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Design and Realizations of Networked Estimators: A Descriptor Model Approach

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ABSTRACT This paper addresses the design and realization of a networked estimator, which requires further digital implementation using several subsystems. Implementation on a digital process entails finite word length effects on the coefficients' representation, and a multiple-subsystem architecture also introduces internal time delays in the information interaction. Dealing with these intrinsic defects requires finding a resilient realization. A corresponding descriptor model-based approach is thus constructed to describe the internal time delays and equivalent realizations with finite word length effects in a unifying framework, which enables simultaneous consideration of design and realization. Based on the obtained descriptor model, a stability analysis condition is deduced and a design method for the estimator is further obtained. An algorithm is also proposed for finding the optimal realization requiring the minimum word length for stabilization. Finally, a simulation with two cooperative robots is considered to illustrate the effectiveness of the results.

INDEX TERMS Estimation, system implementation, finite word length effects, delay systems.

I. INTRODUCTION

Recent advances in decentralized computing and wireless communication technology cause increasingly decentralized implementation of practical applications, and many studies have focused on networked systems [1]–[3]. Such implementation schemes are generally motivated by the following constraints: 1) the plant is spread over a large space, requiring the controller or filter to be implemented with the same structure [4], [5]; 2) the control or filtering problem requires significant computing capacity, and thus the controller or filter must be implemented on several subsystems with limited processing capacity [6]. A decentralized networked system consists of many similar units such as multiple vehicles, agents or mobile sensors equipped with micro-processors containing limited capability and energy. The architecture of decentralized implementation for a given controller or filter is not unique and depends on the capability of each unit as well as how they are distributed and interconnected.

Digital devices entail finite precision, leading to some finite word length (FWL) effects on ensuring the stability and performance of the system. In a decentralized architecture, these effects can be emphasized as: 1) computing devices embedded in such architecture often have reduced computing capabilities; 2) potentially numerous processors can be involved. There are two categories of FWL effects: 1) the roundoff noise due to the rounding of variables in mathematical operations [7] and 2) distortion of parameters resulting from coefficients' representation [8]. The FWL effects generally depend on the arithmetic format (floating-point, fixed-point, etc.) and chosen type of realization. The imperfect communication network also introduces internal time delays in the internal information interaction between subsystems, which can inevitably affect the stability and performance of the implemented system [2], [9]–[12]. Time delays have received attention in networked systems, and therefore several different modeling and analysis methods exist for networks with time delays. However, significant literature results address delays involved in the transmission of control and measurement signals between the controller and the plant, while few studies address internal delays inside a controller or filter.

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This paper deals with the estimating and realization problem and analyzes deteriorations caused by both coefficients' representation with FWL effects and internal transmission delays. A given system can be expressed using equivalent realization forms with different coefficients, such as direct-form I, direct-form II, balanced realization, δ-operator realization, etc. When subjected to coefficients' representation with FWL effects, equivalent realizations become no longer equivalent, and it is thus necessary to select the appropriate realization form according to the scenario; for example, the δ-operator generally has favorable FWL properties with coefficients' representation [13] and in [18], and a new ρ-modal realization is constructed motivated by the ρ-operator for implementing of the filters or controllers with distinct poles in the descriptor model framework. The state-space form can represent most realizations and many realization problems are considered in the state-space framework [14], [15]. Although most realizations can be transformed into the state-space form, this form is not completely generalizable and features several limitations: 1) many realizations require computing intermediate variables that cannot be expressed in a standard state-space form [8], [16]; 2) analyzing the rounding effect of a coefficient in a particular realization form can become difficult after transformation to the state-space form [17], which is furthermore restricted to the single shift operator. To overcome these limitations, this study adopts the descriptor model that includes intermediate variables to provide a generalized description of any realization in a unifying framework that allows analyzing the FWL effects. The descriptor model is first introduced in [17] to analyze deteriorations caused by FWL effects, while [19] adopts the same descriptor model-based framework to address the implementation problem of controllers/filters involving time delays in the internal network among subsystems for information interaction.

The realization problems still involve several challenging issues. To list some, most studies consider a specific constraint only such as time delay [19], round-off noise [7] or coefficients' representation with FWL [8]. In addition, most studies focus on stability analysis, whereby numerical optimization approaches are adopted to seek the optimal realization for a given system with all parameters known. The contribution of this paper mainly focus on the following two aspects: 1) a descriptor model-based method is proposed to describe the design and realization problem subjected to both fixed internal time delays and FWL effects in a unified framework; 2) based on the obtained descriptor model, both the stability analysis condition and design method for the estimator gain are deduced, thereby simultaneously obtaining both the resilient realization and minimal word length for guaranteeing stability. In this paper, the superscript T is the transpose. $\lfloor * \rfloor$ denotes the floor function while $\odot$ denotes the Hadamard product. $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^+$ denote the field of real numbers, field of integral numbers, field of natural numbers and positive integral numbers, respectively.

II. PROBLEM FORMULATION

This paper is concerned with the design and realization problem for the estimator/observer described by the following linear time-invariant (LTI) discrete-time model

$$x(k+1) = Ax(k) + Bu(k) + L[y(k) - \bar{z}(k)], \qquad (1)$$

where $\bar{z}(k) = Cx(k)$, $x(k) \in \mathbb{R}^n$ is the estimator state at time $t = kT_s$ with T_s as the sampling period and $k \in \mathbb{N}$. $u(k) \in \mathbb{R}^m$ is the input vectors, $y(k) \in \mathbb{R}^p$ the measurement from the plant, while the known matrices A, B, C with appropriate dimensions are the state matrix, input matrix, output matrix of the plant, respectively. $L \in \mathbb{R}^{n \times p}$ in (1) is the estimator gain to be determined and the matrix pair (A, C) in (1) is assumed to be observable.

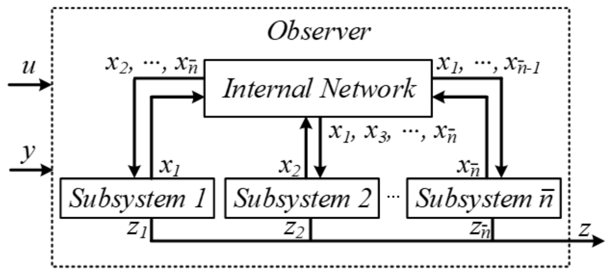

FIGURE 1. Problem setup.

As depicted in Fig. 1, the estimator given in (1) requires digital implementation using $\bar{n}$ SOCs (system on chip) with finite precision, in which case the FWL effects should be considered for the variables and constants involved in (1). Therefore, (1) is partitioned into $\bar{n}$ subsystems according to the following partition

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ A_{\bar{n}1} & \cdots & A_{\bar{n}\bar{n}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ B_{\bar{n}1} & \cdots & B_{\bar{n}\bar{n}} \end{bmatrix},$$
$$C = \begin{bmatrix} C_{11} & \cdots & C_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ C_{\bar{n}1} & \cdots & C_{\bar{n}\bar{n}} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & \cdots & L_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ L_{\bar{n}1} & \cdots & L_{\bar{n}\bar{n}} \end{bmatrix},$$
$$x^T(k) = \begin{bmatrix} x_1^T(k) & \cdots & x_{\bar{n}}^T(k) \end{bmatrix}, \quad u^T(k) = \begin{bmatrix} u_1^T(k) & \cdots & u_{\bar{n}}^T(k) \end{bmatrix},$$
$$y^T(k) = \begin{bmatrix} y_1^T(k) & \cdots & y_{\bar{n}}^T(k) \end{bmatrix}, \quad \bar{z}^T(k) = \begin{bmatrix} \bar{z}_1^T(k) & \cdots & \bar{z}_{\bar{n}}^T(k) \end{bmatrix},$$

where, $x_i(k) \in \mathbb{R}^{n_i}$, $u_i(k) \in \mathbb{R}^{m_i}$, $y_i(k) \in \mathbb{R}^{p_i}$, $\bar{z}_i(k) \in \mathbb{R}^{p_i}$, $\sum_i^{\bar{n}} n_i = n$, $\sum_i^{\bar{n}} m_i = m$, $\sum_i^{\bar{n}} p_i = p$. The matrices $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_{ij} \in \mathbb{R}^{n_i \times m_j}$, $C_{ij} \in \mathbb{R}^{p_i \times n_j}$ and $L_{ii} \in \mathbb{R}^{n_i \times p_j}$ are defined according to the partition of signals $x(k)$, $u(k)$, $y(k)$ and $\bar{z}(k)$.

For the estimator composed of $\bar{n}$ subsystems, a peer-to-peer network is structured for information interaction, where the central communication server does not exist and one communication link is established between each pair of subsystems for peer-to-peer communications. A bipartite-directed graph $\mathscr{G} = (a, \mathscr{X}, \mathscr{A})$ is introduced to represent the topology of the estimator/observer, where $a = 1, 2, \cdots, \bar{n}$ is the set of

$\bar{n}$ subsystems. The set of edges $\mathscr{X} \subset a \times a$ represents the communication topology of these subsystems, while $\mathscr{A} = [a_{ij}]$ with $a_{ij} \in \{0, 1\}$ is the weighted adjacency matrix with adjacency elements a_{ij}. An edge of $\mathscr{G}$ is denoted by (i, j). The adjacency elements associated with the edges of the graph are $a_{ij} = 1 \Leftrightarrow (i, j) \in \mathscr{X}$, which means that the i^{th} subsystem can directly receive information from the j^{th} subsystem. In contrast, $a_{ij} = 0$. It is undoubtedly that $a_{ii} = 1$ for all $i \in a$. The set composed of all the neighbors of node $i \in a$ is denoted by $\mathscr{N}_i = \{j \in a, j \neq i, (i, j) \in \mathscr{X}\}$.

The stability of the estimator (1) must be ensured after implementation. Focused on stability analysis, the input $u(k)$ as well as the measurement $y(k)$ in (1) are temporarily omitted. By considering the communication topology of all the subsystems defined by the weighted adjacency matrix $\mathscr{A} = [a_{ij}]$, (1) can be rewritten as the following system

$$x(k + 1) = (A_D + A_C)x(k) - L_D z(k), \tag{2}$$

where $z(k) = (C_D + C_C)x(k)$, $A_D = \mathrm{d}iag\,[A_{11}, \ \cdots, \ A_{\bar{n}\bar{n}}]$, $L_D = \mathrm{d}iag\,[L_{11}, \ \cdots, \ L_{\bar{n}\bar{n}}]$, $C_D = \mathrm{d}iag\,[C_{11}, \ \cdots, \ C_{\bar{n}\bar{n}}]$, and the matrices

$$A_C = \begin{bmatrix} 0 & a_{12}A_{12} & \cdots & a_{1\bar{n}}A_{1\bar{n}} \\ a_{21}A_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{\bar{n}-1\bar{n}}A_{\bar{n}-1\bar{n}} \\ a_{\bar{n}1}A_{\bar{n}1} & \cdots & a_{\bar{n}\bar{n}-1}A_{\bar{n}\bar{n}-1} & 0 \end{bmatrix},$$

$$C_C = \begin{bmatrix} 0 & a_{12}C_{12} & \cdots & a_{1\bar{n}}C_{1\bar{n}} \\ a_{21}C_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{\bar{n}-1\bar{n}}C_{\bar{n}-1\bar{n}} \\ a_{\bar{n}1}C_{\bar{n}1} & \cdots & a_{\bar{n}\bar{n}-1}C_{\bar{n}\bar{n}-1} & 0 \end{bmatrix}$$

in (2) denote the information interaction among all the subsystems by introducing adjacency elements $a_{ij} \in \{0, 1\}$ defined in the weighted adjacency matrix $\mathscr{A} = [a_{ij}]$. $z(k)$ in (2) is given as $z^T(k) = \left[z_1^T(k) \ \cdots \ z_{\bar{n}}^T(k)\right]$, $z_i(k) \in \mathbb{R}^{p_i}$, $z_i(k) = C_{ii}x_i(k) + \sum_{1 \le j \le \bar{n}, j \neq i} a_{ij}C_{ij}x_j(k)$.

With the peer-to-peer network defined, each subsystem in (2) can only receive the information from its neighbors, which means the evolution of the state $x_i(k + 1)$ requires only $x_i(k)$ and $x_j(k)$, $j \in \mathscr{N}_i$. To meet such constraints, the centralized estimator gain L in (1) is restricted to be block-diagonal as L_D in (2). It should be noted that such block-diagonal structure is not necessary to derive the results proposed in this paper, and its design method is also suitable for L_D with other structure based on the considered communication strategy.

According to the information interaction defined by the weighted adjacency matrix $\mathscr{A}$, the centralized system (2) can be further rewritten as the following subsystems

$$x_i(k + 1) = A_{ii}x_i(k) - L_{ii}z_i(k) + \sum_{1 \le j \le \bar{n}, j \neq i} a_{ij}A_{ij}x_j(k), \tag{3}$$

where $z_i(k) = C_{ii}x_i(k) + \sum_{1 \le j \le \bar{n}, j \neq i} a_{ij}C_{ij}x_j(k)$. With $\mathscr{N}_i = \{j \in a, j \neq i, (i, j) \in \mathscr{X}\}$ defined as the set composing all neighbors of the i^{th} subsystem, (3) is equal to the following subsystems

$$x_i(k + 1) = A_{ii}x_i(k) - L_{ii}z_i(k) + \sum_{j \in \mathscr{N}_i} A_{ij}x_j(k), \tag{4}$$

where $z_i(k) = C_{ii}x_i(k) + \sum_{j \in \mathscr{N}_i} C_{ij}x_j(k)$.

The subsystem (4) represents an estimator realized by the shift operator in the state-space form with the coefficients L_{ii}, $i = 1, 2, \cdots, \bar{n}$ to be determined. The implementation strategy depicted in Fig. 1 introduce time delays in the information interaction among the subsystems, and the matrices A_{ij}, C_{ij}, L_{ii}, $i, j = 1, 2, \cdots, \bar{n}$ in (4) requires digital representation by each SOC with finite precision. In this case, the stability of the estimator (4) cannot be strictly guaranteed following implementation even if designed to be stable. Moreover, the estimator (4) can be described using different operators and equivalent realizations. However, when the parameters are subjected to FWL effects, the realizations are no longer equivalent. Therefore, the FWL effects should be considered along with the equivalent realizations of (4), whose resilience to these defects must be determined. To gain a detailed description of the problem, the characteristics of the internal time delays as well as arithmetic format for coefficients' representation must be described. In addition, the equivalent realizations for (4) should be further defined and considered in a general unifying framework, which will be detailed in the following subsections.

A. NETWORK AND FWL EFFECTS

(4) shows that the evolution of the state $x_i(k+1)$ requires $x_j(k)$, $j \in \mathscr{N}_i$ from neighbors of the i^{th} subsystem. It is assumed that single-packet transmission is adopted by $\mathscr{N}_i$ to send $x_j(k)$ to the i^{th} subsystem, while the packet dropouts are not considered in the transmission process. To describe the network $\mathscr{X}$, the input and output of the channel (i, j) are defined as $\nu_{ij}(k) \in \mathbb{R}^{n_j}$ and $\eta_{ij}(k) \in \mathbb{R}^{n_j}$ respectively. The time delay of each individual communication channel (i, j) is assumed to be independent and fixed as $\tau_{ij}T_s$, $\tau_{ij} \in \mathbb{N}$, where T_s is the sampling period of (1). And it is undoubtedly that $\tau_{ii} = 0$ for $i = 1, 2, \cdots, \bar{n}$. In this case, the input-output characteristic of (i, j) is given as: $\eta_{ij}(k) = \nu_{ij}(k - \tau_{ij})$. By considering the time delays, (4) can be rewritten as:

$$x_i(k + 1) = A_{ii}x_i(k) - L_{ii}z_i(k) + \sum_{j \in \mathscr{N}_i} A_{ij}x_j(k - \tau_{ij}), \tag{5}$$

where $z_i(k) = C_{ii}x_i(k) + \sum_{j \in \mathscr{N}_i} C_{ij}x_j(k - \tau_{ij})$.

For each subsystem (5), the representation of its coefficients A_{ij}, C_{ij} and L_{ii}, $i, j = 1, 2, \cdots, \bar{n}$ should be considered, which depends both on the arithmetic format and word length for representation. In this paper, the fixed-point representation [20] scheme is introduced. A real number $b \in \mathbb{R}$ can be represented in fixed-point format with a total word length $\gamma = \alpha + \beta + 1$ by assigning 1 bit for the sign, α bits for the integer part and β bits for the fraction part of b. In this case, the integer part of a real number can be represented by a sufficiently large word length a without overflow as $\alpha = \log_2 \lfloor a \rfloor$, while its fraction part cannot be exactly represented and the representation error of b is only related to the word length $\beta \in \mathbb{N}^+$. More specifically, after fixed-point representation, b is given as

$$\varphi(b) = b + \Delta, \quad |\Delta| < 2^{-(\beta+1)},$$

where $\varphi(*)$ denotes the function for fixed-point representation.

To consider a real matrix X, let $d(X)$ represent the matrix of the same dimension with elements

$$d(X)_{ij} = \begin{cases} 0, & X_{ij} \in \mathbb{Z}, \\ 1, & X_{ij} \notin \mathbb{Z}, \end{cases} \tag{6}$$

where $d(X)_{ij}$ denotes the element of $d(X)$ in the i^{th} row while j^{th} column and X_{ij} denotes the element of X in the i^{th} row and j^{th} column. After fixed-point representation $\varphi(*)$ with word length β, X is given as

$$\varphi(X) = X + d(X) \odot \Delta_{ij}, \quad |\Delta_{ij}| < 2^{-(\beta+1)}.$$

where $\odot$ denotes the Hadamard product. In this paper, the variables and constants involved in each subsystem (5) are represented by the same fixed-point scheme with the same β bits for the fraction part. Assuming that all uncertainties $|\Delta_{ij}|$ also have the common bound $2^{-(\beta+1)}$, and it can be obtained that

$$|\varphi(X)_{ij} - X_{ij}| < 2^{-(\beta+1)},$$

where $\varphi(X)_{ij}$ denotes the element of $\varphi(X)$ in the i^{th} row and j^{th} column. The bound is also considered in the stability analysis and estimator gain design method proposed in Section III, the bound is considered, and therefore $\varphi(X)$ can be simplified as

$$\varphi(X) = X + d(X)\Delta, \quad |\Delta| < 2^{-(\beta+1)}.$$

In the above-mentioned fixed-point representation scheme, the integer part of the coefficients A_{ij}, C_{ij} and L_{ii}, $i, j = 1, 2, \cdots, \bar{n}$ are assumed to be precisely represented without overflow, while the fraction part of the coefficients is represented with word length β. In this case, the result of coefficients' representation is related only to β, while the word length α for the integer part as well as the bit for the sign are subsequently omitted.

B. PROBLEM DEFINITION

As discussed in Subsection II-A, the coefficients in (5) cannot be precisely represented using word length β, and thus equivalent realizations may result in different properties against the FWL effects. For illustration, we consider an example of realizing the coefficients A_{ij}, C_{ij} and L_{ii}, $i, j = 1, 2, \cdots, \bar{n}$ in (5) with δ-operator as:

$$\delta[x_i(k)] = A^{\delta}_{ii} x_i(k) - L^{\delta}_{ii} z_i(k) + \sum_{j \in \mathcal{N}_i} A^{\delta}_{ij} x_j(k - \tau_{ij}), \tag{7}$$

where $z_i(k) = C^{\delta}_{ii} x_i(k) + \sum_{j \in \mathcal{N}_i} C^{\delta}_{ij} x_j(k - \tau_{ij})$, $A^{\delta}_{ii} = (A_{ii} - I)/\Delta_\delta$, $A^{\delta}_{ij} = A_{ij}/\Delta_\delta$, $L^{\delta}_{ii} = L_{ii}/\Delta_\delta$, $C^{\delta}_{ii} = C_{ii}$, $C^{\delta}_{ij} = C_{ij}$, $\delta = (q-1)/\Delta_\delta$ with Δ_δ as a positive constant and q^{-1} the shift operator [15].

Noticing that (5) and (7) are equivalent realizations with different coefficients and, with coefficients' representation subject to FWL effects, may lead to different properties of stability. Moreover, with $\delta[x_i(k)]$ in expression (7), the classical state-space model is insufficient to describe the form of (7), and thus a more generalized model is introduced in the next section.

The problem under consideration can be paraphrased as finding an appropriate equivalent realization for (5) which is resilient to FWL effects using the fixed-point representation and designing the corresponding estimator gain to guarantee the estimator's stability, which is summarized as the following Problem 1.

Problem 1: For given word length β for the coefficients' representation and network $\mathscr{X}$ with internal time delay τ_{ij}, $i, j = 1, 2, \cdots, \bar{n}$, find an appropriate realization for system (5) and design the estimator to be stable after implementation.

III. MAIN RESULTS

This section details the method to solve Problem 1. In subsection III-A, the descriptor model is first adopted to consider the internal time delay, coefficients representation and the equivalent realizations in the unifying framework. Based on the above modeling method, an analysis condition is then deduced in III-B to evaluate the stability resilience of a given realization against the FWL effects, and the design method of the observer gain is further provided.

A. DESCRIPTOR MODEL REPRESENTATION

To describe the equivalent realizations of (5) within a general unifying framework, the following descriptor model [17] is introduced with the specialized form given as

$$\begin{bmatrix} \mathcal{J} & 0 & 0 \\ -\mathcal{K} & I & 0 \\ -\mathcal{L} & 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{T}(k+1) \\ \mathcal{X}(k+1) \\ \mathcal{Y}(k) \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{M} & \mathcal{N} \\ 0 & \mathcal{P} & \mathcal{Q} \\ 0 & \mathcal{R} & \mathcal{S} \end{bmatrix} \begin{bmatrix} \mathcal{T}(k) \\ \mathcal{X}(k) \\ \mathcal{U}(k) \end{bmatrix}, \tag{8}$$

where $\mathcal{J} \in \mathbb{R}^{l\times l}$, $\mathcal{K} \in \mathbb{R}^{n\times l}$, $\mathcal{L} \in \mathbb{R}^{p\times l}$, $\mathcal{M} \in \mathbb{R}^{l\times n}$, $\mathcal{N} \in \mathbb{R}^{l\times m}$, $\mathcal{P} \in \mathbb{R}^{n\times n}$, $\mathcal{Q} \in \mathbb{R}^{n\times m}$, $\mathcal{R} \in \mathbb{R}^{p\times n}$, $\mathcal{S} \in \mathbb{R}^{p\times m}$, $\mathcal{T}(k) \in \mathbb{R}^l$, $\mathcal{X}(k) \in \mathbb{R}^n$, $\mathcal{U}(k) \in \mathbb{R}^m$, $\mathcal{Y}(k) \in \mathbb{R}^p$, $\mathcal{X}(k)$ is the state vector, $\mathcal{U}(k)$ the input, $\mathcal{Y}(k)$ the output, $\mathcal{T}(k)$ the intermediate variable and $\mathcal{J}$ a lower triangular matrix with 1 on the diagonal. The state-space model can be regarded as a special case of the above model, while the above model can be used to express the state-space system. On the other hand, the intermediate variable $\mathcal{T}(k)$ in the above model enables representing the system (7) realized by the δ-operator, thereby explicitly describing the parametrization and allowing the analysis of the FWL effects in a unifying framework.

The above model takes the form of an implicit state-space system [21]. In (8), the state vector $\mathcal{X}(k+1)$ is the stored state vector and $\mathcal{X}(k)$ is effectively stored between steps to compute $\mathcal{X}(k+1)$ at step k. $\mathcal{T}$ plays a particular role since $\mathcal{T}(k+1)$ is independent of $\mathcal{T}(k)$ and $\mathcal{T}(k)$ is not used for the calculation at step k, which characterizes an intermediate variable. The particular structure of $\mathcal{J}$ allows expressing how the computations are decomposed, and providing intermediate results that could be reused. The computations associated with the above realization are executed in row order, giving the following algorithm:

$$\begin{aligned}\mathcal{J}\mathcal{T}(k+1) &= \mathcal{M}\mathcal{X}(k)+\mathcal{N}\mathcal{U}(k),\\ \mathcal{X}(k+1) &= \mathcal{K}\mathcal{T}(k+1)+\mathcal{P}\mathcal{X}(k)+\mathcal{Q}\mathcal{U}(k),\\ \mathcal{Y}(k) &= \mathcal{L}\mathcal{T}(k+1)+\mathcal{R}\mathcal{X}(k)+\mathcal{S}\mathcal{U}(k).\end{aligned}$$

There is no need to compute $\mathcal{J}^{-1}$ since the computations are executed in row order and $\mathcal{J}$ is a lower triangular with 1 on the diagonal. See [22] for a practical example taking benefits from this descriptor model.

The descriptor model (8) is equivalent in infinite precision to the classical state-space form

$$\begin{bmatrix}\mathcal{T}(k+1)\\ \mathcal{X}(k+1)\\ \mathcal{Y}(k)\end{bmatrix}=\begin{bmatrix}0 & \mathcal{J}^{-1}\mathcal{M} & \mathcal{J}^{-1}\mathcal{N}\\ 0 & \mathcal{A} & \mathcal{B}\\ 0 & \mathcal{C} & \mathcal{D}\end{bmatrix}\begin{bmatrix}\mathcal{T}(k)\\ \mathcal{X}(k)\\ \mathcal{U}(k)\end{bmatrix},$$

where $\mathcal{A}\in\mathbb{R}^{n\times n}$, $\mathcal{B}\in\mathbb{R}^{n\times m}$, $\mathcal{C}\in\mathbb{R}^{p\times n}$, $\mathcal{D}\in\mathbb{R}^{p\times m}$ with

$$\begin{aligned}\mathcal{A}&=\mathcal{K}\mathcal{J}^{-1}\mathcal{M}+\mathcal{P}, & \mathcal{B}&=\mathcal{K}\mathcal{J}^{-1}\mathcal{N}+\mathcal{Q},\\ \mathcal{C}&=\mathcal{L}\mathcal{J}^{-1}\mathcal{M}+\mathcal{R}, & \mathcal{D}&=\mathcal{L}\mathcal{J}^{-1}\mathcal{N}+\mathcal{S}.\end{aligned}$$

The finite-precision implementation of the above model will cause differing numerical deterioration to that of (8).

To match the structure of the above descriptor model, (5) and (7) are rewritten into a general unifying framework as:

$$\begin{cases} J_iT_i(k+1)=M_{ii}x_i(k)-N_{ii}z_i(k)\\ \qquad\qquad +\sum_{j\in\mathcal{N}_i}M_{ij}x_j(k-\tau_{ij}),\\ x_i(k+1)=K_iT_i(k+1)+P_{ii}x_i(k)\\ \qquad\qquad +\sum_{j\in\mathcal{N}_i}P_{ij}x_j(k-\tau_{ij}),\end{cases}\tag{9}$$

where $z_i(k)=C_{ii}x_i(k)+\sum_{j\in\mathcal{N}_i}C_{ij}x_j(k-\tau_{ij})$, $T_i(k)\in\mathbb{R}^{l_i}$ are intermediate variables. In (9), J_i, K_i, M_{ii}, M_{ij}, P_{ii} and P_{ij} are known matrices satisfying

$$A_{ii}=K_iJ_i^{-1}M_{ii}+P_{ii},\quad A_{ij}=K_iJ_i^{-1}M_{ij}+P_{ij},$$

while N_{ii} are matrices to be determined, which satisfy

$$L_{ii}=K_iJ_i^{-1}N_{ii},\quad i=1,\ 2,\ \cdots,\ ,\bar{n}.$$

Compared to the state-space form (5), representation (9) is more general and provides more detailed information on the implementation. The intermediate variables $T_i(k)$ typically enable describing the δ-operator in (7). Specifically, (9) is equivalent in infinite precision to the classical state-space form (5) with shift operators by selecting the parameters in (9) as

$$K_i=I,\quad J_i=I,\ P_{ii}=A_{ii},\ P_{ij}=A_{ij},$$
$$L_{ii}=N_{ii},\quad M_{ii}=0,\ M_{ij}=0.$$

(9) is similarly equivalent in infinite precision to the realization (7) with δ-operator by selecting the parameters as

$$\begin{aligned}J_i&=I, & M_{ii}&=\Delta_\delta^{-1}(A_{ii}-I),\ M_{ij}=\Delta_\delta^{-1}A_{ij},\\ P_{ii}&=I, & P_{ij}&=0,\ K_i=\Delta_\delta I,\ N_{ii}=\Delta_\delta^{-1}L_{ii}.\end{aligned}$$

In this case, (9) is specific given as

$$\begin{cases} T_i(k+1)=\Delta_\delta^{-1}(A_{ii}-I)x_i(k)-\Delta_\delta^{-1}L_{ii}z_i(k)\\ \qquad\qquad +\sum_{j\in\mathcal{N}_i}\Delta_\delta^{-1}A_{ij}x_j(k-\tau_{ij}),\\ x_i(k+1)=\Delta_\delta T_i(k+1)+x_i(k).\end{cases}$$

Compared with (7), (5) realized by the δ-operator as

$$\delta[x_i(k)]=A_{ii}^{\delta}x_i(k)-L_{ii}^{\delta}z_i(k)+\sum_{j\in\mathcal{N}_i}A_{ij}^{\delta}x_j(k-\tau_{ij}),$$

is in evaluated as

$$\begin{aligned}T_i(k+1)=\Delta_\delta^{-1}(A_{ii}-I)x_i(k)-\Delta_\delta^{-1}L_{ii}z_i(k)\\ +\sum_{j\in\mathcal{N}_i}\Delta_\delta^{-1}A_{ij}x_j(k-\tau_{ij}),\end{aligned}$$

where T_i is an intermediate variable, and then

$$x_i(k+1)=\Delta_\delta T_i(k+1)+x_i(k).$$

For the convenience of the follow-up discussion, denote

$T^T(k)=[T_1^T(k)\ \cdots\ T_{\bar{n}}^T(k)]$, $\bar{P}_1=\mathrm{d}iag\,[P_{11},\ \cdots,\ P_{\bar{n}\bar{n}}]$, $\bar{M}_1=\mathrm{d}iag\,[M_{11},\ \cdots,\ M_{\bar{n}\bar{n}}]$, $N=\mathrm{d}iag\,[N_{11},\ \cdots\ N_{\bar{n}\bar{n}}]$, $\bar{C}=\mathrm{d}iag\left[\bar{C}_1,\ \cdots,\ \bar{C}_{\bar{n}}\right]$, $\bar{C}_i=\left[a_{i1}C_{i1}\ \ \cdots\ \ a_{i\bar{n}}C_{i\bar{n}}\right]$, $i=1,2,\cdots,\bar{n}$,

$$\bar{M}_2=\begin{bmatrix}a_{11}M_{11} & \cdots & a_{1\bar{n}}M_{1\bar{n}}\\ \vdots & \ddots & \vdots\\ a_{\bar{n}1}M_{\bar{n}1} & \cdots & a_{\bar{n}\bar{n}}M_{\bar{n}\bar{n}}\end{bmatrix}-\bar{M}_1,$$

$$\bar{P}_2=\begin{bmatrix}a_{11}P_{11} & \cdots & a_{1\bar{n}}P_{1\bar{n}}\\ \vdots & \ddots & \vdots\\ a_{\bar{n}1}P_{\bar{n}1} & \cdots & a_{\bar{n}\bar{n}}P_{\bar{n}\bar{n}}\end{bmatrix}-\bar{P}_1.$$

The item $x_j(k-\tau_{ij})$ in (9) with fixed time delays τ_{ij} does not yet match the form of descriptor model (8), which can be overcome by adopting the similar modeling method proposed in [19], where each communication path (i,j) is represented by the following state-space system:

$$\begin{cases}\kappa_{ij}(k+1)=\Gamma_{ij}\kappa_{ij}(k)+\Pi_{ij}\nu_{ij}(k),\\ \eta_{ij}(k)=\Psi_{ij}\kappa_{ij}(k)+\Sigma_{ij}\nu_{ij}(k),\end{cases}\tag{10}$$

where $\kappa_{ij}(k)\in\mathbb{R}^{(\tau_{ij}+1)n_i}$, $\nu_{ij}(k)\in\mathbb{R}^{n_j}$, $\eta_{ij}(k)\in\mathbb{R}^{n_j}$ are the state, input, output vectors, respectively,

$$\Gamma_{ij}=\begin{bmatrix}0 & & & (0)\\ I_{n_i} & \ddots & & \\ & \ddots & \ddots & \\ (0) & & I_{n_i} & 0\end{bmatrix},\quad \Pi_{ij}=\begin{bmatrix}I_{n_i}\\ 0\\ \vdots\\ 0\end{bmatrix},$$

$$\Psi_{ij}^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n_i} \end{bmatrix}, \quad \Sigma_{ij} = 0_{n_i}\ (if\ \tau_{ij} \neq 0),$$
$$\Sigma_{ij} = I_{n_i} \quad (if\ \tau_{ij} = 0).$$

(10) shows that $\eta_{ij}(k) = \nu_{ij}(k - \tau_{ij})(if\ \tau_{ij} \neq 0)$, $\eta_{ij}(k) = \nu_{ij}(k)(if\ \tau_{ij} = 0)$, and therefore, (10) can be adopted to describe the input-output characteristic of channel (i, j).

By combining all individual communication models (10) for (i, j), the model of the whole communication network $\mathcal{X}$ is defined as:

$$\begin{cases} \kappa(k+1) = \Gamma\kappa(k) + \Pi\nu(k), \\ \eta(k) = \Psi\kappa(k) + \Sigma\nu(k), \end{cases} \tag{11}$$

where

$$\begin{aligned}
\Psi &= diag\left[\Psi_{11}, \cdots, \Psi_{1\bar{n}}, \Psi_{21}, \cdots, \Psi_{ij}, \cdots, \Psi_{\bar{n}\bar{n}}\right],\\
\Pi &= diag\left[\Pi_{11}, \cdots, \Pi_{1\bar{n}}, \Pi_{21}, \cdots, \Pi_{ij}, \cdots, \Pi_{\bar{n}\bar{n}}\right],\\
\Gamma &= diag\left[\Gamma_{11}, \cdots, \Gamma_{1\bar{n}}, \Gamma_{21}, \cdots, \Gamma_{ij}, \cdots, \Gamma_{\bar{n}\bar{n}}\right],\\
\Sigma &= diag\left[\Sigma_{11}, \cdots, \Sigma_{1\bar{n}}, \Sigma_{21}, \cdots, \Sigma_{ij}, \cdots, \Sigma_{\bar{n}\bar{n}}\right],\\
\eta^T(k) &= \left[\eta_{11}^T \ \cdots \ \eta_{1\bar{n}}^T \ \eta_{21}^T \ \cdots \ \eta_{ij}^T \ \cdots \ \eta_{\bar{n}\bar{n}}^T\right],\\
\kappa^T(k) &= \left[\kappa_{11}^T \ \cdots \ \kappa_{1\bar{n}}^T \ \kappa_{21}^T \ \cdots \ \kappa_{ij}^T \ \cdots \ \kappa_{\bar{n}\bar{n}}^T\right],\\
\nu^T(k) &= \left[\nu_{11}^T \ \cdots \ \nu_{1\bar{n}}^T \ \nu_{21}^T \ \cdots \ \nu_{ij}^T \ \cdots \ \nu_{\bar{n}\bar{n}}^T\right],\\
& i, j = 1, 2, \cdots, \bar{n}, \ i \neq j.
\end{aligned}$$

By combining (11) and (9), (9) can be rewritten as the following autonomous system in the form of the descriptor model (8) as

$$\begin{bmatrix} \bar{J} & 0 \\ -\bar{K} & I \end{bmatrix}\begin{bmatrix} \bar{T}(k+1) \\ \bar{x}(k+1) \end{bmatrix} = \begin{bmatrix} 0 & \bar{M} \\ 0 & \bar{P} \end{bmatrix}\begin{bmatrix} \bar{T}(k) \\ \bar{x}(k) \end{bmatrix} \tag{12}$$

where $\bar{T}^T(k) = [T^T(k)\, \nu^T(k)], \bar{x}^T(k) = [\eta^T(k)\, \kappa^T(k)\, x^T(k)]$,

$$\bar{J} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} \bar{M}_2 & 0 & \bar{M}_1 + N\bar{C} \\ 0 & 0 & M^\tau \end{bmatrix},$$
$$\bar{K} = \begin{bmatrix} 0 & \Sigma \\ 0 & \Pi \\ K & 0 \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} 0 & \Psi & 0 \\ 0 & \Gamma & 0 \\ \bar{P}_2 & 0 & \bar{P}_1 \end{bmatrix},$$
$$M^\tau = \begin{bmatrix} M_{11}^\tau & \cdots & M_{1\bar{n}}^\tau & \cdots & M_{ij}^\tau & \cdots & M_{\bar{n}\bar{n}-1}^\tau \end{bmatrix}^T,$$
$$M_{ij}^\tau = \begin{bmatrix} 0_{n_j\times(n_1+\cdots+n_{j-1})} & I_{n_j} & 0_{n_j\times(n_{j+1}+\cdots+\bar{n})} \end{bmatrix}^T.$$

The descriptor model (12) is obtained by combining (9) and (11), where (9) generally describes any realization in a unifying framework while (11) provides a model for the whole communication network, including all individual communication channels. Therefore, (12) provides an overall description by considering both the realization description and time delays.

(9) and (12) are equivalent and therefore (9) can only be determined by the set of matrices $\bar{J}$, $\bar{K}$, $\bar{M}$ and $\bar{P}$, leading to following definition:

Definition 1: A realization $\aleph$ of (9) is defined by the specific set of matrices $\bar{J}$, $\bar{K}$, $\bar{M}$ and $\bar{P}$ as:

$$\aleph \triangleq (\bar{J}, \bar{K}, \bar{M}, \bar{P}).$$

With Definition 1, Problem 1 can be paraphrased as follows:

Problem 2: Find a realization $\aleph = (\bar{J}, \bar{K}, \bar{M}, \bar{P})$ and design the parameters N_{ii}, $i = 1, 2, \cdots, \bar{n}$ so that (9) is stable with its coefficients represented by the given word length β.

B. STABILITY ANALYSIS AND ESTIMATOR GAIN DESIGN

In this subsection, the design method is derived to solve Problem 2, where (9) is rewritten as the descriptor model in (12) and N composed of N_{ii}, $i = 1, 2, \cdots, \bar{n}$ as

$$N = diag\,[N_{11}, N_{22} \cdots, N_{\bar{n}\bar{n}}] \tag{13}$$

is the only parameter to be determined. Therefore, Problem 2 is solved if there exist an appropriate matrix N such that system (12) is stable subject to the coefficients' representation with given word length β.

In (12), the representation of coefficients $\bar{J}$, $\bar{K}$, $\bar{M}$ and $\bar{P}$ should be considered. Section II-A shows that analyzing the FWL effects on the stability of (12) is equivalent to analyse the stability of the following system

$$\begin{bmatrix} \bar{J} + d(\bar{J})\Delta & 0 \\ -\bar{K} + d(\bar{K})\Delta & I \end{bmatrix}\begin{bmatrix} \bar{T}(k+1) \\ \bar{x}(k+1) \end{bmatrix} = \begin{bmatrix} 0 & \bar{M} + d(\bar{M})\Delta \\ 0 & \bar{P} + d(\bar{P})\Delta \end{bmatrix}\begin{bmatrix} \bar{T}(k) \\ \bar{x}(k) \end{bmatrix},$$

where the function $d(*)$ is defined in (6), $|\Delta| < 2^{-(\beta+1)}$ with β being word length to represent the fraction part of the coefficients.

It is difficult to analyze stability due to the uncertainties Δ on both sides of the above system, and thus the above system is further augmented and rewritten as the following singular system:

$$\hat{E}\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{N}(\hat{C} + d(\hat{C})\Delta)\hat{x}(k), \tag{14}$$

where $\hat{x}(k)^T = [\bar{T}^T(k)\, \bar{x}^T(k)\, \epsilon_1^T(k)\, \epsilon_2^T(k)\, \epsilon_3^T(k)]$,

$$\hat{A} = \hat{A}_1 + d(\hat{A}_2)\Delta, \quad \bar{E} = \begin{bmatrix} \bar{J} & 0 \\ -\bar{K} & I \end{bmatrix}, \bar{A} = \begin{bmatrix} 0 & \bar{M} \\ 0 & \bar{P} \end{bmatrix},$$
$$\hat{E} = \begin{bmatrix} \bar{E} & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{A}_1 = \begin{bmatrix} \bar{A} & 0 & I & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix},$$
$$\hat{N} = \begin{bmatrix} \bar{N} \\ 0 \\ 0 \\ I \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} \bar{A} & 0 & 0 & 0 \\ \bar{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{N} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{C}^T = \begin{bmatrix} \bar{C}^T \\ 0 \\ 0 \\ 0 \end{bmatrix},$$
$$\bar{N} = \begin{bmatrix} N \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{C}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{C}_2^T \\ \bar{C}_1^T \end{bmatrix}.$$

The above system (14) is a singular system with a known singular matrix $\hat{E}$, and to analyze its stability, the following singular value decomposition [21], [23] can be introduced for $\hat{E}$ as:

$$\tilde{E} = M_d \hat{E} N_d = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where M_d and N_d are upper triangular and lower triangular non-singular matrices such that

$$\tilde{A}_1 = M_d \hat{A}_1 N_d, \quad \tilde{N} = M_d \hat{N}, \; \tilde{C} = \hat{C} N_d$$

and

$$\tilde{A} = M_d\{\hat{A} + \hat{N}[\hat{C} + d(\hat{C})\Delta]\} N_d = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}. \quad (15)$$

The above decomposition allows rewriting the singular system (14) as:

$$\tilde{E}\hat{x}(k+1) = \tilde{A}\hat{x}(k) + \tilde{N}(\tilde{C} + d(\tilde{C})\Delta)\hat{x}(k). \quad (16)$$

(16) shows that the proposed descriptor model-based method explicitly describes equivalent realizations of the estimator (9) with internal time delays and can also deal with the coupled uncertainties. Therefore, the analysis of FWL effects is achieved via a unifying framework to design the parameter N_{ii} in the estimator (9).

The stability analysis is first considered for (16) with N given as the known matrix, where all parameters in (16) are known besides the representation error Δ. The following Theorem 1 is given for solving this stability analysis problem.

Different from the state-space systems, analyzing singular systems requires considering not only stability but also regularity and causality. A singular system is said to be admissible if it is regular, causal and stable.

Theorem 1: For given scalars $\beta \in \mathbb{N}^+$, $\tau_{ij} \in \mathbb{N}$, $i, j = 1, 2, \cdots, \bar{n}$, the system in (16) is admissible if there exist matrices $\bar{Q}, \bar{R}, \bar{S}, \bar{P} > 0$ and a scalar $\epsilon > 0$, such that

$$\begin{bmatrix} -\frac{1}{2}\bar{Q} - \frac{1}{2}\bar{Q}^T & * & * & * \\ \bar{\Phi}_{21} & \bar{\Phi}_{22} & * & * \\ \bar{P} - \bar{Q} - \frac{1}{2}\bar{Q}^T & \bar{\Phi}_{32} & -\bar{Q} - \bar{Q}^T & * \\ \bar{\Phi}_{41} & \bar{\Phi}_{42} & \bar{\Phi}_{43} & \bar{\Phi}_{44} \end{bmatrix} < 0, \quad (17)$$

where $\bar{\Phi}_{21} = \bar{\Phi}_4 \bar{\Phi}_1^T$, $\bar{\Phi}_{32} = \bar{\Phi}_1 \bar{\Phi}_4^T$, $\bar{\Phi}_{42}^T = [\,\bar{\Phi}_5 \;\bar{\Phi}_2\,]$,

$$\bar{\Phi}_{22} = \bar{\Phi}_2 \bar{\Phi}_4^T + \bar{\Phi}_4 \bar{\Phi}_2^T - \bar{\Phi}_3, \; \bar{\Phi}_{43} = \bar{\Phi}_{41}, \; \bar{\Phi}_{44} = -\epsilon I,$$

$$\bar{\Phi}_{41}^T = [\,0 \; \bar{\Phi}_1\,], \quad \bar{\Phi}_1 = [\,\bar{Q} \quad \bar{R}\,], \; \bar{\Phi}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{S} \end{bmatrix},$$

$$\bar{\Phi}_3 = \begin{bmatrix} \bar{P} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{\Phi}_4^T = \tilde{A}_1 + \tilde{N}\tilde{C}, \; \bar{\beta} = 2^{-(\beta+1)},$$

$$\bar{\Phi}_5^T = \bar{\beta}\epsilon[M_d d(\hat{A}_2) N_d + \tilde{N} d(\tilde{C})].$$

The following lemmas are introduced for proof of Theorem 1.

Lemma 1 [24]: Let Λ and Π be any given real matrices of appropriate dimensions. Then, for any scalar $\epsilon > 0$,

$$\Lambda^T \Pi + \Pi^T \Lambda \leq \epsilon^{-1} \Lambda^T \Lambda + \epsilon \Pi^T \Pi.$$

Lemma 2 [21]: For a matrix $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$ where $\Phi_{11}, \Phi_{12}, \Phi_{21}$ and Φ_{22} are any real matrices with appropriate dimensions such that is invertible and $\Phi + \Phi^T < 0$. Then we have

$$\Phi_{11} + \Phi_{11}^T - \Phi_{12}\Phi_{22}^{-1}\Phi_{21} - \Phi_{21}^T \Phi_{22}^{-T} \Phi_{12}^T < 0.$$

Proof of Theorem 1: Suppose that the inequality (17) holds. By Schur complement, one has

$$\bar{\Phi}_4 + \epsilon^{-1}\bar{\Phi}_5\bar{\Phi}_5^T + \epsilon\bar{\Phi}_6\bar{\Phi}_6^T < 0,$$

where

$$\bar{\Phi}_4 = \begin{bmatrix} -\frac{1}{2}\bar{Q} - \frac{1}{2}\bar{Q}^T & * & * \\ \bar{\Phi}_{21} & \bar{\Phi}_{22} & * \\ \bar{P} - \bar{Q} - \frac{1}{2}\bar{Q}^T & \bar{\Phi}_{32} & -\bar{Q} - \bar{Q}^T \end{bmatrix},$$

$$\bar{\Phi}_5 = \begin{bmatrix} 0 \\ \bar{\beta}[M_d d(\hat{A}_2) N_d + \tilde{N} d(\tilde{C})] \\ 0 \end{bmatrix}, \quad \bar{\Phi}_6 = \begin{bmatrix} \bar{\Phi}_1 \\ \bar{\Phi}_2 \\ \bar{\Phi}_1 \end{bmatrix}.$$

Then applying Lemma 1 leads to

$$\begin{aligned} &\bar{\Phi}_4 + \Delta/\bar{\beta}\bar{\Phi}_5\bar{\Phi}_6^T + \Delta/\bar{\beta}\bar{\Phi}_6\bar{\Phi}_5^T \\ &\leq \bar{\Phi}_4 + \epsilon^{-1}\bar{\Phi}_5\bar{\Phi}_5^T + \epsilon\Delta^2/\bar{\beta}^2\bar{\Phi}_6\bar{\Phi}_6^T \\ &< \bar{\Phi}_4 + \epsilon^{-1}\bar{\Phi}_5\bar{\Phi}_5^T + \epsilon\bar{\Phi}_6\bar{\Phi}_6^T \\ &< 0, \end{aligned}$$

where $\Delta < \bar{\beta}$ is the representation error. By Schur complement, it is obtained from the above inequality that

$$\begin{bmatrix} -\frac{1}{2}\bar{Q} - \frac{1}{2}\bar{Q}^T & * & * \\ \tilde{A}^T \Phi_1^T & \Phi_2\tilde{A} + \tilde{A}^T\Phi_2^T - \Phi_3 & * \\ \bar{P} - \bar{Q} - \frac{1}{2}\bar{Q}^T & \Phi_1\tilde{A} & -\bar{Q} - \bar{Q}^T \end{bmatrix} < 0. \quad (18)$$

With the decompositions given in (15), one has

$$\begin{bmatrix} -\frac{1}{2}\bar{Q} - \frac{1}{2}\bar{Q}^T & * & * & * \\ \Phi_{21} & -\bar{P} & * & * \\ \Phi_{31} & \bar{S}(\tilde{A}_{21} + \Delta d(\tilde{A}_{21})) & \Phi_{33} & * \\ \bar{P} - \bar{Q} - \frac{1}{2}\bar{Q}^T & \Phi_{42} & \Phi_{43} & -\bar{Q} - \bar{Q}^T \end{bmatrix} < 0,$$

where

$$\Phi_{21} = \tilde{A}_{11}^T \bar{Q}^T + \tilde{A}_{21}^T \bar{R}^T, \quad \Phi_{31} = \tilde{A}_{12}^T \bar{Q}^T + \tilde{A}_{22}^T \bar{R}^T,$$
$$\Phi_{33} = \bar{S}\tilde{A}_{22} + \tilde{A}_{22}^T \bar{S}^T, \quad \Phi_{42} = \bar{Q}\tilde{A}_{11} + \bar{R}\tilde{A}_{21},$$
$$\Phi_{43} = \bar{Q}\tilde{A}_{12} + \bar{R}\tilde{A}_{22}.$$

Left- and right-multiplying the above inequality by

$$T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}$$

and its transpose, respectively, lead to

$$W + W^T < 0 \tag{19}$$

with

$$W = \begin{bmatrix} -\frac{1}{2}\bar{Q} & 0 & 0 & 0 \\ W_{21} & -\frac{1}{2}\bar{P} & W_{23} & \tilde{A}_{21}^T\bar{S}^T \\ W_{31} & 0 & -\bar{Q} & 0 \\ W_{41} & 0 & W_{43} & \tilde{A}_{22}^T\bar{S}^T \end{bmatrix},$$

where

$$W_{21} = \tilde{A}_{11}^T\bar{Q}^T + \tilde{A}_{21}^T\bar{R}^T, \quad W_{24} = \tilde{A}_{11}^T\bar{Q}^T + \tilde{A}_{21}^T\bar{R}^T,$$
$$W_{41} = \tilde{A}_{12}^T\bar{Q}^T + \tilde{A}_{22}^T\bar{R}^T, \quad W_{43} = \tilde{A}_{12}^T\bar{Q}^T + \tilde{A}_{22}^T\bar{R}^T,$$
$$W_{31} = \bar{P} - \frac{1}{2}\bar{Q}^T - \bar{Q}.$$

Note that $\tilde{A}_{22}^T\bar{S}^T + \bar{S}\tilde{A}_{22} < 0$ in the above inequality. Using the matrix measurement properties [25], one can claim that the matrices $\tilde{A}_{22}$ and $\bar{S}$ are both non-singular. Hence, the singular system (14) is regular and causal [23]. And it can be reduced to a state-space system

$$x(k+1) = A_r x(k),$$

where $A_r = \tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$.

The above system is stable, if and only if there exists a matrix $\bar{P} > 0$, such that $A_r^T\bar{P}A_r - \bar{P} < 0$. By Schur complement, the inequality $A_r^T\bar{P}A_r - \bar{P} < 0$ is equivalent to

$$\begin{bmatrix} -\bar{P} & * \\ A_r^T\bar{P} & -\bar{P} \end{bmatrix} < 0.$$

The above inequality can be rewritten as $\Xi_\Gamma^T\Xi\Xi_\Gamma < 0$, where

$$\Xi = \begin{bmatrix} 0 & 0 & \bar{P} \\ 0 & -\bar{P} & 0 \\ \bar{P} & 0 & 0 \end{bmatrix}, \quad \Xi_\Gamma = \begin{bmatrix} I & 0 \\ 0 & I \\ -\frac{1}{2}I & A_r \end{bmatrix}.$$

Noting $\bar{P} > 0$, a trivial constraint is introduced as

$$\begin{bmatrix} -\bar{P} & 0 \\ 0 & -2\bar{P} \end{bmatrix} < 0.$$

And this constraint can be rewritten as $\Xi_\Psi^T\Xi\Xi_\Psi < 0$, where

$$\Xi_\Psi^T = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & I \end{bmatrix}.$$

Then the following matrices can be defined

$$\Psi = \begin{bmatrix} I & 0 & I \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -\frac{1}{2}I & A_r & -I \end{bmatrix},$$

such that $\Psi\Xi_\Psi = 0$, $\Gamma\Xi_\Gamma = 0$.

With inequalities $\Xi_\Gamma^T\Xi\Xi_\Gamma < 0$ and $\Xi_\Psi^T\Xi\Xi_\Psi < 0$, applying Projection Lemma [26] leads to

$$\Xi + \Gamma^T\bar{Q}^T\Psi + \Psi^T\bar{Q}\Gamma < 0, \tag{20}$$

On the other hand, applying Lemma 2 to inequality (19) gives

$$\begin{bmatrix} -\frac{1}{2}\bar{Q} - \frac{1}{2}\bar{Q}^T & * & * \\ A_r^T\bar{Q}^T & -\bar{P} & * \\ \bar{P} - \bar{Q} - \frac{1}{2}\bar{Q}^T & \bar{Q}A_r & -\bar{Q} - \bar{Q}^T \end{bmatrix} < 0,$$

which is equivalent to inequality (20). Then, it is observed that A_r is stable. Therefore, the singular system (16) is admissible.

When matrix N in (13) is given known, Theorem 1 provides the stability analysis condition for (16) with given internal time delays τ_{ij}, $i, j = 1, 2, \cdots, \bar{n}$; $i \neq j$ with word length β for the coefficients' representation. When N is unknown and requires determination, the following Theorem 2 is further proposed to design the matrices N_{ii}, $i = 1, 2, \cdots, \bar{n}$ in (13).

Theorem 2: For given scalars $\beta \in \mathbb{N}^+$, $\tau_{ij} \in \mathbb{N}$, $i, j = 1, 2, \cdots, \bar{n}$, the system in (16) is admissible if there exist matrices $\bar{Q} = diag[\bar{Q}_1, \bar{Q}_2]$, $\bar{R}$, $\bar{S}$, $\bar{P} > 0$, $\bar{Y}$ and a scalar $\epsilon > 0$, such that

$$\begin{bmatrix} -\frac{1}{2}\bar{Q} - \frac{1}{2}\bar{Q}^T & * & * & * \\ \hat{\Phi}_{21} & \hat{\Phi}_{22} & * & * \\ \bar{P} - \bar{Q} - \frac{1}{2}\bar{Q}^T & \hat{\Phi}_{32} & -\bar{Q} - \bar{Q}^T & * \\ \hat{\Phi}_{41} & \hat{\Phi}_{42} & \hat{\Phi}_{43} & \hat{\Phi}_{44} \end{bmatrix} < 0, \tag{21}$$

where $\hat{\Phi}_{21} = \tilde{A}_1^T\bar{\Phi}_1^T + \hat{C}^T(Y + \bar{R}\bar{I})^T\bar{\Phi}_1^T$, $\hat{\Phi}_{32} = \hat{\Phi}_{21}^T$,

$$\hat{\Phi}_{22} = \bar{\Phi}_2\hat{\Phi}_4^T + \hat{\Phi}_4\bar{\Phi}_2^T - \bar{\Phi}_3, \quad \hat{\Phi}_{41}^T = \begin{bmatrix} 0 & \bar{\Phi}_1 \end{bmatrix},$$
$$\hat{\Phi}_{44} = -\epsilon I, \quad \bar{I} = [0 \; I_p], \quad \hat{\Phi}_{42}^T = \begin{bmatrix} \hat{\Phi}_5 & \bar{\Phi}_2 \end{bmatrix},$$
$$\hat{\Phi}_{43} = \hat{\Phi}_{41}, \quad \bar{\Phi}_1 = \begin{bmatrix} \bar{Q} & \bar{R} \end{bmatrix}, \quad \bar{Y} = [Y \; 0],$$
$$Y = \begin{bmatrix} a_{11}Y_{11} & \cdots & a_{1\bar{n}}Y_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ a_{\bar{n}1}Y_{\bar{n}1} & \cdots & a_{\bar{n}\bar{n}}Y_{\bar{n}\bar{n}} \end{bmatrix}, \quad Y \in \mathbb{R}^{n \times p},$$
$$\bar{Q}_1 = diag[Q_{11}, \; Q_{12}, \; \cdots, \; Q_{1\bar{n}}], \quad \bar{Q}_1 \in \mathbb{R}^{n \times n},$$
$$Q_{1i} \in \mathbb{R}^{n_i \times n_i}, \quad Y_{1i} \in \mathbb{R}^{n_i \times P_j}, \; i, j = 1, 2, \cdots, \bar{n},$$
$$\bar{\Phi}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{S} \end{bmatrix}, \quad \bar{\Phi}_3 = \begin{bmatrix} \bar{P} & 0 \\ 0 & 0 \end{bmatrix}, \; \bar{\beta} = 2^{-(\beta+1)},$$
$$\hat{\Phi}_5^T = \bar{\beta}\epsilon M_d d(\hat{A}_2)N_d + \bar{\beta}\epsilon(Y + \bar{R}\bar{I})d(\hat{C}),$$
$$\hat{\Phi}_4^T = \tilde{A}_1 + C^T(Y + \bar{R}\bar{I}).$$

Moreover, the the matrix N in (13) is given as

$$\begin{bmatrix} N \\ 0 \end{bmatrix} = M_d^{-1}\begin{bmatrix} \bar{Q}_1^{-1}Y \\ 0 \end{bmatrix},$$

and with the obtained matrix N, a estimator in form of (9) can be constructed, where $N = diag\,[N_{11}, \; N_{22} \; \cdots, \; N_{\bar{n}\bar{n}}]$.

Proof of Theorem 2: In (17), let $\bar{Q} = diag[\bar{Q}_1, \ \bar{Q}_2]$ and rewritten $\bar{Q}_1 N$ as Y, (21) can be obtained. Therefore, (14) is admissible with (21) holding.

Theorem 2 provides the design method of the estimator in (9) with a given realization. However, in some practical application, it may be significant to find the realization which is most resilient to the FWL effects. Therefore, Algorithm 1 is further proposed to identify realizations that minimize the FWL effects.

Algorithm 1 A Search Algorithm for $\aleph^*$ and β^*

Input: A set of $\hat{n}$ alternative realizations $\aleph(i)$,
$i = 1, 2, \cdots, \hat{n}$;
Output: Realization $\aleph^*$; The smallest word length β^*
1: Initialize β;
2: For $i = 1 : \hat{n}$;
3: Solve the inequality (21) for $\aleph(i)$ and β;
4: If (21) is solvable;
5: Set $\beta = \beta - 1$, $\beta^* = \beta$, $\aleph^* = \aleph(i)$, go to step 3;
6: end;
7: end;
Return: $\aleph^*$, β^*.

C. SPECIAL CASE: ESTIMATORS IMPLEMENTED ON ONE SOC

If the estimator is implemented by only one SOC, it can be regarded as a simplified special case of the results proposed in Section III-B, in which case the internal information interaction of the estimator and internal network with time delays no longer require consideration. In this situation, the stability analysis and design method for the corresponding estimator are provided.

By implementation in only one SOC, the information interaction of the estimator is not considered. Therefore, (2) is rewritten as

$$x(k+1) = Ax(k) - L\bar{z}(k),$$

and the descriptor model (9) is correspondingly rewritten as

$$\begin{cases} JT(k+1) = Mx(k) - N\bar{z}(k), \\ x(k+1) = KT(k+1) + Px(k), \end{cases}$$

where $T(k) \in \mathbb{R}^l$ is the intermediate variable. Matrices K, N and P satisfy

$$A = KJ^{-1}M + P,$$

and N are matrix to be determined, which satisfies

$$L = KJ^{-1}N.$$

And, (12) is further rewritten as

$$\begin{bmatrix} J & 0 \\ -K & I \end{bmatrix} \begin{bmatrix} T(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} 0 & M \\ 0 & P \end{bmatrix} \begin{bmatrix} T(k) \\ x(k) \end{bmatrix}.$$

To analyse the FWL effects on the stability of the above system is equivalent to analyse the stability of the system

$$\begin{bmatrix} J + d(J)\Delta & 0 \\ -K + d(K)\Delta & I \end{bmatrix} \begin{bmatrix} T(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} 0 & M + d(M)\Delta \\ 0 & P + d(P)\Delta \end{bmatrix} \begin{bmatrix} T(k) \\ x(k) \end{bmatrix}. \quad (22)$$

For (22), the singular value decomposition is given as

$$\tilde{E}_c = M_{dc}\hat{E}_c N_{dc} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{N}_c = M_{dc}\hat{N}_c, \ \tilde{C}_c = \hat{C}_c N_{dc}$$

$$\tilde{A}_{c1} = M_{dc}\hat{A}_{c1}N_{dc}, \ \tilde{A}_c = M_{dc}\{\hat{A}_c + \hat{N}_c[\hat{C}_c + d(\hat{C}_c)\Delta]\}N_{dc},$$

where $\hat{A}_c = \hat{A}_{c1} + d(\hat{A}_{c2})\Delta$, $\bar{E}_c = \begin{bmatrix} J & 0 \\ -K & I \end{bmatrix}$,

$$\hat{E}_c = \begin{bmatrix} \bar{E}_c & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{A}_{c1} = \begin{bmatrix} \bar{A}_c & 0 & I & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix},$$

$$\hat{N}_c = \begin{bmatrix} \bar{N}_c \\ 0 \\ 0 \\ I \end{bmatrix}, \ \hat{A}_{c2} = \begin{bmatrix} \bar{A}_c & 0 & 0 & 0 \\ \bar{E}_c & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{N}_c \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \bar{N}_c = \begin{bmatrix} N \\ 0 \end{bmatrix},$$

$$\bar{A}_c = \begin{bmatrix} 0 & M \\ 0 & P \end{bmatrix}, \ \bar{C}_c^T = \begin{bmatrix} 0 \\ C^T \end{bmatrix}, \ \hat{C}_c^T = \begin{bmatrix} \bar{C}_c^T \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

M_{dc} and N_{dc} are upper triangular and lower triangular non-singular matrices. By adopting the similar analysis and design method proposed in section III-B, the following corollaries are further proposed. The analysis condition for stability of (22) is given first.

Corollary 1: For given scalars $\beta \in \mathbb{N}^+$, the system in (22) is admissible if there exist matrices $\bar{Q}_c, \bar{R}_c, \bar{S}_c, \bar{P}_c > 0$ and a scalar $\epsilon > 0$, such that

$$\begin{bmatrix} -\frac{1}{2}\bar{Q}_c - \frac{1}{2}\bar{Q}_c^T & * & * & * \\ \check{\Phi}_{21} & \check{\Phi}_{22} & * & * \\ \check{\Phi}_{31} & \check{\Phi}_{32} & -\bar{Q}_c - \bar{Q}_c^T & * \\ \check{\Phi}_{41} & \check{\Phi}_{42} & \check{\Phi}_{43} & \check{\Phi}_{44} \end{bmatrix} < 0, \quad (23)$$

where $\check{\Phi}_{21} = \check{\Phi}_4\check{\Phi}_1^T$, $\check{\Phi}_{32} = \check{\Phi}_1\check{\Phi}_4^T$, $\check{\Phi}_{42}^T = [\,\check{\Phi}_5 \ \check{\Phi}_2\,]$,

$$\check{\Phi}_{22} = \check{\Phi}_2\check{\Phi}_4^T + \check{\Phi}_4\check{\Phi}_2^T - \check{\Phi}_3, \quad \check{\Phi}_{43} = \check{\Phi}_{41}, \ \check{\Phi}_{44} = -\epsilon I,$$

$$\check{\Phi}_{41}^T = [0 \quad \check{\Phi}_1], \ \check{\Phi}_1 = [\,\bar{Q}_c \quad \bar{R}_c\,], \quad \check{\Phi}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{S}_c \end{bmatrix},$$

$$\check{\Phi}_3 = \begin{bmatrix} \bar{P}_c & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{\beta} = 2^{-(\beta+1)}, \quad \check{\Phi}_{31} = \bar{P}_c - \bar{Q}_c - \frac{1}{2}\bar{Q}^T,$$

$$\check{\Phi}_4^T = \tilde{A}_{c1} + \tilde{N}_c\tilde{C}_c, \quad \check{\Phi}_5^T = \bar{\beta}\epsilon[M_{dc}d(\hat{A}_{c2})N_{dc} + \tilde{N}_{dc}(\tilde{C}_c)].$$

Proof of Corollary 1: The proof can be simply achieved according to the proof of Theorem 1 by replacing $\bar{Q}, \bar{R}, \bar{S}, \bar{P}, \tilde{A}, \tilde{A}_1, \tilde{A}_2, \tilde{C}, \tilde{N}, \tilde{M}_d, \tilde{N}_d$ in it with $\bar{Q}_c, \bar{R}_c, \bar{S}_c, \bar{P}_c, \tilde{A}_c, \tilde{A}_{c1}, \tilde{A}_{c2}, \tilde{C}_c, \tilde{N}_c, \tilde{M}_{dc}, \tilde{N}_{dc}$ and is therefore omitted.

Based on Corollary 1, the design method for unknown matrix N in (22) is further given.

Corollary 2: For given scalars $\beta \in \mathbb{N}^+$, the system in (22) is admissible if there exist matrices $\bar{Q}_c = diag[\bar{Q}_{1c}, \ \bar{Q}_{2c}], \bar{R}_c$, $\bar{S}_c, \bar{P}_c > 0$, Y_c and a scalar $\epsilon > 0$, such that

$$\begin{bmatrix} -\frac{1}{2}\bar{Q}_c - \frac{1}{2}\bar{Q}_c^T & * & * & * \\ \tilde{\Phi}_{21} & \tilde{\Phi}_{22} & * & * \\ \tilde{\Phi}_{31} & \tilde{\Phi}_{32} & -\bar{Q}_c - \bar{Q}_c^T & * \\ \tilde{\Phi}_{41} & \tilde{\Phi}_{42} & \tilde{\Phi}_{43} & \tilde{\Phi}_{44} \end{bmatrix} < 0, \quad (24)$$

where $\tilde{\Phi}_{21} = \tilde{A}_{c1}^T \tilde{\Phi}_1^T + \hat{C}_c^T (Y + \bar{R}_c \tilde{I})^T \tilde{\Phi}_1^T$, $\tilde{\Phi}_{32} = \tilde{\Phi}_{21}^T$,

$$\tilde{\Phi}_{22} = \tilde{\Phi}_2 \tilde{\Phi}_4^T + \tilde{\Phi}_4 \tilde{\Phi}_2^T - \tilde{\Phi}_3, \ \tilde{\Phi}_{44} = -\epsilon I, \ \tilde{I} = [0 \ I_p],$$

$$\tilde{\Phi}_{42}^T = \begin{bmatrix} \tilde{\Phi}_5 & \tilde{\Phi}_2 \end{bmatrix}, \quad \tilde{\Phi}_{41}^T = \begin{bmatrix} 0 & \tilde{\Phi}_1 \end{bmatrix}, \ \tilde{\Phi}_{43} = \tilde{\Phi}_{41},$$

$$\tilde{\Phi}_1 = \begin{bmatrix} \bar{Q}_c & \bar{R}_c \end{bmatrix}, \quad \tilde{\Phi}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{S}_c \end{bmatrix}, \quad \bar{\beta} = 2^{-(\beta+1)},$$

$$\tilde{\Phi}_3 = \begin{bmatrix} \bar{P}_c & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{\Phi}_4^T = \tilde{A}_{c1} + C^T (Y_c + \bar{R}_c \tilde{I}),$$

$$\tilde{\Phi}_5^T = \bar{\beta}\epsilon M_d d(\hat{A_{c2}}) N_{dc} + \bar{\beta}\epsilon (Y \tilde{\Phi}_2^T + \bar{R} \tilde{\Phi}_2^T \tilde{I}) d(\hat{C} \tilde{\Phi}_2^T),$$

$$\tilde{\Phi}_{31} = \bar{P}_c - \bar{Q}_c - \frac{1}{2}\bar{Q}_c^T.$$

Moreover, the the matrix N is given as

$$\begin{bmatrix} N \\ 0 \end{bmatrix} = M_{dc}^{-1} \begin{bmatrix} \bar{Q}_{1c}^{-1} Y_c \\ 0 \end{bmatrix},$$

Proof of Corollary 2: In (23), let $\bar{Q}_c = diag[\bar{Q}_{c1}, \ \bar{Q}_{c2}]$ and rewritten $\bar{Q}_{c1} N_c$ as Y_c, (24) can be obtained. Therefore, (22) is admissible with (24) holding.

For searching the realization that minimize the FWL effects, the method proposed in Algorithm 1 can also be adopted for the special case in this section by further replacing the inequality (21) in step 3 and 4 of Algorithm 1 with (24).

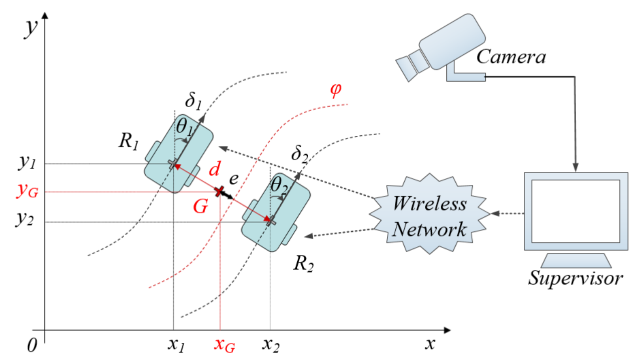

FIGURE 2. Setup of the cooperative robots platform.

IV. EXAMPLE

In this section, the results of Section III is applied to a platform with two cooperative robots to verify results' effectiveness. As depicted in Fig. 2, the platform consists of two cooperative mobile robots, one camera and a supervisor. Using the camera, the supervisor can measure the absolute position and velocity of each robot to calculate and send control signals to each robot through a wireless protocol, which drive the two cooperative robots so that their center of gravity follows a predefined path [27]. The future objective is to remove the supervisor and camera so that the control law can become directly embedded onto the two robots. In this section, we focus on the design and realization of the estimator required to estimate the two robots' position, and thus an estimator with two subsystems is considered, with one subsystem embedded in each robot by utilizing the wireless network. To focus on the results proposed in this paper, only the communication delays between the subsystems are considered, and it is assumed the supervisor can communicate with each robot without any delays or packet dropouts.

A. COOPERATIVE ROBOTS MODELING

Each robot can be modeled by a classic kinematic unicycle model

$$\begin{cases} \dot{x}_i = \delta_i \cos(\theta_i), \\ \dot{y}_i = \delta_i \sin(\theta_i), \\ \dot{\theta}_i = \eta_i, \\ \dot{\delta}_i = \gamma_i, \end{cases}$$

where $x_i, \ y_i$ represent the position of the i^{th} *robot* on the x and y axles, δ_i and θ_i the velocity and its angular orientation, $\dot{x}_i, \ \dot{y}_i$ the velocity on the x and y axles, $\dot{\theta}_i$ the angular velocity, and $\eta_i, \ \gamma_i$ the input of the i^{th} robot, $i = 1, 2$. A classical linearizing feedback control law is implemented [28] in each robot, which leads a new input-output mapping based on the following two decoupled integrator chains:

$$\begin{cases} x_i / a_i^x = 1/s^2, \\ y_i / a_i^y = 1/s^2. \end{cases}$$

The two new control inputs a_i^x and a_i^y for each robot are homogeneous to the robot's acceleration.

After obtaining the above decoupled integrator chains and exact discretization with sampling period $T_s = 0.1s$, the plant model for the two cooperative robots is given by the following system:

$$x_p(k+1) = A_p x_p(k) + B_p u_p(k), \quad (25)$$

with $u_p^T(k) = \left[a_1^x(k) \ a_1^y(k) \ a_2^x(k) \ a_2^y(k)\right]$,

$$x_p^T(k) = [x_1(k) \ \dot{x}_1(k) \ y_1(k) \ \dot{y}_1(k) \ x_2(k) \ \dot{x}_2(k) \ y_2(k) \ \dot{y}_2(k)],$$

$$A_p = \text{diag}\left[A_1^x, A_1^y, A_2^x, A_2^y\right], \ B_p = \text{diag}\left[B_1^x, B_1^y, B_2^x, B_2^y\right],$$

$$A_i^x = A_i^y = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad B_i^x = B_i^y = \begin{bmatrix} 0.005 \\ 0.100 \end{bmatrix}, \ i = 1, 2.$$

System (25) is unstable, and thus a state feedback gain F for system (25) is obtained to ensure the tracking performance as:

$$F = \begin{bmatrix} -0.9832 & -0.0000 & 0.3169 & 0.0000 \\ -1.3835 & 0.0000 & 0.2291 & 0.0000 \\ 0.0000 & -0.9832 & 0.0000 & 0.3169 \\ 0.0000 & -1.3835 & 0.0000 & 0.2291 \\ 0.3169 & 0.0000 & -0.9832 & -0.0000 \\ 0.2291 & 0.0000 & -1.3835 & 0.0000 \\ 0.0000 & 0.3169 & -0.0000 & -0.9832 \\ 0.0000 & 0.2291 & 0.0000 & -1.3835 \end{bmatrix}^T.$$

The control signal is calculated by the supervisor and sent to the robots as $u_p(k) = Fx_p(k)$, which results in the closed-loop as:

$$x_p(k+1) = Ax_p(k),$$

where $A = A_p + B_pF$. The specific design method for F can be found in [27], which is not directly related to the results proposed in this paper and is therefore not mentioned here.

For the above plant, an estimator with the standard model defined in (1) is proposed as

$$x(k+1) = Ax(k) + Bu(k) + L[y(k) - z(k)], \tag{26}$$

where $x(k)$ is the estimator state, $u(k)$ the external input signal, and $y(k) = Cx_p(k)$ the measurement from the robots,

$$z(k) = Cx(k), B = B_p, C = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

The estimator (26) is required to be partitioned into two subsystems and embedded into each robot. To detail the information interaction between them, the weighted adjacency matrix is given as $\mathscr{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, i.e., (the 2^{th} subsystem cannot receive information from the 1^{th} subsystem).

For implementation, the matrices A, C and L in (26) are partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix},$$

with

$$A_{11} \in \mathbb{R}^{4\times4}, \quad A_{12} \in \mathbb{R}^{4\times4}, A_{22} \in \mathbb{R}^{4\times4}, C_{11} \in \mathbb{R}^{2\times4},$$
$$C_{12} \in \mathbb{R}^{2\times4}, \quad C_{22} \in \mathbb{R}^{2\times4}, L_{11} \in \mathbb{R}^{4\times2}, L_{22} \in \mathbb{R}^{4\times2}.$$

Omitting $u(k)$ and $y(k)$ in (26), a estimator in form of (9) can be obtained as

$$\begin{cases} J_1T_1(k+1) = M_{11}x_1(k) - N_{11}z_1(k) \\ \qquad\qquad + M_{12}x_2(k-\tau_{12}), \\ x_1(k+1) = K_1T_1(k+1) + P_{11}x_1(k) \\ \qquad\qquad + P_{12}x_2(k-\tau_{12}), \\ J_2T_2(k+1) = M_{22}x_2(k) - N_{22}z_2(k), \\ x_2(k+1) = K_2T_2(k+1) + P_{22}x_1(k), \end{cases} \tag{27}$$

where τ_{12} is the time delay in channel $(1, 2)$,

$$z_1(k) = C_{11}x_1(k) + C_{12}x_2(k-\tau_{12}), \; z_2(k) = C_{22}x_2(k),$$

$$x^T(k) = \left[x_1^T(k)\, x_2^T(k)\right], \; x_1^T(k) = \left[\hat{x}_1(k)\,\hat{x}_2(k)\,\hat{x}_3(k)\,\hat{x}_4(k)\right],$$
$$x_2^T(k) = \left[\hat{x}_5(k)\,\hat{x}_6(k)\,\hat{x}_7(k)\,\hat{x}_8(k)\right].$$

B. ESTIMATOR DESIGNING AND REALIZATION

In this simulation, (27) requires digital implemented with word length β for the coefficients' representation, and two realization forms are considered respectively:

- realization 1: (27) realized by the shift operator with parameters in (27) being selected as $K_1 = I$, $J_1 = I$, $P_{11} = A_{11}$, $P_{12} = A_{12}$, $N_{11} = L_{11}$, $M_{11} = 0$, $M_{12} = 0$, $K_2 = I$, $J_2 = I$, $P_{22} = A_{22}$, $N_{22} = L_{22}$, $M_{22} = 0$;
- realization 2: (27) realized by the δ-operator with parameters in (27) being selected as $K_1 = \Delta_\delta I$, $J_1 = I$, $P_{11} = I$, $P_{12} = A_{12}$, $N_{11} = \Delta_\delta^{-1}L_{11}$, $M_{11} = \Delta_\delta^{-1}(A_{11} - I)$, $M_{12} = \Delta_\delta^{-1}A_{12}$, $K_2 = \Delta_\delta I$, $J_2 = I$, $P_{22} = I$, $N_{22} = \Delta_\delta^{-1}L_{22}$, $M_{22} = \Delta_\delta^{-1}(A_{22} - I)$, $\Delta_\delta = 2^{-4}$.

The time delay for communication channel $(1, 2)$ is given as $\tau_{12} = 0.1s$. To describe the communication channels, the following state-space model in the form of (10) are constructed as for the channel $(1, 2)$ as:

$$\begin{cases} \kappa_{12}(k+1) = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}\kappa_{12}(k) + \begin{bmatrix} I \\ 0 \end{bmatrix}\nu_{12}(k), \\ \eta_{12}(k) = \begin{bmatrix} 0 & I \end{bmatrix}\kappa_{12}(k). \end{cases}$$

C. SIMULATION RESULTS

With the above state-space model for channel $(1, 2)$, (27) can be rewritten in the form of the descriptor model (8). By solving Theorem 2, the unknown parameters N_{11} and N_{22} in (27) can be obtained respectively for two realizations 1 and 2 defined in the Subsection 4.2.

For (27) realized by the shift operator, N_{11} and N_{22} are obtained by solving Theorem 2 as

$$N_{11} = \begin{bmatrix} -1.2521 & -0.4932 \\ -0.9634 & -0.5476 \\ -1.2440 & -0.4259 \\ -1.0738 & -0.5545 \end{bmatrix},$$
$$N_{22} = \begin{bmatrix} 0.1923 & 1.0249 \\ 0.2308 & 0.6429 \\ 0.1210 & 1.0940 \\ 0.2413 & 0.7599 \end{bmatrix}$$

with $\beta = 5$, while for (27) realized by the δ operator,

$$N_{11} = \begin{bmatrix} -0.1204 & 0.0711 \\ 0.0148 & -0.1589 \\ -0.0418 & 0.0414 \\ -0.0318 & -0.0178 \end{bmatrix},$$
$$N_{22} = \begin{bmatrix} 0.0013 & 0.0198 \\ 0.0007 & -0.0290 \\ -0.0066 & -0.0624 \\ 0.0031 & 0.0888 \end{bmatrix}$$

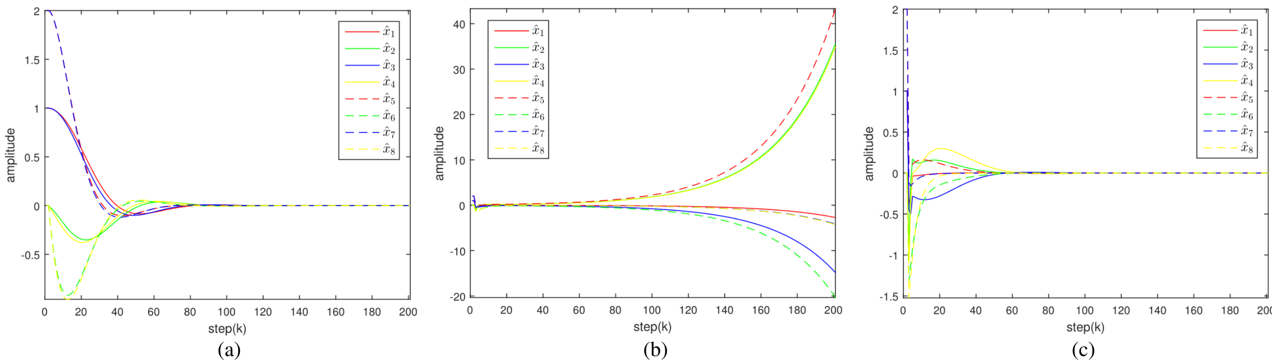

FIGURE 3. State evolution of the estimator realized by the δ operator with $\beta = 3$ (a), by the shift operator with $\beta = 3$ (b) and by the shift operator with $\beta = 5$ (c).

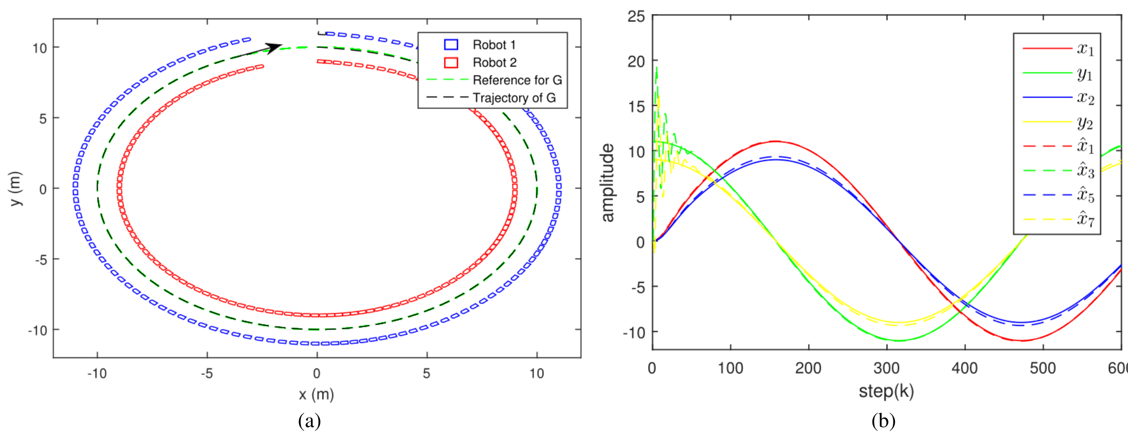

FIGURE 4. Trajectory of two cooperative robots (a) and the estimating performance of the estimator realized by the shift operator with $\beta = 6$ (b).

TABLE 1. Minimum word length β^* for stability.

Realizations	By shift operator	By δ-operator ($\Delta_\delta = 2^{-4}$)
Minimum word length	β^*=5	β^*=3

are obtained with $\beta = 3$. Both realizations 1 and 2 of the estimator (27) have been simulated. With the same initial states $x_1(0) == [1\ 0\ 1\ 0]^T$, $x_2(0) = [2\ 0\ 2\ 0]^T$, the evolutions of the estimator states are depicted in Figures 3(a), 3(b) and 3(c). The simulation results verify the effectiveness of the proposed estimator and show that the δ-operator features better FWL properties with the coefficients' representation compared with the shift operator.

By applying Algorithm 1, the minimum word length β^* for stability is calculated for each realization. According to the results shown in Table 1, the estimator implemented by δ-operator with $\Delta_\delta = 2^{-4}$ requires at least 3 bits for stability, while the estimator implemented with the shift operator requires at least 5. Therefore, Algorithm 1 can be adopted to choose an appropriate realization to reduce the minimum word length required for preserving the stability of the estimator following implementation, which can lead to practical consequences based on the total word length to be manipulated; for example, cheaper SOCs based on an 8-bits architecture with 4 bits for the fraction part could instead use 16 bits with 8 bits for the fraction part.

To further verify the estimating performance, set $\mathscr{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\tau_{12} = 0.1s$, $\tau_{21} = 0.1s$, $\beta = 6$. For (27) realized by the shift operator, N_{11} and N_{22} are calculated by solving Theorem 2 as

$$N_{11} = \begin{bmatrix} -0.9003 & -0.0012 \\ -0.0158 & -0.0106 \\ -0.0012 & -0.9003 \\ -0.0106 & -0.0158 \end{bmatrix},$$

$$N_{22} = \begin{bmatrix} 0.5633 & 0.0006 \\ -0.0124 & 0.0033 \\ 0.0006 & 0.5633 \\ 0.0033 & -0.0124 \end{bmatrix}.$$

The measurement $y(k)$ from the robots and external tracking reference $u(k)$ are imported to the estimator according to (26). The tracking problem is not included in the theoretical results proposed in this paper, and therefore the defects such as time delay and coefficients' representation are not considered for $y(k)$ and $u(k)$. The specific design method for $u(k)$ can be found in [27], and is not reiterated in this paper. The simulation results are shown in Fig. 4(a) and 4(b), with the former depicting the trajectories of two robots as well as the trajectory reference for their geometric center, while the latter shows the estimating performance for the robots' position

x_1, y_1, x_2 and y_2. The above simulation results show that the design and realization method for the proposed estimator can achieve an acceptable estimating performance. This paper is focused on the stabilization of the estimator subject to FWL effects and time delays, because there is currently no considered performance index for the estimator to ensure favorable estimating performance.

V. CONCLUSION

This paper is concerned with the design and realization problem for a networked estimator. To digitally implement all subsystems, the coefficients' representation with FWL effects are considered, and the interconnected architecture also introduces the internal time delays. The descriptor model is therefore adopted to describe the networked system subject to both internal time delays and FWL effects in a general unifying way. Based on the obtained descriptor model, a condition for stability analysis and design method for the estimator gain are then deduced. A search algorithm is also proposed to determine the optimal realization requiring the minimum word length for stabilization. Finally, a simulation based on two cooperative robots is included to demonstrate the effectiveness of the theoretical results.

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