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A New Approach to Solve Fuzzy Data Envelopment Analysis Model Based on Uncertainty

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ABSTRACT Many problems in operations research including problems from management, production planning and scheduling, transportation, location, and many others necessitate decision making in the presence of uncertainty. Therefore, many theories and methodologies have been developed to deal with optimization problems under uncertainty in general. To understand uncertain data envelopment analysis models, the process was started by introducing the definition of uncertain variables, uncertain vectors, fuzzy linear programming (FLP). Selection criteria of an incoming vector in improving uncertain linear programming is established along with its unbounded nature. This paper based on the established theoretical framework proposes an alternative linear programming model that can include some uncertainty information. Finally, input-oriented CCR model with fuzzy variables is developed and an effective approach for measuring efficiency is demonstrated with a numerical example.

INDEX TERMS Uncertain theory, uncertain linear programming, uncertain data envelopment analysis, fuzzy linear programming.

I. INTRODUCTION

For efficiency analysis of business entities or organisation, data envelopment analysis (DEA) is a well-known technique. DEA requires precise inputs and outputs but the data for the real-world problem is imprecise and in the form of qualitative, linguistic. Fuzzy set theory and DEA can be integrated by fuzzy DEA. Fuzzy DEA models take the form of fuzzy linear programming (FLP) and we can find few papers solved by this method.

In the literature of fuzzy DEA, only a few papers have been published on solving these problems. However, some authors have solved fuzzy DEA problems by 1) the tolerance approach, 2) the ranking approach, and 3) the parametric programming approach. The efficient frontier of the DEA model is subject to uncertainty since the observed inputs and observed outputs are usually uncertain. Sengupta described two estimation viewpoints: The Least Absolute Value (LAV) of error criterion, and chance-constrained programming [2]. Sengupta combined a tolerance approach [7], [29] with a fuzzy goal approach to solving fuzzy DEA.

In the case of linear membership functions, the fuzzy DEA model was to maximize the satisfaction level of the constraints in the fuzzy DEA model. Recently, Guo and Tanaka [6] have introduced fuzzy inputs and fuzzy outputs in the input-oriented CCR (Charnes, Cooper and Rhodes) model [3]. They also studied the relationship between Regression Analysis (RA) and DEA. The RA and CCR model were considered as two special cases of a Goal Programming problem. Kao and Liu [11] formulated two DEA models: 1) The model that gives an upper limit efficiency, and 2) the model that gives a lower limit efficiency. Then, an interval-valued efficiency can be constructed from these two extreme efficiencies. Further, Kao and Liu [10] studied fuzzy DEA models in the light of the model proposed by Maeda *et al.* [16]. It falls into the category of parametric programming models. Kahraman and Tolga [8] evaluated two alternative Computer Integrated Manufacturing Systems (CIMS) by using a fuzzy version of the CCR model. They used the tolerance approach from Kahraman and Tolga [8] to solve the model.

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Inaccurate or uncertain data may be the result of non-quantifiable (say, qualitative measurements, expert opinions), incomplete and unavailable information (confidential or missed information). Inaccurate or indeterminate data is often expressed with bounded intervals, sequence data, or fuzzy numbers. In recent years, many researchers have developed fuzzy DEA models to deal with situations where some input and output data are inaccurate or indeterminate. There are a relatively large number of articles in the DEA fuzzy literature. Fuzzy set theory has been widely used to model uncertainty in DEA. Applications of fuzzy set theory in DEA are usually divided into four groups (Lertworasirikul [15], Lertworasirikul [14], Karsak [12]): the tolerance approach, the α-based approach, the fuzzy order approach, and the possible approach. Although most of these approaches are powerful, they usually have some theoretical and/or computational limitations and sometimes relate to a very specific situation (e.g. Olfati *et al.* [27]). The tolerance approach was one of the first fuzzy DEA models developed by Sengupta [26] and further improved by Kahraman *et al.* [9]. In this approach, the main idea is to incorporate uncertainty into DEA models by defining tolerance levels. The approach α-level is one of another popular fuzzy models of DEA. This is evident from the number of documents at the α-level and published in the DEA fuzzy literature. In this approach, the main idea is to convert the fuzzy CCR model into a pair of parametric programs to find the lower and upper limits of the α-level member functions of the efficiency score. The proposed method in this paper is familiar to α-level model but the advantage of this method obtains the result with more easy calculations in a shorter period of time. The fuzzy order approach is also another popular technique that has attracted much attention in the fuzzy DEA literature. In this approach, the main idea is to find the fuzzy efficiency score of the DMU using fuzzy linear programs that require the order of fuzzy sets.

Application of fuzzy DEA to the newspaper preprint insertion manufacturing process was given by Girod and Triantis [5]. Kao and Liu [10] also used the concept of the fuzzy sets theory for representing imprecise data. They were studying the efficiencies of university libraries in Taiwan as a real-world problem. Kao and Liu [10] also provided the projection method to improve the efficiency of an inefficient DMU. After the fuzzy efficiencies for all DMUs were obtained, they used the method of Charnes and Cooper [3] called a simple approach to ranking a group of aggregated fuzzy utilities to rank the fuzzy efficiency scores.

The rest of the paper is organized as follows: Section 2 presents some necessary backgrounds of uncertain theory and particular uncertain parameters is given. Then an uncertain linear programming (ULP) model will be described and some important relevant result will be proved in Section 3. Section 4 presents the application of a new approach to uncertain DEA problem. Section 5 present Fuzzy mathematical programming with application in DEA models. Section 6 introduces one of the convenient kinds of FLP problem. Section 7 will introduce uncertain data envelopment analysis. Section 8 presents a numerical example. Finally, Section 9 concludes the paper.

II. DEFINITIONS AND FUNDAMENTAL BACKGROUNDS

In this section, some concepts including definitions and notation of uncertainty theory which are taken from [17], [18], [24] and [20] are brought.

Definition 1: An uncertain variable is a measurable function X from an uncertainty space $(\Gamma, \Omega, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{X \in B\} = \{\gamma \in \Gamma | X(\gamma) \in B\}$$

Definition 2: The vector $(X_1, X_2, \ldots, X_n)$ is called an uncertain vector if $X_1, X_2, \ldots, X_n$ are uncertain variables.

Definition 3: An uncertainty distribution $\Phi : R \longrightarrow [0, 1]$ of the uncertain variable, X is defined by

$$\Phi(x) = \mathcal{M}\{X \leq x\}$$

Note that usually, we consider linear and normal uncertain variables for our study and the other kind is omitted here.

Definition 4: Let X be an uncertain variable. Then, the expected value of X is defined by

$$\mathrm{E(X)} = \int_0^{\infty} M\{X \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{X \leq r\}dr$$

provided that at least one of the two integrals are finite.

Theorem 1: Let X and Y be denoted the independent uncertain variables with finite expected values. Then, for any real numbers a and b, we have

$$E[aX + bY] = aE[X] + bE[X]$$

Note that a normal uncertain variable $X \sim \mathcal{N}(e, \sigma)$ has an expected value $E(X) = e$.

Now, one of the important tools for solving LP programs with uncertain variables will be discussed. In fact, the ranking of uncertain parameters has a key role to establish a suitable approach for solving these programs. Unlike the real situation, the emphasis is on the point that declares natural order ship on the set of all uncertain variables do not exist. Hence, it is clearly important to focus on job orders for uncertain environments. One of the obvious and convention approaches is ranking criteria using the expected value of the uncertain variables. For this aim, a function from the set of all uncertain variables to the real line is defined, where natural order exists such that

$$E : F(\mathbb{R}) \rightarrow \mathbb{R}$$
$$X \rightarrow E(X)$$

where F($\mathbb{R}$) is the set of all uncertain variables.

Definition 5: Let X and Y be two independent uncertain variables with similar distributions (Both are linear or both are normal and so on).

1) We say $X > Y$ iff $E[X] > E[Y]$
2) We say $X = Y$ iff $E[X] = E[Y]$

Remark 1: We denote $X \geq Y$ if $X = Y$ or $X > Y$.

Lemma 1: Assume that X,Y and Z are uncertain variables. So,

1) $X = X$ (reflexivity)
2) if $X = Y$, then $Y = X$ (symmetry);
3) if $X = Y$, $Y = Z$, then $X = Z$ (transitivity).

Proof: It is straightforward based on the definition.

Remark 2: The relation " $=$" is an equivalence relation on uncertain variables based on above lemma. Furthermore, if X is an uncertain variable, the set of uncertain variables defined by $[X] = \{Y|X = Y\}$ is called equivalence set of X.

Lemma 2: Assume $X > Y$ then $-X < -Y$.

Proof: Since $X > Y$, we have $E[X] > E[Y]$ if and only if $-E[X] < -E[Y]$ if and only if $E[-X] < E[-Y]$ if and only if $-X < -Y$.

Lemma 3: Assume X,Y and Z are uncertain variables. The relation $\leq$ is a partial order on the set of all uncertain variables.

Proof: Since, $E : F(\mathbb{R}) \to \mathbb{R}$ such that $X \to E(X)$, then we will define a linear order in which the partial order is eligible while it is valid naturally in the real line.

Remark 3: We emphasize that the relation $\leq$ is a linear order on the set of all uncertain variables based on above lemma.

Lemma 4: If $X \leq Y$ and $Z \leq \delta$, then $X+Z \leq Y+\delta$, where X, Y, Z and δ are uncertain variables.

Proof: For achieving the mentioned result, we need to investigate the following properties,

1) $X \leq X$ (reflexivity);
2) If $X \leq Y$ and $Y \leq X$, then $X = Y$ (antisymmetry);
3) If $X \leq Y$ and $Y \leq Z$, then $X \leq Z$ (transitivity).

Clearly, the rest of the proof is straightforward.

Remark 4: We emphasize that the relation $\leq$ is a linear order on the set of all uncertain variables based on above lemma.

Lemma 5: If $X \leq Y$ and $Z \leq \delta$, then $X+Z \leq Y+\delta$, where X,Y,Z and δ are uncertain variables.

Proof: It is straightforward.

III. UNCERTAIN LINEAR PROGRAMMING

In this section, first the ULP is introduced and then some concepts, definitions, and theorems and their results are presented which used throughout of the paper [10].

Definition 6: An ULP model is defined as follows:

$$\begin{aligned} &\max \quad tx \\ &\text{s.t.} \prod x \leq \beta \\ &\quad \text{x} \geq 0 \end{aligned} \tag{1}$$

where, $\Pi = (\pi_{ij})_{m\times n}, t = (t_1, \ldots, t_n), \beta = (\beta_i)_{m\times 1}$ such that π_{ij}, t_j, β_i are independent uncertain variables for $i \in I = \{1, \ldots, m\}$ and $j \in J = \{1, \ldots, n\}$.

Definition 7: Any x which satisfies all constraints of ULP is termed as a feasible solution. The set of all feasible solutions of ULP is called feasible space and it is denoted by S.

Definition 8: Let $x^* \in S$. We say x^* is an optimal solution for ULP, if $tx \leq tx^*$ for all $x \in S$.

Theorem 2: The ULP model (1) is equivalent to a crisp model as follows,

$$\max \ cx \quad s.t. \ Px \leq b \quad x \geq 0 \tag{2}$$

where $P = (p_{ij})_{m\times n}, c = (c_j)_{1\times n}, b = (b_i)_{m\times 1}$ so that p_{ij}, c_j, b_i are expected values corresponding to π_{ij}, t_j, β_i, respectively and also $x = (x_j)_{n\times 1}$, where x_j is the real decision making variable ($j = 1, \ldots, n$).

Proof: Let s_1 and s_2 be the feasible spaces of models (1) and (2), respectively. Then $x \in s_1$, if and only if

$$\begin{aligned} \sum_{j=1}^{n} \pi_{ij}x_j \leq \beta_i &\Leftrightarrow E\left[\sum_{j=1}^{n} \pi_{ij}x_j\right] \leq E[\beta_i] \\ &\Leftrightarrow \sum_{j=1}^{n} E[\pi_{ij}]x_j \leq E[\beta_i] \Leftrightarrow \sum_{j=1}^{n} p_{ij}x_j \leq b_i \\ &\Leftrightarrow x \in s_2 \end{aligned} \tag{3}$$

Therefore, $s_1 = s_2$.

Now assume that x^* is an optimal solution for the model (1), then for all x, we have:

$$\begin{aligned} tx^* \geq tx &\Leftrightarrow E[tx^*] \geq E[tx] \\ &\Leftrightarrow E\left[\sum_{j=1}^{n} t_jx_j^*\right] \geq E\left[\sum_{j=1}^{n} t_jx_j\right] \\ &\Leftrightarrow \sum_{j=1}^{n} t_jx_j^* \geq \sum_{j=1}^{n} t_jx_j \end{aligned} \tag{4}$$

It could obviously be concluded that x^* is an optimal solution for the model (2).

In this step, the above theorem will be illustrated by a numerical example.

IV. NUMERICAL EXAMPLE. CONSIDER THE FOLLOWING ULP

Consider the following ULP as follows:

$$\begin{aligned} &\text{Max } £ = \mathcal{N}(3, 1)x_1 + \mathcal{N}(4, 1)x_2 \\ &\text{s.t } \mathcal{N}(2, 1)x_1 + \mathcal{N}(3, 1)x_2 \leq \mathcal{N}(6, 1) \\ &\quad \mathcal{N}(3, 1)x_1 + \mathcal{N}(2, 1)x_2 \leq \mathcal{N}(5, 1) \\ &\quad x_1, x_2 \geq 0 \end{aligned} \tag{5}$$

where $\mathcal{N}(e, \sigma)$ denotes the normal uncertainty distribution. Using Theorem 2, the above program reduces to

$$\begin{aligned} &\text{Max } Z = 3x_1 + 4x_2 \\ &\text{s.t } 2x_1 + 3x_2 \leq 6 \\ &\quad 3x_1 + 2x_2 \leq 5 \\ &\quad x_1, \quad x_2 \geq 0 \end{aligned} \tag{6}$$

The optimal solution is $x^* = (0.6, 0.8)$ and $z^* = 8.2 = E[\mathcal{N}(8.2, 1.4)] = E[£^*]$.

Now the most important concepts in linear programming "basic feasible solution" (BFS), as well as the classical situation will be presented. Here, this concept in an uncertain environment is defined. Consider the ULP program as follows:

$$\begin{aligned} &\text{Max } \varsigma = £x \\ &\text{s.t. } \prod_{x\geq 0} x = \beta \quad x \geq 0 \end{aligned} \tag{7}$$

where all parameters of this program are as well as given in model (1).

Let $P = (p_{ij})_{m\times n} = E\left[\pi_{ij}\right]_{m\times n} = E[\Pi]$ and assume that rank $(P) = m$.

A partition of matrix P as $[B, N]$ where B is as an $m \times m$ is non-singular matrix, a that is where rank $B = m$ such that $B = E[B]$ where B is an $m \times m$ uncertain matrix which is established by the columns of uncertain matrix in to B. Cleary, the basic solution

$$x_B = \left(x_{B_1}, \ldots, x_{B_m}\right)^T = B^{-1}b, \quad x_N = 0$$

is a solution of $Px = b$, where $b = E[\beta]$. Now, based on this definition the uncertain value of the objective function for this solution is mentioned as $\zeta = £_B x_B$. Let $y_j = B^{-1}p_j$ and $\varsigma_j = £_B y_j$ are known for every column p_j of B which is not in B and such that there exists an i with $y_{ij} > 0$. Now, if we select p_{Br} from B by B use of the criterion $\theta = x_{B_r}/y_{rj} = \min\left\{x_{B_r}/y_{ij} | y_{ij} > 0\right\}$ and then replace p_{Br} by p_j in B, thus a new BFS will be achieved as follows:

$$\bar{x}_{B_r} = \begin{cases} x_{B_r}/y_{rj}, & \text{for } i = r \\ x_{B_r} - x_{B_r}y_{ij}/y_{rj}, & \text{for } i \neq r \end{cases}$$

and

$$\bar{p}_i = \begin{cases} p_j, & \text{for } i = r \\ p_i, & \text{for } i \neq r \end{cases} \tag{8}$$

Definition 9: The ULP model is called non-degenerate, where all basic variables which corresponds to every basis B are nonzero and hence positive. The theorem proposes a criterion for the selection of p_j to allow as achieving a better solution.

Theorem 3: If for any column p_k in P, which is not in B, the condition $\varsigma_j < £_j$ where ς_j and $£_j$ are respectively the objective values of the current basic feasible solution and the new basic feasible solution, holds, and if at least one $y_{ik} > 0, i \in I = \{1, \ldots, m\}$ then it is eligible to obtain a new BFS by replacing one of the columns in B by p_k and a new uncertain value for the objective function $\bar{\zeta}$ satisfies $\bar{\varsigma} \geq \varsigma$.

Proof: Since $\overline{£}_{B_i} = £_{B_i}, i \neq r$ and $\overline{£}_{B_r} = £_k$, hence:

$$\begin{aligned} \bar{\varsigma} &= \sum_{i=1}^{m} B_i \left(x_{B_i} - \frac{x_{B_r}y_{ik}}{y_{rk}}\right) + \frac{x_{B_r}}{y_{rk}}k \\ &= \sum_{i=1}^{m} B_i \left(x_{B_i} - \frac{x_{B_r}y_{ik}}{y_{rk}}\right) + \theta k \end{aligned} \tag{9}$$

where $\theta = \frac{x_{Br}}{y_{rk}}$. It is clear if $\theta = 0$, then $\bar{\varsigma} = \varsigma$. On the other hand, in the case $\theta > 0$, we have

$$\begin{aligned} E[\bar{\varsigma}] &= E\left[\sum_{i=1}^{m} £_{B_i}\left(x_{B_i} - \frac{x_{B_r}}{y_{rk}}y_{ik}\right) + \frac{x_{B_r}}{y_{rk}}£_k\right] \\ &= \sum_{i=1}^{m} E\left[£_{B_i}\right]\left(x_{B_i} - \frac{x_{B_r}}{y_{rk}}y_{ik}\right) + \frac{x_{B_r}}{y_{rk}}E\left(£_k\right) \\ &= \sum_{i=1}^{m} E\left[£_{B_i}\right]\left(x_{B_i} - \frac{x_{B_r}}{y_{rk}}y_{ik}\right) + \frac{x_{B_r}}{y_{rk}}E\left(\xi_k\right) \\ &\geq \sum_{i=1}^{m} E\left[£_{B_i}\right]\left(x_{B_i} - \frac{x_{B_r}}{y_{rk}}y_{ik}\right) + \frac{x_{B_r}}{y_{rk}}E\left[\varsigma_k\right] \\ &= E\left[\sum_{i\neq r}^{m} £_{B_i}\left(x_{B_i} - \frac{x_{B_r}}{y_{rk}}y_{ik}\right) + \frac{x_{B_r}}{y_{rk}}\varsigma_k\right] \\ &= E\left[\sum_{\substack{i=1\\i\neq r}}^{m} £_{Bi}\left(x_{B_i} - \frac{x_{B_r}}{y_{rk}}y_{ik}\right) + \frac{x_{B_r}}{y_{rk}}\varsigma_k\right] \\ &= E\left[\sum_{\substack{i=1\\i\neq r}}^{m} £_{B_i}\left(x_{B_i} - \frac{x_{B_r}}{y_{rk}}y_{ik}\right) + \frac{x_{B_r}}{y_{rk}}\sum_{i=1}^{m} £_{B_i}y_{ik}\right] \\ &= E\left[\sum_{\substack{i=1\\i\neq r}}^{m} £_{B_i}\left(x_{B_i} - \frac{x_{B_r}}{y_{rk}}y_{ik}\right)\right. \\ &\qquad \left. + \frac{x_{B_r}}{y_{rk}}\sum_{\substack{i=1\\i\neq r}}^{m} £_{B_i}y_{ik} + \frac{x_{B_r}}{y_{rk}} + £_{B_k}y_{rk}\right] \\ &= E\left[\sum_{\substack{i=1\\i\neq r}}^{m} £_{B_i}x_{B_i} + £_{B_r}x_{B_r}\right] = E\left[\sum_{i=1}^{m} £_{B_i}x_{B_i}\right] = E[\varsigma] \end{aligned} \tag{10}$$

Or $\bar{\zeta} \geq \zeta$. This completes the proof.

Theorem 4: If for some columns p_k which is not in the basis matrix of any basic feasible solution to the model (6) which $\zeta_k < £_k$ and also $y_{ik} \leq 0, i \in I = \{1, \ldots, m\}$, then ULP program is unbounded.

Proof: Assume that x_B is a basic feasible solution such as $x_B = B^{-1}b$, where $B = \left[p_{B_1}, \ldots, p_{B_m}\right]$ is associated basis matrix. Then, from model (3) we have $\sum_{i=1}^{m} x_{B_i}p_{B_i} = b$. Also let $\zeta = £_B x_B$, $\varsigma_k < £_B$ and $y_{ik} \leq 0, i \in I$. So for any scalar $\theta > 0$, we have $\sum_{i=1}^{m} x_{B_i}p_{B_i} - \theta x_{pk} + \theta x_{pk} = b$.

Therefore,

$$\sum_{i=1}^{m} \left(x_{B_i} - \theta y_{ik}\right) p_{B_i} + \theta p_k = b \tag{11}$$

And hence, now there is a solution in which $(m+1)$ variables can be different from zero. Now, the value of ς for the current feasible solution is computed which is not essentially basic to model (10):

$$\begin{aligned}\bar{\varsigma} &= \sum_{l} £_l x_l = \sum_{i=1}^{m} £_{B_i}\left(x_{B_i} - \theta y_{ik}\right) + \theta £_k \\ &= \varsigma - \theta \sum_{i=1}^{m} £_{B_i} y_{ik} - \theta £_k \\ &= \varsigma - \theta \sum_{i=1}^{m} £_{B_i} y_{ik} - \theta £_k \\ &= \varsigma + \theta\left(£_k - \varsigma_k\right)\rho \end{aligned} \tag{12}$$

Clearly if $\theta > 0$ the value for can be made arbitrarily large. As well as we like these columns unbounded solutions.

As we know same as the classical linear programs, it is easy to prove that when there is a suitable (candidate) non-basic variable it can enter the current basis and improve the value of the objective function, we can continue the solving process by inserting a non-basic candidate vector and removing a vector from the current basis matrix, after a finite number iteration to achieve the optimal solution or conclude the unbounded case.

1) $\exists k$ such that $\varsigma_k - £_k < 0, y_{ik} \leq 0, i = 1, \ldots, m$, or
2) $\forall, \varsigma_j$-$£_j \geq 0$

In the first case, the unbounded solution occurs but for the second one, in the following theorem, we prove that an optimal solution will be achieved.

Theorem 5: Assume that $x_B = B^{-1}b$ is a BFS for model (3) such that $\varsigma_j \geq £_j$ for all p_j in p, then x_B is an optimal basic feasible solution (OBFS) for model (4).

Proof: Let $x_j \geq 0, j \in J = \{1, \ldots, n\}$ be any BFS for (3) that is

$$x_1 p_1 + \cdots + x_n p_n = b \tag{13}$$

Therefore, the corresponding uncertain value for its associated objective function, which is denoted by,

$$\varsigma^* = £_1 x_1 + \cdots + £_n x_n \tag{14}$$

On substituting $p_j = \sum_{i=1}^{m} y_{ij} p_{B_i}$, into (12), we obtain

$$\left(\sum_{j=1}^{n} x_j y_{1j}\right) p_{B_i} + \cdots + \left(\sum_{j=1}^{n} x_j y_{mj}\right) p_{B_m} = b \tag{15}$$

Therefore, $x_{B_i} = \sum_{j=1}^{n} x_j y_{ij}, i = 1, \ldots, m.$

Now, let p_j is the $i^t h$ column of B. So,

$$\varsigma_j = £_B y_j = £_B e_i = £_{B_i} = £_j \tag{16}$$

thus, for every column of P we have $\varsigma_j \geq £_j$. Consequently, $\varsigma_j \geq £_j$ using in (13) we see that

$$\varsigma_1 x_1 + \cdots + \varsigma_n x_n \geq \varsigma^* \tag{17}$$

on substituting $\varsigma_j = \sum_{i=1}^{m} £_{Bi} y_{ij}$ into (16), we obtain

$$\left(\sum_{j=1}^{m} x_j y_{1j}\right) £_{B_i} + \cdots + \left(\sum_{j=1}^{m} x_j y_{mj}\right) £_{B_m} \geq \varsigma^* \tag{18}$$

now if $\zeta_0 = £_B x_B$, we have

$$\zeta_0 = \mathrm{x}_{\mathrm{B}_1} £_{\mathrm{B}_1} + \cdots + \mathrm{x}_{\mathrm{B}_\mathrm{m}} £_{\mathrm{B}_\mathrm{m}} \geq \zeta^* \tag{19}$$

is an optimal solution for the model (3).

V. FUZZY MATHEMATICAL PROGRAMMING: APPLICATION IN DEA MODELS

In LP models in the crisp scenario, the objective is to minimize or to maximize a linear goal function under linear limits. However, the decision-maker may not be able to determine the goals or limits, but they can specify in a fuzzy concept. In such conditions, it is better to use some kinds of FLP models in order to achieve more flexibility. Since fuzzy concept can be appeared as an LP program in many cases, determining the fuzzy programming problem is not unique.

A. APPLICATION IN INPUT-BASED CCR MODEL

Nowadays, Fuzzy sets theory has been applied as a method to determine the unknown and uncertain parameters DEA models. Recently, fuzzy version of DEA models has been attracted many interests, see in [17] and [19]. Input multiplier CCR model (19) with fuzzy data is as follows:

$$\begin{aligned} &\text{Max} \sum_{r=1}^{s} u_r \tilde{y}_{rp} \\ &\text{s.t} \sum_{i=1}^{m} v_r \tilde{x}_{ip} \simeq \tilde{1} \\ &\quad \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_r \tilde{x}_{ij} \leq 0 \quad \text{for } j = 1, \ldots, n \\ &\quad u_r \geq 0, \text{ for } \quad r = 1, \ldots, s \\ &\quad v_i \geq 0, \text{ for } \quad i = 1, \ldots, m \end{aligned} \tag{20}$$

where $\tilde{1} \simeq (1, 1, 0, 0)$ and $\tilde{0} \simeq (0, 0, 0, 0)$.

Now if a kind of linear ranking function is applied to factor matrix and a right-side vector, model (19) will be reduced as (20),

$$\begin{aligned} &\text{Max} \sum_{r=1}^{s} u_r \tilde{y}_{rp} \\ &\text{s.t} \sum_{i=1}^{m} v_r \tilde{x}_{ip} = \tilde{1} \\ &\quad \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_r \tilde{x}_{ij} \leq 0 \quad \text{for } j = 1, \ldots, n \\ &\quad u_r \geq 0, \quad \text{for } r = 1, \ldots, s \\ &\quad v_i \geq 0, \quad \text{for } i = 1, \ldots, m \end{aligned} \tag{21}$$

Suppose that (u^*, v^*) is an optimized solution of DEA model mentioned in (20), DUM_p is fuzzy efficient, if $R(u^*, y_p) = 1$, otherwise DUM_p is not considered as fuzzy efficient.

B. APPLICATION IN FUZZY COST-EFFICIENCY

Here a cost-efficiency classic model will be introduced in which costs in each DMU will be known and in cases which costs levels is assumed fuzzy numbers. According to the fuzzy common unit vector $C_p = (c_{1p} \dots, c_{mp})$ for input x_p, the cost efficiency is evaluated as:

$$\begin{aligned} \text{Min}\, \tilde{c}_p x_p &= \sum_{r=1}^{S} \tilde{c}_{ip} x_i^p \\ \text{s.t}\ \sum_{j=1}^{n} \lambda_j x_{ij} &= x_i^p, \quad i = 1, \dots, n \\ \sum_{j=1}^{m} \lambda_j y_{rj} &\geq y_{rp}, \quad j = 1, \dots, m \\ \lambda_j &\geq 0, \quad for\ j = 1, \dots, n \\ x_i^p &\geq 0, \quad for\ i = 1, \dots, m \end{aligned} \tag{22}$$

The current model is in the form of the FLP program. Now, assume that optimized solution is (x^{p*}, λ^*), then costs efficiency in ratio form determined in accordance with:

$$R(\tilde{C}E)_p = \frac{R\left(\tilde{c}_p x^{p*}\right)}{R\left(\tilde{c}_p x_p\right)}$$

We have $0 \leq R(\tilde{C}E)_p \leq 1$ and DMU_p is cost efficient if and only if $R(\tilde{C}E)_p$ is equal to 1.

VI. FUZZY VARIABLES OF LINEAR PROGRAMMING PROBLEMS (FVLP)

In this part, we are going to define one of the convenient kind of FLP problem, where all variables are assumed in the fuzzy numbers from. We will call them as FVLP problems.

Definition 10: An FVLP problem is defined as follows:

$$\begin{aligned} &\text{Min}\, \tilde{z} \simeq c\tilde{x} \\ &\text{s.t}\, A\tilde{x} \geq \tilde{b} \\ &\quad \tilde{x} \geq \tilde{0} \end{aligned} \tag{23}$$

where $\tilde{b} \in (F(\Re))^m, \tilde{x} \in (F(\Re))^n, A = \left(a_{ij}\right)_{m\times n} \in \Re^{m\times n}, c^T \in \Re^n$.

Definition 11: It could be said that vector $\tilde{x} \in (F(\Re)^n)$ which is satisfied all constructions of models (19) is named as a feasible solution.

Definition 12: A feasible solution $\tilde{x}$ is an optimal solution for (22), if for all feasible solution $\tilde{x}$ for (22), we have $c\tilde{x}_* \leq c\tilde{x}$.

VII. UNCERTAIN DATA ENVELOPMENT ANALYSIS

DEA model has many applications for productivity interpretation of the engineering and economic systems. DEA studies organizations called DMUs (Decision-Making Units). Generically as its definition illustrates, DMU is considered as a unit which converts inputs into outputs and its performances can be evaluated. DEA is a methodology with characteristics of non-parametric frontier-estimation based on LP approach. The relative efficiency of a set of DMUs which have a common functional goal is evaluated by DEA.

It assume that there are n number DMU_s, each DMU produces s different outputs consuming m different inputs. Specifically, DMU_j enthrall amounts $Y_j = (y_{rj})$ of outputs for $r \in \Re$ consuming amounts $X_j = (x_{ij})$ of inputs for $i \in I$ and $j \in J$. We assume $X_j \geq 0, X_j \neq 0$ and $Y_j \geq 0, Y_j \neq 0, X_p = \left(x_{1\,p}, \dots, x_{mp}\right)$ and $Y_p = \left(y_{1\,p}, \dots, y_{sp}\right)$ are inputs and outputs values respectively of DMU_p, which is being evaluated.

The PPS determines the DMUs efficiency estimation. Such a model is known as a input-oriented CCR envelopment model. Using the minimization of inputs with outputs, the model gives the efficiency score for a DMU. This model generates efficiency score for a DMU by minimizing inputs over outputs. Each observed DMU_p, an imaginary composite unit is constructed that output forms DMU_p. λ_j there is a proportion which DMU_p is represented by $j = 1, \dots, n$ that $0 < \alpha_p < 1$ is used in the construction of composite unite for DMU_p, and α Refers to the efficiency score for DMU_p, consume at least the same levels of outputs as DMU_p.

In the case of inefficiency, in order to be efficient, DMU_p has to decreases its inputs. Inputs and outputs are assumed to have definite values in ordinary DEA models. But in recent different observations, infinite values of inputs and outputs has been observed in some DEA applications.

Such data are named as ''inaccurate". Different forms of inaccurate data are ordinal, qualitative, ordinal, interval, Probabilistic or fuzzy for which some researchers were theoretically presented on the development of this technique in fuzzy content [20]. Here, we use uncertain data envelopment analysis which provides a means for handling inaccurate data.

For formulation of the uncertain DEA model, x_{ij}, y_{rj} for $i = 1, \dots, m, j = 1, \dots, n, r = 1, \dots, s$ are assumed as uncertain variables. Thus, the uncertain version of DEA model is given as follows:

$$\begin{aligned} &\text{Min}\, \alpha \\ &\text{s.t}\ \sum_{i=1}^{m} \lambda_j \check{x}_{ij} \leq \alpha_p \check{x}_{ip}, \quad i \in I \\ &\quad \sum_{r=1}^{s} \lambda_j y_{rj} \leq y_{rp}, \quad r \in \Re \\ &\quad \lambda > 0, \quad j \in J \end{aligned} \tag{24}$$

By Theorem 2, the current model can be converted to the below LP model and then it can be solved by standard linear programming solvers,

$$\begin{aligned} &\text{Min}\, \alpha \\ &\text{s.t}\ \sum_{i=1}^{m} \lambda_j x_{ij} \leq \alpha_p x_{ip}, \quad i \in I \\ &\quad \sum_{r=1}^{s} \lambda_j y_{rj} \leq y_{rp}, \quad r \in \Re \\ &\quad \lambda > 0, \quad j \in J \end{aligned} \tag{25}$$

where $x_{ij} = E\left[\check{x}_{ij}\right]$ and $y_{rj} = E\left[\check{y}_{rj}\right]$ for $i \in I = \{1, \dots, m\}, r \in R = \{1, \dots, s\}$ and $j \in J = \{1, \dots, n\}$.

TABLE 1. The uncertain inputs and outputs.

DMU_j	$j=1$	$j=2$	$j=3$	$j=4$
x_{1j}	$\mathcal{L}(29,49)$	$\mathcal{L}(27,47)$	$\mathcal{L}(25,47)$	$\mathcal{L}(29,48)$
x_{2j}	$\mathcal{L}(4,5)$	$\mathcal{L}(4,5)$	$\mathcal{L}(4,5.1)$	$\mathcal{L}(4.1,5.1)$
x_{3j}	$\mathcal{L}(32,60)$	$\mathcal{L}(29,55)$	$\mathcal{L}(31,61)$	$\mathcal{L}(28,70)$
x_{4j}	$\mathcal{L}(4210,5200)$	$\mathcal{L}(4530,5670)$	$\mathcal{L}(4310,5330)$	$\mathcal{L}(4230,5360)$
x_{5j}	$\mathcal{L}(184,235)$	$\mathcal{L}(200,256)$	$\mathcal{L}(192,300)$	$\mathcal{L}(182,239)$
y_{1j}	$\mathcal{L}(3700,4015)$	$\mathcal{L}(1912,2300)$	$\mathcal{L}(4011,4400)$	$\mathcal{L}(2150,3000)$
y_{2j}	$\mathcal{L}(3537235,62895029)$	$\mathcal{L}(60550228,81125439)$	$\mathcal{L}(3399540,5125256)$	$\mathcal{L}(1299536,7813564)$
y_{3j}	$\mathcal{L}(111,125)$	$\mathcal{L}(356,500)$	$\mathcal{L}(80,100)$	$\mathcal{L}(114,122)$

TABLE 2. The uncertain inputs and outputs (continued).

DMU_j	$j=5$	$j=6$	$j=7$	$j=8$
x_{1j}	$\mathcal{L}(26,48)$	$\mathcal{L}(28,48)$	$\mathcal{L}(27,48)$	$\mathcal{L}(28,49)$
x_{2j}	$\mathcal{L}(4,5.1)$	$\mathcal{L}(4.1,5.1)$	$\mathcal{L}(4,5.1)$	$\mathcal{L}(4.2,5.1)$
x_{3j}	$\mathcal{L}(20,34)$	$\mathcal{L}(30,60)$	$\mathcal{L}(37,52)$	$\mathcal{L}(38,50)$
x_{4j}	$\mathcal{L}(4350,5350)$	$\mathcal{L}(4110,5270)$	$\mathcal{L}(4590,5560)$	$\mathcal{L}(400,5540)$
x_{5j}	$\mathcal{L}(189,270)$	$\mathcal{L}(190,280)$	$\mathcal{L}(190,285)$	$\mathcal{L}(178,256)$
y_{1j}	$\mathcal{L}(1970,2190)$	$\mathcal{L}(3217,4218)$	$\mathcal{L}(1870,2321)$	$\mathcal{L}(2750,3200)$
y_{2j}	$\mathcal{L}(6684542,9822048)$	$\mathcal{L}(67264574,114623564)$	$\mathcal{L}(92644574,12242458)$	$\mathcal{L}(4567894,101976104)$
y_{3j}	$\mathcal{L}(80,210)$	$\mathcal{L}(250,430)$	$\mathcal{L}(72,80)$	$\mathcal{L}(100,220)$

TABLE 3. The efficiency scores.

DMU_j	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
$Efficiency$	1	1	1	0.691624	0.904622	1	0.526733	0.816155

VIII. NUMERICAL EXAMPLE

Since efficiency has great importance in the banking industry, so we considered 8 Meli bank branches in Iran to clarify the performance of our proposed approach in this study as 8 DMUs. Meli bank has a comprehensive network of over 3,300 branches and 37,000 employees in Iran. Estimation of countrywide coverage in Iran, service quality and experienced multi-lingual staff are important factors of their success. The results of the model optimization are shown in Tables 1 and 2, which contains 5 inputs and 3 outputs, and they are represented by uncertain linear variables $\check{x}_{ij} = \mathcal{L}\left(a_{\check{x}_{ij}}, b_{\check{x}_{ij}}\right), \check{y}_{ij} = \mathcal{L}\left(a_{\check{y}_{ij}}, b_{\check{y}_{ij}}\right), i = 1, \ldots, 5, r = 1, \ldots, 3$ and $j = 1, \ldots, 8$.

The uncertain efficiency scores for each DMU is shown in Table 3.

According to received reports and analyzing data from these branches, it has been cleared that $DMU_p, p = 1, 2, 3$ are 6 efficient units, while $DMU_p, p = 4, 5, 7$ and 8 are inefficient units. These results show that if $DMU_p, p = 4, 5, 7$ and 8 want to change their inferior positions, then they should decrease their inputs accordingly. To optimize the stochastic linear programming, different operational research software programs can be used, e.g., Lingo and Gams. In this research, we use Gams to modify the results.

Notation	description
DEA	Data Envelopment Analysis
FLP	Fuzzy Linear Programming
ULP	Uncertain Linear Programming
DMU	Decision Making Unit
RA	Regression Analysis
LAV	Least Absolute Value
CIMS	Computer Integrated Manufacturing Systems
LP	Linear programming
BFS	Basic Feasible Solution
OBFS	Optimal Basic Feasible Solution
FVLP	Fuzzy Variables of Linear Programming
PPS	Production Possibility Set
CCR	Charnes, Cooper, Rhodes

IX. CONCLUSION

In this study, based on the proposed mathematical model of uncertainty, a new computational approach to obtain the optimal solution for linear programming with indefinite presented variables in the objective and limiting matrix and especially some important results proved. To illustrate the advantages and effectiveness of the proposed approach, several examples are given, in particular its application in uncertain DEA. In particular, the results can be easily extended to other cases of uncertain variables, with a suitable thematic introduction to these types of uncertainty. We also emphasize that this work can be extended two-phase approach to the evaluation and assessment of DEA models in a fuzzy stochastic environment. One of the great advantages of our numerical example is the use of the problem of real cases, wherein the real-world there are many similar cases to evaluate the performance of different branches of banks. Our results have shown that the approaches presented in the banking system provide a sharp assessment, while the information is prepared on the basis of uncertainty. Also, as it was pointed out that fuzzy versions of DEA models attracted many interests, an interesting topic for future study will be Pythagorean fuzzy set, where it is an effective mathematical tool for solving uncertain problems, especially such as Pythagorean fuzzy interaction ability Bonferroni mean aggregation operators when deciding on multiple attributes.

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