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Geometric Algebra Applications in Geospatial Artificial Intelligence and Remote Sensing Image Processing

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ABSTRACT With the increasing demand for multidimensional data processing, Geometric algebra (GA) has attracted more and more attention in the field of geographical information systems. GA unifies and generalizes real numbers and complex, quaternion, and vector algebra, and converts complicated relations and operations into intuitive algebra independent of coordinate systems. It also provides a solution for solving multidimensional information processing with a high correlation among the dimensions and avoids the loss of information. Traditional methods of computer vision and artificial intelligence (AI) provide robust results in multidimensional processing after being combined with GA and give additional feature analysis facility to remote sensing images. In this paper, we provide a detailed review of GA in different fields of AI and computer vision regarding its applications and the current developments in geospatial research. We also discuss the Clifford–Fourier transform (CFT) and quaternions (sub-algebra of GA) because of their necessity in remote sensing image processing. We focus on how GA helps AI and solves classification problems, as well as improving these methods using geometric algebra processing. Finally, we discuss the issues, challenges, and future perspectives of GA with regards to possible research directions.

INDEX TERMS Geometric algebra, Clifford algebra, geometric algebra, computer vision, artificial intelligence, quaternions.

I. INTRODUCTION

Geographic information is an important part of the information industry and has much importance all around the world. Geospatial information has both social and market value as it has a strong correlation with economic, social, and humanistic information, especially the different types of information obtained through technology and integrated using a spatial coordinate system [38]. From the perspective of urban development, the demand from governments and enterprises for spatial information is huge. Management use

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information-based softwares to deal with and analyze the problems of the entire city. The urban basic geospatial information platform will lay the foundation for the construction of urban e-government and build a spatial information platform for enterprise management, decision making, service industry informatization, and e-commerce, for the construction of digital communities and public information query for the community and the public provide spatial information support [39]. The application of spatial information will drive a number of industry chains and promote a new information industry.

Geographical information systems (GIS) are important support platforms for geographic information and assist

regional disaster planning, environmental monitoring, disaster prevention, electricity, urban planning, education, national defense, and other fields. With more big-data features in geographic information data and more wide-ranging services, the spatial data and geographic information calculations faced by traditional GIS are more complicated and diverse [40]. Existing GIS software provides services to application systems but is failing to meet the needs of fast and efficient intensive space computing. The current mainstream GIS application software programs, such as ArcGIS, MYGIS, and GeoStar, are based on server or desktop mode in the application architecture, which use single calculation, and also the computing power of GIS can no longer meet the complex, highly computationally intensive geographic computing needs. Different applications and types of algorithm are being used in the processing of different types of geospatial data, such as 2D and 3D data (see Fig 1). Therefore, a new computing architecture is needed for the intensive computations carried out in GIS.

Geospatial data processing problems are another challenge to information technology (IT) due to geospatial scientific research and its applications. In the field of geospatial science, computationally intensive problems are needed for spatial data mining, feature extraction, geospatial object simulation etc.:

- Earth is a huge and complicated dynamic system. It is composed of many interdependent subsystems, including the ecosystem, the atmospheric system, the rock system, the social and economic systems, etc. The modeling and geospatial analysis of the Earth's systems in geographic science are inherently computationally complex. The interaction between any two systems in the space-time dimension makes the research and calculation problems more complicated, including spatial data mining, object extraction, physical data simulation and multimodel simulation. Many highly computationally intensive applications are used to study the interactions and relationships between different earth subsystems in time series and spatial relationship, e.g. the global carbon cycle and climate and the Hybrid Single Particle Lagrangian Integrated Trajectory Model (HYSPPLIT).
- Remote sensing images have particular geometric features and specific geometric properties. The extraction of elements in geographic scientific research usually requires the implementation of complex geospatial algorithms with large amounts of geographical observation data to obtain the relevant geographical elements. The implementation of such complex geospatial algorithms makes the extraction of elements highly computationally intensive, e.g. objects from satellite images, the denoising of high-resolution satellite images, the storage and processing of water, and energy and land-use data. These types of data come from different sensors and time-data sources, and the traditional single-service processing model struggles to meet the associated data storage and processing demands.
- Geospatial simulation of the dynamic changes in the Earth's systems has a high degree of computational complexity. For example, simulation of the surface-water cycle includes a variety of complex system calculations, such as measuring ocean tide softness, earthquakes, dust storms, and rivers. This periodic simulation requires multiple iterations according to the performance of the data in different time series and often requires a long calculation time. Therefore, researchers often need high-performance computing methods to speed up geospatially computationally intensive problems. Calculation of the model, which obtains the calculation result within an acceptable time.

Geometric algebra (GA) provide a unified and concise homogeneous algebra framework based on advanced geometric invariants, projection geometry, affine geometry, etc., [1], [30]. It can efficiently solve the geospatial data processing problems described above due to its advanced geometrical vector-based data processing. GA has extremely important applications in many fields, including geometry, theoretical physics, and digital image analysis. It is at the forefront of international geometric algebra research and its applications are growing in every field of science, including physics, geography, electronics, and computing [2], [30]. Hildenbrand *et al.* developed GAALOPWeb for Matlab which is helpful for industrial applications of serial robotic arms [49]. In field of electronics, Lin *et al.* shows the advantage of GA in which the rotor formalism shows how complex coordinate transformations can be obtained intuitively and directly. Another example from his formulates the Maxwell's equations, in which the electric and magnetic fields can be represented as part of a 'Faraday bivector' and the four equations reduced to one [50].

In the field of computer science field, GA is an emerging powerful tool, the applied research in computer graphics, computer-aided design, computer vision, animation, robotics, and other high-tech fields has become important for future developments [3], [30]. Its feature of multidimensional-unified representation makes GA a hot research in geography. Yuan *et al.* proposed a multidimensional-unified data model based on GA and developed a prototype software system based on unified spatial-temporal analysis (CAUSTA) for investigating and modeling the distribution characteristics and dynamic processes of complex geographical phenomena [41], [43].

GA has been widely employed as a new mathematical tool for multidimensional-unified representation and computation. For example, Luo *et al.* developed a new data structure to support the unified organization and computation of geometrical primitives. This can reduce the complexity of data architecture and improve the processing ability of computer graphic software, however, the extra two dimensions in conformal geometric algebra could lead to low efficiency [44]. Yu *et al.* implemented multidimensional representation for 3D vector data and calculated the intersection relations

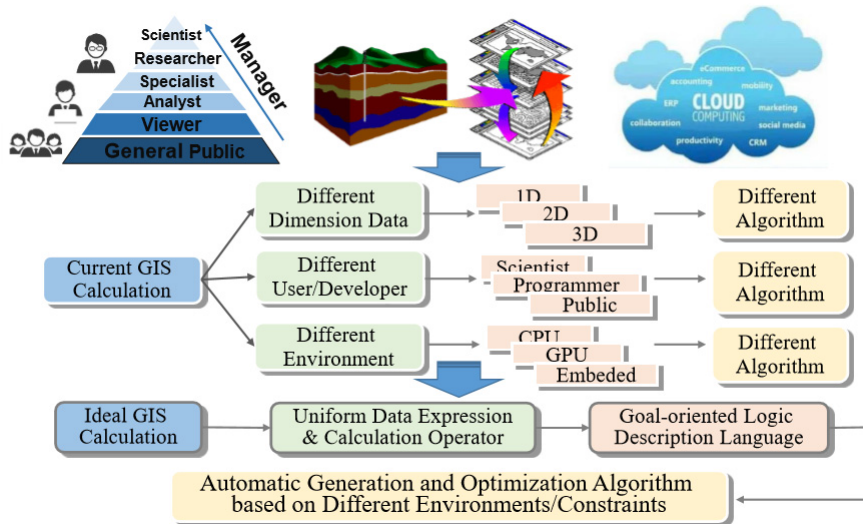


FIGURE 1. Geospatial data processing issues and needs.

between Delaunay triangulated irregular networks (DTINs) with a meet operator. They conducted parallel computation using a graphics processing unit (GPU) to improve computing efficiency [42].

Feature extraction of a given image is a key step in many computer vision and image analysis tasks, such as satellite-image denoising, remote sensing image object identification, image fusion, super-resolution reconstruction, and target recognition [4]. At present, there are many solutions, but due to mathematical limitations, these methods mostly deal with grayscale images, and the matching of color images is rarely studied. The common method used for color image matching is to convert the images into a grayscale images and then use the gray image method to match them. Converting a color image into a grayscale image leads to the vector attributes of the color and some important color information being lost, however, which can cause matching failures. GA provides a solution for this via its sub-algebra quaternions [5], [30]. Fig 2 shows the overall architecture of the features of GA that help process multidimensional data.

In this review, we provide an overview of the theory and applications of GA, mainly in the areas of geospatial artificial intelligence, satellite image and signal processing, and computer and robot vision. The enormous range of applications that have been developed during the past few decades makes a complete overview next to impossible, therefore, we restricted the review to the fields that are most focused in the last few years: image processing and artificial intelligence (AI).

The paper is organized as follows: Section II gives a brief introduction of GA, quaternions, and basic geometric operations. Starting from the expansion of vector algebra, it introduces the basic concepts and basic operations in GA before introducing the geometric points, such as points, lines, and surfaces, and the geometric relations of the intersection,

union, and duality in the geometric algebra of homogeneous space. Section III gives a brief introduction of the modern implementations of GA in image processing and AI, and the final section discusses the importance of CA, the latest development, and the future perspectives.

II. BACKGROUND

Clifford Algebra (CA) was introduced by W. K Clifford as a mathematical tool [7] that can be used for both theoretical research and practical engineering. CA is also called Geometric Algebra and since its appearance, many physicists have applied it to physics to deal with time and space problems. After over 100 years of hard work by physicists and mathematicians, GA evolved into a more mature geometric theory [8]. With the development of computer technology, some researchers have applied GA to the field of information processing, including computer vision, robotics, image processing, etc., and achieved significant results. In this section, we provide the basic mathematical operations of Clifford-based algebra with the Fourier transform and the implementation of sub-algebra quaternions.

Before explaining geometric operations, we need to introduce bivectors, trivectors, and k-blades to better understand GA. In GA, there is a geometric operator called an outer product, which is different from the product extending from one vector to another. This operator is represented by \wedge (wedge product). The outer product $a \wedge b$ of the two vectors is shown in Fig 3.

The result of the outer product operation is a two-dimensional subspace, also called a bivector or a plane quantity. The area of the plane quantity is the size of the parallelogram. This parallelogram has 'a' and 'b' as sides and a clockwise arc along the way and 'b' as the direction. The plane quantity has no shape. The reason why it is described

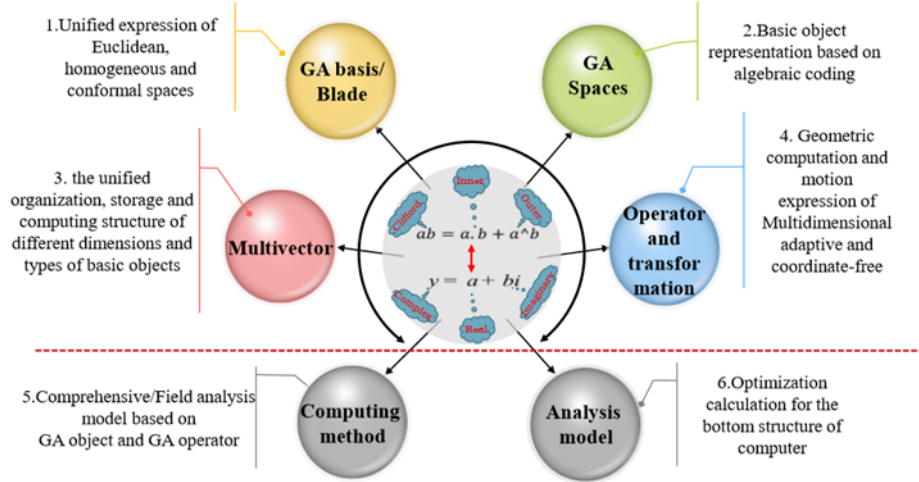


FIGURE 2. GA model for solving high dimensional data.

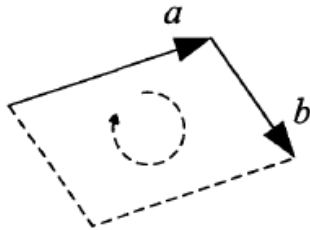


FIGURE 3. The vector a extends to b.

as a parallelogram is to make this area more visual and easy to understand. Mathematically, if the ‘b’ vector extends to the ‘a’ vector, the area of the resulting planar quantity is the same but the direction is opposite (counterclockwise). The outer product is anticommutative, that is:

$$a \wedge b = -b \wedge a \tag{1}$$

Further, get:

$$a \wedge a = 0 \tag{2}$$

Also, the outer product also has some properties:

$$(\lambda a) \wedge b = \lambda(a \wedge b) \tag{3}$$

$$\lambda (a \wedge b) = (a \wedge b)\lambda \tag{4}$$

$$a \wedge (b + c) = (a \wedge b) + (a \wedge c) \tag{5}$$

A. CLIFFORD ALGEBRA SPACE

A set of substrates is generated in all spaces. These substrates constitute the GA subspace, and the symbol is denoted as G_n . First, take the base of G_2 as an example:

$$\left\{ \begin{array}{lll} 1, & e_1 e_2, & I \\ \text{Bases calar} & \text{Base vector} & \text{Base double} \end{array} \right\}$$

Here, 0 and 1 are based scalar, and I is a representation of basis bivector. In Clifford algebra, any element of G_2 can be represented as a linear combination of these substrates.

TABLE 1. Comparison table of the total number of substrates.

G_n	Substrate	Total
G_0	{1}	$2^0 = 1$
G_1	{1; e_1 }	$2^1 = 2$
G_2	{1; e_1, e_2, e_{12} }	$2^2 = 4$
G_3	{1; $e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}$ }	$2^3 = 8$
G_4	{1; $e_1, e_2, e_3, e_4, e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}, e_{123}, e_{124}, e_{134}, e_{234}$ }	$2^4 = 16$

The total number of substrates of a geometric number is the sum of the number of all substrate k products:

$$\sum_{k=0}^n \binom{n}{k} = 2^n \tag{6}$$

Using eq. (6), the following are the geometric numbers generated in Table 1:

B. GEOMETRIC PRODUCT, MULTIPLE VECTORS, AND INNER PRODUCT

The geometric product of any two vectors a and b can be calculated as follows:

$$ab = a \bullet b + a \wedge b \tag{7}$$

We know that complex numbers consist of real and imaginary numbers. Similarly, it can be seen from the above formula that ab is composed of a number and a bivector, and product combination becomes multiple vectors.

Multiple vectors are linear combinations of different kinds of k-slice products. For example, in a two-dimensional space, it contains a bivector of numbers and vector traces:

$$\underbrace{\alpha_1}_{\text{Scalar part}} + \underbrace{\alpha_2 e_1 + \alpha_3 e_2}_{\text{vector part}} + \underbrace{\alpha_4 I}_{\text{bivector part}}$$

α_i is a real number and $i=1,2,3,4$: 4 real numbers are needed to represent complete multiple vectors in 2D space. Three-dimensional and four-dimensional space can refer to the above Table 1

Multiple vectors as a linear combination of subspaces have derived many different concepts in the geometric product. Generally speaking, geometry is an operator of multiple vectors, which has the following characteristics:

$$\begin{aligned} (AB)C &= A(BC) \\ \lambda A &= A\lambda \\ A(B+C) &= AB+AC \end{aligned} \tag{8}$$

What needs to be said is that there is no commutative law, that is, $AB \neq BA$, nor anti-commutative. Select any multiple vectors A and B in G_2 :

$$A = \alpha_1 + \alpha_2 e_1 + \alpha_3 e_2 + \alpha_4 I \tag{9}$$

$$B = \beta_1 + \beta_2 e_1 + \beta_3 e_2 + \beta_4 I \tag{10}$$

The geometric product of A makes B multiplied by:

$$AB = (\alpha_1 + \alpha_2 e_1 + \alpha_3 e_2 + \alpha_4 I)B \tag{11}$$

By the law of distribution:

$$AB = \alpha_1 B + \alpha_2 e_1 B + \alpha_3 e_2 B + \alpha_4 IB \tag{12}$$

Substitute B and arrange the order of multiplication to get:

$$\begin{aligned} AB &= \alpha_1 \beta_1 + \alpha_1 \beta_2 e_1 + \alpha_1 \beta_3 e_2 + \alpha_1 \beta_4 I \\ &+ \alpha_2 \beta_2 + \alpha_2 \beta_2 e_1 e_1 + \alpha_2 \beta_3 e_1 e_2 + \alpha_2 \beta_4 e_1 I \\ &+ \alpha_3 \beta_1 + \alpha_3 \beta_2 e_2 e_1 + \alpha_3 \beta_3 e_2 e_2 + \alpha_3 \beta_4 e_2 I + \alpha_4 \beta_1 I \\ &+ \alpha_4 \beta_2 I e_1 + \alpha_4 \beta_3 I e_2 + \alpha_4 \beta_4 II \end{aligned} \tag{13}$$

It can be seen that the linear combination of the geometric product of the substrates can represent any multivectors because the geometric product is enough. The following is the geometric product operation for different combinations:

$$e_1 e_1 = e_1 \bullet e_1 + e_1 \wedge e_1 \tag{14}$$

According to the previous description of the outer product, we can know $e_1 \wedge e_1 = 0$, and the dot product of a vector and itself is equal to the square of its size. If the size of e_1 and e_2 is 1, the result of the previous calculation can be simplified to:

$$\begin{aligned} e_1 e_1 &= e_1 \bullet e_1 + e_1 \wedge e_1 \\ &= 1 + 0 \\ &= 1 \end{aligned} \tag{15}$$

Another combination of $e_1 e_2$, because e_1 and e_2 are perpendicular, so the dot product is 0, so:

$$\begin{aligned} e_1 e_2 &= e_1 \bullet e_2 + e_1 \wedge e_2 \\ &= 0 + I \\ &= I \end{aligned} \tag{16}$$

There is also a more complicated combination, the geometric product of e_1 and I. The previous example has shown that $I = e_{12}$ is equal to $e_1 e_2$, so $e_1 I$ is calculated:

$$\begin{aligned} e_1 I &= e_1 e_{12} \\ &= e_1 (e_1 e_2) \\ &= (e_1 e_1) e_2 \\ &= (1) e_2 \\ &= e_2 \end{aligned} \tag{17}$$

Because the substrate is vertical, neither dot product nor outer product is meaningful, so this produces certain rules for simplifying the result of the geometric product as shown in Table 2:

TABLE 2. Comparison table of substrate products in G_2 .

	1	e_1	e_2	I
1	1	e_1	e_2	I
e_1	e_1	1	I	e_2
e_2	e_2	-I	1	$-e_1$
I	I	$-e_2$	e_1	1

C. COMMON GEOMETRIC OPERATIONS OF CA

In CA, to operate rotation, we can use the spin. Taking a bivector object as an example, suppose there is a bivector B so that it is around an axis a or a bivector $a^* = A$ related to a , then it is expressed as follows.

The spin is:

$$R = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} A \tag{18}$$

The spin can not only rotate vectors and bivectors but also apply to arbitrary multivectors such as trivectors and quatrivectors and so on. It is also convenient to use it to rotate a plane.

III. GA AS A REPRESENTATION STRUCTURE FOR GEOSPATIAL DATA

With the advancement of geographic data collection, geographic phenomenon observation, and geographic modeling methods, more and more attention has been paid to the expression, analysis, and modeling of geographic phenomena and processes. GIS should be able to express and analyze discrete space and continuous space and discrete processes and continuous processes uniformly and realize the reproduction and simulation of geographic objects' own characteristics, the

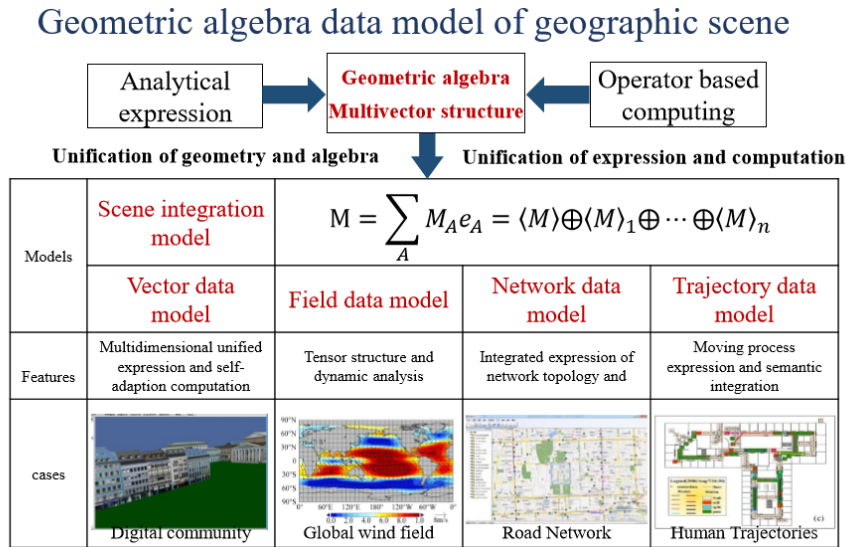


FIGURE 4. Geospatial data analysis model using GA.

relationship between geographic objects, and their spatiotemporal changes. The complexity, diversity, and continuity of real-world geographic entities and geographic phenomena have led to the complexity of GIS data models, however. It is now important to determine a way to strengthen the ability of GIS spatial data models to express and analyze complex geographic objects and continuous geographic phenomena, and develop corresponding GIS spatial methods on this basis, to better reveal the evolution process and characteristics of geographic phenomena.

The scope of GIS applications continues to expand, making the expansion from 2D GIS to 3D GIS and temporal GIS an inevitable trend of GIS development. The transformation from processing 2D objects to 3D and even high-dimensional objects not only means an increase in the amount of data, but has also led to changes in object type and spatial relationships. The existing GIS data model cannot directly expand the dimension, or at least finds it difficult to do this. When facing multidimensional spatio-temporal analysis, the model has to deal with the complexity of dimensional expansion, the inconsistency and asymmetry of different dimensional operations, the ambiguity of spatio-temporal features, and the inconsistency of semantics.

Moreover, most of the existing GIS spatial data models only focus on the expression of geometric objects, such as measurement, orientation, and topological relationship, which need to be obtained through real-time calculations and make the query, analysis, and calculation efficiency relatively low in complex scenarios. Innovating on the basis of the underlying mathematics, establishing a unified expression and calculation framework of different dimensions is a possible way of GIS data model innovation at this stage.

Yuan *et al.* [43] proposed a solution for geospatial data analysis using GA. The study indicated that GA can provide a new mathematical tool for the development of GIS

characterized as multidimension-unified expression and computation. For the development of geographical analysis methods, GA can conveniently represent multidimensional spatio-temporal changes. The effective integration of GIS spatial analysis and the geographic model is an important direction of the current development.

The lack of an underlying mathematical theoretical foundation is an existing bottleneck in spatial analysis and geographic model integration. Existing data models lack unity in terms of multidimensional object expression, storage, and maintenance of topological relationships. The separation of the expression and operation of objects of different dimensions not only increases the complexity of data models, analysis algorithms, and system architectures but also makes it difficult to support the computational needs of complex geographic temporal models.

On the other hand, due to the spatial and temporal multi-scale characteristics of geographic phenomena and processes, complex geographic phenomenon expression and model analysis often need to deal with multiple coordinate systems, e.g. Cartesian coordinates, spherical coordinates, and polar coordinates. The existing processing of objects in different coordinate systems is mostly achieved through complex coordinate system conversion, however, which increases processing complexity and uncertainty.

IV. GA AS A FEATURE EXTRACTION TOOL FOR GEOSPATIAL

Most of our reality exists in high data volume, high dimensionality, non-equilibrium, non-linear, unstructured information systems. Therefore, finding a way to discover and learn the inherent regularity in a large amount of data has become a problem in the field of machine learning research [6]. We generally think of models as vectors or points in

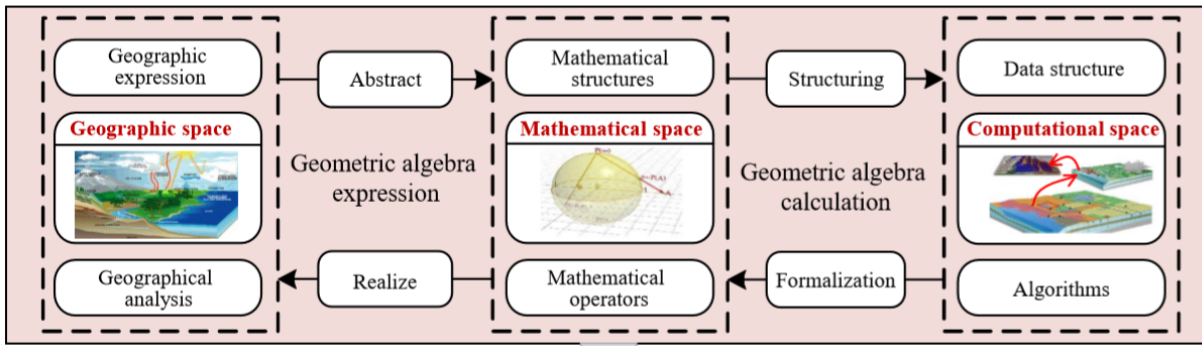


FIGURE 5. GA formalization of the geographical space into mathematical space.

finite-dimensional Euclidean space and the entire dataset as represented by a data matrix.

With the deepening development of research topics in pattern recognition, the problem of feature representation has been given more and more attention. The key to solving this problem is finding a mathematical model that can effectively represent the correlation between pattern features and higher-order structures. GA mainly uses two core concepts, geometric product and multivectors, to perform subspace representation and geometric calculations. This paper also reviews the advantages of GA for pattern classification and feature extraction of pattern recognition. AI applications in pattern recognition technology are to express a certain pattern to be classified as a computer language, which is a form acceptable to computing. The process uses computer processing and data description to encode objects in the objective world.

Every geographical scene can be classified as a geometric structure, as shown in Fig 5. From that structure, data analysis based on the features of each subset can be easily achieved. GA provides a unique way of transforming geographical scenes into vector-based data structures to process objects and extract information. Most of our reality exists in high data volume, high dimensionality, non-equilibrium, non-linear, unstructured information system. Therefore, how to discover and learn the inherent regularity from a large amount of data set information has become a difficult problem in the research process of the field of machine learning research.

During the last 15 years, new algorithms for pattern recognition, classification, and object extraction have been developed in the AI field. Tuan *et al.* proposed a face-detection algorithm that used AdaBoost with GA to identify the geometric features of human faces [45]. This small subset of features can be used to achieve accurate results and to develop a strong classifier with comparable performance and accuracy.

Principal component regression (PCR) is useful for extracting features of hyperplanes [47]. If the data is not distributed along hyperplanes (e.g. it is distributed on hyperspheres like rotation objects), however, the PCR cannot extract the good features to solve the classification problems. Combining PCR with GA can solve this problem, and a new feature extraction

method is proposed by calculating the conformal eigenvectors in conformal geometric algebra (CGA) space to find the approximated hyperplanes or hyperspheres that fit the data using the least squares approach [47]. Another approach uses entropy theory for feature extraction in 3D flow fields using GA [46]. Feature extraction is also helpful for extracting geometric invariant features with GA from spatial vector data [48]. Minh *et al.* developed a semi-supervised kernel to measure the similarity between two series of spatial vectors based on hidden Markov models and used it to find the patterns in online handwritten digits.

We generally think of models as vectors or points in finite-dimensional Euclidean space and the entire dataset as represented by a data matrix. With the deepening development of research topics in pattern recognition, the problem of feature representation has been paid more and more attention. The key to solving the problem is to find a mathematical model that can effectively represent the correlation information between pattern features and higher-order structures. GA mainly uses two core concepts: geometric product and multi-vector to perform subspace representation and geometric calculation. GA is being used in AI to solve the traditional classification problems for vectors and multiple inputs.

A. QUATERNION

Hamilton invented the Quaternion which is a non-commutative extension of complex numbers. If the set of complex numbers is considered to be a multi-cone real number space, then the complex number represents a two-dimensional space. The general quaternion of space is expressed as follows, and the symbol “H” is used to express the quaternion space.

$$H = \gamma_0 + i \bullet \gamma_1 + j \bullet \gamma_2 + k \bullet \gamma_3 \tag{19}$$

A quaternion has 4 components, γ_0 is the real part, and the remaining three are the imaginary parts, where $\gamma_1, \gamma_2, \gamma_3 \in R$ (real number set), i, j, k are the corresponding number factors, and satisfy the relationship:

$$\begin{aligned} i^2 = j^2 = k^2 = ijk = -1 \\ ij = -ji = k, jk = -kj = i, ki = -ik = j \end{aligned} \tag{20}$$

When the parameter $\gamma_0 = 0$, in this special case, the quaternion q is called pure quaternion. For the general quaternion q there are the following calculation properties.

Totally:

$$\begin{aligned} \bar{q} &= \overline{\gamma_0 + i \bullet \gamma_1 + j \bullet \gamma_2 + k \bullet \gamma_3} \\ &= \gamma_0 - i \bullet \gamma_1 - j \bullet \gamma_2 - k \bullet \gamma_3 \end{aligned} \quad (21)$$

Modulus:

$$|q| = \sqrt{\sum_{i=0}^3 \gamma_i^2} \quad (22)$$

When the modulus is 1, the quaternion q is called the unit quaternion.

Meet the multiplication distribution law:

$$q_1 q_2 q_3 = (q_1 q_2) q_3 = q_1 (q_2 q_3) \quad (23)$$

Satisfying the law of multiplication

$$q_1 (q_2 + q_3) = q_1 q_2 + q_1 q_3 \quad (24)$$

$\lambda (q_1 + q_2) = \lambda q_1 + \lambda q_2$, λ is a real number (25) It is worth pointing out here that quaternion multiplication does not satisfy the commutative law, i.e. $q_1 \bullet q_2 \neq q_2 \bullet q_1$.

For a given two quaternions $q_1, q_2 \in H$, i.e.

$$q_1 = a_0 + i \bullet a_1 + j \bullet a_2 + k \bullet a_3 \quad (25)$$

$$q_2 = b_0 + i \bullet b_1 + j \bullet b_2 + k \bullet b_3 \quad (26)$$

Addition and Subtraction can be done as follows:

$$\begin{aligned} q_1 \pm q_2 &= (a_0 \pm b_0) + (a_1 \pm b_1) \bullet i \\ &+ (a_2 \pm b_2) \bullet j + (a_3 \pm b_3) \bullet k \end{aligned} \quad (27)$$

Multiplication:

$$\begin{aligned} q_1 q_2 &= (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) \\ &+ (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) \bullet i \\ &+ (a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3) \bullet j \\ &+ (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1) \bullet k \end{aligned} \quad (28)$$

Dot product:

$$q_1 \bullet q_2 = a_0 \bullet b_0 + a_1 \bullet b_1 + a_2 \bullet b_2 + a_3 \bullet b_3 \quad (29)$$

$$q_1 \bullet q_2 = \frac{\bar{q}_1 q_2 + \bar{q}_2 q_1}{2} \quad (30)$$

The dot product of quaternions is an commutative product.

Inverse operation:

$$q_1^{-1} = \frac{\bar{q}_1}{q_1 \bullet 2} \quad (31)$$

Due to the non-commutability of quaternions, generally

$$q_1^{-1} q_2 \neq q_2 q_1^{-1} \quad (32)$$

This means that expressions like q_2/q_1 cannot be used unless q_1 is a scalar.

B. CLIFFORD FOURIER TRANSFORMATION (CFT)

Clifford Fourier transformation (CFT) provides frequency-based embedding of a watermark in any type of image.

The CFT of G_3 space can be expressed as follows:

For a vector worker $x \in \mathbb{R}^3$, $f : E^3 \rightarrow G_3$, with Clifford-Fourier Conversion:

$$F\{f\}(u) = \int_{E^3} f(x) e^{(-2\pi i_3(x,u))} |dx| \quad (33)$$

Among them, E^3 is a 3-dimensional real number domain vector space, arithmetic, $x, x', u \in E^3$, $i_3 = e_1 e_2 e_3$, e_i is the unit vector.

Clifford-Fourier inverse conversion (ICFT)

$$F^{-1}\{f\}(x) = \int_{E^3} f(u) e^{(2\pi i_3(x,u))} |du| \quad (34)$$

According to the above formula, it can be obtained that the CFT kernel $e^{(-2\pi i_3(x,u))}$ is a multi-vector value, which contains a scalar and a pseudo-scalar.

For $f : E^3 \rightarrow G_3$ there are:

$$\begin{aligned} f &= f_0 e_1 + f_1 e_1 + f_2 e_2 + f_3 e_3 + f_{23} e_{23} \\ &+ f_{31} e_{31} + f_{12} e_{12} + f_{123} e_{123} \\ &= [f_0 + f_{123} i_3] 1 + [f_1 + f_{123} i_3] e_1 \\ &+ [f_2 + f_{13} i_3] e_2 + [f_3 + f_{12} i_3] e_3 \end{aligned} \quad (35)$$

According to the linear characteristics of CFT, it can be concluded that:

$$\begin{aligned} F\{f\}(u) &= [F\{f_0(x) + f_{123}(x) i_3\}(u)] 1 \\ &+ [F\{f_1(x) + f_{23}(x) i_3\}(u)] e_1 \\ &+ [F\{f_2(x) + f_{31}(x) i_3\}(u)] e_2 \\ &+ [F\{f_3(x) + f_{12}(x) i_3\}(u)] e_3 \end{aligned} \quad (36)$$

Clifford Fourier transform allows a frequency analysis of vector fields and of the behavior of vector valued filters. In frequency space, vectors are transformed into general multivectors of the Clifford Algebra. Many basic vector valued pattern such as source, sink, saddle points and potential vortices can be described by a few multivectors in frequency space [60].

V. GA AS AN INTELLIGENT COMPUTATIONAL ACCELERATOR FOR GEOSPATIAL DATA PROCESSING

A. GA WITH A SUPPORT VECTOR MACHINE

A support vector machine (SVM) is a machine learning method based on statistical learning theory. This method seeks to minimize structured risk to improve the generalization ability of the learning machine and to minimize the experience risk and confidence range, to achieve In the case of small statistical sample size, the purpose of good statistical law can also be obtained. Generally speaking, it is a two-class classification model, and its basic model is defined as a linear classifier with the largest interval on the feature space. In other words, the learning strategy of SVMs is to maximize the interval, which can be converted into convex quadratic solving planning problems.

There is a lot of traditional method of SVM development like twin SVM [51] which used for running SVM two times but not fast. On this basis, the V-twin SVM [52], least squares twin SVM [53], smooth twins SVM [54], and least squares projection twin SVM [55] are derived, which is the progressive timely development is solving a classification problem. Another change is the fuzzy SVM (FSVM) [56], which uses a fuzzy math function to overcome the influence of noisy data on the SVM. Later on, the fuzzy least square SVM (FLS-SVM) [57] based on the least square function and FSVM, mainly to solve the unclassifiable part. To overcome the shortcomings of the mixed noise, such as singular points and Gaussian noise, a new type of FSVM, called a fuzzy robust v-SVM (FRv-SVM) [58], the combination of triangle fuzzy theory, v-SVM, and robustness can effectively punish these mixed noises.

A CA SVM (CSVM) was developed based on the real number SVM. The latter solves the multi-classification problem, mainly constructs multiple classifiers based on real number support vector machine, so there are inevitably many problems, such as calling the quadratic programming function too many times [59]. The major benefit of using the CSVM is that it takes multiple types of vectors in the input side and helps perform multi-classification. While the other types of SVMs need a large amount of memory storage to perform training multiple times and the recall phase also requires the same time.

By using a SVM, Bayro-Corrochano *et al.* [9] were able to generalize the real-valued and complex-valued SVM for multi-classification of hypercomplex SVM (CSVM), propose CSVMs based on multiple-input multiple-output (MIMO), and use CSVM for classification, regression and recurrency.

Later on, a quaternion SVM (QSVM) [10] was developed that provides a bridge between the CSVM [9] and the complex-valued SVM [11]. The QSVM uses the sign functions to evaluate the quaternion given by the classification process and decide which class the input vector corresponds to:

$$y = qsign_m \left[\sum_{j=1}^l (a_j \bullet_j) (k(x_n, X) + b) \right] \quad (37)$$

where the quaternion function is denoted by q, they belong to H ($y \in H$), m is a number of classes considered for classification, those being 2^m . The output $y \in H$ can classify up to $2^4 = 16$ classes. The QSVM is mainly used for pattern recognition and classification. Table 3 shows the detailed applications of CSVMs.

B. GA WITH NEURAL NETWORKS

During the past few decades, the stability of real-valued neural networks (NN) has matured and they have achieved many outstanding results [16]. At present, deep learning (DL) is very popular in the field of algorithms. The NNs mentioned here are not biological NNs, therefore, we call them artificial NNs (ANNs), which seems more reasonable. NNs were the

TABLE 3. Clifford based SVM application.

Application	Technique
3D rotation prediction with the help of computer vision data.	MIMO SVM with GA [12].
Prediction of 3D transformations.	DQ regression and SVR and ridge regression to approximate the error metric for the blending of unit dual quaternions [13][15].
Classification of colors and objects.	QSVM [10].
Linear and nonlinear channel equalization.	CSVM and quaternary classification [14].

earliest algorithms or models to be developed in the field of AI. More recently, NNs have developed into a multi-disciplinary and interdisciplinary field. It has also been re-emphasized and respected with the progress of DL.

ANNs combine the knowledge of a biological NN with a mathematical–statistical model and realizes it with the help of mathematical–statistical tools [17]. In the AI field of artificial perception, mathematical statistics are used to enable NN to have human-like decision-making abilities and simple judgment abilities. This method is a further extension of traditional logic calculations.

Recently, as an extension of the real-valued NN model, the Clifford NN has become a popular research area. NNs with function approximation capabilities require operations such as enhancement, rotation, and expansion, such as backpropagation (BP) NNs. Although these operations are limited by the Euclidean metric in a real-valued NN, in the Clifford-valued NN, GA has the characteristics of coordinate less frame and projective splitting, which means that the metric is feasible. Therefore, these operations can be performed effectively in Clifford NNs.

Pearson *et al.* first used the Clifford NN [18] by extending the traditional multi-layer perceptrons to allow activation, threshold, and weight values to take on complex values instead of real values. Later, conventional real-valued models of recurrent NNs were extended into the domain defined by GA [19]. The main focus was proposing models of fully connected recurrent NNs, which are extensions of the real-valued Hopfield-type NNs to the domain of CA. Due to the pioneering work of Bayro *et al.* [63], formulated the first geometric neuron and the Clifford forward neural network.

Buchholz [20] introduced basic Clifford neurons (BCN), another multi-layered Clifford NN but with a split-type activation function, which is different from the one Pearson adopted. He showed that single neurons already described geometric transformations and studied the spinor Clifford neurons (SCN), whose weights act like rotors from two sides. Shen *et al.* [21] proposed fuzzy cellular Clifford

NNs to avoid the inconvenience of the non-commutative multiplication of Clifford numbers, which decompose the considered n -dimensional Clifford-valued systems into $2m$ n -dimensional real-valued systems. Then, by using the Banach fixed-point theorem and a proof by contradiction, they established sufficient conditions for ensuring the existence, uniqueness, and global exponential stability of Sp-almost periodic solutions for the considered NNs. Takahashi *et al.* [22] proposed recurrent quaternion NNs (RQNN), which aid in designing an adaptive-type controller, and also investigated their usefulness in servo-level controller applications.

VI. GA APPLICATIONS IN IMAGE PROCESSING

A. GA AND IMAGE WATERMARKING

In the early traditional watermarking algorithm, the watermark information was embedded in the lowest position of the pixel in the spatial domain of the image. The robustness of this method is not high, however. In recent years, many researchers have proposed a method of watermark embedding in the transform domain. Due to the characteristics of the energy distribution in the transform domain, the watermark embedding algorithm can guarantee the invisibility of the watermark. This algorithm first transforms the image (using a discrete cosine transform, Fourier–Mellin transform, wavelet transform, etc.) and then embeds the watermark information into the changed coefficients for the transformed situation. As this method improves the robustness of the watermark embedding algorithm, it has proven popular. There are also many other watermark embedding algorithms, such as fractal-based and NN-based.

Affes *et al.* [23] designed a CFT content-based watermark, which employs the Harris detector to find the area of interest, then uses the CFT to embed it in the frequency of the image. In our other work [1], we used the quaternion Fourier transform (QFT) to watermark color images as RGB components are perfectly handled in quaternion-based CA. Hsu *et al.* [24] used quaternion the discrete Fourier transform (QDFT) via the modulation technique, while Hosney *et al.* [24] used quaternion Legendre–Fourier moments (QLFMs) to embed a watermark in polar coordinates. In the latter's new work, they used the polar complex exponential transform (PCET) and quaternion PCET (QPCET) [26] to embed watermarks in medical images.

The quaternion polar harmonic transform (QPHT) [27] is useful for lossless watermarking after embedding it with the chaotic encryption to improve the security. The Clifford analytical Fourier–Mellin transform (CAFMT) [28] improves upon the analytical Fourier–Mellin transform (AFMT) and makes it applicable to color images due to its invariant property against planar similarities. In our recent work, we proposed QFT-based watermarking using advanced encryption of watermarks with R, G, and B colors being handled separately [1]. Fig 6 shows our proposed approach for embedding watermark after QFT and then restoring image by IQFT with better results of embedding.

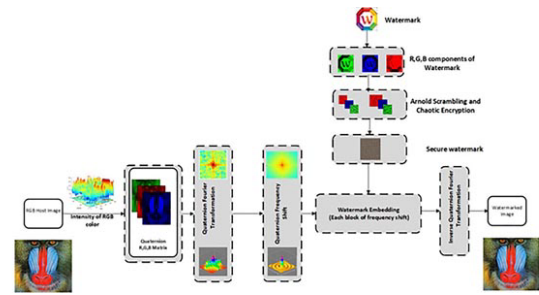


FIGURE 6. Watermarking with QFT [1].

B. GA AND IMAGE EDGE DETECTION

Extracting information from numerous and diverse image data is the key to this data's usability. The edge reflects the main structure, main outline, and skeleton of the image, it is a relatively direct feeling for people to interpret the image, and is also the basis for image segmentation and target recognition. The Wavelet and Clifford wavelet [30], [64] has the advantages of multidimensional unified expression and time-frequency localized analysis and can analyze images according to scale, phase, and direction. Pixel-based data mining the multi-dimensional characteristics of image information in the dimension space of Clifford wavelet to realize the unified expression and processing of the three components of the color image. Extracting the phase information in different directions, realize the fusion of wavelet filtering methods in multiple directions.

Wenming *et al.* [61] use Clifford differential and the Clifford gradient multispectral image for edge detection and fusion algorithm. The algorithm first calculates the Clifford gradient of each pixel, and then obtains the Clifford gradient norm; then, based on this, judge whether the pixel is a boundary point, and then obtain multiple images. Edge detection image; finally, these images are fused to obtain the final edge image. Another similar approach is tested by Xu *et al.* [62] for multispectral images in which comparison of performance is done with maximal entropy edge detection algorithm reveals that the edge detection based on Clifford gradient is better at retaining and identifying edge information of the multispectral image than the maximal entropy edge detection algorithm.

Saqwine *et al.* [29] provided edge detection using the QFT. Quaternion fractional differential (QFD) [31] and quaternion fractional directional differentiation [32] helpful in edge detection by the generalization of general fractional differentiation and quaternion. Yasmin *et al.* [33] developed a multi-directional edge detector for color images by using quaternions of GA with a filter mask in four dimensions ($\pi/2$, $5\pi/2$, $3\pi/2$, and $7\pi/2$), as shown in Fig 7. For color images, a quaternion Hardy filter (QHF) and Di Zenzo gradient operator [34] via the QDFT in the frequency domain improves performance. For remote sense images, edge detection is able to provide valuable benefits by using the QFT and a masking filter.

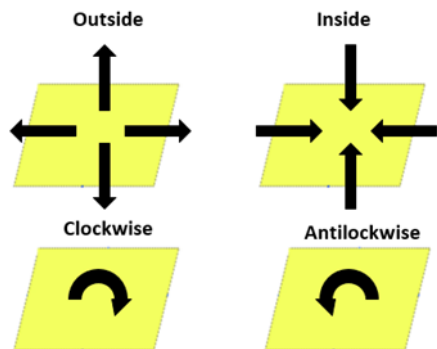


FIGURE 7. Filter mask implementation at different angles.



FIGURE 8. Edge detection using quaternion Hardy filter [34].

Wenshan *et al.* [34] tested the results of edge detection with different noise addition and results were satisfactory against all types of noise on image as shown in Fig 8.

C. GA AND IMAGE SEGMENTATION

Image segmentation is the most basic and most important step of image processing. Through the development of image segmentation technology in the field of image processing, a detailed and accurate basis for subsequent image processing has been provided. By separating the target features in the image, the identification of the target in the image is improved, to make them more efficient, accurate, and targeted analysis of the processed image and make more comprehensive use of the image.

Image segmentation is one of the most important and challenging problems for low-level vision in the field of computer vision. In image processing, segmentation as the basic step, the results directly affect the final analysis results, but also the image conveys the information that has a very important impact. It is also important in image engineering as it is a critical step that needs to be taken before the images are analyzed. Image segmentation also plays an important role in target detection, face recognition, computer vision, AI, and other fields [30]. Fig 9 shows the way color can be segmented by CA.

Khan *et al.* used GA with a masking filter to perform image segmentation and used that technique for agriculture image segmentation as one of application [35]. A quaternion operator with the Fourier transform was implemented to perform segmentation with a masking filter to filter the colors in an RGB image. Another Clifford-based application of image segmentation is blood vessel segmentation in human eyes for the diagnosis of diabetes and hypertension [36]. Using

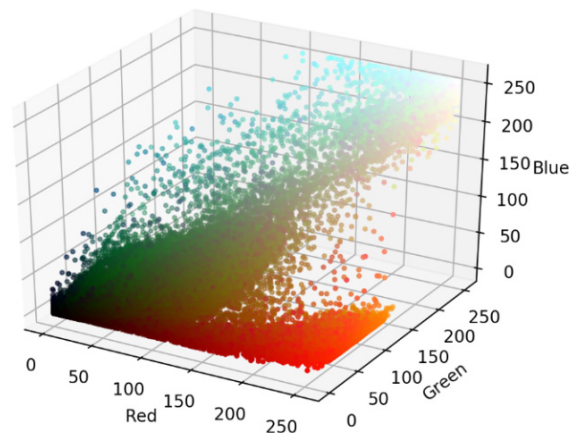


FIGURE 9. Color segmentation by GA in the 3D cube of RGB.

image vectorization, the image is converted from image space to Clifford space. The next step then introduces a Clifford-matched filter for the extraction of a retinal blood vessel via a masking function. The third and final step of this method is a Clifford convolution operation. A 3D biquaternion Clifford analytic signal (CAS) [37] is proposed, which helps in breast lesion segmentation by creating a 3D Clifford temporal image (CTI).

VII. CONCLUSION

The main limitations of the traditional approach are that its primitives are too low-level and all geometric concepts have to be represented by vectors and matrices. This creates a separation between geometric reasoning and matrix-based algorithms, which in turn leads to implementation errors. GA is broad and profound and has links to many fields in mathematics and physics. It is widely used in general relativity, quantum mechanics, quantum field theory, projective geometry, differential geometry, conformal geometry, etc. GA recently became an important tool in the field of computing due to its vast applications in robotics, computer vision, and machine learning.

In this paper, we have discussed the advantages of GA in multidimensional signal processing, geometric quantities, geometric relationship modeling, AI, and geometric calculations with regards to its application in computer vision. We began with an introduction of GA in different fields of science to emphasize its importance in computer science. Then, we discussed the basic geometric operations of GA and showed how it is different from traditional algebra. We also discussed its applications in AI and the latest innovations using GA and quaternions, as well as its application in computer vision.

GA is important in medical science and computer vision due to its application in handling color by using vector and its ability to solve AI classification problems. The main work of GA is as follows:

- Geometry is the foundation of computer vision and the basis for modeling and calculating various problems.

This paper discussed the basic geometric operations of GA and provided an overview of how it is different from traditional algebra and geometric operations. We analyzed the GA representation of geometric points, such as points, lines, and surfaces in projective geometry, as well as geometric algebraic representations, such as intersection, union, and the duality of geometric bodies.

- Image segmentation, edge detection, and registration of image are key steps in many image analyses and computer vision tasks. There are currently many solutions, but due to mathematical limitations, most of these methods process grayscale images only. This article first analyzed the definition of Clifford geometrical operations and quaternions, and then the properties of spatial translation, rotation, scaling, etc. Advance implementation of the CFT was discussed regarding its real-life applications.

The implementation and advantages of SVMs and AI were discussed regarding the real-time difference between the non-geometric SVM and NNs and the Clifford SVM and NNs. Future studies for application of GA will be on advance machine learning algorithm such as capsule neural networks. Clifford-based AI methods are useful for solving big-data problems quickly and more efficiently.

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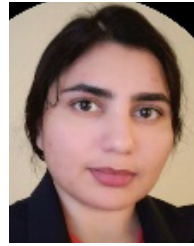
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