}_k\},\{\mathbf{y}_k\})$, taken over all $\mathbf{y}_{k-1} \in Y_{k-1}$.

D. EVOLUTION OF THE MPMM PAIR $(\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\})$

It was shown in [9, Sec. 2.4] that $G^E[h_k, g_k|\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\}]$ in (54) is the p.g.fl. of the bivariate multitarget probability distribution $f(X_k, Y_k|\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\})$ that characterizes the following intuitive dynamics model for $|X_k|, |Y_k| \leq 1$:

1. If $\mathbf{x}_{k-1}$ evolves to $\mathbf{x}_k$ and measurement $\mathbf{y}_k$ is collected from $\mathbf{x}_k$ then the probability (density) that this event will occur is:

$$\begin{aligned} &f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}) \\ &\quad = p_S(\mathbf{x}_{k-1})\cdot p_D(\mathbf{x}_k)\cdot f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}). \end{aligned} \tag{65}$$

2. If $\mathbf{x}_{k-1}$ evolves to $\mathbf{x}_k$ but $\mathbf{x}_k$ is not detected, then the probability (density) that this event will occur is, if $f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ is the marginal of the PMM density $f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$:

$$\begin{aligned} &f(\{\mathbf{x}_k\}, \emptyset|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}) \\ &\quad = p_S(\mathbf{x}_{k-1})\cdot(1-p_D(\mathbf{x}_k))\cdot f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}). \end{aligned} \tag{66}$$

3. If $\mathbf{x}_{k-1}$ does not survive to time t_k then no (nonempty) measurement can be collected from it and so the probability that this event will occur is:

$$f(\emptyset, \emptyset|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}) = 1 - p_S(\mathbf{x}_{k-1}). \tag{67}$$

4. If $\mathbf{x}_{k-1}$ does not survive to time t_k and yet measurement $\mathbf{y}_k$ is collected from it, this is an impossibility and so the probability that this event will occur is:

$$f(\emptyset, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}) = 0. \tag{68}$$

Subsections V-E through V-H will address generalizations and extensions of this basic evolution model.

E. GENERAL EVOLUTION OF $(\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\})$

The dynamics model $f(X_k, Y_k|\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\})$ of the previous section is a special case of a more general model, deduced from an arbitrary MPMM density $f(X_k, Y_k|\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\})$ assuming only that $|X_k|, |Y_k| \leq 1$. Since a nonexistent target cannot generate a nonempty measurement, we may assume that $f(\emptyset,\{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\},\{\mathbf{y}_{k-1}\}) = 0$ identically. Define

$$\begin{aligned} &p_S(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ &\quad := \int f(\{\mathbf{x}_k\}, Y_k|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\})\delta Y_k d\mathbf{x}_k \end{aligned} \tag{69}$$

$$\begin{aligned} &p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ &\quad := \frac{\int f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\})d\mathbf{y}_k}{\int f(\{\mathbf{x}_k\}, Y_k|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\})\delta Y_k} \end{aligned} \tag{70}$$

$$\begin{aligned} &f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ &\quad := \frac{f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\})}{\int f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\})d\mathbf{y}_k} \\ &\qquad \cdot \frac{\int f(\{\mathbf{x}_k\}, Y_k|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\})\delta Y_k}{\int f(\{\mathbf{x}_k\}, Y_k|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\})\delta Y_k d\mathbf{x}_k}. \end{aligned} \tag{71}$$

Abbreviate $\tilde{p}_S(\mathbf{x}_{k-1}) != p_S(\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ and $\tilde{p}_D(\mathbf{x}_k) != p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$. Then it is easily shown that

$$f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}) = \tilde{p}_S(\mathbf{x}_{k-1}) \cdot \tilde{p}_D(\mathbf{x}_k) \cdot f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \quad (72)$$

$$f(\{\mathbf{x}_k\}, \emptyset|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}) = \tilde{p}_S(\mathbf{x}_{k-1}) \cdot (1 - \tilde{p}_D(\mathbf{x}_k)) \cdot f(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \quad (73)$$

$$f(\emptyset, \emptyset|\{\mathbf{x}_{k-1}\}, \{\mathbf{y}_{k-1}\}) = 1 - \tilde{p}_S(\mathbf{x}_{k-1}). \quad (74)$$

This reduces to the model of Section V-D if $p_S(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = p_S(\mathbf{x}_{k-1})$ and $p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = p_D(\mathbf{x}_k)$.

Remark 3: To simplify notation, this is what will be assumed later in Section VI-C (though this assumption is not a necessity).

F. EVOLUTION OF ($\{\mathbf{x}_{k-1}\}, \emptyset$)

Now consider $f(X_k, Y_k|\{\mathbf{x}_{k-1}\}, \emptyset)$. It can be presumed that $f(\emptyset, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \emptyset\}) = 0$. Define:

$$p_S(\mathbf{x}_{k-1}, \emptyset) := \int f(\{\mathbf{x}_k\}, Y_k|\{\mathbf{x}_{k-1}\}, \emptyset)\delta Y_k d\mathbf{x}_k \quad (75)$$

$$f(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset) := \frac{\int f(\{\mathbf{x}_k\}, Y_k|\{\mathbf{x}_{k-1}\}, \emptyset)\delta Y_k}{\int f(\{\mathbf{x}_k\}, Y_k|\{\mathbf{x}_{k-1}\}, \emptyset)\delta Y_k d\mathbf{x}_k} \quad (76)$$

$$p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset) := \frac{\int f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \emptyset) d\mathbf{y}_k}{\int f(\{\mathbf{x}_k\}, Y_k|\{\mathbf{x}_{k-1}\}, \emptyset)\delta Y_k} \quad (77)$$

$$f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \emptyset) := \frac{f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \emptyset)}{\int f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \emptyset) d\mathbf{y}_k} \quad (78)$$

where we abbreviate $\tilde{p}_S(\mathbf{x}_{k-1}) != p_S(\mathbf{x}_{k-1}, \emptyset)$ and $\tilde{f}(\mathbf{x}_k|\mathbf{x}_{k-1}) != f(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset)$ and $\tilde{p}_D(\mathbf{x}_k) != p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset)$ and $\tilde{f}(\mathbf{y}_k|\mathbf{x}_k) != f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \emptyset)$.

Remark 4: To simplify notation, it will be assumed in Section VI-C that $p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset)$ does not depend on $\mathbf{x}_{k-1}$ and that $f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \emptyset)$ does not depend on $\mathbf{x}_{k-1}$ or $\emptyset$; in which case $p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset) = p_D(\mathbf{x}_k)$, $f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \emptyset) = f(\mathbf{y}_k|\mathbf{x}_k)$.

Then it is easily shown that

$$f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \emptyset) = \tilde{p}_S(\mathbf{x}_{k-1}) \cdot \tilde{p}_D(\mathbf{x}_k) \cdot \tilde{f}(\mathbf{x}_k|\mathbf{x}_{k-1}) \cdot \tilde{f}(\mathbf{y}_k|\mathbf{x}_k) \quad (79)$$

$$f(\{\mathbf{x}_k\}, \emptyset|\{\mathbf{x}_{k-1}\}, \emptyset) = \tilde{p}_S(\mathbf{x}_{k-1}) \cdot (1 - \tilde{p}_D(\mathbf{x}_k)) \cdot \tilde{f}(\mathbf{x}_k|\mathbf{x}_{k-1}) \quad (80)$$

$$f(\emptyset, \emptyset|\{\mathbf{x}_{k-1}\}, \emptyset) = 1 - \tilde{p}_S(\mathbf{x}_{k-1}). \quad (81)$$

For future reference, the p.g.fl. of $f(X_k, Y_k|\{\mathbf{x}_{k-1}\}, \emptyset)$ is

$$\tilde{G}^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, \emptyset] = 1 - \tilde{p}_S(\mathbf{x}_{k-1}) + \tilde{p}_S(\mathbf{x}_{k-1}) \cdot \tilde{M}_{h_k(1-\tilde{p}_D+\tilde{p}_D\tilde{L}_{g_k})}(\mathbf{x}_{k-1}) \quad (82)$$

where $\tilde{M}_{h_k}(\mathbf{x}_{k-1}) := \int h_k(\mathbf{x}_k) \cdot \tilde{f}(\mathbf{x}_k|\mathbf{x}_{k-1}) d\mathbf{x}_k$ and $\tilde{L}_{g_k}(\mathbf{x}_{k-1}) := \int g_k(\mathbf{y}_k) \cdot \tilde{f}(\mathbf{y}_k|\mathbf{x}_k) d\mathbf{y}_k$. For,

$$\begin{aligned} &\tilde{G}^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, \emptyset] \\ &= \int h_k^{X_k} \cdot g_k^{Y_k} \cdot f(X_k, Y_k|\{\mathbf{x}_{k-1}\}, \emptyset)\delta X_k \delta Y_k \quad (83) \\ &= f(\emptyset, \emptyset|\{\mathbf{x}_{k-1}\}, \emptyset) \quad (84) \\ &\quad + \int h_k(\mathbf{x}_k) \cdot f(\{\mathbf{x}_k\}, \emptyset|\{\mathbf{x}_{k-1}\}, \emptyset) d\mathbf{x}_k \\ &\quad + \int g_k(\mathbf{y}_k) \cdot f(\emptyset, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \emptyset) d\mathbf{y}_k \\ &\quad + \int h_k(\mathbf{x}_k) \cdot g_k(\mathbf{y}_k) \cdot f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\{\mathbf{x}_{k-1}\}, \emptyset) d\mathbf{x}_k d\mathbf{y}_k \\ &= 1 - \tilde{p}_S(\mathbf{x}_{k-1}) \\ &\quad + \tilde{p}_S(\mathbf{x}_{k-1}) \int h_k(\mathbf{x}_k) \cdot (1 - \tilde{p}_D(\mathbf{x}_k)) \cdot \tilde{f}(\mathbf{x}_k|\mathbf{x}_{k-1}) d\mathbf{x}_k \\ &\quad + \tilde{p}_S(\mathbf{x}_{k-1}) \int h_k(\mathbf{x}_k) \cdot \tilde{p}_D(\mathbf{x}_k) \left(\int g_k(\mathbf{y}_k) f(\mathbf{y}_k|\mathbf{x}_k) d\mathbf{y}_k \right) \\ &\quad \cdot \tilde{f}(\mathbf{x}_k|\mathbf{x}_{k-1}) d\mathbf{x}_k. \quad (85) \end{aligned}$$

G. EVOLUTION OF ($\emptyset, \emptyset$)

Consider the MPMM transition $f(X_k, Y_k|\emptyset, \emptyset)$ where, as usual, $f(\emptyset, \{\mathbf{y}_k\}|\emptyset, \emptyset) = 0$. The evolutions $(\emptyset, \emptyset) \to (\{\mathbf{x}_k\}, \{\mathbf{y}_k\})$ or $(\emptyset, \emptyset) \to (\{\mathbf{x}_k\}, \emptyset)$ describe the target's first appearance or its subsequent reappearance. Define

$$q_k^B(\emptyset, \emptyset) := \int f(\{\mathbf{x}_k\}, Y_k|\emptyset, \emptyset)\delta Y_k d\mathbf{x}_k \quad (86)$$

$$s_k^B(\mathbf{x}_k|\emptyset, \emptyset) := \frac{\int f(\{\mathbf{x}_k\}, Y_k|\emptyset, \emptyset)\delta Y_k}{\int f(\{\mathbf{x}_k\}, Y_k|\emptyset, \emptyset)\delta Y_k d\mathbf{x}_k} \quad (87)$$

$$p_D(\mathbf{x}_k|\emptyset, \emptyset) := \frac{\int f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\emptyset, \emptyset) d\mathbf{y}_k}{\int f(\{\mathbf{x}_k\}, Y_k|\emptyset, \emptyset)\delta Y_k} \quad (88)$$

$$f(\mathbf{y}_k|\mathbf{x}_k, \emptyset, \emptyset) := \frac{f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\emptyset, \emptyset)}{\int f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\emptyset, \emptyset) d\mathbf{y}_k} \quad (89)$$

where we abbreviate $\tilde{q}_k^B != q_k^B(\emptyset, \emptyset)$ and $\tilde{s}_k^B(\mathbf{x}_k) != s_k^B(\mathbf{x}_k|\emptyset, \emptyset)$ and $\tilde{p}_D(\mathbf{x}_k) != p_D(\mathbf{x}_k|\emptyset, \emptyset)$ and $\tilde{f}(\mathbf{y}_k|\mathbf{x}_k) != f(\mathbf{y}_k|\mathbf{x}_k, \emptyset, \emptyset)$.

Remark 5: To simplify notation, it will later be assumed in Section VI-C that $q_k^B(\emptyset, \emptyset) = q_k^B$ and $s_k^B(\mathbf{x}_k|\emptyset, \emptyset) = s_k^B(\mathbf{x}_k)$ and $p_D(\mathbf{x}_k|\emptyset, \emptyset) = p_D(\mathbf{x}_k)$ and $f(\mathbf{y}_k|\mathbf{x}_k, \emptyset, \emptyset) = f(\mathbf{y}_k|\mathbf{x}_k)$.

Then it is easily shown that

$$f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\emptyset, \emptyset) = \tilde{q}_k^B \tilde{s}_k^B(\mathbf{x}_k) \cdot \tilde{p}_D(\mathbf{x}_k) \cdot \tilde{f}(\mathbf{y}_k|\mathbf{x}_k) \quad (90)$$

$$f(\{\mathbf{x}_k\}, \emptyset\}|\emptyset, \emptyset) = \tilde{q}_k^B \tilde{s}_k^B(\mathbf{x}_k) \cdot (1 - \tilde{p}_D(\mathbf{x}_k)) \quad (91)$$

$$f(\emptyset, \emptyset\}|\emptyset, \emptyset) = 1 - \tilde{q}_k^B. \quad (92)$$

Here, $\tilde{q}_k^B$ is the "birth" probability—i.e., the nonexistent target $\emptyset$ at time t_{k-1} transitions to a target with state $\mathbf{x}_k$ at time t_k—and $s_k^B(\mathbf{x}_k)$ is its spatial distribution.

Remark 6: The obvious choice for $q_k^B s_k^B(\mathbf{x}_k)$ is the multiobject version of (40):

$$q_k^B s_k^B(\mathbf{x}_k) := f(\{\mathbf{x}_k\}|Y_{1:k-1}) = \int f(\{\mathbf{x}_k\}, Y_k|Y_{1:k-1})\delta Y_k. \quad (93)$$

This might seem theoretically questionable since from (86,87) it would seem to imply that $f(\{\mathbf{x}_k\}, Y_k|\emptyset, \emptyset)$ depends on $Y_{1:k-1}$. However, the choice of $q_k^B s_k^B(\mathbf{x}_k)$ is arbitrary and so we can choose (93) as we please.

For future reference, the p.g.fl. of $f(X_k, Y_k|\emptyset, \emptyset)$ is

$$\tilde{G}^E[h_k, g_k|\emptyset, \emptyset] = 1 - q_k^B + q_k^B s_k^B[h_k(1 - \tilde{p}_D + \tilde{p}_D \tilde{L}_{g_k})] \tag{94}$$

where $\tilde{L}_{g_k}(\mathbf{x}_k) = \int g_k(\mathbf{y}_k) \cdot \tilde{f}(\mathbf{y}_k|\mathbf{x}_k) d\mathbf{y}_k$. For,

$$\tilde{G}^E[h_k, g_k|\emptyset, \emptyset]$$
$$= \int h_k^{X_k} \cdot g_k^{Y_k} \cdot f(X_k, Y_k|\emptyset, \emptyset)\delta X_k \delta Y_k \tag{95}$$
$$= f(\emptyset, \emptyset|\emptyset, \emptyset)_k \tag{96}$$
$$+ \int h_k(\mathbf{x}_k) \cdot f(\{\mathbf{x}_k\}, Y_k|\emptyset, \emptyset) d\mathbf{x}_k$$
$$+ \int g_k(\mathbf{y}_k) \cdot f(\emptyset, \{\mathbf{y}_k\}|\emptyset, \emptyset) d\mathbf{y}_k$$
$$+ \int h_k(\mathbf{x}_k) \cdot g_k(\mathbf{y}_k) \cdot f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\emptyset, \emptyset) d\mathbf{x}_k d\mathbf{y}_k$$
$$= 1 - q_k^B + q_k^B s_k^B[h_k(1 - \tilde{p}_D)] + q_k^B s_k^B[h_k \tilde{p}_D \tilde{L}_{g_k}] \tag{97}$$
$$= 1 - q_k^B + q_k^B s_k^B[h_k(1 - \tilde{p}_D + \tilde{p}_D \tilde{L}_{g_k})]. \tag{98}$$

H. EVOLUTION OF $(\emptyset, \{\mathbf{y}_{k-1}\})$

Consider the MPMM transition $f(X_k, Y_k|\emptyset, \{\mathbf{y}_{k-1}\})$ where, as usual, $f(\emptyset, \{\mathbf{y}_k\}|\emptyset, \{\mathbf{y}_{k-1}\}) = 0$. As in Section V-G, the transitions $(\emptyset, \{\mathbf{y}_{k-1}\}) \to (\{\mathbf{x}_k\}, \{\mathbf{y}_k\})$ or $(\emptyset, \{\mathbf{y}_{k-1}\}) \to (\{\mathbf{x}_k\}, \emptyset)$ describe the target's first appearance or its reappearance after having disappeared. Thus the reasoning in this section is the same as in Section V-G. Define

$$q_k^B(\emptyset, \mathbf{y}_{k-1}) := \int f(\{\mathbf{x}_k\}, Y_k|\emptyset, \{\mathbf{y}_{k-1}\})\delta Y_k d\mathbf{x}_k \tag{99}$$

$$s_k^B(\mathbf{x}_k|\emptyset, \mathbf{y}_{k-1}) := \frac{\int f(\{\mathbf{x}_k\}, Y_k|\emptyset, \{\mathbf{y}_{k-1}\})\delta Y_k}{\int f(\{\mathbf{x}_k\}, Y_k|\emptyset, \{\mathbf{y}_k\})\delta Y_k d\mathbf{x}_k} \tag{100}$$

$$p_D(\mathbf{x}_k|\emptyset, \mathbf{y}_{k-1}) := \frac{\int f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\emptyset, \{\mathbf{y}_{k-1}\}) d\mathbf{y}_k}{\int f(\{\mathbf{x}_k\}, Y_k|\emptyset, \{\mathbf{y}_{k-1}\})\delta Y_k} \tag{101}$$

$$f(\mathbf{y}_k|\mathbf{x}_k, \emptyset, \mathbf{y}_{k-1}) := \frac{f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\emptyset, \{\mathbf{y}_{k-1}\})}{\int f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\emptyset, \{\mathbf{y}_{k-1}\}) d\mathbf{y}_k} \tag{102}$$

where we abbreviate $\tilde{q}_k^B! = q_k^B(\emptyset, \mathbf{y}_{k-1})$ and $\tilde{s}_k^B(\mathbf{x}_k)! = s_k^B(\mathbf{x}_k|\emptyset, \mathbf{y}_{k-1})$ and $\tilde{p}_D(\mathbf{x}_k)! = p_D(\mathbf{x}_k|\emptyset, \mathbf{y}_{k-1})$ and $\tilde{f}(\mathbf{y}_k|\mathbf{x}_k)! = f(\mathbf{y}_k|\mathbf{x}_k, \emptyset, \mathbf{y}_{k-1})$.

Remark 7: To simplify notation, it will later be assumed in Section VI-C that $q_k^B(\emptyset, \mathbf{y}_{k-1}) = q_k^B$ and $s_k^B(\mathbf{x}_k|\emptyset, \mathbf{y}_{k-1}) = s_k^B(\mathbf{x}_k)$ and $p_D(\mathbf{x}_k|\emptyset, \mathbf{y}_{k-1}) = p_D(\mathbf{x}_k)$ and $f(\mathbf{y}_k|\mathbf{x}_k, \emptyset, \mathbf{y}_{k-1}) = f(\mathbf{y}_k|\mathbf{x}_k)$.

As in Section V-G it follows that

$$f(\{\mathbf{x}_k\}, \{\mathbf{y}_k\}|\emptyset, \{\mathbf{y}_{k-1}\}) = \tilde{q}_k^B \tilde{s}_k^B(\mathbf{x}_k) \cdot \tilde{p}_D(\mathbf{x}_k) \cdot \tilde{f}(\mathbf{y}_k|\mathbf{x}_k) \tag{103}$$

$$f(\{\mathbf{x}_k\}, \emptyset\}|\emptyset, \{\mathbf{y}_{k-1}\}) = \tilde{q}_k^B \tilde{s}_k^B(\mathbf{x}_k) \cdot (1 - \tilde{p}_D(\mathbf{x}_k)) \tag{104}$$

$$f(\emptyset, \emptyset\}|\emptyset, \{\mathbf{y}_{k-1}\}) = 1 - \tilde{q}_k^B \tag{105}$$

and that the corresponding p.g.fl. is

$$\tilde{G}^E[h_k, g_k|\emptyset, \{\mathbf{y}_{k-1}\}] = 1 - q_k^B + q_k^B s_k^B[h_k(1 - \tilde{p}_D + \tilde{p}_D \tilde{L}_{g_k})] \tag{106}$$

from which we conclude that

$$\tilde{G}^E[h_k, g_k|\emptyset, Y_{k-1}] = 1 - q_k^B + q_k^B s_k^B[h_k(1 - \tilde{p}_D + \tilde{p}_D \tilde{L}_{g_k})]. \tag{107}$$

VI. THE BERNOULLI MPMM FILTER

The section is organized as follows: the Bernoulli filter (Section VI-A); the Bernoulli MPMM filter (Section VI-B); transition p.g.fl.'s for the Bernoulli MPMM filter (Section VI-C); summary of the Bernoulli MPMM filter (Section VI-D); derivation of the Bernoulli MPMM filter update when $Y_{k-1} = \emptyset$ (Section VI-E); and derivation of the Bernoulli MPMM filter update when $Y_{k-1} \neq \emptyset$ (Section VI-F).

A. THE BERNOULLI FILTER

The Bernoulli filter [1], [2, Sec. 14.7] is the special case of the multitarget Bayes filter

$$\ldots \to f(X_{k-1}|Z_{1:k-1}) \to f(X_k|Z_{1:k-1}) \to f(X_k|Z_{1:k}) \to \ldots$$

when at most a single target is present—i.e., when $|X_{k-1}|, |X_k| \leq 1$ for all $k \geq 1$. Since

$$f(\emptyset|Z_{1:k-1}) = 1 - \int f(\{\mathbf{x}_k\}|Z_{1:k-1}) d\mathbf{x}_k,$$
$$f(\emptyset|Z_{1:k}) = 1 - \int f(\{\mathbf{x}_k\}|Z_{1:k}) d\mathbf{x}_k, \tag{108}$$

the Bernoulli filter is mathematically equivalent to a filter that propagates the PHD's $D_{k|k-1}(\mathbf{x}_k) != D(\mathbf{x}_k|Z_{1:k-1}) != f(\{\mathbf{x}_k\}|Z_{1:k-1})$ and $D_{k|k}(\mathbf{x}_k) != D(\mathbf{x}_k|Z_{1:k}) != f(\{\mathbf{x}_k\}|Z_{1:k})$. The time-update equation and measurement-update equation are, respectively,

$$D_{k|k-1}(\mathbf{x}_k) = q_k^B s_k^B(\mathbf{x}_k) \cdot (1 - D_{k-1|k-1}[1]) + D_{k-1|k-1}[p_S M_{\mathbf{x}_k}] \tag{109}$$

$$D_{k|k}(\mathbf{x}_k) = \frac{\ell_{Z_k}(\mathbf{x}_k) \cdot D_{k|k-1}(\mathbf{x}_k)}{1 - D_{k|k-1}[1] + D_{k|k-1}[\ell_{Z_k}]}. \tag{110}$$

where ℓ_{Z_k} was defined in (2).

Note that, as presented in [2, Sec. 14.7], the Bernoulli filter propagates two items, not one: the probability of target existence $p_{k|k} = \int f(\{\mathbf{x}_k\}|Z_{1:k}) d\mathbf{x}_k$ and the target spatial distribution $f_{k|k}(\mathbf{x}_k) = f(\{\mathbf{x}_k\}|Z_{1:k})/p_{k|k}$. But it is clear that the filter in (109,110) differs from that in [2, Sec. 14.7] only by a change of notation (although the former is significantly simpler in form). A tutorial on the (original) Bernoulli filter can be found in [22].

State estimation using $D_{k|k}$ is as in [2]. The target exists if $p_{k|k} > \tau$ for some threshold $\tau > 1/2$; and if it exists, its state is the MAP estimate $\text{argsup}_{\mathbf{x}} D_{k|k}(\mathbf{x})$.

Eqs. (109,110) can be consolidated by substitution into the single-step update equation, (111), as shown at the bottom of the next page, where $M_{\ell_{Z_k}}(\mathbf{x}_{k-1})$ was defined in (8).

B. THE BERNOULLI MPMM FILTER

This is a special case of the MPMM filter

$$\ldots \to f(X_{k-1}, Y_{k-1}|Y_{1:k-2}) \to f(X_k, Y_k|Z_{1:k-1}) \to \ldots$$

when $|X_{k-1}|, |X_k| \leq 1$ for all $k \geq 1$. Section V-G described a simplified target-appearance model. This model will allow us to avoid the factor G^B in (61) by assuming that $G^B[h_k] = 1$ identically.

Remark 8: Note that this simplified target-appearance model would not be acceptable in the multitarget case, since then the number of targets could never increase.

Given that $G^B[h_k] = 1$, the p.g.fl. (61) of the MPMM transition density reduces to:

$$G[h_k, g_k|X_{k-1}, Y_{k-1}] = G^\kappa[g_k] \cdot \tilde{G}^E[h_k, g_k|X_{k-1}, Y_{k-1}] \tag{112}$$

where either $X_{k-1} = \emptyset$ or $X_{k-1} = \{\mathbf{x}_{k-1}\}$ for all $k \geq 1$. From (44), the p.g.fl. update for the Bernoulli MPMM filter is

$$G[h_k, g_k|Y_{1:k-1}] = \frac{\int G[h_k, g_k|X_{k-1}, Y_{k-1}] \cdot f(X_{k-1}, Y_{k-1}|Y_{1:k-2})\delta X_{k-1}}{\int f(X_{k-1}, Y_{k-1}|Y_{1:k-2})\delta X_{k-1}} \tag{113}$$

where the numerator is

$$\begin{aligned}\int G[h_k, g_k|X_{k-1}, Y_{k-1}] \cdot f(X_{k-1}, Y_{k-1}|Y_{1:k-2})\delta X_{k-1} \\ = G^\kappa[g_k] \cdot \tilde{G}^E[h_k, g_k|\emptyset, Y_{k-1}] \cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) \\ +G^\kappa[g_k] \int \tilde{G}^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, Y_{k-1}] \\ \cdot f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2}) d\mathbf{x}_{k-1}\end{aligned} \tag{114}$$

and the denominator is

$$\int f(X_{k-1}, Y_{k-1}|Y_{1:k-2})\delta X_{k-1} = f(\emptyset, Y_{k-1}|Y_{1:k-2}) + \int f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2}) d\mathbf{x}_{k-1}. \tag{115}$$

C. STATE-TRANSITION P.G.FL.'s FOR THE BERNOULLI MPMM FILTER

We therefore need formulas for $\tilde{G}^E[h_k, g_k|X_{k-1}, Y_{k-1}]$ in the following four cases:

1. $X_{k-1} = \emptyset$ and $Y_{k-1} = \emptyset$: By (94),

$$\tilde{G}^E[h_k, g_k|\emptyset, \emptyset] = 1 - q_k^B + q_k^B s_k^B[h_k(1-\tilde{p}_D+\tilde{p}_D\tilde{L}_{g_k})] \tag{116}$$

   where (see (86-89)): $\tilde{p}_D(\mathbf{x}_k)! = p_D(\mathbf{x}_k|\emptyset, \emptyset)$ and $\tilde{f}(\mathbf{y}_k|\mathbf{x}_k)! = f(\mathbf{y}_k|\mathbf{x}_k, \emptyset, \emptyset)$ and $\tilde{L}_{g_k}(\mathbf{x}_k) = \int g_k(\mathbf{y}_k)\cdot \tilde{f}(\mathbf{y}_k|\mathbf{x}_k)d\mathbf{y}_k$. To simplify notation, in what follows we will:

   a. further abbreviate $p_D(\mathbf{x}_k) != p_D(\mathbf{x}_k|\emptyset, \emptyset)$ and $f(\mathbf{x}_k|\mathbf{x}_{k-1}) != f(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset, \emptyset)$.

2. $X_{k-1} = \{\mathbf{x}_{k-1}\}$ and $Y_{k-1} = \emptyset$: By (82),

$$\tilde{G}^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, \emptyset] = 1 - \tilde{p}_S(\mathbf{x}_{k-1}) + \tilde{p}_S(\mathbf{x}_{k-1}) \cdot \tilde{M}_{h_k(1-\tilde{p}_D+\tilde{p}_D\tilde{L}_{g_k})}(\mathbf{x}_{k-1}) \tag{117}$$

   where (see (75-78): $\tilde{p}_S(\mathbf{x}_{k-1})! = p_S(\mathbf{x}_{k-1}|\emptyset)$, $\tilde{f}(\mathbf{x}_k|\mathbf{x}_k)! = f(\mathbf{x}_k|\mathbf{x}_k, \emptyset)$, $\tilde{p}_D(\mathbf{x}_k)! = p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset)$, $\tilde{f}(\mathbf{y}_k|\mathbf{x}_k)! = f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \emptyset)$, $\tilde{M}_{h_k}(\mathbf{x}_{k-1}) = \int h_k(\mathbf{x}_k) \cdot \tilde{f}(\mathbf{x}_k|\mathbf{x}_{k-1})d\mathbf{x}_k$, $\tilde{L}_{g_k}(\mathbf{x}_k) = \int g_k(\mathbf{y}_k) \cdot \tilde{f}(\mathbf{y}_k|\mathbf{x}_k)d\mathbf{y}_k$. To simplify notation, in what follows we will:

   a. further abbreviate $p_S(\mathbf{x}_{k-1}) != p_S(\mathbf{x}_{k-1}, \emptyset)$ and $f(\mathbf{x}_k|\mathbf{x}_{k-1}) != f(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset)$; and

   b. assume that $p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset) = p_D(\mathbf{x}_k|\emptyset, \emptyset) = p_D(\mathbf{x}_k)$ and $f(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \emptyset) = f(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset, \emptyset) = f(\mathbf{x}_k|\mathbf{x}_{k-1})$.

3. $X_{k-1} = \emptyset$ and $Y_{k-1} \neq \emptyset$: By (107),

$$\tilde{G}^E[h_k, g_k|\emptyset, Y_{k-1}] = 1 - q_k^B + q_k^B s_k^B[h_k(1 - \tilde{p}_D +\tilde{p}_D\tilde{L}_{g_k})]. \tag{118}$$

4. $X_{k-1} = \{\mathbf{x}_{k-1}\}$ and $Y_{k-1} \neq \emptyset$: By (62),

$$G^E[h_k, g_k|\{\mathbf{x}_{k-1}\}, Y_{k-1}] = \bar{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, Y_{k-1}) \tag{119}$$

   where (see (69-71): $\tilde{p}_S(\mathbf{x}_{k-1})! = p_S(\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ and $\tilde{p}_D(\mathbf{x}_k)! = p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ and where $\bar{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, Y_{k-1})$ was defined in (59). To simplify notation we will:

   a. assume that $p_S(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = p_S(\mathbf{x}_{k-1}, \emptyset) = p_S(\mathbf{x}_{k-1})$ and $p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) = p_D(\mathbf{x}_k|\mathbf{x}_{k-1}, \emptyset) = p_D(\mathbf{x}_k|\emptyset, \emptyset) = p_D(\mathbf{x}_k)$.

D. SUMMARY OF THE BERNOULLI MPMM FILTER

We are given $f(\emptyset, Y_{k-1}|Z_{1:k-2})$, $f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Z_{1:k-2})$ and that

$$K_{Y_{k-1}} := f(\emptyset, Y_{k-1}|Z_{1:k-2}) + \int f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Z_{1:k-2}) d\mathbf{x}_{k-1}. \tag{120}$$

Then the updates $f(\emptyset, Y_k|Z_{1:k-1})$ and $f(\{\mathbf{x}_k\}, Y_k|Z_{1:k-1})$ are given by the following two recursive formulas:

1. If $Z_{k-1} = \emptyset$ is collected then:

$$\begin{aligned}f(\emptyset, Y_k|Z_{1:k-1}) \\ = A_{Y_k}(\emptyset) \cdot f(\emptyset, \emptyset|Z_{1:k-2}) \\ + \int B_{Y_k}(\mathbf{x}_{k-1}) \cdot f(\{\mathbf{x}_{k-1}\}, \emptyset|Z_{1:k-2}) d\mathbf{x}_{k-1}\end{aligned} \tag{121}$$

$$D_{k|k}(\mathbf{x}_k) = \frac{\ell_{Z_k}(\mathbf{x}_k) \cdot \left(q_k^B s_k^B(\mathbf{x}_k) \cdot (1 - D_{k-1|k-1}[1]) + D_{k-1|k-1}[p_S M_{\mathbf{x}_k}]\right)}{\left(\begin{array}{l}(1-q_k^B s_k^B+q_k^B s_k^B[\ell_{Z_k}]) \cdot (1-D_{k-1|k-1}[1]) \\ +D_{k-1|k-1}[1 - p_S + p_S M_{\ell_{Z_k}}]\end{array}\right)} \tag{111}$$

$$\begin{aligned}f(\{\mathbf{x}_k\}, Y_k|Z_{1:k-1}) &= A_{Y_k}(\mathbf{x}_k)\cdot f(\emptyset,\emptyset|Z_{1:k-2}) \\ &\quad + \int B_{Y_k}(\mathbf{x}_k|\mathbf{x}_{k-1})\cdot f(\{\mathbf{x}_{k-1}\},\emptyset|Z_{1:k-2})d\mathbf{x}_{k-1}\end{aligned} \tag{122}$$

where

$$A_{Y_k}(\emptyset) = K_\emptyset^{-1}\kappa_k(Y_k)\cdot(1-q_k^B) \tag{123}$$

$$B_{Y_k}(\mathbf{x}_{k-1}) = K_\emptyset^{-1}\kappa_k(Y_k)\cdot(1-p_S(\mathbf{x}_{k-1})) \tag{124}$$

$$A_{Y_k}(\mathbf{x}_k) = K_\emptyset^{-1}\kappa_k(Y_k)\cdot\ell_{Y_k}(\mathbf{x}_k)\cdot q_k^B s_k^B(\mathbf{x}_k) \tag{125}$$

$$B_{Y_k}(\mathbf{x}_k|\mathbf{x}_{k-1}) = K_\emptyset^{-1}\kappa_k(Y_k)\cdot\ell_{Y_k}(\mathbf{x}_k)\cdot p_S(\mathbf{x}_{k-1}) \cdot f(\mathbf{x}_k|\mathbf{x}_{k-1}) \tag{126}$$

$$\ell_{Y_k}(\mathbf{x}_k) = 1 - p_D(\mathbf{x}_k) + p_D(\mathbf{x}_k) \times \sum_{\mathbf{y}_k\in Y_k} L_{\mathbf{y}_k}(\mathbf{x}_k)\cdot\frac{\kappa_k(Y_k-\{\mathbf{y}_k\})}{\kappa_k(Y_k)}. \tag{127}$$

2. If $Z_{k-1}\neq\emptyset$ is collected then:

$$\begin{aligned}f(\emptyset, Y_k|Z_{1:k-1}) &= K_{Z_{k-1}}^{-1}\kappa_k(Y_k)\cdot(1-q_k^B)\cdot f(\emptyset, Z_{k-1}|Y_{1:k-2}) \\ &\quad + K_{Z_{k-1}}^{-1}\kappa_k(Y_k) \\ &\quad \cdot\int(1-p_S(\mathbf{x}_{k-1}))\cdot f(\{\mathbf{x}_{k-1}\}, Z_{k-1}|Z_{1:k-2})d\mathbf{x}_{k-1}\end{aligned} \tag{128}$$

$$\begin{aligned}f(\{\mathbf{x}_k\}, Y_k|Z_{1:k-1}) &= K_{Z_{k-1}}^{-1}\kappa_k(Y_k)\cdot\ell_{Y_k}(\mathbf{x}_k) \\ &\quad \cdot q_k^B s_k^B(\mathbf{x}_k)\cdot f(\emptyset, Z_{k-1}|Y_{1:k-2}) \\ &\quad + K_{Z_{k-1}}^{-1}\kappa_k(Y_k) \\ &\quad \cdot\int p_S(\mathbf{x}_{k-1})\cdot\ell_{Y_k,Z_{k-1}}(\mathbf{x}_k|\mathbf{x}_{k-1}) \\ &\quad \cdot f(\{\mathbf{x}_{k-1}\}, Z_{k-1}|Z_{1:k-2})d\mathbf{x}_{k-1}\end{aligned} \tag{129}$$

These equations are derived in Appendix A.

Remark 9: In regard to (129), consider the following special case: $\kappa_k(Y_k) = 0$ identically *(no clutter);* $p_S(\mathbf{x}_{k-1}) = 1$ *(target never disappears); and* $p_D(\mathbf{x}_k) = 1$ *(perfect detection); in which case* $|X_k| = |Y_k| = 1$ *for all* $k \geq 1$. *Then Eq. (129) should reduce to (31)—which is indeed the case.*

VII. CONCLUSION

The Bernoulli filter is a general solution for tracking a single disappearing and reappearing target, using a sensor whose observations are corrupted by missed detections and a general, known clutter process. The Bernoulli filter presumes restrictive independence assumptions, namely a hidden Markov model (HMM) structure. That is, the current target state depends only on the previous target state; and the measurement that it generates depends only on its current state.

Pieczynski's pairwise Markov model (PMM) relaxes these restrictions. In it, the current target state can additionally depend on the previous measurement; and the current measurement can additionally depend on the previous measurement and the previous target state.

In this paper we: (i) generalized PMMs to the multitarget case (MPMM); (ii) devised a theoretically rigorous formula for the "standard" MPMM transition density (see (60,61)); (iii) derived transition models for the elementary MPMM pairs (X_k, Y_k) with $|X_k|, |Y_k| \leq 1$ (Sections V-D through V-H); (iv) used them to derive the Bernoulli MPMM filter (an MPMM generalization of the Bernoulli filter, Section VI); and then used it to derive the PMM Bernoulli filter (a generalization of the Bernoulli filter that obeys PMM rather than HMM sensor and target statistics.

Future work will be devoted to generalization of the PMM Bernoulli filter to multiple correlated sensors.

APPENDIX A
DERIVATION OF THE BERNOULLI MPMM FILTER

The derivation has two parts: when $Y_{k-1} = \emptyset$ (Appendix A.1) and when when $Y_{k-1}\neq\emptyset$ (Appendix A.2).

1) DERIVATION OF THE BERNOULLI MPMM FILTER UPDATE WHEN $Y_{k-1} = \emptyset$

Let us turn to the derivation of the Bernoulli MPMM filter update when $Y_{k-1} = \emptyset$. Let

$$\begin{aligned}K &\overset{!}{=} \int f(X_{k-1},\emptyset|Y_{1:k-2})\delta X_{k-1} \\ &= f(\emptyset,\emptyset|Y_{1:k-2}) + \int f(\{\mathbf{x}_{k-1}\},\emptyset|Y_{1:k-2})d\mathbf{x}_{k-1}.\end{aligned} \tag{130}$$

Then from (114), (116), and (117),

$$\begin{aligned}K\cdot G[h_k, g_k|Y_{1:k-1}] &= G^\kappa[g_k]\cdot\tilde{G}^E[h_k, g_k|\emptyset,\emptyset]\cdot f(\emptyset,\emptyset|Y_{1:k-2}) \\ &\quad + G^\kappa[g_k]\int\tilde{G}^E[h_k, g_k|\{\mathbf{x}_{k-1}\},\emptyset]\cdot f(\{\mathbf{x}_{k-1}\}, \\ &\quad \emptyset|Y_{1:k-2})d\mathbf{x}_{k-1}\end{aligned} \tag{131}$$

$$\begin{aligned}&= G^\kappa[g_k]\cdot\begin{pmatrix}1-q_k^B \\ +q_k^B s_k^B[h_k(1-p_D+p_D L_{g_k})]\end{pmatrix} \\ &\quad \cdot f(\emptyset,\emptyset|Y_{1:k-2}) \\ &\quad + G^\kappa[g_k]\int\begin{pmatrix}1-p_S(\mathbf{x}_{k-1}) \\ +p_S(\mathbf{x}_{k-1})\cdot M_{h_k(1-p_D+p_D L_{g_k})}(\mathbf{x}_{k-1})\end{pmatrix} \\ &\quad \cdot f(\{\mathbf{x}_{k-1}\},\emptyset|Y_{1:k-2})d\mathbf{x}_{k-1}.\end{aligned} \tag{132}$$

For fixed h_k, abbreviate

$$\begin{aligned}L[g_k] &\overset{!}{=} \left(1-q_k^B+q_k^B s_k^B[h_k(1-p_D+p_D L_{g_k})]\right) \\ &\quad \cdot f(\emptyset,\emptyset|Y_{1:k-2}) \\ &\quad + \int\begin{pmatrix}1-p_S(\mathbf{x}_{k-1}) \\ +p_S(\mathbf{x}_{k-1})\cdot M_{h_k(1-p_D+p_D L_{g_k})}(\mathbf{x}_{k-1})\end{pmatrix} \\ &\quad \cdot f(\{\mathbf{x}_{k-1}\},\emptyset|Y_{1:k-2})d\mathbf{x}_{k-1}\end{aligned} \tag{133}$$

in which case

$$K\cdot G[h_k, g_k|Y_{1:k-1}] = G^\kappa[g_k]\cdot L[g_k]. \tag{134}$$

For $W \subseteq Y_k$, note that

$$\frac{\delta L}{\delta W}[g_k] = \begin{cases} L[g_k], & \text{if} \quad W = \emptyset \\ l(\mathbf{y}), & \text{if} \quad W = \{\mathbf{y}\} \\ 0, & \text{if} \quad |W| > 1 \end{cases} \tag{135}$$

where

$$l(\mathbf{y}) \stackrel{\text{abbr.}}{=} q_k^B s_k^B[h_k p_D L_{\mathbf{y}}] \cdot f(\emptyset, \emptyset|Y_{1:k-2}) + \int p_S(\mathbf{x}_{k-1}) \cdot M_{h_k p_D L_{\mathbf{y}}}(\mathbf{x}_{k-1}) \cdot f(\{\mathbf{x}_{k-1}\}, \emptyset|Y_{1:k-2}) d\mathbf{x}_{k-1} \tag{136}$$

and where $L_{\mathbf{y}}(\mathbf{x}) \stackrel{\text{abbr.}}{=} f(\mathbf{y}|\mathbf{x})$. Thus from the product rule for functional derivatives [2, p. 389],

$$\begin{aligned} K \cdot \frac{\delta G}{\bullet \delta Y_k}[h_k, g_k|Y_{1:k-1}] &= \sum_{W \subseteq Y_k} \left(\frac{\delta}{\delta(Y_k - W)} G^\kappa[g_k] \right) \cdot \frac{\delta L}{\delta W}[g_k] & (137) \\ &= \sum_{W \subseteq Y_k} \left(\frac{\delta G^\kappa}{\delta(Y_k - W)}[g_k] \right) \cdot \frac{\delta L}{\delta W}[g_k] & (138) \\ &= \sum_{W \subseteq Y_k : |W| \le 1} \frac{\delta G^\kappa}{\delta(Y_k - W)}[g_k] \cdot \frac{\delta L}{\delta W}[g_k] & (139) \\ &= \frac{\delta G^\kappa}{\delta Y_k}[g_k] \cdot L[g_k] + \sum_{\mathbf{y}_k \in Y_k} \frac{\delta G^\kappa}{\delta(Y_k - \{\mathbf{y}_k\})}[g_k] \cdot l(\mathbf{y}_k) & (140) \\ &= \frac{\delta G^\kappa}{\delta Y_k}[g_k] \cdot \left(L[g_k] + \sum_{\mathbf{y}_k \in Y_k} \frac{\frac{\delta G^\kappa}{\delta(Y_k - \{\mathbf{y}_k\})}[g_k] \cdot l(\mathbf{y}_k)}{\frac{\delta G^\kappa}{\delta Y_k}[g_k]} \right) & (141) \end{aligned}$$

and so substituting $g_k = 0$ and using the fact that

$$\frac{\delta G^\kappa}{\delta Y_k}[0] = \kappa_k(Y_k), \qquad \frac{\delta G^\kappa}{\delta(Y_k - \{\mathbf{y}\})}[0] = \kappa_k(Y_k - \{\mathbf{y}\}) \tag{142}$$

we get

$$K \cdot \frac{\delta G}{\bullet \delta Y_k}[h_k, 0|Y_{1:k-1}] = \kappa_k(Y_k) \cdot \left(L[0] + \sum_{\mathbf{y}_k \in Y_k} l(\mathbf{y}) \cdot \frac{\kappa_k(Y_k - \{\mathbf{y}_k\})}{\kappa_k(Y_k)} \right) \tag{143}$$

where

$$L[0] = \left(1 - q_k^B + q_k^B s_k^B[h_k(1-p_D)]\right) \cdot f(\emptyset, \emptyset|Y_{1:k-2}) + \int \begin{pmatrix} 1 - p_S(\mathbf{x}_{k-1}) \\ + p_S(\mathbf{x}_{k-1}) \cdot M_{h_k(1-p_D)}(\mathbf{x}_{k-1}) \end{pmatrix} \cdot f(\{\mathbf{x}_{k-1}\}, \emptyset|Y_{1:k-2}) d\mathbf{x}_{k-1} \tag{144}$$

and

$$l(\mathbf{y}) = q_k^B s_k^B[h_k p_D L_{\mathbf{y}}] \cdot f(\emptyset, \emptyset|Y_{1:k-2}) + \int p_S(\mathbf{x}_{k-1}) \cdot M_{h_k p_D L_{\mathbf{y}}}(\mathbf{x}_{k-1}) \cdot f(\{\mathbf{x}_{k-1}\}, \emptyset|Y_{1:k-2}) d\mathbf{x}_{k-1}. \tag{145}$$

Thus after substitution and collection of like terms we get:

$$\frac{\delta G}{\bullet \delta Y_k}[h_k, 0|Y_{1:k-1}] = A_{h_k} \cdot f(\emptyset, \emptyset|Y_{1:k-2}) + \int B_{h_k}(\mathbf{x}_{k-1}) \cdot f(\{\mathbf{x}_{k-1}\}, \emptyset|Y_{1:k-2}) d\mathbf{x}_{k-1} \tag{146}$$

where

$$A_{h_k} = K^{-1}\kappa_k(Y_k) \cdot \begin{pmatrix} 1 - q_k^B + q_k^B s_k^B[h_k(1-p_D)] \\ + \sum_{\mathbf{y}_k \in Y_k} q_k^B s_k^B[h_k p_D L_{\mathbf{y}_k}] \cdot \frac{\kappa_k(Y_k - \{\mathbf{y}_k\})}{\kappa_k(Y_k)} \end{pmatrix} \tag{147}$$

$$B_{h_k}(\mathbf{x}_{k-1}) = K^{-1}\kappa_k(Y_k) \cdot \begin{pmatrix} 1 - p_S(\mathbf{x}_{k-1}) \\ + p_S(\mathbf{x}_{k-1}) \cdot M_{h_k(1-p_D)}(\mathbf{x}_{k-1}) \\ + p_S(\mathbf{x}_{k-1}) \sum_{\mathbf{y}_k \in Y_k} M_{h_k p_D L_{\mathbf{y}_k}}(\mathbf{x}_{k-1})] \frac{\kappa_k(Y_k - \{\mathbf{y}_k\})}{\kappa_k(Y_k)} \end{pmatrix}. \tag{148}$$

Consequently, and as claimed,

$$f(\emptyset, Y_k|Z_{1:k-1}) = A_{Y_k}(\emptyset) \cdot f(\emptyset, \emptyset|Z_{1:k-2}) + \int B_{Y_k}(\mathbf{x}_{k-1}) \cdot f(\{\mathbf{x}_{k-1}\}, \emptyset|Z_{1:k-2}) d\mathbf{x}_{k-1} \tag{149}$$

$$f(\{\mathbf{x}_k\}, Y_k|Z_{1:k-1}) = A_{Y_k}(\mathbf{x}_k) \cdot f(\emptyset, \emptyset|Z_{1:k-2}) + \int B_{Y_k}(\mathbf{x}_k|\mathbf{x}_{k-1}) \cdot f(\{\mathbf{x}_{k-1}\}, \emptyset|Z_{1:k-2}) d\mathbf{x}_{k-1} \tag{150}$$

where

$$A_{Y_k}(\emptyset) = K^{-1}\kappa_k(Y_k) \cdot (1 - q_k^B) \tag{151}$$

$$B_{Y_k}(\mathbf{x}_{k-1}) = K^{-1}\kappa_k(Y_k) \cdot (1 - p_S(\mathbf{x}_{k-1})) \tag{152}$$

$$A_{Y_k}(\mathbf{x}_k) = K^{-1}\kappa_k(Y_k) \cdot \ell_{Y_k}(\mathbf{x}_k) \cdot q_k^B s_k^B(\mathbf{x}_k) \tag{153}$$

$$B_{Y_k}(\mathbf{x}_k|\mathbf{x}_{k-1}) = K^{-1}\kappa_k(Y_k) \cdot \ell_{Y_k}(\mathbf{x}_k) \cdot p_S(\mathbf{x}_{k-1}) \cdot M_{\mathbf{x}_k}(\mathbf{x}_{k-1}) \tag{154}$$

and

$$\ell_{Y_k}(\mathbf{x}_k) = 1 - p_D(\mathbf{x}_k) + p_D(\mathbf{x}_k) \sum_{\mathbf{y}_k \in Y_k} L_{\mathbf{y}_k}(\mathbf{x}_k) \cdot \frac{\kappa_k(Y_k - \{\mathbf{y}_k\})}{\kappa_k(Y_k)}. \tag{155}$$

2) DERIVATION OF THE BERNOULLI MPMM FILTER UPDATE WHEN $Y_{k-1} \neq \emptyset$

Now turn to the derivation of the Bernoulli MPMM filter update when $Y_{k-1} \neq \emptyset$. From (114), (119), and (59) we have the following:

$$\begin{aligned} K \cdot G[h_k, g_k|Y_{1:k-1}] &= G^\kappa[g_k] \cdot \tilde{G}^E[h_k, g_k|\emptyset, Y_{k-1}] \cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) \\ &+ \frac{G^\kappa[g_k]}{|Y_{k-1}|} \times \int \left(\sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \ddot{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \right) \cdot f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2}) d\mathbf{x}_{k-1} \end{aligned} \tag{156}$$

with normalization factor

$$K = \int f(X_{k-1}, Y_{k-1}|Y_{1:k-2})\delta X_{k-1} = f(\emptyset, Y_{k-1}|Y_{1:k-2}) + \int f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1} \quad (157)$$

and where, by (118), $G^E[h_k, g_k|\emptyset, Y_{k-1}] = 1 - q_k^B + q_k^B s_k^B[h_k(1 - p_D + p_D L_{g_k})]$. Thus

$$\begin{aligned} &K \cdot G[h_k, g_k|Y_{1:k-1}] \\ &= G^\kappa[g_k] \cdot (1 - q_k^B + q_k^B s_k^B[h_k(1 - p_D + p_D L_{g_k})]) \\ &\cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) \\ &K \cdot G[h_k, g_k|Y_{1:k-1}] \\ &= G^\kappa[g_k] \cdot (1 - q_k^B + q_k^B s_k^B[h_k(1 - p_D + p_D L_{g_k})]) \\ &\cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}). \end{aligned} \quad (158)$$

First note that, setting $h_k = 0$,

$$\begin{aligned} &K \cdot G[0, g_k|Y_{1:k-1}] \\ &= G^\kappa[g_k] \cdot (1 - q_k^B) \cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) \\ &+ \frac{G^\kappa[g_k]}{|Y_{k-1}|} \int \left(\sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \ddot{M}_{1-p_S}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \right) \\ &\cdot f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1} \end{aligned} \quad (159)$$

where

$$\begin{aligned} &\ddot{M}_{1-p_S}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ &= \int (1 - p_S(\mathbf{x}_{k-1})) \cdot f(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})d\mathbf{x}_k d\mathbf{y}_k \\ &= 1 - p_S(\mathbf{x}_{k-1}) \end{aligned} \quad (160)$$

and so

$$\begin{aligned} &K \cdot G[0, g_k|Y_{1:k-1}] \\ &= G^\kappa[g_k] \cdot (1 - q_k^B) \cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) \\ &+ G^\kappa[g_k] \int (1 - p_S(\mathbf{x}_{k-1})) \cdot f(\{\mathbf{x}_{k-1}\}, \\ &Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1}. \end{aligned} \quad (161)$$

Taking $\delta/\delta Y_k$ of both sides with respect to g_k and then setting $g_k = 0$ we get:

$$\begin{aligned} &K \cdot f(\emptyset, Y_k|Y_{1:k-1}) \\ &= \kappa_k(Y_k) \\ &\cdot \begin{pmatrix} (1 - q_k^B) \cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) \\ + \int (1 - p_S(\mathbf{x}_{k-1})) \cdot f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1} \end{pmatrix}. \end{aligned} \quad (162)$$

Now note that

$$\begin{aligned} &K \cdot \frac{\delta G}{\delta \mathbf{x}_k^\bullet}[h_k, g_k|Y_{1:k-1}] \\ &= G^\kappa[g_k] \cdot q_k^B s_k^B[\delta_{\mathbf{x}_k}(1 - p_D + p_D L_{g_k})] \cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) \\ &+ \frac{G^\kappa[g_k]}{|Y_{k-1}|} \\ &\times \int \left(\sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \frac{\delta}{\delta \mathbf{x}_k^\bullet} \ddot{M}_{1-p_S+p_S h_k(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \right) \end{aligned} \quad (163)$$

$$\begin{aligned} &\cdot f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1} \\ &= \frac{G^\kappa[g_k]}{|Y_{k-1}|} \int \left(\sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \ddot{M}_{p_S \delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \right) \\ &\cdot f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1} \end{aligned} \quad (164)$$

and so

$$\begin{aligned} &K \cdot \frac{\delta G}{\delta \mathbf{x}_k^\bullet}[0, g_k|Y_{1:k-1}] \\ &= G^\kappa[g_k] \cdot q_k^B s_k^B[\delta_{\mathbf{x}_k}(1 - p_D + p_D L_{g_k})] \cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) \\ &+ \frac{G^\kappa[g_k]}{|Y_{k-1}|} \int \left(\sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \ddot{M}_{p_S \delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \right) \\ &\cdot f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1}. \end{aligned} \quad (165)$$

Thus

$$\begin{aligned} &K \cdot \frac{\delta G}{\delta \mathbf{x}_k^\bullet \delta Y_k}[0, g_k|Y_{1:k-1}] \\ &= \left(\frac{\delta}{\delta Y_k} \left(G^\kappa[g_k] \cdot q_k^B s_k^B[\delta_{\mathbf{x}_k}(1 - p_D + p_D L_{g_k})] \right) \right) \\ &\cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) \\ &+ \frac{1}{|Y_{k-1}|} \sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \int \frac{\delta}{\delta Y_k} \\ &\times \left(G^\kappa[g_k] \cdot \ddot{M}_{p_S \delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \right) \\ &\cdot f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1}. \end{aligned} \quad (166)$$

For the first term in this sum, note that

$$\begin{aligned} &\frac{\delta}{\delta Y_k} \left(G^\kappa[g_k] \cdot q_k^B s_k^B[\delta_{\mathbf{x}_k}(1 - p_D + p_D L_{g_k})] \right) \\ &= \sum_{W \subseteq Y_k} \left(\frac{\delta G^\kappa}{\delta(Y_k - W)}[g_k] \right) \\ &\times \left(\frac{\delta}{\delta W} q_k^B s_k^B[\delta_{\mathbf{x}_k}(1 - p_D + p_D L_{g_k})] \right) \\ &+ \frac{1}{|Y_{k-1}|} \sum_{\mathbf{y}_{k-1} \in Y_{k-1}} \int \frac{\delta}{\delta Y_k} \\ &\times \left(G^\kappa[g_k] \cdot \ddot{M}_{p_S \delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \right) \\ &\cdot f(\{\mathbf{x}_{k-1}\}, Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1} \end{aligned} \quad (167)$$

where, for $W \subseteq Y_k$,

$$\begin{aligned} &\frac{\delta}{\delta W} q_k^B s_k^B[\delta_{\mathbf{x}_k}(1 - p_D + p_D L_{g_k})] \\ &= \begin{cases} q_k^B s_k^B[\delta_{\mathbf{x}_k}(1 - p_D + p_D L_{g_k})], & \textit{if} \quad W = \emptyset \\ q_k^B s_k^B[\delta_{\mathbf{x}_k} p_D L_{\mathbf{y}_k}], & \textit{if} \quad W = \{\mathbf{y}_k\} \\ 0, & \textit{if} \quad \textit{otherwise} \end{cases} \end{aligned} \quad (168)$$

and so

$$\left[\frac{\delta}{\delta Y_k}\left(G^\kappa[g_k]\cdot q_k^B s_k^B[\delta_{\mathbf{x}_k}(1-p_D+p_D L_{g_k})]\right)\right]_{g_k=0}$$
$$= G^\kappa[g_k]\cdot q_k^B s_k^B[\delta_{\mathbf{x}_k}(1-p_D)] + \sum_{\mathbf{y}_k\in Y_k}\kappa_k(Y_k-\{\mathbf{y}_k\})\cdot q_k^B s_k^B[\delta_{\mathbf{x}_k}p_D L_{\mathbf{y}_k}] \tag{169}$$

where

$$q_k^B s_k^B[\delta_{\mathbf{x}_k}(1-p_D)] = (1-p_D(\mathbf{x}_k))\cdot q_k^B s_k^B(\mathbf{x}_k) \tag{170}$$
$$q_k^B s_k^B[\delta_{\mathbf{x}_k}p_D L_{\mathbf{y}_k}] = p_D(\mathbf{x}_k)\cdot L_{\mathbf{y}_k}(\mathbf{x}_k)\cdot q_k^B s_k^B(\mathbf{x}_k) \tag{171}$$

and so

$$\left[\frac{\delta}{\delta Y_k}\left(G^\kappa[g_k]\cdot q_k^B s_k^B[\delta_{\mathbf{x}_k}(1-p_D+p_D L_{g_k})]\right)\right]_{g_k=0}$$
$$= \kappa_k(Y_k) \cdot\left(1-p_D(\mathbf{x}_k)+p_D(\mathbf{x}_k)\sum_{\mathbf{y}_k\in Y_k}L_{\mathbf{y}_k}(\mathbf{x}_k)\cdot\frac{\kappa_k(Y_k-\{\mathbf{y}_k\})}{\kappa_k(Y_k)}\right) \cdot q_k^B s_k^B(\mathbf{x}_k) \tag{172}$$
$$= \kappa_k(Y_k)\cdot \ell_{Y_k}(\mathbf{x}_k)\cdot q_k^B s_k^B(\mathbf{x}_k). \tag{173}$$

For the second term in (166), note that

$$\frac{\delta}{\delta Y_k}\left(G^\kappa[g_k]\cdot \ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1})\right)$$
$$= \sum_{W\subseteq Y_k}\left(\frac{\delta G^\kappa}{\delta(Y_k-W)}[g_k]\right) \times\left(\frac{\delta}{\delta W}\ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1})\right) \tag{174}$$

and where for $W\subseteq Y_k$,

$$\frac{\delta}{\delta W}\ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}) = \begin{cases}\ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}), & \text{if} \quad W=\emptyset\\ \ddot{M}_{p_S\delta_{\mathbf{x}_k}p_D\delta_{\mathbf{y}_k}}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}), & \text{if}\quad W=\{\mathbf{y}_k\}\\ 0, & \text{if}\quad \text{otherwise}.\end{cases} \tag{175}$$

Thus

$$\frac{\delta}{\delta Y_k}\left(G^\kappa[g_k]\cdot\ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1})\right)$$
$$= \frac{\delta G^\kappa}{\delta Y_k}[g_k]\cdot\ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}) + \sum_{\mathbf{y}_k\in Y_k}\frac{\delta G^\kappa}{\delta(Y_k-\{\mathbf{y}_k\})}[g_k]\cdot\ddot{M}_{p_S\delta_{\mathbf{x}_k}p_D\delta_{\mathbf{y}_k}}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \tag{176}$$

$$\left[\frac{\delta}{\delta Y_k}\left(G^\kappa[g_k]\cdot\ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D+p_D g_k)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1})\right)\right]_{g_k=0}$$
$$= \kappa_k(Y_k)\cdot\ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}) + \sum_{\mathbf{y}_k\in Y_k}\kappa_k(Y_k-\{\mathbf{y}_k\})\cdot\ddot{M}_{p_S\delta_{\mathbf{x}_k}p_D\delta_{\mathbf{y}_k}}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \tag{177}$$

where

$$\ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}) = \int p_S(\mathbf{x}_{k-1})\cdot\delta_{\mathbf{x}_k}(\mathbf{u}_k)\cdot(1-p_D(\mathbf{u}_k)) \cdot f(\mathbf{u}_k,\mathbf{v}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})d\mathbf{u}_k d\mathbf{v}_k \tag{178}$$
$$= p_S(\mathbf{x}_{k-1})\cdot(1-p_D(\mathbf{x}_k))\cdot f(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \tag{179}$$

$$f(\{\mathbf{x}_k\},Y_k|Y_{1:k-1}) = \kappa_k(Y_k)\cdot\ell_{Y_k}(\mathbf{x}_k)\cdot q_k^B s_k^B(\mathbf{x}_k)\cdot f(\emptyset,Y_{k-1}|Y_{1:k-2}) + \frac{K^{-1}\kappa_k(Y_k)}{|Y_{k-1}|} \cdot\sum_{\mathbf{y}_{k-1}\in Y_{k-1}}\int\left(\begin{array}{l} p_S(\mathbf{x}_{k-1})\cdot(1-p_D(\mathbf{x}_k))\cdot f(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})\\ +\sum_{\mathbf{y}_k\in Y_k}\frac{\left(\begin{array}{l}f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})\\ \cdot p_S(\mathbf{x}_{k-1})\cdot p_D(\mathbf{x}_k)\cdot\kappa_k(Y_k-\{\mathbf{y}_k\})\end{array}\right)}{\kappa_k(Y_k)}\end{array}\right) \tag{183}$$

$$\cdot f(\{\mathbf{x}_{k-1}\},Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1}. = \kappa_k(Y_k)\cdot\ell_{Y_k}(\mathbf{x}_k)\cdot q_k^B s_k^B(\mathbf{x}_k)\cdot f(\emptyset,Y_{k-1}|Y_{1:k-2}) + K^{-1}\kappa_k(Y_k)\int p_S(\mathbf{x}_{k-1}) \cdot\left(\begin{array}{l}\frac{1-p_D(\mathbf{x}_k)}{|Y_{k-1}|}\sum_{\mathbf{y}_{k-1}\in Y_{k-1}}f(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})\\ +\frac{p_D(\mathbf{x}_k)}{|Y_{k-1}|}\sum_{\mathbf{y}_{k-1}\in Y_{k-1}}\sum_{\mathbf{y}_k\in Y_k}f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})\cdot\frac{\kappa_k(Y_k-\{\mathbf{y}_k\})}{\kappa_k(\mathbf{y}_k)}\end{array}\right) \cdot f(\{\mathbf{x}_{k-1}\},Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1} \tag{184}$$

$$= \kappa_k(Y_k)\cdot\ell_{Y_k}(\mathbf{x}_k)\cdot q_k^B s_k^B(\mathbf{x}_k)\cdot f(\emptyset,Y_{k-1}|Y_{1:k-2}) + K^{-1}\kappa_k(Y_k)\int p_S(\mathbf{x}_{k-1})\cdot\ell_{Y_k,Y_{k-1}}(\mathbf{x}_k|\mathbf{x}_{k-1}) \cdot f(\{\mathbf{x}_{k-1}\},Y_{k-1}|Y_{1:k-2})d\mathbf{x}_{k-1} \tag{185}$$

$$\ddot{M}_{p_S\delta_{\mathbf{x}_k}p_D\delta_{\mathbf{y}_k}}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}) = \int p_S(\mathbf{x}_{k-1})\cdot\delta_{\mathbf{x}_k}(\mathbf{u}_k)\cdot p_D(\mathbf{u}_k)\cdot\delta_{\mathbf{y}_k}(\mathbf{v}_k) \cdot f(\mathbf{u}_k,\mathbf{v}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})d\mathbf{u}_k d\mathbf{v}_k \tag{180}$$

$$= p_S(\mathbf{x}_{k-1})\cdot p_D(\mathbf{x}_k)\cdot f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \tag{181}$$

and so

$$\left[\frac{\delta}{\delta Y_k}\left(G^\kappa[g_k]\cdot\ddot{M}_{p_S\delta_{\mathbf{x}_k}(1-p_D+p_Dg_k)}(\mathbf{x}_{k-1},\mathbf{y}_{k-1})\right)\right]_{g_k=0} = \kappa_k(Y_k)\cdot p_S(\mathbf{x}_{k-1}) \cdot\begin{pmatrix}(1-p_D(\mathbf{x}_k))\cdot f(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})\\ +p_D(\mathbf{x}_k)\sum_{\mathbf{y}_i\in Y_k} f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})\cdot\frac{\kappa_k(Y_k-\{\mathbf{y}_k\})}{\kappa_k(Y_k)}\end{pmatrix}. \tag{182}$$

Thus setting $g_k=0$ in (166) and substituting (173) and (182), we get (129), (183)–(185), as shown at the bottom of the previous page, where

$$\ell_{Y_k,Y_{k-1}}(\mathbf{x}_k|\mathbf{x}_{k-1}) = \frac{1-p_D(\mathbf{x}_k)}{|Y_{k-1}|}\sum_{\mathbf{y}_{k-1}\in Y_{k-1}} f(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1}) +\frac{p_D(\mathbf{x}_k)}{|Y_{k-1}|}\sum_{\mathbf{y}_{k-1}\in Y_{k-1}}\sum_{\mathbf{y}_k\in Y_k} f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \cdot\frac{\kappa_k(Y_k-\{\mathbf{y}_k\})}{\kappa_k(Y_k)}. \tag{186}$$

APPENDIX B
DERIVATION OF THE PMM BERNOULLI FILTER

The derivation has two parts: when $Z_{k-1}=\emptyset$ (Appendix B.1) and when $Z_{k-1}\neq\emptyset$ (Appendix B.2).

1) DERIVATION OF THE THE PMM BERNOULLI FILTER WHEN $Y_{k-1}=\emptyset$

We are to verify (10) assuming that $Z_{k-1}=\emptyset$. The multitarget version of (38) is

$$f(X_k|Z_{1:k}) = \frac{f(X_k,Z_k|Z_{1:k-1})}{\int f(X_k,Z_k|Z_{1:k-1})\delta X_k} = \frac{f(X_k,Z_k|Y_{1:k-1})}{f(Z_k|Z_{k-1})}. \tag{187}$$

For a Bernoulli filter, $|X_k|\leq 1$ and so the updated PHD is

$$D_{k|k}(\mathbf{x}_k) = f(\{\mathbf{x}_k\}|Z_{1:k}) = \frac{f(\{\mathbf{x}_k\},Z_k|Z_{1:k-1})}{f(\emptyset,Z_k|Z_{1:k-1})+\int f(\{\mathbf{x}_k\},Z_k|Z_{1:k-1})d\mathbf{x}_k}. \tag{188}$$

Recall from (120) that

$$K_{Z_{k-1}} = f(\emptyset,Z_{k-1}|Z_{1:k-2}) + \int f(\{\mathbf{x}_{k-1}\},Z_{k-1}|Z_{1:k-2})d\mathbf{x}_{k-1}. \tag{189}$$

Thus on the one hand, (121) can be written as:

$$K_\emptyset\cdot f(\emptyset,\emptyset|Z_{1:k-1}) = \kappa_k(Y_k)\begin{pmatrix}(1-q_k^B)\cdot f(\emptyset,\emptyset|Z_{1:k-2})\\ +\int(1-p_S(\mathbf{x}_{k-1}))\cdot f(\{\mathbf{x}_{k-1}\},\emptyset|Z_{1:k-2})d\mathbf{x}_{k-1}\end{pmatrix} \tag{190}$$

$$= \kappa_k(Y_k)\cdot\begin{pmatrix}(1-q_k^B)\cdot f(\emptyset|Z_{1:k-1})\\ +\int(1-p_S(\mathbf{x}_{k-1}))\cdot f(\{\mathbf{x}_{k-1}\}|Z_{1:k-1})d\mathbf{x}_{k-1}\end{pmatrix} \tag{191}$$

$$= \kappa_k(Y_k)\cdot\begin{pmatrix}(1-q_k^B)\cdot f(\emptyset|Z_{1:k-1})\\ +\int(1-p_S(\mathbf{x}_{k-1}))\cdot D_{k-1|k-1}(\mathbf{x}_{k-1})d\mathbf{x}_{k-1}\end{pmatrix} \tag{192}$$

$$= \kappa_k(Y_k)\cdot\Big((1-q_k^B)\cdot(1-D_{k-1|k-1}[1]) +D_{k-1|k-1}[1-p_S]\Big). \tag{193}$$

On the other hand, (122) can be written as:

$$K_\emptyset\cdot f(\{\mathbf{x}_k\},Z_k|Z_{1:k-1}) = \kappa_k(Y_k)\cdot\ell_{Z_k}(\mathbf{x}_k) \tag{194}$$

$$\cdot\begin{pmatrix}q_k^Bs_k^B(\mathbf{x}_k)\cdot f(\emptyset,\emptyset|Z_{1:k-2})\\ +\int p_S(\mathbf{x}_{k-1})\cdot M_{\mathbf{x}_k}(\mathbf{x}_{k-1})\cdot f(\{\mathbf{x}_{k-1}\},\emptyset|Z_{1:k-2})d\mathbf{x}_{k-1}\end{pmatrix}$$

$$= \kappa_k(Y_k)\cdot\ell_{Z_k}(\mathbf{x}_k)\cdot\begin{pmatrix}q_k^Bs_k^B(\mathbf{x}_k)\cdot(1-D_{k-1|k-1}[1])\\ +D_{k-1|k-1}[p_DM_{\mathbf{x}_k}]\end{pmatrix}. \tag{195}$$

Note that

$$K_\emptyset\int f(\{\mathbf{x}_k\},Z_k|Z_{1:k-1})d\mathbf{x}_k = \kappa_k(Y_k)\cdot\Big(q_k^Bs_k^B[\ell_{Y_k}]\cdot(1-D_{k-1|k-1}[1]) +D_{k-1|k-1}[p_DM_{\ell_{Z_k}}]\Big). \tag{196}$$

Thus adding (193) and (195) we get:

$$K_\emptyset\cdot f(Z_k|Z_{1:k-1}) = \kappa_k(Y_k)\cdot\begin{pmatrix}(1-q_k^B)\cdot(1-D_{k-1|k-1}[1])\\ +D_{k-1|k-1}[1-p_S]\\ +q_k^Bs_k^B[\ell_{Y_k}]\cdot(1-D_{k-1|k-1}[1])\\ +D_{k-1|k-1}[p_DM_{\ell_{Z_k}}]\end{pmatrix} \tag{197}$$

$$= \kappa_k(Y_k)\cdot\begin{pmatrix}(1-q_k^B+q_k^Bs_k^B[\ell_{Z_k}])\cdot(1-D_{k-1|k-1}[1])\\ +D_{k-1|k-1}[1-p_D+p_DM_{\ell_{Z_k}}]\end{pmatrix}. \tag{198}$$

Consequently, the single-step PHD update is, (199), as shown at the bottom of the page, which is identical to (111),

$$D_{k|k}(\mathbf{x}_k) = \frac{\ell_{Z_k}(\mathbf{x}_k)\cdot(q_k^Bs_k^B(\mathbf{x}_k)\cdot(1-D_{k-1|k-1}[1])+D_{k-1|k-1}[p_SM_{\mathbf{x}_k}])}{\begin{pmatrix}(1-q_k^Bs_k^B+q_k^Bs_k^B[\ell_{Z_k}])\cdot(1-D_{k-1|k-1}[1])\\ +D_{k-1|k-1}[1-p_S+p_SM_{\ell_{Z_k}}]\end{pmatrix}} \tag{199}$$

$$K_{Z_{k-1}} \cdot f(\{\mathbf{x}_k\}, Z_k|Z_{1:k-1}) = \kappa_k(Y_k) \cdot \begin{pmatrix} \ell_{Z_k}(\mathbf{x}_k) \cdot q_k^B s_k^B(\mathbf{x}_k) \cdot f(\emptyset, Z_{k-1}|Y_{1:k-2}) \\ + \int (p_S(\mathbf{x}_{k-1}) \cdot \ell_{Z_k,Z_{k-1}}(\mathbf{x}_k|\mathbf{x}_{k-1}) \cdot f(\{\mathbf{x}_{k-1}\}, Z_{k-1}|Z_{1:k-2}) d\mathbf{x}_{k-1} \end{pmatrix} \tag{204}$$

$$= \kappa_k(Y_k) \cdot \begin{pmatrix} \ell_{Z_k}(\mathbf{x}_k) \cdot q_k^B s_k^B(\mathbf{x}_k) \cdot f(\emptyset|Z_{1:k-1}) \\ +D_{k-1|k-1}[p_S \ell_{Z_k,Z_{k-1},\mathbf{x}_k}] \end{pmatrix} \tag{205}$$

$$= \kappa_k(Y_k) \cdot \begin{pmatrix} \ell_{Z_k}(\mathbf{x}_k) \cdot q_k^B s_k^B(\mathbf{x}_k) \cdot (1 - D_{k-1|k-1}[1]) \\ +D_{k-1|k-1}[p_S \ell_{Z_k,Z_{k-1},\mathbf{x}_k}] \end{pmatrix} \tag{206}$$

$$D_{k|k}(\mathbf{x}_k) = \frac{\ell_{Z_k}(\mathbf{x}_k) \cdot \left(q_k^B s_k^B(\mathbf{x}_k) \cdot (1 - D_{k-1|k-1}[1]) + D_{k-1|k-1}[p_S \ell_{Z_k,Z_{k-1},\mathbf{x}_k}]\right)}{\begin{pmatrix} (1 - q_k^B s_k^B + q_k^B s_k^B[\ell_{Z_k}]) \cdot (1 - D_{k-1|k-1}[1]) \\ +D_{k-1|k-1}[1 - p_S + p_S \tilde{\ell}_{Z_k,Z_{k-1}}] \end{pmatrix}}. \tag{210}$$

the conventional Bernoulli filter single-step update. But when $Z_{k-1} = \emptyset$,

$$\begin{aligned} &\ell_{Z_k,Z_{k-1},\mathbf{x}_k}(\mathbf{x}_{k-1}) \\ &\quad = \ell_{Z_k,Z_{k-1}}(\mathbf{x}_k|\mathbf{x}_{k-1}) = \ell_{Z_k}(\mathbf{x}_k) \cdot M_{\mathbf{x}_k}(\mathbf{x}_{k-1}), \\ &\quad\quad \tilde{\ell}_{Z_k,Z_{k-1}}(\mathbf{x}_{k-1}) \\ &\quad = M_{\ell_{Z_k}}(\mathbf{x}_{k-1}) \end{aligned} \tag{200}$$

and thus (10) reduces to (119) when $Z_{k-1} = \emptyset$.

2) DERIVATION OF THE PMM BERNOULLI FILTER WHEN $Y_{k-1} \neq \emptyset$

We are to verify (10) assuming that $Z_{k-1} \neq \emptyset$. On the one hand, (138) can be written as:

$$\begin{aligned} &K_{Z_{k-1}} \cdot f(\emptyset, Z_k|Z_{1:k-1}) \\ &= \kappa_k(Y_k) \\ &\quad \cdot \begin{pmatrix} (1 - q_k^B) \cdot f(\emptyset, Z_{k-1}|Y_{1:k-2}) \\ + \int (1 - p_S(\mathbf{x}_{k-1})) \cdot f(\{\mathbf{x}_{k-1}\}, Z_{k-1}|Z_{1:k-2}) d\mathbf{x}_{k-1} \end{pmatrix} \end{aligned} \tag{201}$$

$$\begin{aligned} &= \kappa_k(Y_k) \\ &\quad \cdot \begin{pmatrix} (1 - q_k^B) \cdot f(\emptyset|Z_{1:k-1}) \\ + \int (1 - p_S(\mathbf{x}_{k-1})) \cdot D_{k-1|k-1}(\mathbf{x}_{k-1}) d\mathbf{x}_{k-1} \end{pmatrix} \end{aligned} \tag{202}$$

$$\begin{aligned} &= \kappa_k(Y_k) \cdot \Big((1 - q_k^B) \cdot (1 - D_{k-1|k-1}[1]) \\ &\quad + D_{k-1|k-1}[1 - p_S]\Big). \end{aligned} \tag{203}$$

On the other hand, (139) can be written as (204)–(206), shown at the top of the page, and so

$$\begin{aligned} &\int f(\{\mathbf{x}_k\}, Z_k|Z_{1:k-1}) d\mathbf{x}_k \\ &\quad = \kappa_k(Y_k) \cdot \begin{pmatrix} q_k^B s_k^B[\ell_{Z_k}] \cdot (1 - D_{k-1|k-1}[1]) \\ +D_{k-1|k-1}[p_S \tilde{\ell}_{Z_k,Z_{k-1}}] \end{pmatrix}. \end{aligned} \tag{207}$$

Thus adding (203) and (207) we get:

$$\begin{aligned} &f(Z_k|Z_{k-1}) \\ &= \kappa_k(Y_k) \cdot \begin{pmatrix} (1 - q_k^B) \cdot (1 - D_{k-1|k-1}[1]) \\ +D_{k-1|k-1}[1 - p_S] \\ +q_k^B s_k^B[\ell_{Z_k}] \cdot (1 - D_{k-1|k-1}[1]) \\ +D_{k-1|k-1}[p_S \tilde{\ell}_{Z_k,Z_{k-1}}] \end{pmatrix} \end{aligned} \tag{208}$$

$$= \kappa_k(Y_k) \cdot \begin{pmatrix} \left(1 - q_k^B + q_k^B s_k^B[\ell_{Z_k}]\right) \cdot (1 - D_{k-1|k-1}[1]) \\ +D_{k-1|k-1}[1 - p_S + p_S \tilde{\ell}_{Z_k,Z_{k-1}}] \end{pmatrix}. \tag{209}$$

Thus the single-step PHD update does result in (10), (210), as shown at the top of the page.

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