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SPECIAL SECTION ON RECENT ADVANCES ON HYBRID COMPLEX NETWORKS: ANALYSIS AND CONTROL

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Pinning Stabilization of Probabilistic Boolean Networks With Time Delays

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ABSTRACT In this article, the stabilization issues for probabilistic Boolean Networks (PBNs) with time delays are discussed. This article's objective is designing an efficient algorithm to choose suitable nodes to be pinning controlled for PBNs with time delays. By using the semi-tensor product (STP) of matrices, a PBN with time delays can be converted into a discrete-time linear system, and the transition matrix also can be obtained. Then, the necessary and sufficient conditions in the form of algebraic expression are given for the existence and solvability of the pinning feedback controllers with minimum pinning nodes for PBNs with time delays. Besides, three algorithms are proposed for designing and solving minimum pinning controllers.

INDEX TERMS Probabilistic Boolean networks, pinning control, time delays, semi-tensor product.

I. INTRODUCTION

Boolean Networks (BNs), which were first proposed by Kauffman in 1969 [1], are a kind of logical dynamical models to describe gene regulatory networks (GRNs) [2]. As we all know, in a gene regulatory network, each gene can be expressed (1) or not expressed (0), which corresponds to binary state variables. A BN is a deterministic model to simulate the evolution of binary state variables. What's more, Boolean Networks have been widely studied in state estimation [3], logical networks [4], neural networks [5], etc. Recently, the STP of matrices was introduced by Cheng's team. With the help of STP, a BN can be transformed into a discrete-time linear system. Moreover, a logic function can be represented by an algebraic form with STP. This new matrix product was introduced to the study of BNs in many fields, such as the controllability, event-triggered control, ect., which have been studied in [6]–[11].

To better handle of biological system uncertainty, Shmulevich etc. in [12] generalized the concept of BNs for application to probabilistic Boolean Networks (PBNs). In general, the PBNs can be seen as a kind of randomly switched BNs in given sets of BNs. Every BN is chosen with an definite probability. Many interesting results have been obtained for PBNs and probabilistic Boolean control networks (PBCNs), such as stability and stabilization [13], optimal control [4], controllability [14], and pinning control [15], etc.

Stability and stabilization are two important problems in BNs. For example, the apoptotic pathway can be activated to allow an organism to clear damaged or unwanted cells by combining with tumor necrosis factor (TNF) to death receptor tumor necrosis factor receptor 1 (TNFR1) [16]. Without TNF, cells can be bistable in two different states: survival and initiation of apoptosis [17]. However, the decision on one state or the other mainly depends on the initial conditions of random variation in each cell, and it can be seen as a stability problem in PBNs. Meanwhile, time delays are unavoidable in many real world systems, such as biological, physiological systems, and economic, and so on [18]–[20]. For GRNs, the direction of gene evolution is uncertain due to the possibility of gene mutation. Hence, PBNs with time delays can be better to simulate the real biological systems and GRNs in some cases. Thus, in this article, we will discuss the stabilization of PBNs with time delays.

In [21], BNs realize stabilization via state feedback control. Different from feedback control, only a small part of nodes are selected to be pinning controlled, which reduce the cost of the control effectively. A natural question in pinning control is how to select the nodes to be pinned. In [22], an algorithm is proposed to solve the minimum number of pinning

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controllers. Recently, minimizing controlled nodes to realize stabilization of BNs has been investigated deeply in [23]. Moreover, the stochastic networks can realize stabilization via minimum pinning controlled nodes, which has been studied in [24]. In [25], BNs with time delays realize stabilization via the pinning control. Thus, for a PBNs with time delays, how to stabilize via the pinning control and how to solve the minimum number pinning nodes to stabilize system are worth considering. Inspired by above works, the stabilization of PBNs with time delays via pinning control is investigated in this article.

The difficulties of this article are mainly two folds. 1) How to select pinning nodes for a PBN with time delays? Since a PBN with time delays is a dynamics system with random variables and multiple time delays, which make the pinning control problem for PBNs with time delays more complicated and challenging than that of BNs. 2) How to solve the minimum number of pinned nodes for a PBN with time delays through the algebraic method? In [22] and [26], the stabilization issue of BNs via minimum pinned nodes is solved through the graph theory method, rather than the algebraic method. Thus, it is a challenge to obtain corresponding results via the algebraic method. To overcome these difficult problems, inspired by the work of [24] and [25], we will take three steps to solve these difficulties. (i) Changing columns of the structure matrix to obtain the desired structure matrix. Thus, we propose a new algorithm to obtain the desired structure matrix. (ii) Selecting the pinning nodes via the columns of new structure matrix directly. The existence of the pinning feedback controllers for PBNs with time delays is considered, and the corresponding necessary and sufficient conditions in the form of algebraic expression are given. (iii) Choosing the minimal number pinning nodes by an efficient way. Moreover, an effective algorithm is proposed to calculate the minimum number of pinning controllers.

Notations: $\Delta_h := \{\delta_h^k \mid 1 \leq k \leq h\}$, where δ_h^k is the kth column of the identity matrix I_h. $D := \{1, 0\}$. 1_n and 0_n denote the column vector of length n, where all of the elements are equal to 1 and 0 respectively. $M_{r\times h}$ stands for the set of all $r \times h$ matrices and $M^k_{r\times h}$ stands for the set of matrix A where $A_{ij} = (a_1, a_2, \ldots, a_k)^T$, $1 \leq i \leq r$ and $1 \leq j \leq h$. We denote $Row_f(W)(Col_f(W))$ stands for the fth row(column) of matrix $W_{r\times h}$ and $Row(W)(Col(W))$ is the set of rows(columns) of matrix $W_{r\times h}$. A matrix $W \in M_{r\times h}$ is called a logical matrix if its columns $Col(W) \subset \Delta_r$. Moreover, we define the set of $r \times h$ logical matrices as $\mathcal{L}_{r\times h}$. $W = [\delta_r^{k_1}, \delta_r^{k_2}, \ldots, \delta_r^{k_h}]$ is denoted by $W := \delta_r[k_1, k_2, \ldots, k_h]$. $p = (k_1, k_2, \ldots, k_h)^T$ is called a h-dimensional probabilistic vector if $k_r \geq 0$, $r = 1, 2, \ldots, h$ and $\sum_{r=1}^h k_r = 1$. we define the set of h-dimensional probabilistic vectors as $\mathcal{P}_h$. For a probabilistic vector $p = (k_1, k_2, \ldots, k_h)^T$, we denote a operator $\langle p\rangle = \{\delta_k^r \mid k_r > 0,\ r = 1, 2, \ldots, h\}$. For a matrix $W \in M_{r\times h}$, if its columns $Col(W) \subset \mathcal{P}_r$, then this matrix is called a probabilistic matrix. Moreover, we define the set of $r \times h$ probabilistic matrices as $\mathcal{P}_{r\times h}$.

II. PRELIMINARIES

A. STP OF MATRICES

Definition 1: [27] For matrices $W \in M_{r\times h}$ and $Q \in M_{s\times t}$. Then, the STP of W and Q is

$$W \ltimes Q = (W \otimes I_{q/h})(Q \otimes I_{q/s}).$$

Here $\otimes$ is the Kronecker product of matrices and q is the least common multiple of h and s ($q = lcm\{h, s\}$).

Remark 1: Since STP is a generalization of the general matrix products, this notation $\ltimes$ can be omitted in the following discussion if no confusion arises.

Lemma 1: [27]

1) Let $X \in \mathbb{R}^t$ be a row vector and a matrix $A \in M_{m\times n}$, we have $A \ltimes X = X \ltimes (I_t \otimes A)$;
2) Let $X \in \mathbb{R}^t$ be a column vector and a matrix $A \in M_{m\times n}$, we have $X \ltimes A = (I_t \otimes A) \ltimes X$.

Definition 2: [27] Define a matrix:

$$\begin{aligned} Q_{[r,h]} = \delta_{rh}[&1, r+1, \ldots, (h-1)r+1, \\ &2, r+2, \ldots, (h-1)r+2, \\ &\ldots \\ &r, 2r, \ldots, rh] \in \mathcal{L}_{rh\times rh}. \end{aligned} \tag{1}$$

then for column vectors $a \in \mathbb{R}^r$ and $b \in \mathbb{R}^h$, we have $Q_{[h,r]} \ltimes b \ltimes a = a \ltimes b$.

Lemma 2: [27] Define a logical matrix

$$\Phi_n = \mathrm{diag}(\delta_{2^n}^1, \delta_{2^n}^2, \ldots, \delta_{2^n}^{2^n}) = \delta_{2^{2n}}[1, 2^n+2, \ldots, 2^{2^n}],$$

and let $X \in \Delta_{2^n}$. Then, $X \ltimes X = \Phi_n X$.

B. ALGEBRAIC REPRESENTATIONS OF PROBABILISTIC BOOLEAN NETWORKS WITH TIME DELAYS

Letting True $= 1 \sim \delta_2^1$, False $= 0 \sim \delta_2^2$. Then we can express the logical function by using STP of matrices.

Lemma 3: [27] A logical function $h(L_1, \ldots, L_r)$ with logical arguments $L_1, \ldots, L_r \in \Delta_2$ can be expressed in a multi-linear form as

$$h(L_1, \ldots, L_r) = M_h L_1 L_2 \ldots L_r,$$

where $M_h \in \mathcal{L}_{2\times 2^r}$ is unique. Moreover, we define the matrix M_h as the structure matrix of h.

A PBN with time delays is described as

$$\begin{aligned} x_h(t+1) = f_h(&x_1(t), \ldots, x_r(t), x_1(t-1), \ldots, x_r(t-1), \\ &\ldots, x_1(t-\tau), \ldots, x_r(t-\tau)),\ h = 1, \ldots, r, \end{aligned} \tag{2}$$

where $x_h(t) \in D$ is the state of node h at time t, $h = 1, 2, \ldots, r$, and τ is a positive integer.

In system (2), f_h is randomly selected from a given finite set of Boolean functions $\Omega = \{f_h^1, f_h^2, \ldots, f_h^{k_h}\}$, and $f_h^s : D^{r(\tau+1)} \mapsto D$, $s = 1, 2, \ldots, k_h$.

We suppose that $P(f_h = f_h^s) = p_h^s,\ s = 1, 2, \ldots, k_h$, where $\sum_{s=1}^{k_h} p_h^s = 1$, and $x(t) = \ltimes_{h=1}^{r} x_h(t)$. Then, system (2) can be converted into an algebraic form as

$$x_h(t+1) = M_h x(t)x(t-1)\ldots x(t-\tau),$$

where $M_h \in \mathcal{L}_{2\times 2^{r(\tau+1)}}$ is selected from a matrices' set $\{M_h^1, M_h^2, \ldots, M_h^{k_h}\}$ and M_h^s are logical matrices of $f_h^s,\ s = 1, 2, \ldots, k_h$, respectively. Then, $P(M_h = M_h^s) = p_h^s,\ s = 1, 2, \ldots, k_h$.

Based on the above discussion, the evolution of the state expectation can be obtained as follows

$$\begin{aligned}
E&[x_h(t+1)]\\
&= E[E[x_h(t+1) \mid M_h]]\\
&= E[E[M_h x(t)\ldots x(t-\tau) \mid M_h]]\\
&= \sum_{s=1}^{k_h} E[M_h x(t)\ldots x(t-\tau) \mid M_h = M_h^s]P(M_h = M_h^s)\\
&= \sum_{s=1}^{k_h} E[M_h^s x(t)\ldots x(t-\tau)]p_h^s\\
&= \sum_{s=1}^{k_h} M_h^s p_h^s E[x(t)\ldots x(t-\tau)]\\
&:= \bar{M}_h E[x(t)\ldots x(t-\tau)], \quad h = 1, 2, \ldots, r, \qquad (3)
\end{aligned}$$

where $\bar{M}_h = \sum_{s=1}^{k_h} M_h^s p_h^s \in \mathcal{P}_{2\times 2^{r(\tau+1)}}$. As a result, the following eauqtion holds

$$E[x(t+1)] = ME[x(t)\ldots x(t-\tau)], \qquad (4)$$

where $M = \bar{M}_1 * \ldots * \bar{M}_r \in \mathcal{P}_{2^r\times 2^{r(\tau+1)}}$ and $*$ is Khatri-Rao product.

Definition 3: A probabilistic Boolean networks with time delays is globally stable with probability one to a state $\bar{x} = \delta_{2^r}^q$ $(1 \le q \le 2^r)$, for any sequence of initial states $x_0, x_{-1}, \ldots, x_{-\tau} \in \Delta_{2^r}$, if there exists a positive integer $T \in Z_+$, such that $P(x(t) = \bar{x} \mid x(0) = x_0, \ldots, x(-\tau) = x_{-\tau}) = 1$ for all $t \ge T$.

Based on the discussion of remark 1 of [28], for obtaining the condition of the globally stable to a state with probability one of PBNs (2), we only need to study the globally stable of system (4).

III. MAIN RESULTS

In this section, the stabilization of PBCNs with time delays is considered by designing pinning controllers. We need to design a algorithm to get some suitable controlled nodes. In the end, we discuss how to stabilize system (2) to a given state $\delta_{2^r}^q$ by minimum pinned nodes.

Suppose that the first l nodes are pinned and state feedback controllers are as follows,

$$\begin{aligned}
u_s(t) &= \varphi_s(x_1(t), \ldots, x_r(t), x_1(t-1), \ldots, x_r(t-1), \ldots,\\
&\qquad x_1(t-\tau), \ldots, x_r(t-\tau)), \quad s = 1, 2, \ldots, l\ (l \le r). \qquad (5)
\end{aligned}$$

Then system (2) with controllers becomes the following system

$$\begin{cases}
x_s(t+1) = F_s(u_s(t), f_s(x_1(t), \ldots, x_r(t), x_1(t-1),\\
\ldots, x_r(t-1), \ldots, x_1(t-\tau), \ldots, x_r(t-\tau))),\\
s = 1, 2, \ldots, l,\\
x_v(t+1) = f_v(x_1(t), \ldots, x_r(t), x_1(t-1), \ldots,\\
x_r(t-1), \ldots, x_1(t-\tau), \ldots, x_r(t-\tau))),\\
v = l+1, l+2, \ldots, r,
\end{cases} \qquad (6)$$

where F_s is a logical function of variables $u_s(t)$ and f_s, $u_s(t)$ is the state feedback controller of $x(t), \ldots, x(t-\tau)$, $s = 1, 2\ldots, l$.

According to Lemma 3, there exists a matrix $H_s \in \mathcal{L}_{2\times 2^{r(\tau+1)}}$, where $u_s(t) = H_s x(t)\ldots x(t-\tau)$. Furthermore, since F_s is a logical function of variables $u_s(t)$ and f_s, there exists a logical matrix $L_s \in \mathcal{L}_{2\times 4}$, where $x_s(t+1) = L_s H_s x(t)\ldots x(t-\tau)M_s x(t)\ldots x(t-\tau) = L_s H_s(I_{2^{r(\tau+1)}} \otimes M_s)\Phi_{r(\tau+1)} x(t)\ldots x(t-\tau)$, where $P(M_s = M_s^q) = P_s^q$. From (3), the following formula can be obtained

$$\begin{aligned}
E&[x_s(t+1)]\\
&= E[E[x_s(t+1) \mid M_s]]\\
&= \sum_{q=1}^{k_s} E[L_s H_s(I_{2^{r(\tau+1)}} \otimes M_s^q)\Phi_{r(\tau+1)} x(t)\ldots x(t-\tau)]p_s^q\\
&= \sum_{q=1}^{k_s} p_s^q L_s H_s(I_{2^{r(\tau+1)}} \otimes M_s^q)\Phi_{r(\tau+1)} E[x(t)\ldots x(t-\tau)]\\
&= L_s H_s(I_{2^{r(\tau+1)}} \otimes \bar{M}_s)\Phi_{r(\tau+1)} E[x(t)\ldots x(t-\tau)]\\
&:= \widehat{M}_s E[x(t)\ldots x(t-\tau)], \quad s = 1, 2, \ldots, l. \qquad (7)
\end{aligned}$$

Hence, systems (6) can be transformed to be

$$\begin{cases}
E[x_s(t+1)] = \widehat{M}_s E[x(t)\ldots x(t-\tau)]\\
\quad s = 1, 2, \ldots, l.\\
E[x_s(t+1)] = \bar{M}_s E[x(t)\ldots x(t-\tau)]\\
\quad s = l+1, l+2, \ldots, r.
\end{cases} \qquad (8)$$

Therefore, we can get the following results

$$E[x(t+1)] = \widehat{M} E[x(t)\ldots x(t-\tau)], \qquad (9)$$

where $\widehat{M} = \widehat{M}_1 * \widehat{M}_2 * \ldots * \widehat{M}_r$, and

$$\widehat{M}_s = \begin{cases} \widehat{M}_s, & s = 1, \ldots, l,\\ \bar{M}_s, & s = l+1, \ldots, r. \end{cases}$$

A. THE PINNED NODES OF PBNS WITH DELAYS

Definition 4: For two probabilistic vectors $\psi, \omega \in \mathcal{P}_r$, where $\psi = (\psi_1, \psi_2, \ldots, \psi_r)^T$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_r)^T$. Then,

$$Row_s(\psi \circ \omega) = \psi_s \oplus \omega_s,\ s = 1, \ldots, r,$$

where

$$\psi_s \oplus \omega_s = \begin{cases} 1, & \psi_s\omega_s > 0\\ 0, & else. \end{cases}$$

Let $\bar{x} = \delta_{2^r}^q$ be the pinning objective state and $\overbrace{\delta_{2^r}^q \ltimes \delta_{2^r}^q \ltimes \ldots \ltimes \delta_{2^r}^q}^{\tau+1} = \delta_{2^{r(\tau+1)}}^h$. First, we define a sequence of set $\{\Xi_k \mid k = 1, 2, \ldots\}$ as follows

$$\begin{aligned}\Xi_1(\delta_{2^{r(\tau+1)}}^h) &= \{\delta_{2^{r(\tau+1)}}^s \mid M\delta_{2^{r(\tau+1)}}^s, \quad s = 1, \ldots, 2^{r(\tau+1)}\}\\ \Xi_{k+1}(\delta_{2^{r(\tau+1)}}^h) &= \{\delta_{2^{r(\tau+1)}}^s \mid \Gamma \circ (1_{2^{r(\tau+1)}} \\ &\quad - \sum_{\delta_{2^{r(\tau+1)}}^i \in \Xi_k(\delta_{2^{r(\tau+1)}}^h)} \delta_{2^{r(\tau+1)}}^i) = 0_{2^{r(\tau+1)}},\\ &\quad \Gamma = Y_{\tau+1} \ltimes Y_\tau \ltimes \ldots \ltimes Y_1, \quad s = 1, \ldots, 2^{r(\tau+1)}\},\end{aligned} \tag{10}$$

where $Y_1 = M\delta_{2^{r(\tau+1)}}^s$, $Y_i = M\overbrace{Y_{i-1}\ldots Y_1\delta_{2^r}^{a_1}\ldots\delta_{2^r}^{a_{\tau+2-i}}}^{\tau+1}$, $i = 2, \ldots, \tau+1$, and $\delta_{2^{r(\tau+1)}}^s = \ltimes_{j=1}^{\tau+1}\delta_{2^r}^{a_j}$.

Remark 2: Notice that there are finite state totally and $\Xi_k(\delta_{2^{r(\tau+1)}}^h) \subseteq \Xi_{k+1}(\delta_{2^{r(\tau+1)}}^h)$. Therefore, there exists a positive integer T such that $\Xi_{T+1}(\delta_{2^{r(\tau+1)}}^h) = \Xi_T(\delta_{2^{r(\tau+1)}}^h)$. Then, the set $\Xi_{T+1}(\delta_{2^{r(\tau+1)}}^h)$ can not be larger any more and we can obtain a finial set $\Xi(\delta_{2^{r(\tau+1)}}^h) = \Xi_T(\delta_{2^{r(\tau+1)}}^h)$.

Now we can give an algorithm to obtain the desired new structure matrix $\widehat{M}$.

Algorithm 1 Obtain the Desired Structure Matrix $\widehat{M}$

Input: q, M
Output: $\widehat{M}$
1: Initialize $\widehat{M} = 0$
2: If $Col_{(q-1)2^{r\tau}+(q-1)2^{r(\tau-1)}+\ldots+q}(M) \neq \delta_{2^r}^q$
3: then change $Col_{(q-1)2^{r\tau}+(q-1)2^{r(\tau-1)}+\ldots+q}(M)$ to $\delta_{2^r}^q$.
4: end if
5: for $s = 1, 2 \ldots, 2^{r(\tau+1)}$ do
6: Calculate $\Xi_s(\delta_{2^{r(\tau+1)}}^h)$.
7: end for
8: Calculate $\Xi(\delta_{2^{r(\tau+1)}}^h) = \cup_{s=1}^{2^{r(\tau+1)}}\Xi_s(\delta_{2^{r(\tau+1)}}^h)$.
9: for $s = 1, 2 \ldots, 2^{r(\tau+1)}$ do
10: if $\delta_{2^{r(\tau+1)}}^s \in \Delta_{2^{r(\tau+1)}} \setminus \Xi(\delta_{2^{r(\tau+1)}}^h)$
11: then change the sth column of M to $\delta_{2^r}^q$.
12: end if
13: end for
14: $\widehat{M} = M$
15: return $\widehat{M}$

Theorem 1: Suppose that the structure matrix of (4) is M, and M is changed to $\widehat{M}$ by the above algorithm. Then, the PBN with delays is globally stabilized to $\delta_{2^r}^q$ with probaility one.

Proof: For all initial states $\ltimes_{i=-\tau}^0 x_i = \delta_{2^{r(\tau+1)}}^j \in \Delta_{2^{r(\tau+1)}}$, $j \neq h$, using the above algorithm, such that $\delta_{2^{r(\tau+1)}}^j \in \Xi(\delta_{2^{r(\tau+1)}}^h)$. The result implies that there exist $T \in \mathbb{Z}_+$ such that $\delta_{2^{r(\tau+1)}}^j \in \Xi_T(\delta_{2^{r(\tau+1)}}^h)$. Next, we will prove that $\delta_{2^{r(\tau+1)}}^j$ can reach $\delta_{2^{r(\tau+1)}}^h$ in T steps by mathematical induction.

For $T = 1$, $\delta_{2^{r(\tau+1)}}^j \in \Xi_1(\delta_{2^{r(\tau+1)}}^h)$, then $\widehat{M}\delta_{2^{r(\tau+1)}}^j = \delta_{2^r}^q$, where $\widehat{M}$ is the new structure matrix form M.

Assuming that when $T = t-1, t > 2$, the Theorem 1 holds. For $T = t$, assuming $\delta_{2^{r(\tau+1)}}^j \in \Xi_t(\delta_{2^{r(\tau+1)}}^h)$, then

$$\Gamma \circ (1_{2^{r(\tau+1)}} - \sum_{\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)} \delta_{2^{r(\tau+1)}}^s) = 0_{2^{r(\tau+1)}},$$

where Γ is a $2^{r(\tau+1)}$-dimensional probabilistic vector and $1_{2^{r(\tau+1)}} - \sum_{\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)} \delta_{2^{r(\tau+1)}}^s$ is a column vector with all elements are belonging to D. Thus, Γ can be a linear combination of $\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)$. Thus, there exists $r_s \in \{0, 1\}$ such that

$$\Gamma = \sum_{\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)} r_s\delta_{2^{r(\tau+1)}}^s,$$

where $\sum_{\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)} r_s = 1$.

Then, it holds that

$$\begin{aligned}\widehat{M}\Gamma &= \widehat{M}(\sum_{\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)} r_s\delta_{2^{r(\tau+1)}}^s)\\ &= \sum_{\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)} r_s\widehat{M}\delta_{2^{r(\tau+1)}}^s\\ &= \sum_{\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)} r_s \sum_{\delta_{2^{r(\tau+1)}}^i \in \Xi_{t-2}(\delta_{2^{r(\tau+1)}}^h)}\\ &\quad p_i\widehat{M}\delta_{2^{r(\tau+1)}}^i \quad (\sum_{\delta_{2^{r(\tau+1)}}^i \in \Xi_{t-2}(\delta_{2^{r(\tau+1)}}^h)} p_i = 1)\\ &= \sum_{\delta_{2^{r(\tau+1)}}^i \in \Xi_{t-2}(\delta_{2^{r(\tau+1)}}^h)} p_i \sum_{\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)}\\ &\quad r_s\widehat{M}\delta_{2^{r(\tau+1)}}^i\\ &= \sum_{\delta_{2^{r(\tau+1)}}^i \in \Xi_{t-2}(\delta_{2^{r(\tau+1)}}^h)} p_i(\sum_{\delta_{2^{r(\tau+1)}}^s \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)} r_s)\\ &\quad \widehat{M}\delta_{2^{r(\tau+1)}}^i\\ &= \widehat{M}\sum_{\delta_{2^{r(\tau+1)}}^i \in \Xi_{t-2}(\delta_{2^{r(\tau+1)}}^h)} p_i\delta_{2^{r(\tau+1)}}^i,\end{aligned} \tag{11}$$

which implies that $\Gamma \in \Xi_{t-1}(\delta_{2^{r(\tau+1)}}^h)$. That is to say, Γ can reach $\delta_{2^{r(\tau+1)}}^h$ in $t-1$ steps. From (10), it holds that $\delta_{2^{r(\tau+1)}}^j$ can reach $\delta_{2^{r(\tau+1)}}^h$ in t steps. ■

B. THE DESIGN OF PINNING FEEDBACK CONTROLLERS

From (6), we can get the following expressions:

$$\begin{cases}\widehat{M}_1 = L_1H_1(I_{2^{r(\tau+1)}} \otimes \bar{M}_1)\Phi_{r(\tau+1)}\\ \widehat{M}_2 = L_2H_2(I_{2^{r(\tau+1)}} \otimes \bar{M}_2)\Phi_{r(\tau+1)}\\ \ldots\\ \widehat{M}_l = L_lH_l(I_{2^{r(\tau+1)}} \otimes \bar{M}_l)\Phi_{r(\tau+1)}.\end{cases} \tag{12}$$

Thus, if we can solve L_i, H_i, $i = 1, 2, \ldots, l$ from (12), then the logical functions F_i, $i = 1, 2, \ldots, l$ and feedback controllers $u_i(t),\ i = 1, 2, \ldots, l$ can be solved. Consequently, system (6) can be globally stabilized with probability one to the desired state $\delta_{2^r}^{q}$ by Theorem 1.

Theorem 2: System of equation (12) is solvable if and only if the columns of $\widehat{M}_i\ (i = 1, 2, \ldots, l)$ belong to two sets of $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ at most, where $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ as follows

$$\begin{cases} \Psi_1 = \{Col_j(\widehat{M}_i) \mid Col_j(\widehat{M}_i) = Col_j(\bar{M}_i) \neq (0,1)^T \text{ or } \\ \quad (1,0)^T\}, \\ \Psi_2 = \{Col_j(\widehat{M}_i) \mid Col_j(\widehat{M}_i) = (1,1)^T - Col_j(\bar{M}_i)\}, \\ \Psi_3 = \{Col_j(\widehat{M}_i) \mid Col_j(\widehat{M}_i) = (0,1)^T\}, \\ \Psi_4 = \{Col_j(\widehat{M}_i) \mid Col_j(\widehat{M}_i) = (1,0)^T\}. \end{cases}$$

Proof: (Necessity) Assume that

$$L_i = [\begin{smallmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ 1-\eta_1 & 1-\eta_2 & 1-\eta_3 & 1-\eta_4 \end{smallmatrix}],$$
$$H_i = [\begin{smallmatrix} \theta_1 & \theta_2 & \cdots & \theta_{2^{r(\tau+1)}} \\ 1-\theta_1 & 1-\theta_2 & \cdots & 1-\theta_{2^{r(\tau+1)}} \end{smallmatrix}],$$
$$\bar{M}_i = [\begin{smallmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{2^{r(\tau+1)}} \\ 1-\lambda_1 & 1-\lambda_2 & \cdots & 1-\lambda_{2^{r(\tau+1)}} \end{smallmatrix}],$$
$$\widehat{M}_i = [\begin{smallmatrix} \mu_1 & \mu_2 & \cdots & \mu_{2^{r(\tau+1)}} \\ 1-\mu_1 & 1-\mu_2 & \cdots & 1-\mu_{2^{r(\tau+1)}} \end{smallmatrix}],$$

where $\eta_s, \theta_q \in D = \{0,1\},\ s = 1,2,3,4,\ q = 1,\ldots,2^{r(\tau+1)}$ and $\lambda_s, \mu_s \in [0,1],\ s = 1,\ldots,2^{r(\tau+1)}$. Then,

$$\begin{aligned} & L_s H_s (I_{2^{r(\tau+1)}} \otimes \bar{M}_s)\Phi_{r(\tau+1)} \\ &= L_s (H_s \otimes I_2)(I_{2^{r(\tau+1)}} \otimes \bar{M}_s)\Phi_{r(\tau+1)} \\ &= L_s (H_s I_{2^{r(\tau+1)}} \otimes I_2 \bar{M}_s)\Phi_{r(\tau+1)} \\ &= L_s (H_s \otimes \bar{M}_s)\Phi_{r(\tau+1)} \\ &= \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ 1-\eta_1 & 1-\eta_2 & 1-\eta_3 & 1-\eta_4 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \theta_1\lambda_1 & \theta_2\lambda_2 & \cdots & \theta_{2^{r(\tau+1)}}\lambda_{2^{r(\tau+1)}} \\ \theta_1(1-\lambda_1) & \theta_2(1-\lambda_2) & \cdots & \theta_{2^{r(\tau+1)}}(1-\lambda_{2^{r(\tau+1)}}) \\ (1-\theta_1)\lambda_1 & (1-\theta_2)\lambda_2 & \cdots & (1-\theta_{2^{r(\tau+1)}})\lambda_{2^{r(\tau+1)}} \\ (1-\theta_1)(1-\lambda_1) & (1-\theta_2)(1-\lambda_2) & \cdots & (1-\theta_{2^{r(\tau+1)}})(1-\lambda_{2^{r(\tau+1)}}) \end{bmatrix} \\ &= \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_{2^{r(\tau+1)}} \\ 1-\mu_1 & 1-\mu_2 & \cdots & 1-\mu_{2^{r(\tau+1)}} \end{bmatrix}, \\ &\quad s = 1,\ldots,l. \end{aligned} \tag{13}$$

Thus, we have

$$\begin{cases} \eta_1\theta_1\lambda_1 + \eta_2\theta_1(1-\lambda_1) + \eta_3(1-\theta_1)\lambda_1 \\ +\eta_4(1-\theta_1)(1-\lambda_1) = \mu_1 \\ \eta_1\theta_2\lambda_2 + \eta_2\theta_2(1-\lambda_2) + \eta_3(1-\theta_2)\lambda_2 \\ +\eta_4(1-\theta_2)(1-\lambda_2) = \mu_2 \\ \cdots \\ \eta_1\theta_{2^{r(\tau+1)}}\lambda_{2^{r(\tau+1)}} + \eta_2\theta_{2^{r(\tau+1)}}(1-\lambda_{2^{r(\tau+1)}}) \\ +\eta_3(1-\theta_{2^{r(\tau+1)}})\lambda_{2^{r(\tau+1)}} + \eta_4(1-\theta_{2^{r(\tau+1)}}) \\ (1-\lambda_{2^{r(\tau+1)}}) = \mu_{2^{r(\tau+1)}} \end{cases} \tag{14}$$

Then, these $2^{r(\tau+1)}$ equations can be classified into 3 parts:

$$\begin{cases} (1):\ \mu_s = \lambda_s \\ (2):\ \mu_s = 1-\lambda_s \qquad s = 1,2,\ldots,2^{r(\tau+1)} \\ (3):\ \mu_s \neq \lambda_s \text{ and } \mu_s \neq 1-\lambda_s \end{cases} \tag{15}$$

For (1), if $\theta_s = 0$, then $\eta_3 = 1, \eta_4 = 0$. If $\theta_s = 1$, then $\eta_1 = 1, \eta_2 = 0$.

For (2), if $\theta_s = 0$, then $\eta_3 = 0, \eta_4 = 1$. If $\theta_s = 1$, then $\eta_1 = 0, \eta_2 = 1$.

For (3), if $\theta_s = 0$, then $\eta_3\lambda_s + \eta_4(1-\lambda_s) = \mu_s$. If $\theta_s = 1$, then $\eta_1\lambda_s + \eta_2(1-\lambda_s) = \mu_s$. Thus, we can obtain the following results:

$$\begin{cases} \theta_s = 0 \begin{cases} \mu_s = 0,\ \eta_3 = \eta_4 = 0, \\ \mu_s = 1,\ \eta_3 = \eta_4 = 1, \end{cases} \\ \theta_s = 1 \begin{cases} \mu_s = 0,\ \eta_1 = \eta_2 = 0, \\ \mu_s = 1,\ \eta_1 = \eta_2 = 1. \end{cases} \end{cases}$$

If μ_s satisfies three conditions, a counter example can be given as follows: let $\mu_s = \lambda_s,\ \mu_j = 1,\ \mu_k = 0$, there's no intersection for the solutions in this case, thus $\eta_1, \eta_2, \eta_3, \eta_4$ can't be solved.

If μ_s satisfies two conditions, $\mu_s = \lambda_s$ and $\mu_j = 1$, when $\theta_s = 0$ and $\theta_j = 1$ respectively, then $\eta_1 = \eta_2 = \eta_3 = 1, \eta_4 = 0$. Therefore, μ_s satisfies two conditions at most.

(Sufficiency) If $(\mu_i, 1-\mu_i)^T,\ i = 1,2,\ldots,2^{r(\tau+1)}$ belong to one of $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, according to the above derivation, it is obvious that Equations (12) can be solved.

If $(\mu_i, 1-\mu_i)^T,\ i = 1,2,\ldots,2^{r(\tau+1)}$ belong to two of $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, they can be classified into 6 parts:

$$\begin{aligned} &\begin{cases} (\mu_i, 1-\mu_i)^T \in \Psi_1, & \theta_i = 0, \\ (\mu_j, 1-\mu_j)^T \in \Psi_2, & \theta_j = 1, \end{cases} \Rightarrow \begin{cases} \eta_1 = 0, & \eta_2 = 1, \\ \eta_3 = 1, & \eta_4 = 0, \end{cases} \\ &\begin{cases} (\mu_i, 1-\mu_i)^T \in \Psi_1, & \theta_i = 0, \\ (\mu_j, 1-\mu_j)^T \in \Psi_3, & \theta_j = 1, \end{cases} \Rightarrow \begin{cases} \eta_1 = 0, & \eta_2 = 0, \\ \eta_3 = 1, & \eta_4 = 0, \end{cases} \\ &\begin{cases} (\mu_i, 1-\mu_i)^T \in \Psi_1, & \theta_i = 0, \\ (\mu_j, 1-\mu_j)^T \in \Psi_4, & \theta_j = 1, \end{cases} \Rightarrow \begin{cases} \eta_1 = 1, & \eta_2 = 1, \\ \eta_3 = 1, & \eta_4 = 0, \end{cases} \\ &\begin{cases} (\mu_i, 1-\mu_i)^T \in \Psi_2, & \theta_i = 0, \\ (\mu_j, 1-\mu_j)^T \in \Psi_3, & \theta_j = 1, \end{cases} \Rightarrow \begin{cases} \eta_1 = 0, & \eta_2 = 1, \\ \eta_3 = 0, & \eta_4 = 0, \end{cases} \\ &\begin{cases} (\mu_i, 1-\mu_i)^T \in \Psi_2, & \theta_i = 0, \\ (\mu_j, 1-\mu_j)^T \in \Psi_4, & \theta_j = 1, \end{cases} \Rightarrow \begin{cases} \eta_1 = 0, & \eta_2 = 1, \\ \eta_3 = 1, & \eta_4 = 1, \end{cases} \\ &\begin{cases} (\mu_i, 1-\mu_i)^T \in \Psi_3, & \theta_i = 0, \\ (\mu_j, 1-\mu_j)^T \in \Psi_4, & \theta_j = 1, \end{cases} \Rightarrow \begin{cases} \eta_1 = 1, & \eta_2 = 1, \\ \eta_3 = 0, & \eta_4 = 0, \end{cases} \end{aligned} \tag{16}$$

where the solutions may have many combinations under the above conditions. For example, when $(\mu_i, 1-\mu_i)^T \in \Psi_1$ and $(\mu_j, 1-\mu_j)^T \in \Psi_2$, if $\theta_i = 1$ and $\theta_j = 0$, then $\eta_1 = 0, \eta_2 = \eta_3 = 1, \eta_4 = 0$.

In conclusion, $\widehat{M}_s = L_s H_s (I_{2^{r(\tau+1)}} \otimes \bar{M}_s)\Phi_{r(\tau+1)}$ is solvable, $s = 1,2,\ldots,l$. ■

Remark 3: The above Theorem 2 provides a sufficient and necessary condition for the solvability of the pinning feedback controllers for PBNs with time delays, which generalizes the results of [25]. In other words, if the transition matrix is a determined matrix, Theorem 2 is degenerated to be Proposition 3.2 of [25]. Once the system of equation (12) is solvable, then L_s and H_s, $s = 1, 2, \ldots, l$, can be obtained according to the proof of Theorem 2.

The above results can be summed up as the Algorithm 2 to design pinning controllers for a PBN with time delays.

Algorithm 2 Design Pinning Controllers

Input: $q,\ M$

Output: $F_i,\ \varphi_i,\ i = 1, \ldots, l$

1: Using Algorithm 1 to change M to the desired structure matrix $\widehat{M}$.
2: for i = 1,…,l do
3: Calculate L_i, H_i from (12) by using Theorem 2.
4: end for
5: for i = 1,…,l do
6: Reconstruct the PBN with time delays from its structural matrices L_i and H_i to its logical expressions F_i and φ_i by using the methods in [7].
7: end for
8: return $F_i,\ \varphi_i,\ i = 1, \ldots, l$

C. MINIMUM NUMBER OF PINNING NODES

Based on the above discussion, the elements of set $\Xi(\delta^h_{2^{r(\tau+1)}})$ can be steered to the desired state $\delta^q_{2^r}$. However, we want to get the minimum number nodes to be pinned. Suppose that $\Xi^c(\delta^h_{2^{r(\tau+1)}}) := \Delta_{2^{r(\tau+1)}} \setminus \Xi(\delta^h_{2^{r(\tau+1)}}) = \{\delta^{k_1}_{2^{r(\tau+1)}}, \delta^{k_2}_{2^{r(\tau+1)}}, \ldots, \delta^{k_l}_{2^{r(\tau+1)}}\}$. Next we will stabilize elements in $\Xi^c(\delta^h_{2^{r(\tau+1)}})$ by finding minimum number pinning nodes. Then, we can get a new matrix from M as follows

$$C = \begin{bmatrix} \begin{bmatrix} a^1_{k_1} \\ 1-a^1_{k_1} \end{bmatrix} & \begin{bmatrix} a^2_{k_1} \\ 1-a^2_{k_1} \end{bmatrix} & \cdots & \begin{bmatrix} a^{r(\tau+1)}_{k_1} \\ 1-a^{r(\tau+1)}_{k_1} \end{bmatrix} \\ \begin{bmatrix} a^1_{k_2} \\ 1-a^1_{k_2} \end{bmatrix} & \begin{bmatrix} a^2_{k_2} \\ 1-a^2_{k_2} \end{bmatrix} & \cdots & \begin{bmatrix} a^{r(\tau+1)}_{k_2} \\ 1-a^{r(\tau+1)}_{k_2} \end{bmatrix} \\ \vdots & \vdots & \vdots & \vdots \\ \begin{bmatrix} a^1_{k_l} \\ 1-a^1_{k_l} \end{bmatrix} & \begin{bmatrix} a^2_{k_l} \\ 1-a^2_{k_l} \end{bmatrix} & \cdots & \begin{bmatrix} a^{r(\tau+1)}_{k_l} \\ 1-a^{r(\tau+1)}_{k_l} \end{bmatrix} \end{bmatrix},$$

where $Col_j(M) = \ltimes^{r(\tau+1)}_{i=1}(a^i_j, 1-a^i_j)^T$, $a^i_j \in [0, 1]$, $j = k_1, k_2, \ldots, k_l$, and $C \in R^2_{l\times r(\tau+1)}$. The following transformations are based on matrix C.

For the transition matrix $M = \bar{M}_1 * \ldots * \bar{M}_r$ of system (4), where $Col_j(M) = \ltimes^{r(\tau+1)}_{i=1}(a^i_j, 1-a^i_j)^T$, we define two matrix sets that $M^\star = M^\star_1 * \ldots * M^\star_r$ and $M^{\star\star} = M^{\star\star}_1 * \ldots * M^{\star\star}_r$, where $M^\star_s, M^{\star\star}_s \in \mathcal{P}_{2\times r(\tau+1)}$, $s = 1, \ldots, r(\tau+1)$. Then, these matrix sets can be expressed as follows

$$\Theta[Col_i(C)] = \{M^\star\}, \tag{17}$$

where if $\delta^j_{2^{r(\tau+1)}} \in \Xi(\Delta^h_{2^{r(\tau+1)}})$, then $Col_j(M^\star) = Col_j(M)$. If $\delta^j_{2^{r(\tau+1)}} \in \Xi^c(\Delta^h_{2^{r(\tau+1)}})$, then $Col_j(M^\star_i) \in \Delta_2$.

$$\bar{\Theta}[Col_i(C)] = \{M^{\star\star}\}, \tag{18}$$

where if $Col(M^\star_i)$ belongs to two of Ψ_1, Ψ_3, Ψ_4, then $M^{\star\star} = M^\star$. If $Col_j(M^\star_i) = \delta^1_2$, then $Col_j(M^\star_i) \in \Psi_1$, and if $Col_j(M^\star_i) \notin \Psi_1$, then $Col_j(M^{\star\star}_i) = Col_j(M^\star_i)$.

Lemma 4: If $\ltimes^{i-1}_{j=1}\begin{pmatrix} a_j \\ 1-a_j \end{pmatrix} \ltimes \begin{pmatrix} \alpha_i \\ 1-\alpha_i \end{pmatrix} \ltimes^{r(\tau+1)}_{j=i+1} \begin{pmatrix} a_j \\ 1-a_j \end{pmatrix}$ can be steered to $\Xi(\delta^h_{2^{r(\tau+1)}})$, where $\alpha_i \in (0, 1) \subseteq R$, then, $\ltimes^{i-1}_{j=1}\begin{pmatrix} a_j \\ 1-a_j \end{pmatrix} \ltimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ltimes^{r(\tau+1)}_{j=i+1} \begin{pmatrix} a_j \\ 1-a_j \end{pmatrix}$ or $\ltimes^{i-1}_{j=1}\begin{pmatrix} a_j \\ 1-a_j \end{pmatrix} \ltimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ltimes^{r(\tau+1)}_{j=i+1} \begin{pmatrix} a_j \\ 1-a_j \end{pmatrix}$ can also be steered to $\Xi(\delta^h_{2^{r(\tau+1)}})$.

Proof: Since $\ltimes^{i-1}_{j=1}\begin{pmatrix} a_j \\ 1-a_j \end{pmatrix} \ltimes \begin{pmatrix} \alpha_i \\ 1-\alpha_i \end{pmatrix} \ltimes^{r(\tau+1)}_{j=i+1} \begin{pmatrix} a_j \\ 1-a_j \end{pmatrix}$ can be steered to $\Xi(\delta^h_{2^{r(\tau+1)}})$, then $\langle \ltimes^{i-1}_{j=1}\begin{pmatrix} a_j \\ 1-a_j \end{pmatrix} \ltimes \begin{pmatrix} \alpha_i \\ 1-\alpha_i \end{pmatrix} \ltimes^{r(\tau+1)}_{j=i+1} \begin{pmatrix} a_j \\ 1-a_j \end{pmatrix}\rangle \subseteq \Xi(\delta^h_{2^{r(\tau+1)}})$.

Then, it holds that

$$\begin{aligned} &\langle \ltimes^{i-1}_{j=1}\begin{pmatrix} a_j \\ 1-a_j \end{pmatrix} \ltimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ltimes^{r(\tau+1)}_{j=i+1} \begin{pmatrix} a_j \\ 1-a_j \end{pmatrix}\rangle \\ &\subseteq \langle \ltimes^{i-1}_{j=1}\begin{pmatrix} a_j \\ 1-a_j \end{pmatrix} \ltimes \begin{pmatrix} \alpha_i \\ 1-\alpha_i \end{pmatrix} \ltimes^{r(\tau+1)}_{j=i+1} \begin{pmatrix} a_j \\ 1-a_j \end{pmatrix}\rangle \\ &\subseteq \Xi(\delta^b_{2^{r(\tau+1)}}), \end{aligned}$$

and

$$\begin{aligned} &\langle \ltimes^{i-1}_{j=1}\begin{pmatrix} a_j \\ 1-a_j \end{pmatrix} \ltimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ltimes^{r(\tau+1)}_{j=i+1} \begin{pmatrix} a_j \\ 1-a_j \end{pmatrix}\rangle \\ &\subseteq \langle \ltimes^{i-1}_{j=1}\begin{pmatrix} a_j \\ 1-a_j \end{pmatrix} \ltimes \begin{pmatrix} \alpha_i \\ 1-\alpha_i \end{pmatrix} \ltimes^{r(\tau+1)}_{j=i+1} \begin{pmatrix} a_j \\ 1-a_j \end{pmatrix}\rangle \\ &\subseteq \Xi(\delta^h_{2^{r(\tau+1)}}). \end{aligned}$$

■

Theorem 3: System (4) can be stabilized via only one pinning node if and only if there exist $j \in \{1, \ldots, r(\tau+1)\}$ and a transition matrix B, such that $B \subseteq \bar{\Theta}[Col_j(C)]$ where $\Xi(\delta^h_{2^{r(\tau+1)}}) = \Delta_{2^{r(\tau+1)}}$ and $\overbrace{\delta^q_{2^r} \ltimes \delta^q_{2^r} \ltimes \ldots \ltimes \delta^q_{2^r}}^{\tau+1} = \delta^h_{2^{r(\tau+1)}}$.

Proof: (Necessity)

Suppose system (4) can be stabilized by pinning one node j. According to Algorithm 1, we can get the set $\Lambda = \{Col_j(M), j = k_1, k_2, \ldots, k_l\}$, the elements of which need to be changed. By changing the columns in set Λ, system (4) is changed into $x(t+1) = \widehat{M}x(t)\ldots x(t-\tau)$ and can realize globally stable.

Note that $M = \bar{M}_1 * \ldots * \bar{M}_r$ and $\widehat{M} = \bar{M}_1 * \ldots * \widehat{M}_j * \ldots * \bar{M}_r$. For $i = k_1, k_2, \ldots, k_l$, it holds that $Col_i(M) = \ltimes^{r(\tau+1)}_{m=1}\begin{bmatrix} a^m_i \\ 1-a^m_i \end{bmatrix}$. Hence, $Col_i(\widehat{M}) = \ltimes^{j-1}_{m=1}\begin{bmatrix} a^m_i \\ 1-a^m_i \end{bmatrix} \ltimes \begin{bmatrix} \alpha^j_i \\ 1-\alpha^j_i \end{bmatrix} \ltimes^{r(\tau+1)}_{m=j+1}\begin{bmatrix} a^m_i \\ 1-a^m_i \end{bmatrix}$, where $\alpha^j_i \in [0, 1]$.

Thus, it holds that $\widehat{M}_j$, as shown at the bottom of the next page.

For all $\alpha_i^j, i \in \{k_1, k_2, \ldots, k_l\}$, they can be classified into 7 parts as follows

(a) $\alpha_i^j = 0,\ i = k_1, k_2, \ldots, k_l$,
(b) $\alpha_i^j = 1,\ i = k_1, k_2, \ldots, k_l$,
(c) $\alpha_i^j \notin D = \{0, 1\},\ i = k_1, k_2, \ldots, k_l$,
(d) some $\alpha_i^j = 0$, some $\alpha_i^j = 1$,
(e) some $\alpha_i^j = 0$, some $\alpha_i^j \notin D = \{0, 1\}$,
(f) some $\alpha_i^j = 1$, some $\alpha_i^j \notin D = \{0, 1\}$,
(g) some $\alpha_i^j = 0$, some $\alpha_i^j = 1$, some $\alpha_i^j \notin D = \{0, 1\}$.

Suppose there is a new matrix $B_j \in \mathcal{P}_{2\times 2^{r(\tau+1)}}$. The elements in B_j not mentioned in the following discussion are the same as the corresponding elements in matrix $\widehat{M}_j$.

For (a), (b) and (c), let $B_j = \widehat{M}_j$.

For (d), if $i \in \{1, 2, \ldots, 2^{r(\tau+1)}\} \setminus \{k_1, k_2, \ldots, k_l\}$ and $a_i^j \notin D = \{0, 1\}$, then let $a_i^j = 1$ in matrix B_j.

For (e), if $\alpha_i^j \notin D = \{0, 1\}$, then let $\alpha_i^j = 0$ in matrix B_j.

For (f), if $\alpha_i^j \notin D = \{0, 1\}$, then let $\alpha_i^j = 1$ in matrix B_j.

For (g), if $\alpha_i^j \notin D = \{0, 1\}$, then let $\alpha_i^j = 1$ in matrix B_j. If $i \in \{1, 2, \ldots, 2^{r(\tau+1)}\} \setminus \{k_1, k_2, \ldots, k_l\}$ and $a_i^j \notin D = \{0, 1\}$, then let $a_i^j = 1$ in matrix B_j.

Thus, according to Lemma 4, $B = \bar{M}_1 * \ldots * B_j * \ldots \bar{M}_r$ can steer all initial states to the desire state $\delta_{2^r}^q$. It also holds that $B \subseteq \bar{\Theta}[Col_j(C)]$.

(Sufficiency)

Let the transition matrix $B \subseteq \bar{\Theta}[Col_j(C)]$ and $\Xi(\delta^h_{2^{r(\tau+1)}}) = \Delta_{2^{r(\tau+1)}}$, where $B = \bar{M}_1 * \ldots * B_j * \ldots \bar{M}_r$. Then, $Col(B_i)$ belongs to two of Ψ_1, Ψ_3, Ψ_4 at most. Thus, there exists a state feedback control $u(t) = Hx(t) \ldots x(t-\tau)$ satisfying $B = H(I_{2^{r(\tau+1)}} \otimes \bar{M}_j)\Phi_r$. Therefore, the one pinned node is j and the pinning controller with delays can be solved from (12) by using Theorem 2. ■

Next, the above results can be further generalized as follows

$$\Theta[Col_{i_1,i_2,\ldots,i_t}(C)] = \{M^\star\}, \tag{19}$$

where the matrix $M^\star$ satisfies the following conditions: if $\delta^j_{2^{r(\tau+1)}} \in \Xi(\Delta^h_{2^{r(\tau+1)}})$, then $Col_j(M^\star) = Col_j(M)$, and if $\delta^j_{2^{r(\tau+1)}} \in \Xi^c(\Delta^h_{2^{r(\tau+1)}})$, then $Col_j(M^\star_{i_1} * M^\star_{i_2} * \ldots * M^\star_{i_t}) \in \Delta_{2^t}$, $j = 1, 2, \ldots, r(\tau+1)$.

$$\bar{\Theta}[Col_{i_1,i_2,\ldots,i_t}(C)] = \{M^{\star\star}\}, \tag{20}$$

where the matrix $M^{\star\star}$ satisfies the conditions: if $Col(M_i^\star)$ belongs to two of $\{\Psi_1, \Psi_3, \Psi_4\}$ at most and $M_i^\star \in \Theta[Col_{i_1,i_2,\ldots,i_t}(C)]$, then $M_i^{\star\star} = M_i^\star$, and when $i \in \{i_1, i_2, \ldots, i_t\}$ and $j \in \{1, 2, \ldots, 2^{r(\tau+1)}\} \setminus \{k_1, k_2, \ldots, k_l\}$, it holds that

$$Col_j(M_i^{\star\star}) = \begin{cases} \delta_2^1, & \text{if } Col_j(M_i^\star) \in \Psi_1, \\ Col_j(M_i^\star), & \text{if } Col_j(M_i^\star) \notin \Psi_1. \end{cases}$$

Then, the above theorem can be further generalized as follows.

Theorem 4: System (4) achieves stability via t pinning nodes if and only if there exists $i_1, \ldots, i_t \in \{1, \ldots, r(\tau+1)\}$ and a transition matrix $\widehat{M}$, such that $\widehat{M} \in \bar{\Theta}[Col_{i_1,i_2,\ldots,i_t}(A)]$, where $\Xi(\delta^h_{2^{r(\tau+1)}}) = \Delta_{2^{r(\tau+1)}}$ and $\overbrace{\delta^q_{2^r} \ltimes \delta^q_{2^r} \ltimes \ldots \ltimes \delta^q_{2^r}}^{\tau+1} = \delta^h_{2^{r(\tau+1)}}$.

Proof: Using Algorithm 1, the structure matrix M can be changed to $\widehat{M}$, and these matrices can be decomposed to $\widehat{M} = \widehat{M}_1\widehat{M}_2 \ldots \widehat{M}_r$ and $M = M_1M_2 \ldots M_r$. Comparing $\widehat{M}_i$ with M_i, $i = 1, 2, \ldots, r$, we can know that $\widehat{M}_j$ and M_j are different for $j = s_1, s_2, \ldots, s_t$. For $j \in \{s_1, s_2, \ldots, s_t\}$, $\widehat{M}_j$ are changed by using Theorem 3. The rest of proof is similar to Theorem 3, and it is omitted here. ■

Algorithm 3 Calculate the Minimal Number of Pinning Controllers

Input: $q, \widehat{M}, M$
Output: $\Sigma(N, \Omega)$
Initialize $P = \varnothing$
for $i = 1, 2, \ldots, r(\tau+1)$ do
  Calculate $\bar{\Theta}[Col_i(C)]$ for the elements in this set as $M_i^1, M_i^2, \ldots, M_i^{2^c}$
  for $j = 1, 2, \ldots, 2^c$ do
    if $\Xi(\delta^h_{2^{r(\tau+1)}}) = \Delta_{2^{r(\tau+1)}}$ when matrix M_i^j is transition matrix then
      return $\Sigma(1, \{i\})$
    end if
  end for
end for
for $t = 2, 3, \ldots, r(\tau+1)$ do
  choose t nodes: $i_1, i_2, \ldots, i_t$ as a set $\lambda_k = \{i_1, i_2, \ldots, i_t\}$, $k = 1, 2, 3, \ldots, C^t_{r(\tau+1)}$
  for $j = 1, 2, \ldots, C^t_{r(\tau+1)} = \frac{r(\tau+1)!}{t![r(\tau+1)-t]!}$ do
    Calculate $\bar{\Theta}[Col_{i_1 i_2 \ldots i_t}(C)]$ for the elements in this set as $M_i^1, M_i^2, \ldots, M_i^{2^{t+c}}$
    for $l = 1, 2, \ldots, 2^{t+c}$ do
      if $\Xi(\delta^h_{2^{r(\tau+1)}}) = \Delta_{2^{r(\tau+1)}}$ when matrix M_j^l is transition matrix then
        return $\Sigma(t, \{i_1, i_2, \ldots, i_t\})$
      end if
    end for
  end for
end for

$$\widehat{M}_j = \begin{bmatrix} a_1^j & \ldots & \alpha^j_{k_1} & \ldots & \alpha^j_{k_l} & \ldots & a^j_{2^{r(\tau+1)}} \\ 1 - a_1^j & \ldots & 1 - \alpha^j_{k_1} & \ldots & 1 - \alpha^j_{k_l} & \ldots & 1 - a^j_{2^{r(\tau+1)}} \end{bmatrix}.$$

From Theorem 3 and 4, the existence of minimum pinning nodes are discussed, and the necessary and sufficient conditions are obtained about exact number of pinning controllers. Next, the following Algorithm 3 is given to stabilize system (4) to the objective state δ_{2r}^{q} via minimum pinning controllers, which is based on these two theorems, and the minimum number of pinning controllers is solved by traversal.

In the following Algorithm 3, we denote the set $\Sigma(N, \Omega)$, where N stands for the minimum number and Ω stands for the set of pinning controllers. And t of i_t stands for the number of pinning controllers. Since there are finite nodes totally, if t adds to r, then the corresponding results will be returned.

IV. EXAMPLES

Example 1: Consider the following PBNs with time delays

$$\begin{cases}\rho_1(t+1) = f_1(\rho_1(t), \rho_2(t), \rho_1(t-1), \rho_2(t-1)),\\ \rho_2(t+1) = f_2(\rho_1(t), \rho_2(t), \rho_1(t-1), \rho_2(t-1)),\end{cases} \tag{21}$$

where $f_1 \in \{f_1^1, f_1^2\}$, $f_2 \in \{f_2^1, f_2^2\}$, and $P(f_1 = f_1^1) = \frac{1}{2}$, $P(f_1 = f_1^2) = \frac{1}{2}$, $P(f_2 = f_2^1) = \frac{2}{3}$, $P(f_2 = f_2^2) = \frac{1}{3}$.

These boolean functions are as follows

$$\begin{cases}f_1^1 = \rho_1(t) \vee \rho_1(t-1) \wedge \rho_2(t-1)\\ f_1^2 = \rho_2(t) \vee \rho_1(t-1) \wedge \rho_2(t-1)\\ f_2^1 = \rho_1(t) \vee \rho_1(t-1) \bar{\vee} \rho_2(t-1)\\ f_2^2 = \rho_2(t) \vee \rho_1(t-1) \bar{\vee} \rho_2(t-1),\end{cases}$$

Then, we can obtain

$$\begin{aligned} M_1^1 &= M_\wedge M_\vee (I_2 \otimes 1_2^T)\\ &= \delta_{16}[1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 2, 2, 1, 2, 2, 2],\\ M_1^2 &= M_\wedge M_\vee (1_2^T \otimes I_2)\\ &= \delta_{16}[1, 2, 1, 2, 1, 2, 2, 2, 1, 2, 1, 2, 1, 2, 2, 2],\\ M_2^1 &= M_{\bar{\vee}} M_\vee (I_2 \otimes 1_2^T)\\ &= \delta_{16}[2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 1, 2, 2, 1, 1, 2],\\ M_2^2 &= M_{\bar{\vee}} M_\vee (1_2^T \otimes I_2)\\ &= \delta_{16}[2, 1, 2, 1, 2, 1, 1, 2, 2, 1, 2, 1, 2, 1, 1, 2].\end{aligned}$$

Hence, we have $\bar{M}_1$, $\bar{M}_2$, and M, as shown at the bottom of the page.

Next, we use Algorithm 1 to design the pinning feedback controllers to steer the PBNs (23) to the objective state δ_4^3 in probability 1, and $\delta_4^3 \ltimes \delta_4^3 = \delta_{16}^{11}$.

Firstly, since $Col_{11}(M) = (\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6})^T$, we change $Col_{11}(M)$ to δ_4^3.

Secondly, calculate $\Xi(\delta_{16}^{11}) = \cup_{i=1}^{16} \Xi_i(\delta_{16}^{11})$. We find that

$$\begin{aligned}\Xi_1(\delta_{16}^{11}) &= \{\delta_{16}^2, \delta_{16}^4, \delta_{16}^6, \delta_{16}^{11}, \delta_{16}^{14}, \delta_{16}^{15}\} \text{ and}\\ \Xi_2(\delta_{16}^{11}) &= \Xi_3(\delta_{16}^{11}) = \{\delta_{16}^1, \delta_{16}^2, \delta_{16}^3, \delta_{16}^4, \delta_{16}^5, \delta_{16}^6, \delta_{16}^9, \delta_{16}^{10},\\ &\qquad \delta_{16}^{11}, \delta_{16}^{13}, \delta_{16}^{14}, \delta_{16}^{15}\}.\end{aligned}$$

Hence, it holds that

$$\Xi(\delta_{16}^{11}) = \{\delta_{16}^1, \delta_{16}^2, \delta_{16}^3, \delta_{16}^4, \delta_{16}^5, \delta_{16}^6, \delta_{16}^9, \delta_{16}^{10}, \delta_{16}^{11}, \delta_{16}^{13}, \delta_{16}^{14}, \delta_{16}^{15}\}.$$

Thirdly, change the 7th, 8th, 12th, 16th columns to δ_4^3. Then, M is changed into $\widehat{M}$ as follows

Thus, we have $\widehat{M}$, $\widehat{M}_1$, and $\widehat{M}_2$ as shown at the top of the next page.

Since $\widehat{M}_1 \neq \bar{M}_1$ and $\widehat{M}_2 \neq \bar{M}_2$, there exist F_1 and F_2 such that $\rho_1(t+1) = F_1(u_1(t), f_1)$ and $\rho_2(t+1) = F_2(u_2(t), f_2)$.

$$\begin{aligned}\bar{M}_1 &= \frac{1}{2}M_1^1 + \frac{1}{2}M_1^2\\ &= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \frac{1}{2} & 0 & 1 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{2} & 1 & 0 & 1 & \frac{1}{2} & 1 & 0 & 1 & 1 & 1\end{bmatrix},\\ \bar{M}_2 &= \frac{2}{3}M_1^1 + \frac{1}{3}M_1^2\\ &= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{3} & \frac{2}{3} & 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 1 & 1 & 0\\ 1 & 0 & 1 & 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 & 1\end{bmatrix},\\ M &= \bar{M}_1 * \bar{M}_2\\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 1 & 0 & \frac{1}{3} & 0 & 1 & 0 & \frac{1}{6} & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{6} & \frac{2}{3} & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned}$$

$$\widehat{M} = \widehat{M}_1 * \widehat{M}_2 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0 \\ 0&1&0&1&0&1&1&1&0&1&1&1&0&1&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

$$\widehat{M}_1 = \begin{bmatrix} 1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0 \\ 0&1&0&1&0&1&1&1&0&1&1&1&0&1&1&1 \end{bmatrix},$$
$$\widehat{M}_2 = \begin{bmatrix} 0&1&0&1&0&1&1&1&0&1&1&1&0&1&1&1 \\ 1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0 \end{bmatrix}.$$

Then, according to the proof of Theorem 2, $L_1 = \delta_2[2, 2, 1, 2]$ and $L_2 = \delta_2[1, 1, 1, 2]$. Hence, the logical relationship between f_j and u_j for $j = 1, 2$ are $F_1 = u_1 \wedge f_1$ and $F_2 = u_2 \vee f_2$. Furthermore, $H_1 = \delta_2[2, 2, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 2]$ and $H_2 = \delta_2[2, 2, 2, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 2, 1]$ can be obtained.

Thus, the feedback controllers can be designed as follows

$$\begin{cases} u_1 = [\rho_1(t) \wedge (\neg\rho_2(t)) \wedge (\neg\rho_1(t-1)) \wedge \rho_2(t-1)] \vee \\ [\neg\rho_1(t) \wedge \rho_2(t) \wedge (\neg\rho_1(t-1)) \wedge \rho_2(t-1)] \\ u_2 = [\rho_1(t) \wedge (\neg\rho_2(t)) \wedge \rho_2(t-1)] \vee [\neg\rho_1(t) \wedge [(\rho_2(t) \wedge \\ \rho_2(t-1)) \vee (\neg\rho_2(t) \wedge (\neg\rho_1(t-1)) \wedge (\neg\rho_2(t-1)))]]. \end{cases} \quad (22)$$

Then, we can use Algorithm 3 to calculate the minimum number of pinning controllers. It can be found that the minimum number of pinning controllers is 2 for (21).

V. CONCLUSION

In this article, for PBNs with time delays, the stabilization issue has been discussed. With the help of STP, the transition matrix of a PBN with time delays can be obtained, and the model is converted into a discrete-time linear system. Then, the necessary and sufficient conditions in the form of the algebraic expression for the pinning feedback controllers' existence and solvability are given. Moreover, the existence of minimum pinning nodes is discussed and the corresponding algorithm is designed. In the future, we will extend the results of this article to PBNs with more communication constraints, such as impulsive effects, stochastic perturbations, etc.

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