Generalized Multiple Dependent State Sampling Plans in Presence of Measurement Data

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ABSTRACT This article introduces a generalized version of the multiple dependent state sampling plan based on the measurement data, that is, when the quality characteristic is measured in a numerical scale. The conditions of application and the operating procedures of the proposed plan are discussed sequentially. A few important performance measures, such as operating characteristic curve, average total inspection and average sample number, are developed. The advantage of the proposed plan over the multiple dependent state sampling and single sampling plan is demonstrated through numerical example. Finally, one real-life data set is analyzed to illustrate the application of the proposed sampling plan.

INDEX TERMS Consumer's risk, hybrid censoring, life-testing, producer's risk, quality control, reliability sampling plan.

I. INTRODUCTION

Testing and inspection of the finished products are done to see either the manufactured product meets the given specifications or not. In addition, this type of activity alerts the producer to enhance the quality of the product. At the stage of inspection or testing, it is costly to check all the items in the lot. Therefore, only a proportion of a product is selected at random from the lot of products as sample and inspected. A lot of the products is accepted if the experimenter found the items in the sample are according to given specifications, otherwise, the lot of the products will be rejected. The acceptance sampling scheme provides a guideline for accepting or rejecting a lot of the products. According to [16], the acceptance sampling plan protects both the producer and the customer by minimizing the risk of rejecting good lots and by minimizing the risk of accepting bad lots, respectively. The choice of sampling plan depends on the nature of the data. The variable sampling plan is applied when the quality characteristic under consideration is measured in a numerical scale. This situation is occurred when the quality characteristic follows a probability distribution. Variable sampling plan has more information than the attribute sampling plan. Variable sampling plan has an advantage over attribute sampling plan in the sense that same operating characteristic curve can be obtained

by using less number of samples (see [24, p.671]). A rich literature is available on both types of sampling plans, see for examples, [4], [8], [11], [14], [17]–[20], [25], [28] and [34].

The single sampling plan (SSP) is common to use for testing and inspection of the product due to its easy operational procedure. But, it is indicated by [8] that a single sampling plan needs large sample size for the inspection of the product. Therefore, the use of a single sampling plan for testing and inception will be costly. Reference [9] showed that the multiple dependent state (MDS) sampling plan is an efficient sampling plan and is operated on small sample size as compared to the single sampling plan. In MDS sampling, a set of samples from a lot is selected and tested for accepting or rejecting the lot. The final decision about lot acceptance is made by taking into account the lot acceptance information from all samples. More applications of the MDS sampling plans can be seen in [1], [2], [5], [6], [8], [29], [32], [33] and [36].

Recently, [27] proposed the generalization of the MDS plan which is called GMDS and found that it is efficient in minimizing sample size as compared to the MDS plan. In GMDS plan, for in-decision state, the lot of the product is accepted if at least k out of m lots are accepted, where m is previously accepted lots. Reference [3] used the similar approach and discussed the economic aspect of designing GMDS plan. However, both the articles used the framework of attribute sampling plan. By exploring the literature and

best of the authors' knowledge, there is no work on GMDS in presence of measurement data. This article introduces a variables GMDS sampling plan. Recently, [10] proposed a SSP by considering lifetime-performance index as the quality characteristic. In this article, lifetime-performance index is considered as the quality characteristic to construct the proposed GMDS sampling plan when it is assumed that the lifetime follows an exponential model. The advantage of the proposed plan over the SSP and MDS plan is established by analyzing the operating characteristic (OC) curves. In order to implement the proposed plan, the experimenter needs to conduct a life-testing experiment. Due to time and budget constraints, life-testing experiments are usually conducted under censoring setup. Hybrid censoring, proposed by [15], is widely used in reliability sampling plans (see [23]). The censoring scheme is defined as follows. The experimentation starts with n testing units. The experiment continues until when a pre-defined number of failures r or a pre-defined time bound X_0, whichever is observed earlier. This scheme is a generalized version of Type-I and Type-II censoring schemes. By setting $X_0 = \infty$ or $r = n$, hybrid censoring reduces to Type-I or Type-II censoring, respectively. Moreover, by setting together $X_0 = \infty$ and $r = n$, hybrid censoring becomes a complete data case, that is, no censoring. In the present article, GMDS sampling plan is developed in presence of hybrid censoring along with Type-I censoring, Type-II censoring and no censoring are being its special cases. The aim of the article is two-fold. First, the GMDS sampling plan is developed under a generalized censoring scheme (that is, hybrid censoring) so that it can be easily extended to its sub-cases (that is, Type-I censoring, Type-II censoring and no censoring). Secondly, the exact distribution of the estimator of the unknown exponential distribution is used to construct the sampling plan. As a consequence, the computed sampling plans are exact.

The rest of the article is organized as follows. The lifetime-performance index is defined and the associated statistical results under hybrid censoring scheme is presented in Section II. Section III is devoted for the development of proposed GMDS sapling plan. The conditions of applicability of the proposed plan, its operating procedures and three measures of performance are also discussed here. Section IV is devoted for numerical illustrations. A comparative study between the proposed plan and SSP and MDS sampling plans is carried out in Section V. Finally, a real-life data analysis based on the proposed sampling plan is presented in Section VI and, at the end, few concluding remarks are made in Section VII.

II. ASSOCIATED STATISTICAL RESULTS
This section describes few statistical tools those will be used to construct the sampling plan in the subsequent sections.

A. LIFETIME-PERFORMANCE INDEX
Suppose X be a random variable which represents the lifetime of a product. In practice, the realizations on X always need not

to be clock-time or chronological. For instance, in automobile industries, often the lifetime of an automobile component is measured in the scale of distance unit (see [21]). However, it is easy to notice that the lifetime can be classified as a larger-the-better type quality characteristic. This means that the products having a longer lifetime are quantified as the better quality products. As a consequence, a lower specification limit, denoted by L, is generally associated with lifetime. The lifetime-performance index, a dimensionless quantity, is a process capability index which measures the larger-the-better type quality characteristic (see [30]). It is defined as

$$C_L = \frac{\mu_X - L}{\sigma_X}, \tag{1}$$

where μ_X and σ_X represent the mean and variance of the quality characteristic. In this article, lifetime is considered as the product quality characteristic.

Assume that X follows an exponential distribution with probability distribution function is defined as

$$F_X(x; \theta) = 1 - e^{-\frac{x}{\theta}}, \quad x \geq 0, \theta > 0. \tag{2}$$

The choice of exponential lifetime model is due to two primary reasons. Firstly, exponential distribution has a wide range of applications in reliability study (see [7]). Secondly, the explicit expression of maximum likelihood estimator of the exponential parameter is available in the literature and the exact sampling distribution of this estimator can be derived. Under the exponential lifetime model in (2), $\mu_X = \sigma_X = \theta$. Therefore, the lifetime-performance index C_L in (1) can be expressed as

$$C_L = 1 - \frac{L}{\theta}, \tag{3}$$

where $-\infty < C_L < 1$. Let us define a quantity p as

$$\begin{aligned} p &= \Pr(X < L) \\ &= 1 - e^{-\frac{L}{\theta}}. \end{aligned} \tag{4}$$

In literature, p is called the lifetime non-conforming rate which quantifies if the lifetime of the product X achieves its pre-specified lower quality specification L. Using the equations (3) and (4), the relationship between C_L and p can be seen as

$$C_L = 1 + \ln(1 - p). \tag{5}$$

B. ESTIMATION PROCEDURES OF C_L
In this section, we discuss the results on the estimation procedures of the quantity C_L. Note that the estimation of the quantity C_L depends on the estimation of the quantity θ based on the observed data. In this article, hybrid censored data are considered.

Let us assume that $X_{1:n} \leq \cdots \leq X_{n:n}$ be the ordered observations on failure time of n testing units. In the framework of hybrid censoring, let us define two random variables D and ξ which represent the number of failed items and the duration of testing, respectively. Therefore, $\xi = \min(X_{r:n}, X_0)$ and

the possible values of D are $0, 1, \cdots, r$. Therefore, based on the hybrid censored data, the likelihood function can be formulated as

$$L(\theta) \propto \prod_{i=1}^{d} f_X(x_{i:n}; \theta)(1 - F_X(\xi_0; \theta))^{n-d}, \qquad (6)$$

where $x_{i:n}$, d and ξ_0 denote the observed value of $X_{i:n}$, D and ξ. Here $f_X(\cdot; \theta)$ is the probability density function of X. Upon maximizing the function (6) with respect to θ, the maximum likelihood estimate (MLE) of θ can be computed as

$$\hat{\theta} = \frac{\sum_{i=1}^{d} x_{i:n} + (n-d)\xi_0}{d}. \qquad (7)$$

From the equations (3) and (7), the estimates of C_L can be obtained as

$$\hat{C}_L = 1 - \frac{Ld}{\sum_{i=1}^{d} x_{i:n} + (n-d)\xi_0}. \qquad (8)$$

Note that the MLE $\hat{C}_L$ in (8) is the biased estimate of C_L because the MLE $\hat{\theta}$ in (7) is the biased estimate of θ (see [13]). Now, one result on the distribution of $\hat{\theta}$ is presented which will be required to develop the sampling plan in Section 3.2. The distribution of $\hat{\theta}$ follows from the results due to [13].

Result 1 (Theorem 2.2 of [13]): Conditional on $D \geq 1$, the pdf of $\hat{\theta}$ under hybrid censoring scheme is given by

$$f_{\hat{\theta}}(x)$$
$$= (1-q^n)^{-1} \left[\sum_{i=1}^{r-1} \sum_{k=0}^{i} \mathcal{C}_{k,i} \mathcal{G}\left(x - T_{k,i}^*; \frac{i}{\theta}, i\right) + \mathcal{G}\left(x; \frac{r}{\theta}, r\right) \right.$$
$$\left. + r \binom{n}{r} \sum_{k=1}^{r} \frac{(-1)^k q^{n-r+k}}{n-r+k} \binom{r-1}{k-1} \mathcal{G}\left(x - T_{k,r}^*; \frac{r}{\theta}, r\right) \right],$$
$$0 < x < nX_0, \qquad (9)$$

where $q = e^{-X_0/\theta}$, $T_{k,i}^* = (n-i+k)X_0/i$, $\mathcal{C}_{k,i} = (-1)^k \binom{n}{i}\binom{i}{k} q^{n-i+k}$, and

$$\mathcal{G}(x; \gamma; \delta) = \frac{\gamma^\delta}{\Gamma(\delta)} x^{\delta-1} e^{-\gamma x}, \quad x > 0.$$

where γ and δ are the rate and shape parameters of a gamma distribution, respectively.

III. DEVELOPMENT OF GENERALIZED MULTIPLE DEPENDENT STATE SAMPLING PLAN

In this section, the development of generalized multiple dependent state (GMDS) sampling plan is discussed. First we provide the necessary conditions on the applicability of the proposed GMDS sampling plan. Then, in subsequent sections, we introduce the operating procedures of the sampling plan. Few performance measures and numerical illustrations are also furnished to validate the proposed sampling plan.

A. ASSUMPTIONS FOR IMPLEMENTING GMDS SAMPLING PLAN

While implementing the proposed GMDS sampling plan in practice, the following assumptions are required to be valid in the production process (see [9]).

 (i) The inspection strategy consists of taking successive lots produced from a continuing production process. That means the lots are submitted for inspection serially in the order in which they had been produced in the production process.
 (ii) All the submitted lots for inspection has essentially the same quality level. This means that the production process has a constant proportion nonconforming.
(iii) The consumer has confidence in the integrity of producer's production process. This means that there should not be any reason to believe that any particular lot quality is poorer than the preceding lots.
(iv) The quality characteristic under consideration follows an exponential distribution.

B. OPERATING PROCEDURE OF THE SAMPLING PLAN

Suppose that the quality characteristic of interest has a lower specification limit L and it follows an exponential distribution as in (2). The parameters r and X_0 of hybrid censoring scheme are also specified. Often, r is expressed in terms of degree of censoring $q = (n-r)/n$. Then, the operating procedures of the proposed GMDS sampling plan are

Step 1: Choose a random sample of size n from the submitted lot. Place the selected products on a life-test under hybrid censoring with the specified parameters. Based on the observed data from the life-test, estimate $\hat{C}_L$, lifetime performance index, using (8).

Step 2: Accept the lot if
$$\hat{C}_L \geq k_a,$$
otherwise reject the lot if
$$\hat{C}_L < k_b.$$
If $k_b \leq \hat{C}_L \leq k_a$, then accept the present lot under the condition that k out of preceding m lots have been accepted with the condition $\hat{C}_L \geq k_a$, otherwise reject the lot.

Therefore, GMDS sampling plan is characterized by the five parameters n, k, m, k_a and k_b. When $k = m$, it is multiple dependent state sampling plan proposed by [31].

C. MEASURE OF PERFORMANCE

In this section, two measures of performance of the proposed GMDS sampling plan are derived. They are the probability of lot acceptance and average total inspection.

1) THE PROBABILITY OF LOT ACCEPTANCE

The probability of lot acceptance is an important measure to access the performance of a sampling plan. It quantifies the

proportion of lots that are expected to be accepted for a given product quality level p. In the context of statistical quality control, it is also called operating characteristic (OC) curve (see [24]). The OC curve under GMDS sampling plan can be derived as

$$
\begin{aligned}
\mathrm{P}_a(p) = {} & \Pr(\hat{C}_L \geq k_a|p) \\
& + \Pr(k_b \leq \hat{C}_L \leq k_a|p) \\
& \times \sum_{j=k}^{m} \binom{m}{j} \left[\Pr(\hat{C}_L \geq k_a|p)\right]^j \left[1-\Pr(\hat{C}_L \geq k_a|p)\right]^{m-j}.
\end{aligned}
\tag{10}
$$

The derivation of the above formula is straight forward. It consists of two parts. The first part is the probability of lot acceptance when the estimate of the lifetime performance index exceeds unconditional lot acceptance value. The second part is the probability of lot acceptance when the estimate of the lifetime performance index lies between conditional and unconditional lot acceptance values. The probabilities in (10) can be simplified as

$$
\begin{aligned}
\Pr(\hat{C}_L \geq k_a|p) &= \Pr\left(1 - \frac{L}{\hat{\theta}} \geq k_a \mid p\right) \\
&= \Pr\left(\hat{\theta} \geq \frac{L}{1 - k_a} \mid p\right).
\end{aligned}
\tag{11}
$$

Using equations (3) and (5) in (11), $\Pr(\hat{C}_L \geq k_a|p)$ can be written as

$$
\Pr(\hat{C}_L \geq k_a|p) = \Pr\left(\hat{\theta} \geq \frac{\theta \ln(1 - p)}{k_a - 1}\right).
\tag{12}
$$

Similarly, $\Pr(k_b \leq \hat{C}_L \leq k_a|p)$ can be computed as

$$
\Pr(k_b \leq \hat{C}_L \leq k_a|p) = \Pr\left(\frac{\theta \ln(1 - p)}{k_b - 1} \leq \hat{\theta} \leq \frac{\theta \ln(1-p)}{k_a - 1}\right).
\tag{13}
$$

The probabilities in (12) and (13) can be computed by using Result 1 and, therefore, $\mathrm{P}_a(p)$ in (10) can also be computed by substituting the former probabilities.

2) AVERAGE TOTAL INSPECTION
The average total inspection is defined as the average number of products inspected per lot based on the sample size for the accepted lot and all inspected products in non-accepted lots. It is computed as

$$
\mathrm{ATI}(p) = n + \{1 - \mathrm{P}_a(p)\}(N - n),
$$

where $\mathrm{P}_a(p)$ is the probability of lot acceptance under GMDS sampling plan and N is the lot size.

In addition to the above mentioned two measures, average sample number (ASN), another common performance measure, is introduced. It is defined as the average number of sampled units per lot used for lot sentencing. Therefore,

$$
\mathrm{ASN}(p) = n.
$$

It is worth to mention that all the performance measures under GMDS sampling plan can be reduced to performance measures under MDS sampling plan when $k = m$. Also, by setting $k_a = k_b, k = m$ and $m \to \infty$, all these performance measures reduce to performance measures for single sampling plan.

D. DESIGN METHODOLOGY OF GMDS SAMPLING PLAN
In order to design a sampling plan by variable, first one needs to choose two points on the OC curve. Mathematically, a well-designed sampling plan must pass through two points, namely $(p_\alpha, 1 - \alpha)$ and (p_β, β) on the OC curve. The quantity p_α is termed as acceptable quality level (AQL) at which the sampling plan has the high probability of acceptance $1 - \alpha$. On the other hand, the quantity p_β is termed as limiting quality level (LQL) at which the sampling plan has the low probability of acceptance. The quantities α and β are termed as producer's risk and consumer's risk, respectively. In general, a sampling plan with less $\mathrm{ASN}(p)$ is most desirable. In order to achieve this goal, a non-linear optimization problem is formulated as follows.

$$
\underset{n,k,m,k_a,k_b}{\text{Minimize ASN}}(p) = n
$$

Subject to

$$
\begin{aligned}
\mathrm{P}_a(p_\alpha) &\geq 1 - \alpha \\
\mathrm{P}_a(p_\beta) &\leq \beta \\
k_b &\leq k_a < 1.
\end{aligned}
\tag{14}
$$

Note that $\mathrm{P}_a(p_\alpha)$ and $\mathrm{P}_a(p_\beta)$ are the probabilities of lot acceptance under GMDS sampling plan at AQL and LQL, respectively. The constraints in the above optimization problem are formulated based on the interpretation of the quantities AQL and LQL.

IV. NUMERICAL ILLUSTRATION
In this section, we compute few sampling plans in order to illustrate the propose methodology. Note that the optimization problem in (14) is a mixed integer programming problem where n, k, m are discrete variables and k_a, k_b are continuous variables. To reduce the complexity of the optimization problem, we assume the values of k and m are known and we need to solve the optimization problem in (14) only for n, k_a, k_b. To solve the optimization problem, a routine "NlcOptim" in the R software was used, which is based on sequential quadratic programming algorithm due to [26]. The algorithm do not always provide integer solution for n, in which case the nearest integer satisfying the constraints was taken as the solution.

In practice, the values of p_α and p_β are chosen by the mutual agreement between the consumer and the producer. However, for the illustrative purpose, sampling plans are computed with $(p_\alpha, p_\beta) = (0.00284, 0.03110)$, $(0.00654, 0.04260)$, $(0.01090, 0.05350)$, $(0.02090, 0.07420)$ and $(0.03190, 0.09420)$. These values are chosen to match the specifications in MIL-STD-105D ([22]). This MIL-STD-105D is a United States defense standard which is widely referred by the quality engineering professionals. Table 1 reports the computed sampling plans, namely (n, k_a, k_b) for the known values of $(m, k) = (3, 2)$

TABLE 1. Summary of computed sampling plans under hybrid censoring scheme with $m = 3$ and $k = 2$.

X_0	$(\alpha, \beta) = (0.05, 0.1)$		Degree of censoring (q)								
				0.2			0.5			0.8	
	p_α	p_β	n	k_a	k_b	n	k_a	k_b	n	k_a	k_b
0.5	0.00284	0.03110	8	0.985	0.153	8	0.984	0.778	16	0.985	0.157
	0.00654	0.04260	8	0.986	0.061	8	0.979	0.132	16	0.981	0.162
	0.01090	0.05350	8	0.971	0.665	9	0.971	0.302	16	0.969	0.365
	0.02090	0.07420	9	0.957	0.230	10	0.959	0.114	21	0.963	0.242
	0.03190	0.09420	10	0.942	0.336	10	0.943	0.006	21	0.942	0.449
1	0.00284	0.03110	5	0.983	0.055	6	0.984	0.734	16	0.990	0.715
	0.00654	0.04260	5	0.982	0.165	6	0.976	0.030	16	0.982	0.244
	0.01090	0.05350	5	0.971	0.065	8	0.974	0.246	16	0.969	0.394
	0.02090	0.07420	6	0.956	0.356	8	0.956	0.322	21	0.962	0.766
	0.03190	0.09420	7	0.944	0.034	8	0.943	0.365	21	0.943	0.241
3	0.00284	0.03110	5	0.989	0.275	6	0.985	0.414	16	0.991	0.609
	0.00654	0.04260	5	0.978	0.598	6	0.979	0.765	16	0.976	0.185
	0.01090	0.05350	5	0.977	0.544	6	0.971	0.320	16	0.971	0.333
	0.02090	0.07420	5	0.960	0.421	8	0.955	0.082	21	0.956	0.363
	0.03190	0.09420	5	0.943	0.542	8	0.942	0.196	21	0.943	0.479

X_0	$(\alpha, \beta) = (0.05, 0.05)$		Degree of censoring (q)								
				0.1			0.6			0.9	
	p_α	p_β	n	k_a	k_b	n	k_a	k_b	n	k_a	k_b
0.5	0.00284	0.03110	7	0.984	0.011	10	0.986	0.186	31	0.988	0.533
	0.00654	0.04260	8	0.977	0.383	10	0.981	0.593	31	0.978	0.207
	0.01090	0.05350	8	0.973	0.020	10	0.974	0.141	31	0.972	0.129
	0.02090	0.07420	9	0.960	0.252	11	0.960	0.337	41	0.964	0.388
	0.03190	0.09420	11	0.945	0.510	13	0.946	0.654	41	0.946	0.066
1	0.00284	0.03110	5	0.987	0.103	8	0.990	0.503	31	0.987	0.770
	0.00654	0.04260	5	0.979	0.805	8	0.982	0.085	31	0.979	0.642
	0.01090	0.05350	5	0.972	0.003	10	0.980	0.847	31	0.972	0.044
	0.02090	0.07420	6	0.961	0.601	10	0.958	0.735	41	0.963	0.737
	0.03190	0.09420	7	0.947	0.349	13	0.944	0.357	41	0.946	0.015
3	0.00284	0.03110	5	0.994	0.453	8	0.987	0.271	31	0.984	0.808
	0.00654	0.04260	5	0.977	0.499	8	0.978	0.777	31	0.978	0.172
	0.01090	0.05350	5	0.978	0.502	9	0.972	0.227	32	0.972	0.011
	0.02090	0.07420	5	0.958	0.063	10	0.961	0.378	41	0.960	0.088
	0.03190	0.09420	6	0.948	0.380	10	0.946	0.014	41	0.946	0.099

with $(\alpha, \beta) = (0.05, 0.1)$ and $(0.05, 0.05)$. We considered various levels of censoring $q = 0.1, 0.2, 0.5, 0.6,$ $0.8, 0.9$ and three values of X_0 given by $X_0 = 0.5,$ 1 and 3. From Table 1, the following observations are noticed.

(a) The computed sample size increases with increasing X_0 for fixed q. Intuitively this is because of increasing X_0 for fixed q allows more duration in life-test and, hence, observing more failures. Note that expected duration of life-test is increasing with X_0 for fixed q (see [11, Result 2]).

(b) The computed sample size increases with increasing degree of censoring q for fixed X_0. Intuitively, higher degree of censoring indicates lesser number of failures are observed and, as a consequence, a larger sample size is required for lot sentencing.

In order to investigate the effect of m and k on the sampling plans, we present Table 2. The sampling plans are computed for various combinations of (m, k) with $(p_\alpha, \alpha) = (0.03190, 0.05)$, $(p_\beta, \beta) = (0.0942, 0.05)$. The table shows that the computed sample size increases with k for the fixed value of m, as expected.

V. PERFORMANCES AND COMPARISONS OF SAMPLING PLANS

In order to measure the performance of the GMDS sampling plan, the OC curves of SSP, MDS sampling plan and GMDS sampling plan are shown in Figure 1. The parameters considered are: $n = 41$ and $k_a = 0.946$ for SSP; $n = 41$, $m = 3$, $k_a = 0.946$ and $k_b = 0.099$ for MDS sampling plan; $n = 41$, $k = 2$, $m = 3$, $k_a = 0.946$ and $k_b = 0.099$ for the proposed GMDS sampling plan. These plans are selected to satisfy $(p_\alpha, 1 - \alpha) = (0.03190, 0.95)$ and $(p_\beta, \beta) = (0.09420, 0.05)$. The figure clearly shows that the proposed GMDS sampling plan has higher probability of acceptance when the lot fraction defective is less and it converges with the OC curves of others when the lot fraction defective increases. This indicates that the proposed GMDS sampling plan has more protection in comparison with the SSP and MDS sampling plan for identical parameters.

Next, we compare the ATI(p) performance of the proposed GMDS sampling plan with SSP and MDS sampling plan. To satisfy $(p_\alpha, 1 - \alpha) = (0.03190, 0.95)$ and $(p_\beta, \beta) = (0.09420, 0.05)$, the following parameters were considered for comparison

TABLE 2. Summary of computed sampling plans for various values of m and k with $(p_\alpha, \alpha) = (0.03190, 0.05)$, $(p_\beta, \beta) = (0.0942, 0.05)$ and $X_0 = 3$.

| | | | | | | Degree of censoring (q) | | | | | |
| | | | 0.1 | | | 0.5 | | | 0.9 | |
m	k	n	k_a	k_b	n	k_a	k_b	n	k_a	k_b
1	1	7	0.948	0.024	12	0.944	0.450	41	0.949	0.269
2	1	5	0.953	0.046	8	0.954	0.396	41	0.952	0.238
	2	7	0.941	0.473	12	0.941	0.448	41	0.942	0.401
3	1	5	0.953	0.072	8	0.954	0.608	41	0.953	0.749
	2	6	0.947	0.510	8	0.946	0.016	41	0.946	0.076
	3	8	0.942	0.445	14	0.940	0.568	36	0.943	0.533
4	1	5	0.956	0.499	8	0.965	0.288	41	0.958	0.198
	2	5	0.955	0.039	8	0.947	0.714	41	0.952	0.478
	3	6	0.946	0.686	10	0.947	0.477	26	0.943	0.693
	4	8	0.939	0.407	14	0.939	0.421	36	0.939	0.524
5	1	5	0.960	0.649	6	0.960	0.067	31	0.960	0.108
	2	5	0.959	0.307	8	0.950	0.568	41	0.949	0.664
	3	5	0.947	0.509	8	0.950	0.019	41	0.950	0.291
	4	6	0.944	0.119	10	0.943	0.017	26	0.943	0.298
	5	9	0.941	0.820	15	0.939	0.025	38	0.939	0.062
10	1	5	0.972	0.141	6	0.962	0.242	31	0.967	0.029
	2	5	0.959	0.559	6	0.965	0.108	31	0.963	0.149
	3	5	0.952	0.636	6	0.957	0.143	31	0.959	0.141
	4	5	0.952	0.111	6	0.952	0.056	31	0.950	0.150
	5	5	0.949	0.433	8	0.945	0.149	41	0.952	0.068
	6	5	0.947	0.636	8	0.946	0.087	41	0.948	0.318
	7	6	0.944	0.158	10	0.950	0.222	26	0.943	0.446
	8	7	0.943	0.580	12	0.942	0.942	31	0.946	0.093
	9	8	0.942	0.378	14	0.939	0.603	36	0.941	0.275
	10	10	0.939	0.643	18	0.937	0.081	46	0.939	0.228

TABLE 3. Summary of sampling plans under Type-I censoring.

| | | | | | | X_0 | | | | | |
| $(\alpha, \beta) = (0.05, 0.1)$ | | | 0.5 | | | 1 | | | 3 | |
p_α	p_β	n	k_a	k_b	n	k_a	k_b	n	k_a	k_b
0.00284	0.03110	7	0.983	0.221	5	0.992	0.665	5	0.987	0.453
0.00654	0.04260	8	0.976	0.129	5	0.982	0.674	5	0.977	0.025
0.01090	0.05350	8	0.971	0.615	5	0.970	0.333	5	0.977	0.230
0.02090	0.07420	9	0.960	0.361	6	0.961	0.017	6	0.965	0.650
0.03190	0.09420	10	0.944	0.228	6	0.943	0.109	6	0.950	0.109

| | | | | | | X_0 | | | | | |
| $(\alpha, \beta) = (0.05, 0.05)$ | | | 0.5 | | | 1 | | | 3 | |
p_α	p_β	n	k_a	k_b	n	k_a	k_b	n	k_a	k_b
0.00284	0.03110	7	0.985	0.058	5	0.984	0.682	5	0.995	0.152
0.00654	0.04260	8	0.979	0.417	5	0.983	0.495	5	0.978	0.840
0.01090	0.05350	8	0.973	0.042	5	0.974	0.387	5	0.973	0.506
0.02090	0.07420	9	0.961	0.101	6	0.961	0.346	6	0.959	0.140
0.03190	0.09420	11	0.945	0.290	7	0.946	0.425	7	0.944	0.467

(i) $n = 41$, $k = 2$, $m = 3$, $k_a = 0.946$ and $k_b = 0.099$ for the proposed GMDS sampling plan

(ii) $n = 45$, $m = 3$, $k_a = 0.649$ and $k_b = 0.185$ for the MDS sampling plan

(iii) $n = 56$ and $k_a = 0.742$ for SSP (see [10])

In Figure 2, it can be clearly seen that the GMDS sampling plan has the minimum ATI in comparison with others indicating the better performance.

So far we have presented all the numerical illustrations in presence of hybrid censoring. As the Type-I and Type-II censoring schemes are the special cases of hybrid censoring, the proposed GMDS sampling plan methodology can be easily extended to those censoring schemes. To illustrate this, we carry out the following further investigation.

(a) **Type I censoring:** In Table 3, we reported the computed GMDS sampling plans by setting $q = 0$, that is, $r = n$. From Table 3, it is observed that the computed sampling plan decreases with X_0 indicating the fact that the duration of the experiment tends to decrease with X_0.

(b) **Type-II censoring:** In Table 4, we reported the computed GMDS sampling plans by setting $X_0 = \infty$. From Table 4, it is observed that the computed sampling plan increases with degree of censoring. This is probably because, intuitively, when the degree of censoring is high, expected number of failures tends to increase in order to increase the precision of the estimate to render reasonably good lot-sentencing.

TABLE 4. Summary of sampling plans under Type-II censoring.

$(\alpha, \beta) = (0.05, 0.1)$		Degree of censoring								
		0.1			0.6			0.9		
p_α	p_β	n	k_a	k_b	n	k_a	k_b	n	k_a	k_b
0.00284	0.03110	5	0.995	0.868	8	0.984	0.024	31	0.990	0.831
0.00654	0.04260	5	0.986	0.753	8	0.982	0.276	31	0.982	0.470
0.01090	0.05350	5	0.972	0.445	8	0.970	0.449	31	0.970	0.397
0.02090	0.07420	5	0.955	0.198	10	0.955	0.761	41	0.962	0.712
0.03190	0.09420	5	0.943	0.211	10	0.946	0.031	41	0.944	0.542
$(\alpha, \beta) = (0.05, 0.05)$		Degree of censoring								
		0.2			0.5			0.8		
p_α	p_β	n	k_a	k_b	n	k_a	k_b	n	k_a	k_b
0.00284	0.03110	5	0.994	0.750	6	0.985	0.866	16	0.992	0.785
0.00654	0.04260	5	0.980	0.265	6	0.983	0.326	16	0.978	0.408
0.01090	0.05350	5	0.980	0.023	6	0.972	0.029	16	0.972	0.021
0.02090	0.07420	5	0.960	0.760	8	0.964	0.506	21	0.964	0.347
0.03190	0.09420	5	0.946	0.070	8	0.946	0.090	22	0.946	0.031

TABLE 5. Summary of sampling plans without censoring.

p_α	p_β	$(\alpha, \beta) = (0.05, 0.1)$			$(\alpha, \beta) = (0.05, 0.05)$		
		n	k_a	k_b	n	k_a	k_b
0.00284	0.03110	5	0.987	0.793	5	0.995	0.172
0.00654	0.04260	5	0.973	0.575	5	0.979	0.498
0.01090	0.05350	5	0.977	0.252	5	0.983	0.419
0.02090	0.07420	5	0.966	0.583	5	0.957	0.614
0.03190	0.09420	5	0.945	0.571	5	0.946	0.545

TABLE 6. Breaking strength of jute fiber of gauge length 10 mm from [35].

693.73	704.66	323.83	778.17	123.06	637.66	383.43	151.48
108.94	50.16	671.49	183.16	257.44	727.23	291.27	101.15
376.42	163.40	141.38	700.74	262.90	353.24	422.11	43.93
590.48	212.13	303.90	506.60	530.55	177.25		

(c) **No censoring:** In Table 5, we reported the computed GMDS sampling plans by setting $q = 0$ and $X_0 = \infty$.

VI. REAL-LIFE DATA EXAMPLE

This section describes an analysis of real-life data in order to demonstrate the application of the proposed sampling plan. The data for the analysis are taken from [35] which consists of the breaking strengths of jute fiber of gauge length 10mm. The data are presented in Table 6. By using a goodness-of-fit test, the data were found to be fitted with an exponential distribution reasonably well. The Kolmogorov-Smirnov (KS) distance statistic value between the empirical distribution functions and the fitted distribution functions was found to be 0.174 and the associated p-value was 0.283. For the analytical expressions of p-value and KS distance statistic, the readers are referred to the book [12]. On the basis of p-value, there is not enough evidence that one can reject the hypothesis "The data follow an exponential distribution". The MLE of the exponential parameter $\hat{\theta}$ is computed as 365.729. Moreover, the P-P plot of the data in Figure 3 advocates that the exponential model fits the data well.

To demonstrate the application of the GMDS sampling plan, we first compute the sampling plan as described in Sections 3.4 and 3.5 with the following specifications: $m = 3$, $k = 2$, $q = 0.9$, $X_0 = 1000$, $\theta = 365.729$, $((p_\alpha, 1 - \alpha) = (0.00, 0.95))$ and $(p_\beta, \beta) = (0.03110, 0.05)$. The computed GMDS sampling plan is $(n, k_a, k_b) = (30, 0.945, 0.349)$. Now the data in Table 6 are used in accordance with the computed sampling plan to lot-sentencing. The working principle of lot-sentencing by the GMDS sampling plan is described step wise as follows.

Step 1: The required quality standards $(p_\alpha, p_\beta) = (0.0319, 0.09420)$ and the associated risks $(\alpha, \beta) = (0.05, 0.05)$ are set for the lifetime characteristic X with a specified lower lifetime limit $L = 38.533$. For the illustrative purpose, L is set at 0.1th quantile of the exponential lifetime distribution with $\theta = 365.729$.

Step 2: Suppose that the GMDS sampling plan is carried out under hybrid censoring with pre-specified values of $q = 0.9$ and $X_0 = 1000$. Therefore, the required sample size $n = 30$ along with the
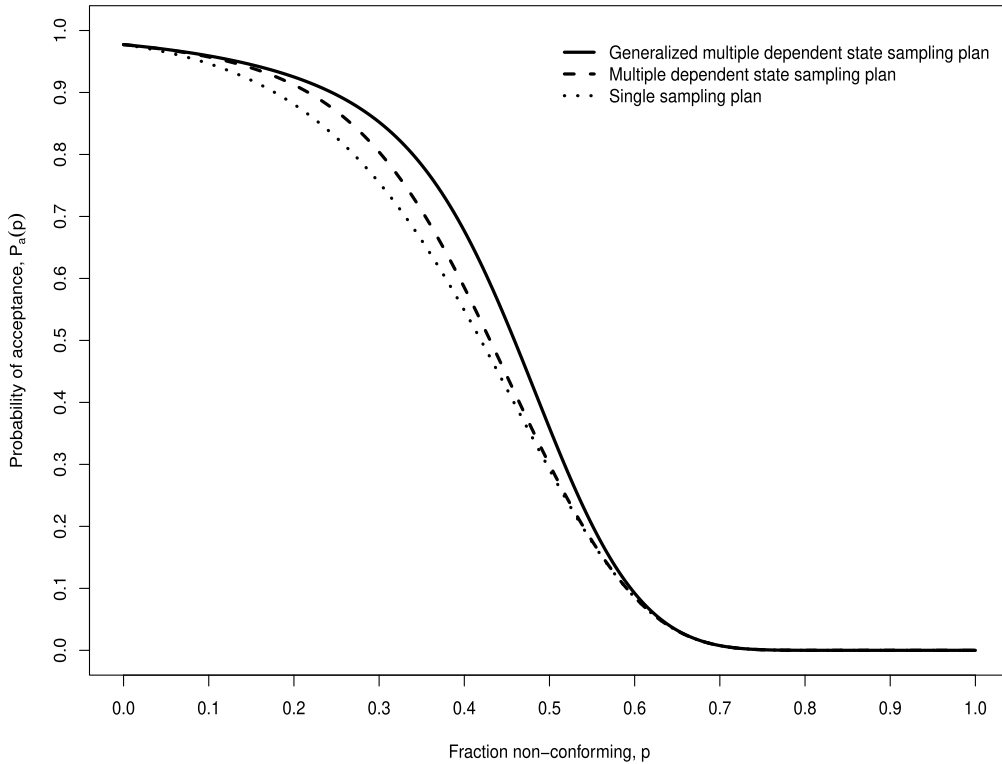
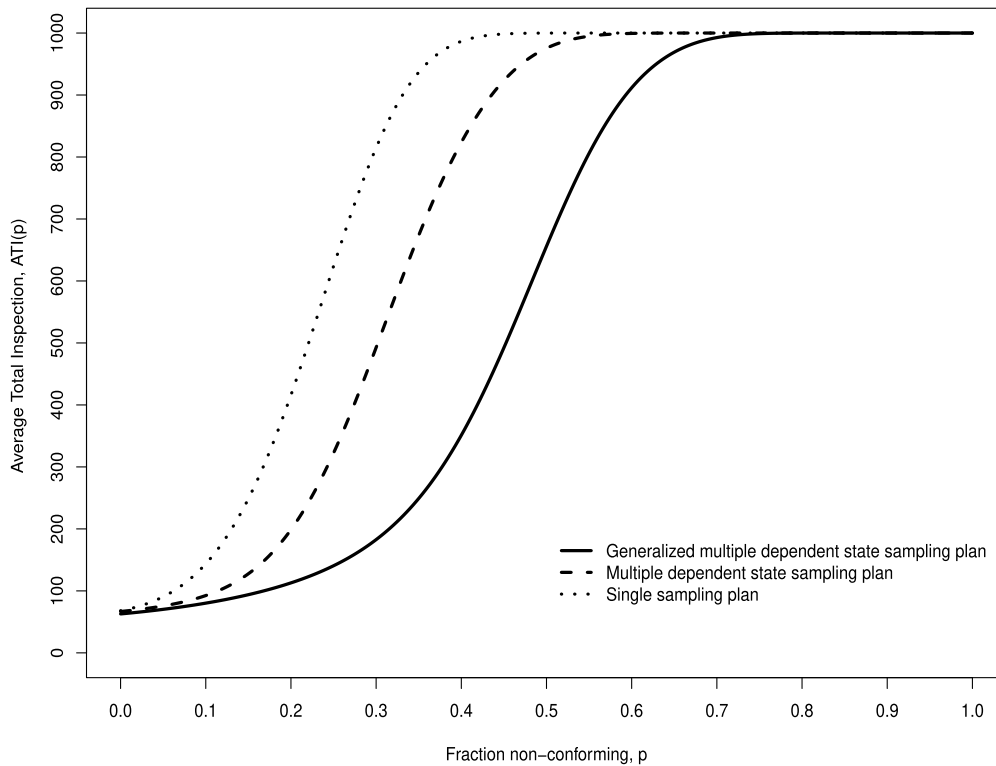**FIGURE 1.** OC curves under different sampling plans.

FIGURE 2. ATI curves under different sampling plans.

critical values $k_a = 0.945$ and $k_b = 0.349$ are found for the acceptance/rejection decision.

Step 3: Using these sampling plan specifications $(n, r, X_0) = (30, 3, 1000)$ and data set, the
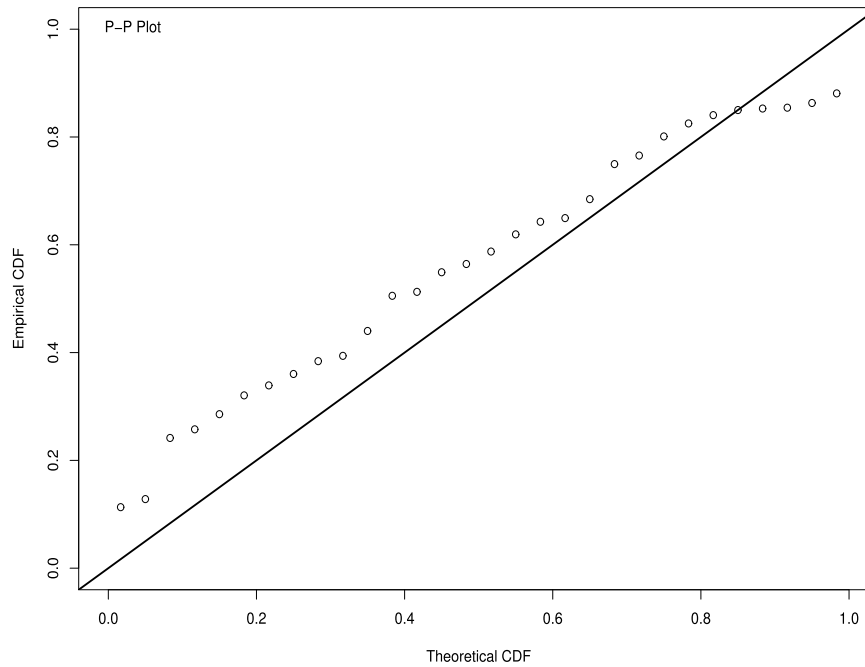
FIGURE 3. Probability plot of 30 breaking strengths of jute fiber of gauge length 10mm from [35].

following hybrid censored observations are generated: 43.93, 50.16 and 101.15.

Step 3: Calculate the estimated value of C_L using (8) as

$$\hat{C}_L = 1 - \frac{38.533 * 3}{(43.93 + 50.16 + 101.15) + (30 - 3)101.15}$$
$$= 1 - \frac{115.599}{2926.29}$$
$$= 0.960$$

Step 4: In this case, the lot will be accepted since $\hat{C}_L = 0.960 > k_a = 0.945$.

It may be noted that, in the above example, the lot is accepted based on the first sample. It would require more samples (in this case $m = 3$) if the estimate $\hat{C}_L$ would lie between k_b and k_a.

VII. CONCLUDING REMARKS

In this article, a new sampling plan, generalized multiple dependent state sampling, is developed in which the quality characteristic follows an exponential distribution. Multiple dependent state sampling and the single sampling plans are the two special cases of the proposed sampling plan. It has been shown that the proposed plan has more protection than the other two. Also, the proposed plan has higher probability of acceptance when the lot quality is good. While constructing the sampling plan, we assumed that the data come from a hybrid censored life-test. A further research could be to apply such proposed sampling plan in presence of more general censoring schemes such as progressive and progressively hybrid censoring.

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