Consistency Analysis and Priorities Deriving for Pythagorean Fuzzy Preference Relation in the "Computing in Memory"

LI ZHANG[ID][1], LIGANG ZHOU[ID][2], AND KAIFANG YANG[1]
[1]School of Electronics and Information Engineering, Anhui University, Hefei 230601, China
[2]School of Mathematical Sciences, Anhui University, Hefei 230601, China

Corresponding author: Li Zhang (04068@ahu.edu.cn)

ABSTRACT Pythagorean fuzzy set, characterized by membership function and non-membership function, has received increasing attention in recent years. In this paper, a new approach to decision making is proposed based on Pythagorean fuzzy preference relation and its additive consistency. Firstly, the concepts of Pythagorean fuzzy preference relations and its additive consistency are introduced, and followed by a discussion of their desirable properties. Then, a linear goal programming model is proposed to determine the consistency of PFPRs. For the PFPRs that does not satisfy the consistency, the consistency index is defined to measure the degree of consistency, and a consistency adjustment algorithm is proposed. Finally, based on the additive consistency, a new algorithm for decision making is presented. An example of CIM(Computing In Memory) is provided, and in comparison with other methods, the validity and rationality of the proposed method are verified.

INDEX TERMS Pythagorean fuzzy preference relations, additive consistency, consistency adjustment, decision making, computing in memory.

I. INTRODUCTION

In multiple criteria decision making (MCDM) process, a decision maker (DM) need to make a pairwise comparison of alternatives or criteria, so as to propose his/her preference in a set of *n* alternatives or criteria, and establish a preference relation to reflect the DM's judgment. Analytic hierarchy process (AHP) [1] is one of the most commonly used and most powerful methods for solving MCDM. It chooses the optimal solution from multiple alternatives according to the preferences provided by decision makers. The AHP provides a convenient framework for the derivation of multiplicative preference relations based on pairwise comparisons. In recent years, with the introduction of fuzzy logic and fuzzy methods into AHP, fuzzy preference relations have received more and more attention [2]–[4].

Due to the complexity of objective things and the uncertainty of actual problems, it is often difficult for DM to express his/her judgments with precise numerical values. In order to express this ambiguity and uncertainty, different uncertain preference relations are proposed. such as interval

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fuzzy preference relations [5]–[7], interval multiplicative preference relations [8], reciprocal fuzzy preference relations [9], and triangular fuzzy preference relations [10], trapezoidal fuzzy preference relations [54]–[57], hesitant fuzzy linguistic preference relation [61]. Saaty [8] introduce interval multiplicative preference relations and propose a Monte Carlo simulation method to generate priority weights from the interval multiplicative preference relations. Many methods have been proposed to derive priority weights from interval multiplicative preference relations, such as goal programming models [11], [12] and convex combination method [13]. Xu and Chen [14] give additive and multiplicative transitivity conditions for interval fuzzy preference relations based on normalized crisp weights and present some linear programming models for deriving priority weights. hou [60] propose an optimal group continuous logarithm compatibility measure for interval multiplicative preference relations based on the COWGA operator. By using interval arithmetic, Wang and Li [15] introduce new definitions of additive consistent, multiplicative consistent and weakly transitive interval fuzzy preference relations. Xu [16] made a survey on different kinds of preference relations and discussed their properties.

As an extension of the fuzzy set [17], Atanassov [18], [19] introduced the concept of intuitionistic fuzzy sets (IFSs), the sum of its membership degree and non-membership degree is less than or equal to 1. Up to now, IFSs have been widely applied in real-life MCDM problems, the studies of methods of MCDM problems with IFSs have received extensive attentions [20]–[24], and the applications of MCDM problems based on IFSs have attracted widespread attention of researchers. Szmidt [25] generalize fuzzy preference relations to intuitionistic fuzzy preference relations and discuss how to reach consensus with intuitionistic fuzzy preference relations in group decision making. Xu and Yager [26] introduce a new similarity measure between IFSs and apply it to consensus analysis in group decision making with intuitionistic fuzzy preference relations. Xu [27] defines multiplicative consistent intuitionistic fuzzy preference relations based on intuitionistic fuzzy number operations, and develops a new group decision making method by using intuitionistic fuzzy aggregation operators. Xu *et al.* [28] by directly employing the membership and non-membership degrees in intuitionistic fuzzy judgments, they propose a new definition of multiplicative consistent intuitionistic fuzzy preference relations and develop two algorithms to estimate missing values for incomplete intuitionistic preference relations. Xu [29] presents an error-analysis-based approach to determine priority interval weights for both consistent and inconsistent intuitionistic fuzzy preference relations. Gong *et al.* [30] give another multiplicative consistency definition for intuitionistic fuzzy preference relations based on the corresponding membership degree interval fuzzy relations with multiplicative consistency and propose goal programming approaches to obtain priority weights. Gong *et al.* [31] further define additive consistent intuitionistic fuzzy preference relations and establish some optimization models to obtain intuitionistic fuzzy weights from intuitionistic fuzzy preference relations.

In recent years, Pythagorean fuzzy sets (PFSs) proposed by Yager [32], [33] is an useful extension of the concept of Atanassov's intuitionistic fuzzy sets (IFSs) [18]. PFSs have more powerful abilities than IFSs do in modeling the uncertainty of practical decision making problems, because it satisfies the condition that the square sum of its membership degree and non-membership degree is equal to or less than 1. Yager [33] gave an example to illustrate this situation: the membership degree and the non-membership degree of one alternative in a criterion are $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$, it is easily seen that $\frac{\sqrt{3}}{2} + \frac{1}{2} \geq 1$, thus this situation cannot be described by using the IFSs, but $(\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2 \leq 1$ holds. Obviously, PFSs have more capability than IFSs do in modeling the vagueness of practical multiple attribute decision making (MADM) problems. Yager [32], [33] proposed a series of aggregation operators: Pythagorean fuzzy weighted average (PFWA) operator, Pythagorean fuzzy weighted geometric average (PFWG) operator, Pythagorean fuzzy weighted power average (PFWPA) operator, Pythagorean fuzzy weighted power geometric (PFWPG) operator, and

applied them to MADM problems. Peng and Yang [34] discussed their relationships, and also proposed a method called superiority and inferiority ranking (SIR) multiple attribute group decision making (MAGDM). At the same time, inspired by soft set theory [35] and linguistic set theory [36], they proposed Pythagorean fuzzy soft sets [37] and Pythagorean fuzzy linguistic sets [38], respectively. Yager and Abbasov [32] studied the relationship between the Pythagorean fuzzy numbers (PFNs) and the complex numbers and concluded that Pythagorean degrees are a subclass of complex numbers. Wang and Li [39] presented Pythagorean fuzzy interaction power Bonferroni mean aggregation operators in multiple attribute decision making. Xu [40] defined the algorithms to detect and rectify multiplicative and ordinal inconsistencies of fuzzy preference relations. Zhang and Xu [42] presented a technique for finding the best alternative based on its ideal solution under the Pythagorean fuzzy environment. Xu [41] propose the algorithms to identify and rectify ordinal inconsistencies for incomplete fuzzy linguistic preference relations. Chen [43] defined an extended ELECTRE approach in Pythagorean fuzzy sets. Later on, Garg [44] presented a novel accuracy function for interval-valued PFSs and apply it to solve the decision-making problem. A correlation coefficient between the two PFSs has been proposed by Garg [45] by showing the advantages as compared to the existing correlation coefficients under IFSs environment.

This paper focuses on PFPRs and its additive consistency, and a linear goal programming model is developed to determine whether the PFPR has additive consistency. For PFPRs that do not meet the consistency, a consistency index is defined to measure its consistency degree, and an algorithm for consistency adjustment is presented to adjust the PFPR until its consistency reaches an acceptable range. The remainder of this paper is organized as follows. In Section 2, we will briefly review some basic concepts and operations, including IFSs, IFPRs, PFSs, and so on. In section 3, a linear goal programming model is defined to determine the additive consistency of PFPRs, for the PFPRs that does not meet the consistency, a consistency adjustment algorithm is proposed. In section 4, we develop an approach to decision making based on Pythagorean fuzzy preference relation (PFPR) and its additive consistency. And in Section 5, we will provide two practical examples to illustrate the developed approaches respectively. Section 6 ends this paper with some concluding remarks.

II. PRELIMINARIES
A. FPRS AND IFPRS
In this section, we briefly review some basic concepts, including fuzzy preference relation (FPR) and its additive consistency, intuitionistic fuzzy preference relation (IFPR) and its additive consistency and Pythagorean fuzzy set (PFS).

For a decision-making problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives, where $x_i(i = 1, 2, \ldots, n)$ denote the ith alternatives. In the decision-making process,

a DM need to provide his/her preferences for each pair of alternatives, and then construct a preference relation matrix, which can be defined as follows.

Definition 2.1 [5]: A preference relation P on set X is characterized by a function $\mu_P : X \times X \to D$, where D is the domain of representation of preference degrees. These preference relations can be mainly classed into three categories: multiplicative preference relations, fuzzy preference relations, linguistic preference relations.

Definition 2.2 [46]: A fuzzy preference relation R on set X is represented by a complementary matrix $R = (r_{ij})_{n \times n} \subset X \times X$ with

$$r_{ij} \geq 0, r_{ij} + r_{ji} = 1, \quad r_{ii} = 0.5$$
$$\text{for all } i, j = 1, 2, \ldots, n. \quad (1)$$

where r_{ij} denotes the degree that the alternatives x_i is preferred to x_j. In particular, $r_{ij} = 0.5$ indicates that there is no difference between alternative x_i and x_j; $r_{ij} > 0.5$ indicates that the alternative x_i is preferred to x_j, especially, $r_{ij} = 1$ means that the alternative x_i is absolutely preferred to x_j; and $r_{ij} < 0.5$ indicates that the alternative x_j is preferred to x_i, especially, $r_{ij} = 0$ means that the alternative x_j is absolutely preferred to x_i.

Definition 2.3 [47]: A fuzzy preference relation $R = (r_{ij})_{n \times n}$ is called an additive consistency FPR, if it satisfies the following condition:

$$r_{ik} = r_{ij} + r_{jk} - 0.5,$$
$$\text{for all } i, j, k = 1, 2, \cdots, n. \quad (2)$$

Since $r_{ij} = 1 - r_{ji}$ for all $i, j, k = 1, 2, \cdots, n$, it follows from (2) that $1 - r_{ji} = (1 - r_{ki}) + (1 - r_{kj}) + 0.5$. Then, we have $r_{ki} = r_{kj} + r_{ji} - 0.5$, add it to (2), we can get

$$r_{ij} + r_{jk} + r_{ki} = r_{kj} + r_{ji} + r_{ik},$$
$$\text{for all } i, j, k = 1, 2, \cdots, n. \quad (3)$$

For a fuzzy preference relation $R = (r_{ij})_{n \times n}$, if there exists a normalized crisp weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that

$$r_{ij} = 0.5(\omega_i - \omega_j) + 0.5,$$
$$\text{for all } i, j, k = 1, 2, \cdots, n. \quad (4)$$

where $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$ for $i = 1, 2, \cdots, n$, then R is additive consistent [22], [23], [48].

Due to the complexity, ambiguity and uncertainty of decision-making problems, it is difficult to be convinced to express the DM's preference information with exact numbers. For this case, Atanassov [7] proposed intuitionistic fuzzy sets (IFSs) composed of membership function and non-membership function.

Definition 2.4. [18], [19]: Let a set X be fixed. An IFS A in X is shown as follows:

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}.$$

Which is characterized by a membership function $\mu_A : A \to [0, 1]$ and a non-membership function $\nu_A : A \to [0, 1]$ with the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \text{for all } x \in X.$$

where $\mu_A(x)$ and $\nu_A(x)$ represent the membership degree and the non-membership degree of the element $x \in X$ to the set A, respectively. For each IFS A on X, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the indeterminacy degree of the membership of the element $x \in X$ to the set A.

According to the concept of IFS, the concept of IFPR is defined as follows.

Definition 2.5 [27]: An intuitionistic fuzzy preference relation (IFPR) B on set X is represented by a matrix $B = (b_{ij})_{n \times n} \subset X \times X$ with $b_{ij} =< (x_i, x_j), \mu(x_i, x_j), \nu(x_i, x_j) >$, for all $i, j = 1, 2, \ldots, n$. For convenience, for all $i, j = 1, 2, \ldots, n$, let $b_{ij} = (\mu_{ij}, \nu_{ij})$, where b_{ij} is an intuitionistic fuzzy number composed by the certainty degree μ_{ij} to which x_i is preferred to x_j and the certain degree ν_{ij} to which x_i is non-preferred to x_j, and $\pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$ is interpreted as the uncertainty degree to which x_i is preferred to x_j.

Furthermore, for all $i, j = 1, 2, \ldots, n$, μ_{ij} and ν_{ij} satisfy the following characteristics:

$$\mu_{ij}, \nu_{ij} \in [0, 1], \quad 0 \leq \mu_{ij} + \nu_{ij} \leq 1,$$
$$\mu_{ji} = \nu_{ij}, \mu_{ii} = \nu_{ii} = 0.5. \quad (5)$$

Consistency is a very important issue for all kinds of preference relations, and the lack of consistency in a preference relation may result in unreasonable conclusions. For IFPR, several different consistency have been proposed, of which there are two main types: the additive consistency and the multiplicative consistency. Xu [49], Gong *et al.* [31], and Wang [48] proposed some different definitions of additive consistent IFPR, respectively.

Definition 2.6 [50]: Let $B = (b_{ij})_{n \times n}$ be an IFPR with $b_{ij} = (\mu_{ij}, \nu_{ij})$, $(i, j = 1, 2, \cdots, n)$, if there exists a vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that

$$\mu_{ij} \leq 0.5(\omega_i - \omega_j + 1) \leq 1 - \nu_{ij}, \text{ for all}$$
$$i, j = 1, 2, \ldots, n. \quad (6)$$

where $\omega_i \in [0, 1](i = 1, 2, \cdots, n)$ that $\sum_{i=1}^{n} \omega_i = 1$. Thus, B is called an additive consistent IFPR.

Definition 2.7 [48]: An intuitionistic fuzzy preference relation $B = (b_{ij})_{n \times n}$ with $b_{ij} = (\mu_{ij}, \nu_{ij})$ is called additive consistent, if it satisfies the following transitivity:

$$\mu_{ij} + \mu_{jk} + \mu_{ki} = \mu_{kj} + \mu_{ji} + \mu_{ik},$$
$$\text{for all } i, j, k = 1, 2, \cdots, n. \quad (7)$$

where $\mu_{ij} = \nu_{ji}$ for all $i, j = 1, 2, \ldots, n$, it follows from (2.7) that

$$\nu_{ij} + \nu_{jk} + \nu_{ki} = \nu_{kj} + \nu_{ji} + \nu_{ik},$$
$$\text{for all } i, j, k = 1, 2, \cdots, n. \quad (8)$$

B. PYTHAGOREAN FUZZY SETS(PFSS) AND ITS BASIC OPERATIONS

Definition 2.8. [51]: Let S be a fixed set, A Pythagorean fuzzy set (PFS) P on S can be represented as the following mathematical symbol:

$$P = \{< s, P(\mu_p(s), v_p(s)) > | s \in S \}.$$

where $\mu_p(s) : S \rightarrow [0, 1]$ and $v_p(s) : S \rightarrow [0, 1]$ are the membership degree and non-membership degree of s to P, respectively. For each $s \in S$, it satisfies the condition: $0 \leq (\mu_p(s))^2 + (v_p(s))^2 \leq 1$. The degree of indeterminacy of s to P is $\pi_P(s) = \sqrt{1 - (\mu_p(s))^2 - (v_p(s))^2}$. For simplicity, Zhang and Xu [31] called $P(\mu_p(s), v_p(s))$ as a Pythagorean fuzzy number (PFN), denoted by $\beta = P(\mu_\beta, v_\beta)$, where $\mu_\beta, v_\beta \in [0, 1]$, $\pi_\beta = \sqrt{1 - (\mu_\beta)^2 - (v_\beta)^2}$ and $(\mu_\beta)^2 + (v_\beta)^2 \leq 1$.

Definition 2.9. [42]: For any PFN $p_i = (\mu_i, v_i)(i = 1, 2, \ldots, n)$, the score function of p_i is defined as follows:

$$s(p_i) = (\mu_i)^2 - (v_i)^2. \quad (9)$$

where $s(p_i) \in [-1, 1]$.

Definition 2.10. [34]: For any PFN $p_i = (\mu_i, v_i)$ $(i = 1, 2, \ldots, n)$, the accuracy function of p_i is defined as follows:

$$a(p_i) = (\mu_i)^2 + (v_i)^2. \quad (10)$$

where $a(p_i) \in [0, 1]$.

By use the score function and accuracy function, we can compare the size of two different PFNs, and for any two PFNs p_1, p_2,

(1) if $s(p_1) > s(p_2)$, then $p_1 \succ p_2$;
(2) if $s(p_1) < s(p_2)$, then $p_1 \prec p_2$;
(3) if $s(p_1) = s(p_2)$, then,

    1) if $a(p_1) > a(p_2)$, then $p_1 \succ p_2$;
    2) if $a(p_1) < a(p_2)$, then $p_1 \prec p_2$;
    3) if $a(p_1) = a(p_2)$, then $p_1 \sim p_2$.

Definition 2.11. [52]: Let $P_j = (\mu_j, v_j)(j = 1, 2)$ be two PFNs, then the distance between P_1 and P_2 can be defined as follows:

$$d(P_1, P_2) = \frac{1}{2}(|\mu_1^2 - \mu_2^2| + |v_1^2 - v_2^2| + |\pi_1^2 - \pi_2^2|). \quad (11)$$

Theorem 2.1: The distance $d(P_1, P_2)$ between two PFNs P_1 and P_2 satisfies the following properties:

1. $d(P_1, P_2) \geq 0$;
2. $d(P_1, P_2) = 0$ if and only if $P_1 = P_2$;
3. $d(P_1, P_2) = d(P_2, P_1)$.

Definition 2.12. [53]: Let $p_i = (\mu_{p_i}, v_{p_i})$ $(i = 1, 2, \cdots, n)$ be a collection of PFNs and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weight vector of $p_i(i = 1, 2, \cdots, n)$, with $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$, $\lambda > 0$, then a generalized Pythagorean fuzzy weighted average (GPFWA) operator is a mapping PFWA: $p^n \rightarrow p$, where

$$GPFWA\,(p_1, p_2, \cdots, p_n)$$
$$= \left((\sum_{i=1}^{n} \omega_i \mu_{pi}^\lambda)^{1/\lambda}, (\sum_{i=1}^{n} \omega_i v_{pi}^\lambda)^{1/\lambda} \right). \quad (12)$$

Especially, if $\lambda = 1$, the GPFWA operator reduces to the Pythagorean fuzzy weighted averaging (PFWA) operator [53]; if $\lambda = 2$ the GPFWA operator reduces to the Pythagorean fuzzy power weighted averaging (PFPWA) operator [53]; And if $\omega_i = \frac{1}{n}$, the Pythagorean fuzzy weighted average (PFWA) operator becomes Pythagorean fuzzy arithmetic average (PFAA) operator, where

$$PFAA(p_1, p_2, \cdots, p_n) = (\frac{1}{n} \sum_{i=1}^{n} \mu_i, \frac{1}{n} \sum_{i=1}^{n} v_i) \quad (13)$$

Theorem 2.2 (Idempotency): Let $p_i = (\mu_{p_i}, v_{p_i})$ $(i = 1, 2, \cdots, n)$ be a collection of PFNs, and $\lambda > 0$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weight vector of them, $\sum_{i=1}^{n} \omega_i = 1$. If all $p_i(i = 1, 2, \cdots, n)$ are equal, i.e., $\forall i$, $p_i = p$, then

$$GPFWA\,(p_1, p_2, \cdots, p_n) = p.$$

Theorem 2.3 (Boundedness): Let $p_i = (\mu_{p_i}, v_{p_i})$ $(i = 1, 2, \cdots, n)$ be a collection of PFNs, and $\lambda > 0$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weight vector of them, $\sum_{i=1}^{n} \omega_i = 1$. Assume that $p^- = (\min_i \mu_{p_i}, \max_i v_{p_i})$, $p^+ = (\max_i \mu_{p_i}, \min_i v_{p_i})$, then

$$p^- \leq GPFWA\,(p_1, p_2, \cdots, p_n) \leq p^+.$$

Theorem 2.4 (Monotonicity): Let $p_i = (\mu_{p_i}, v_{p_i})$ $(i = 1, 2, \cdots, n)$ and $p_i' = (\mu_{p_i'}, v_{p_i'})$ be two collections of PFNs, and $\lambda > 0$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weight vector of them, $\sum_{i=1}^{n} \omega_i = 1$. If $\forall i$, $\mu_{p_i} \leq \mu_{p_i'}$, $v_{p_i} \leq v_{p_i'}$, then

$$GPFWA\,(p_1, p_2, \cdots, p_n) \leq GPFWA\,(p_1', p_2', \cdots, p_n').$$

III. ADDITIVE CONSISTENCY IMPROVING APPROACH

In this section, PFPR and its additive consistency are defined. Based on the additive consistency of PFPR, a model for determining Pythagorean weights is presented. Based on Pythagorean fuzzy weight information, a PFPR with additive consistency is constructed. In addition, this section also proposes a consistency index to measure the Pythagorean fuzzy preference relation matrix, in order to judge whether the PFPRs has acceptable consistency. For the PFPRs with unacceptable consistency, the algorithm of consistency adjustment is proposed.

A. ADDITIVE CONSISTENCY OF PYTHAGOREAN FUZZY PREFERENCE RELATIONS

Definition 3.1: A Pythagorean fuzzy preference relation (PFPR) P on the set X is represented by a matrix $P = (p_{ij})_{n \times n} \subset X \times X$ with $p_{ij} = < (x_i, x_j), \mu(x_i, x_j), v(x_i, x_j) >$ for all $i, j = 1, 2, \cdots, n$. For convenience, we let $p_{ij} = (\mu_{ij}, v_{ij})$, for all $i, j = 1, 2, \cdots, n$, where p_{ij} is a Pythagorean fuzzy value composed by the certainty degree μ_{ij} to which x_i is preferred to x_j and the certainty degree v_{ij} to which x_i is non-preferred to x_j. Furthermore, μ_{ij} and v_{ij} satisfy the following conditions:

$$0 \leq \mu_{ij}^2 + v_{ij}^2 \leq 1, \mu_{ij} = v_{ji},$$
$$v_{ij} = \mu_{ji}, \mu_{ii} = v_{ii} = 0.5 \quad \text{for all } i, j = 1, 2, \cdots, n.$$

Definition 3.2: A PFPR $P = (p_{ij})_{n \times n}$ with $p_{ij} = (\mu_{ij}, v_{ij})$ is called an order consistent PFPR, if it satisfies $p_{is} \geq p_{it}$ for all $i \in \{1, 2, \cdots, n\}$, where $s \in \{1, 2, \cdots, n\}$ and $t \in \{1, 2, \cdots, n\}$.

Example 3.1: Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of alternatives, suppose that there is a PFPR on X, which is shown as follows:

$$P = \left\{ \begin{array}{cccc} (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) & (0.4, 0.3) & (0.5, 0.3) & (0.7, 0.4) \\ (0.3, 0.4) & (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) & (0.6, 0.5) & (0.8, 0.6) \\ (0.3, 0.5) & (0.5, 0.6) & (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) & (0.5, 0.4) \\ (0.4, 0.7) & (0.6, 0.8) & (0.4, 0.5) & (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \end{array} \right\}.$$

According to the PFPR P, it is obvious that $p_{is} \geq p_{it}$ for all $i \in \{1, 2, \cdots, n\}$, where $s \in \{1, 2, \cdots, n\}$ and $t \in \{1, 2, \cdots, n\}$. Since $p_{i2} \geq p_{i1}, p_{i3} \geq p_{i2}$ and $p_{i4} \geq p_{i3}$ for all $i \in \{1, 2, \cdots, n\}$, which denote the alternative x_2 is better than x_1, the alternative x_3 is better than x_2, the alternative x_4 is better than x_3, then PFPR P indicating that the ranking is that $x_4 \succ x_3 \succ x_2 \succ x_1$.

We can see that intuitionistic fuzzy preference relations can be viewed as a degeneration of Pythagorean fuzzy preference relation. Based on the additive consistency of intuitionistic fuzzy preference relation, a new definition of additive consistency is introduced by directly employing the membership and non-membership degrees in a Pythagorean fuzzy preference relation.

Definition 3.3: For a Pythagorean fuzzy preference relation (PFPR) $P = (p_{ij})_{n \times n}$ with $p_{ij} = (\mu_{ij}, v_{ij})(i, j = 1, 2, \cdots, n)$, if there exists a s vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that

$$\mu_{ij} \leq \frac{\sqrt{2}}{2}(\omega_i - \omega_j + 1) \leq \sqrt{1 - (v_{ij})^2},$$
$$\text{for all } i, j = 1, 2, \cdots, n. \tag{14}$$

where $\omega_i \in [0, 1](i = 1, 2, \cdots, n)$ that $\sum_{i=1}^{n} \omega_i = 1$. Then, P is called an additive consistent PFPR.

Definition 3.4: A Pythagorean fuzzy preference relation $P = (p_{ij})_{n \times n}$ with $p_{ij} = (\mu_{ij}, v_{ij})$ is called additive consistent, if it satisfies the following additive transitivity:

$$\mu_{ij}^2 + \mu_{jk}^2 + \mu_{ki}^2 = \mu_{kj}^2 + \mu_{ji}^2 + \mu_{ik}^2$$
$$\text{for all } i, j, k = 1, 2, \cdots, n.$$

As $\mu_{ij} = v_{ji}, \mu_{ji} = v_{ij}$ for all $i, j = 1, 2, \cdots, n$, it follows from (3.1) that

$$v_{ij}^2 + v_{jk}^2 + v_{ki}^2 = v_{kj}^2 + v_{ji}^2 + v_{ik}^2$$
$$\text{for all } i, j, k = 1, 2, \cdots, n.$$

And we can also get the following equation

$$\mu_{ij}^2 + \mu_{jk}^2 + \mu_{ki}^2 = v_{ij}^2 + v_{jk}^2 + v_{ki}^2$$
$$\text{for all } i, j, k = 1, 2, \cdots, n. \tag{15}$$

Theorem 3.1: A Pythagorean fuzzy preference relation $P = (p_{ij})_{n \times n}$ with $p_{ij} = (\mu_{ij}, v_{ij})$ is additive consistent if and only if $s(p_{ij}) = s(p_{ik}) - s(p_{jk})$ for all $i, j, k = 1, 2, \cdots, n$.

Proof: If $P = (p_{ij})_{n \times n}$ is an additive consistent Pythagorean fuzzy preference relation, the according to **Definition 3.4**, we have $\mu_{ij}^2 + \mu_{jk}^2 + \mu_{ki}^2 = \mu_{kj}^2 + \mu_{ji}^2 + \mu_{ik}^2$ for all $i, j, k = 1, 2, \cdots, n$. As $\mu_{ji} = v_{ij}$ for all $i, j, k = 1, 2, \cdots, n$, we can get $\mu_{ij}^2 - v_{ij}^2 = \mu_{ik}^2 - v_{ik}^2 - (\mu_{jk}^2 - v_{jk}^2)$. Therefore, it follows from that $s(p_{ij}) = s(p_{ik}) - s(p_{jk})$ for all $i, j, k = 1, 2, \cdots, n$. In other words, if $s(p_{ij}) = s(p_{ik}) - s(p_{jk})$ for all $i, j, k = 1, 2, \cdots, n$, then by reversing the aforesaid proof of the necessary condition, we can get $\mu_{ij}^2 + \mu_{jk}^2 + \mu_{ki}^2 = \mu_{kj}^2 + \mu_{ji}^2 + \mu_{ik}^2$ for all $i, j, k = 1, 2, \cdots, n$.

Definition 3.5: A Pythagorean fuzzy preference relation $P = (p_{ij})_{n \times n}$ is weak transitive if $s(p_{ij}) \geq 0$ and $s(p_{jk}) \geq 0$ imply $s(p_{ik}) \geq 0$, for all $i, j, k = 1, 2, \cdots, n$.

Theorem 3.2: If a Pythagorean fuzzy preference relation $P = (p_{ij})_{n \times n}$ is additive consistent, then P is weakly transitive.

Proof: If $s(p_{ij}) \geq 0$ and $s(p_{jk}) \geq 0$, we have $\mu_{ij}^2 - v_{ij}^2 \geq 0$ and $\mu_{jk}^2 - v_{jk}^2 \geq 0$. Since $P = (p_{ij})_{n \times n}$ is additive consistent, it follows from **Theorem 3.1** that $s(p_{ik}) = s(p_{ij}) + s(p_{jk}) = (\mu_{ij}^2 - v_{ij}^2) + (\mu_{jk}^2 - v_{jk}^2)$ for all $i, j, k = 1, 2, \cdots, n$. Therefore, one can obtain $s(p_{ik}) \geq 0$. By **Definition 3.5**, the proof of **Theorem 3.2** is completed.

Definition 3.6: A Pythagorean fuzzy weight vector $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)^T$ with $\tilde{\omega}_i = (\omega_i^\mu, \omega_i^v), \omega_i^\mu, \omega_i^v \in [0, 1]$, and $(\omega_i^\mu)^2 + (\omega_i^v)^2 \leq 1$ for $i = 1, 2, \cdots, n$ is said to be normalized if it satisfies the following conditions:

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \omega_j^\mu + \sqrt{1 - (\omega_i^v)^2} \leq 1, \quad \sum_{\substack{j=1 \\ j \neq i}}^{n} \sqrt{1 - (\omega_j^v)^2} + \omega_i^\mu \geq 1$$
$$\text{for } i = 1, 2, \cdots, n. \tag{16}$$

Let

$$\tilde{p}_{ij} = (\tilde{p}_{ij}^\mu, \tilde{p}_{ij}^v) = \begin{cases} \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) & i = j \\ \left(\sqrt{\frac{(\omega_i^\mu)^2 + (\omega_j^v)^2}{2}}, \sqrt{\frac{(\omega_i^v)^2 + (\omega_j^\mu)^2}{2}}\right) & i \neq j \end{cases} \tag{17}$$

Then the following result is obtained.

Theorem 3.3: Assume that the elements of the matrix $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ are defined by Eq. **(17)**, then $\tilde{P}$ is a Pythagorean fuzzy preference relation.

Proof: It is obvious that $\tilde{p}_{ij}^\mu = \tilde{p}_{ji}^v, \tilde{p}_{ji}^\mu = \tilde{p}_{ij}^v$ for all $i, j = 1, 2, \cdots, n$.

Since $\omega_i^\mu, \omega_i^v \in [0, 1]$ and $(\omega_i^\mu)^2 + (\omega_i^v)^2 \leq 1$, it follows that

$$0 \leq \sqrt{\frac{(\omega_i^\mu)^2 + (\omega_j^v)^2}{2}} \leq 1,$$

$$0 \leq \sqrt{\frac{(\omega_i^v)^2 + (\omega_j^\mu)^2}{2}} \leq 1.$$

and

$$\left(\sqrt{\frac{(\omega_i^\mu)^2+(\omega_j^v)^2}{2}}\right)^2 + \left(\sqrt{\frac{(\omega_i^v)^2+(\omega_j^\mu)^2}{2}}\right)^2$$

$$= \frac{(\omega_i^\mu)^2+(\omega_j^v)^2}{2} + \frac{(\omega_i^v)^2+(\omega_j^\mu)^2}{2}$$

$$= \frac{(\omega_i^\mu)^2+(\omega_j^v)^2+(\omega_i^v)^2+(\omega_j^\mu)^2}{2}$$

$$\le \frac{1+1}{2} = 1.$$

thus:

$$0 \le \left(\sqrt{\frac{(\omega_i^\mu)^2+(\omega_j^v)^2}{2}}\right)^2 + \left(\sqrt{\frac{(\omega_i^v)^2+(\omega_j^\mu)^2}{2}}\right)^2 \le 1.$$

According to **Definition 3.1**, $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ is an PFPR, which completes the proof.

Theorem 3.4: The PFPR $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ in which the elements $\tilde{p}_{ij}(i,j = 1, 2, \cdots, n)$ are defined as in **Eq. (17)** is additive consistent.

Proof: From **Eq. (17)**,

$$(\tilde{p}_{ij}^\mu)^2 + (\tilde{p}_{jk}^\mu)^2 + (\tilde{p}_{ki}^\mu)^2$$

$$= \frac{(\omega_i^\mu)^2+(\omega_j^v)^2}{2} + \frac{(\omega_j^\mu)^2+(\omega_k^v)^2}{2} + \frac{(\omega_k^\mu)^2+(\omega_i^v)^2}{2}$$

$$= \frac{(\omega_i^\mu)^2+(\omega_i^v)^2+(\omega_j^\mu)^2+(\omega_j^v)^2+(\omega_k^\mu)^2+(\omega_k^v)^2}{2}$$

$$(\tilde{p}_{ij}^v)^2 + (\tilde{p}_{jk}^v)^2 + (\tilde{p}_{ki}^v)^2$$

$$= \frac{(\omega_i^v)^2+(\omega_j^\mu)^2}{2} + \frac{(\omega_j^v)^2+(\omega_k^\mu)^2}{2} + \frac{(\omega_k^v)^2+(\omega_i^\mu)^2}{2}$$

$$= \frac{(\omega_i^\mu)^2+(\omega_i^v)^2+(\omega_j^\mu)^2+(\omega_j^v)^2+(\omega_k^\mu)^2+(\omega_k^v)^2}{2}$$

It is obvious that $(\tilde{p}_{ij}^\mu)^2 + (\tilde{p}_{jk}^\mu)^2 + (\tilde{p}_{ki}^\mu)^2 = (\tilde{p}_{ij}^v)^2 + (\tilde{p}_{jk}^v)^2 + (\tilde{p}_{ki}^v)^2$, According to **Definition 3.4**, it is certified that the PFPR $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ is additive consistent, which completes the proof.

From **Theorem 3.3**, one can easily obtain the following corollary.

Corollary 3.1: For a PFPR $P = (p_{ij})_{n\times n}$ in which $p_{ij} = (\mu_{ij}, v_{ij})$, if there exists a normalized Pythagorean fuzzy weight vector $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)^T$ such that

$$p_{ij} = (\mu_{ij}, v_{ij})$$

$$= \begin{cases} (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) & \text{if } i=j \\ \left(\sqrt{\frac{(\omega_i^\mu)^2+(\omega_j^v)^2}{2}}, \sqrt{\frac{(\omega_i^v)^2+(\omega_j^\mu)^2}{2}}\right) & \text{if } i \ne j \end{cases} \quad (18)$$

where $\tilde{\omega}_i = (\omega_i^\mu, \omega_i^v)$, $\omega_i^\mu, \omega_i^v \in [0, 1]$, $(\omega_i^\mu)^2+(\omega_i^v)^2 \le 1$, and $\sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^\mu + \sqrt{1-(\omega_i^v)^2} \le 1$, $\sum_{\substack{j=1 \\ j \ne i}}^{n} \sqrt{1-(\omega_j^v)^2} + \omega_i^\mu \ge 1$

for $i = 1, 2, \cdots, n$, then $P = (p_{ij})_{n\times n}$ is an additive consistent Pythagorean fuzzy preference relation.

Inspired by **Corollary 3.1**, we can develop a method to derive the priority weight vector from PFPRs.

B. LINEAR GOAL PROGRAM MING MODEL FOR GENERATING PYTHAGOREA FUZZY WEIGHT

This section develops a linear goal programming model for deriving Pythagorean fuzzy weights from PFPR.

As presented above, for the purpose of obtaining a reasonable result, the PFPR given by the DM should satisfy additive consistency which can be expressed as **Eq. (18)** according to **Corollary 1**. However, in practical situations of decision making, it is too difficult for a DM to construct such an additive consistent PFPR. Hence, it is expected that the deviation between the given PFPR and its corresponding additive consistent PFPR should be as small as possible. As a result, we introduce the deviation variables as follows:

$$\varepsilon_{ij} = \sqrt{\frac{(\omega_i^\mu)^2+(\omega_j^v)^2}{2}} - \mu_{ij}, i, j = 1, 2, \ldots, n, j \ne i, \quad (19)$$

$$\eta_{ij} = \sqrt{\frac{(\omega_i^v)^2+(\omega_j^\mu)^2}{2}} - v_{ij}, i, j = 1, 2, \ldots, n, j \ne i. \quad (20)$$

As can be seen, the smaller the absolute deviations are, the more exact the results are. Thus, the following fractional programming model can be established to derive the Pythagorean fuzzy weights (M1), as shown at the bottom of the next page.

Theorem 3.5: **M1** is equivalent to the following **M2**, as shown at the bottom of the next page.

Proof: Because of $\mu_{ij} = v_{ji}$ and $\mu_{ji} = v_{ij}$, the deviation of the upper diagonal elements is equal to the deviation of the lower diagonal elements. Hence, we only need to consider the deviation of the upper (or lower) diagonal elements. That is to say, the objective functions of **M1** and **M2** are equivalent, which completes the proof.

In order to simplify the calculation, in the following, we will use **M2** for further discussion.

If $\varepsilon_{ij}^+ = \frac{|\varepsilon_{ij}|+\varepsilon_{ij}}{2}$ and $\varepsilon_{ij}^- = \frac{|\varepsilon_{ij}|-\varepsilon_{ij}}{2}$, then $\varepsilon_{ij} = \varepsilon_{ij}^+ - \varepsilon_{ij}^-$ and $|\varepsilon_{ij}| = \varepsilon_{ij}^+ + \varepsilon_{ij}^-$, where $\varepsilon_{ij}^+ \ge 0$, $\varepsilon_{ij}^- \ge 0$ and $\varepsilon_{ij}^+ \cdot \varepsilon_{ij}^- = 0$. Similarly, if $\eta_{ij}^+ = \frac{|\eta_{ij}|+\eta_{ij}}{2}$ and $\eta_{ij}^- = \frac{|\eta_{ij}|-\eta_{ij}}{2}$, then $\eta_{ij} = \eta_{ij}^+ - \eta_{ij}^-$ and $|\eta_{ij}| = \eta_{ij}^+ + \eta_{ij}^-$, where $\eta_{ij}^+ \ge 0$, $\eta_{ij}^- \ge 0$ and $\eta_{ij}^+ \cdot \eta_{ij}^- = 0$.

Thus, **M2** can be further expressed and simplified as follows:

Theorem 3.6: The PFPR $P = (p_{ij})_{n\times n}$ is an additive consistent preference relation if and only if $J^* = 0$, where J^* is the optimal value of the objective function.

Proof: If $P = (p_{ij})_{n\times n}$ is an additive consistent PFPR, then the deviation of both the membership and non-membership degrees should be equal to 0, which indicates that $J^* = 0$. If $J^* = 0$, i.e., $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \varepsilon_{ij}^+ + \varepsilon_{ij}^- + \eta_{ij}^+ + \eta_{ij}^- = 0$, since $\varepsilon_{ij}^+ \ge 0$, $\varepsilon_{ij}^- \ge 0$, $\eta_{ij}^+ \ge 0$, $\eta_{ij}^- \ge 0$, for all $i,j = 1, 2, \ldots, n$,

then we can derive that $\varepsilon_{ij}^+ = \varepsilon_{ij}^- = \eta_{ij}^+ = \eta_{ij}^- = 0$ for all $i, j = 1, 2, \ldots, n$. That is to say, the PFPR $P = (p_{ij})_{n \times n}$ is equal to its corresponding additive consistent PFPR, which completes the proof.

M3, as shown at the bottom of the page, can be solved by using some optimization computer packages, such as MATLAB and LINGO. Thus, the optimal objective function value J^* and the optimal Pythagorean fuzzy weight vector can be yielded. If $J^* = 0$, the given PFPR $P = (p_{ij})_{n \times n}$ is additive consistent in which case the derived Pythagorean fuzzy weight vector is reasonable. If $J^* \neq 0$, P does not have additive consistency, so we need to consider whether

P have acceptable consistency. If P has acceptable additive consistency, the next calculation can be performed; if P does not have acceptable additive consistency, we will adjust it with an additive consistency adjustment algorithm until it reaches acceptable additive consistency. Once the result is consistent with decision maker's preferences, the calculation process ends. Otherwise, the decision maker should reevaluate the alternatives to construct a more acceptable consistent PFPR $P = (p_{ij})_{n \times n}$, or the process will stop as the repetition times reach the maximum number which we specified previously. The adjustment algorithm for additive consistency will be introduced as below.

$$
\text{Min } J = \sum_{i=1}^{n} \sum_{j=1}^{n} |\varepsilon_{ij}| + |\eta_{ij}|
$$

$$
s.t. \begin{cases}
\sqrt{\dfrac{(\omega_i^\mu)^2 + (\omega_j^\nu)^2}{2}} - \mu_{ij} - \varepsilon_{ij} = 0, & i, j = 1, 2, \ldots, n; \ i \neq j \\[3mm]
\sqrt{\dfrac{(\omega_i^\nu)^2 + (\omega_j^\mu)^2}{2}} - \nu_{ij} - \eta_{ij} = 0, & i, j = 1, 2, \ldots, n; \ i \neq j \\[3mm]
\omega_i^\mu, \omega_i^\nu \in [0, 1], (\omega_i^\mu)^2 + (\omega_i^\nu)^2 \leq 1, & i = 1, 2, \ldots, n \\[3mm]
\sum_{\substack{j=1 \\ j \neq i}}^{n} \omega_i^\mu + \sqrt{1 - (\omega_i^\nu)^2} \leq 1, \ \sum_{\substack{j=1 \\ j \neq i}}^{n} \sqrt{1 - (\omega_j^\nu)^2} + \omega_i^\mu \geq 1, & i = 1, 2, \ldots, n
\end{cases}
\tag{M1}
$$

$$
\text{Min } J = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\varepsilon_{ij}| + |\eta_{ij}|
$$

$$
s.t. \begin{cases}
\sqrt{\dfrac{(\omega_i^\mu)^2 + (\omega_j^\nu)^2}{2}} - \mu_{ij} - \varepsilon_{ij} = 0, & i, j = 1, 2, \ldots, n; \ i \neq j \\[3mm]
\sqrt{\dfrac{(\omega_i^\nu)^2 + (\omega_j^\mu)^2}{2}} - \nu_{ij} - \eta_{ij} = 0, & i, j = 1, 2, \ldots, n; \ i \neq j \\[3mm]
\omega_i^\mu, \omega_i^\nu \in [0, 1], (\omega_i^\mu)^2 + (\omega_i^\nu)^2 \leq 1, & i = 1, 2, \ldots, n \\[3mm]
\sum_{\substack{j=1 \\ j \neq i}}^{n} \omega_i^\mu + \sqrt{1 - (\omega_i^\nu)^2} \leq 1, \ \sum_{\substack{j=1 \\ j \neq i}}^{n} \sqrt{1 - (\omega_j^\nu)^2} + \omega_i^\mu \geq 1, & i = 1, 2, \ldots, n
\end{cases}
\tag{M2}
$$

$$
\text{Min } J = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\varepsilon_{ij}^+ + \varepsilon_{ij}^- + \eta_{ij}^+ + \eta_{ij}^- \right)
$$

$$
s.t. \begin{cases}
\sqrt{\dfrac{(\omega_i^\mu)^2 + (\omega_j^\nu)^2}{2}} - \mu_{ij} - \varepsilon_{ij}^+ + \varepsilon_{ij}^- = 0, & i = 1, 2, \ldots, n-1; j = i+1, \ldots, n \\[3mm]
\sqrt{\dfrac{(\omega_i^\nu)^2 + (\omega_j^\mu)^2}{2}} - \nu_{ij} - \eta_{ij}^+ + \eta_{ij}^- = 0, & i = 1, 2, \ldots, n-1; j = i+1, \ldots, n \\[3mm]
\omega_i^\mu, \omega_i^\nu \in [0, 1], (\omega_i^\mu)^2 + (\omega_i^\nu)^2 \leq 1, & i = 1, 2, \ldots, n-1 \\[3mm]
\sum_{\substack{j=1 \\ j \neq i}}^{n} \omega_i^\mu + \sqrt{1 - (\omega_i^\nu)^2} \leq 1, \ \sum_{\substack{j=1 \\ j \neq i}}^{n} \sqrt{1 - (\omega_j^\nu)^2} + \omega_i^\mu \geq 1, & i = 1, 2, \ldots, n-1 \\[3mm]
\varepsilon_{ij}^+ \geq 0, \varepsilon_{ij}^- \geq 0, \eta_{ij}^+ \geq 0, \eta_{ij}^- \geq 0, & i = 1, 2, \ldots, n-1; j = i+1, \ldots, n
\end{cases}
\tag{M3}
$$

C. ALGORITHM FOR IMPROVING ADDITIVE CONSISTENCY

In this section, we define the consistency index of PFPR and develop a feasible algorithm for improving consistency degree of PFPR without achieving acceptable consistency.

Definition 3.7: Assume two PFPRs $P^1 = (p_{ij}^1)_{n \times n} = (\mu_{ij}^1, v_{ij}^1)_{n \times n}$ and $P^2 = (p_{ij}^2)_{n \times n} = (\mu_{ij}^2, v_{ij}^2)_{n \times n}$, then

$$d(P^1, P^2) = \frac{1}{n(n-1)/2} \sum_{i<j} \frac{1}{2} (|(\mu_{ij}^1)^2 - (\mu_{ij}^2)^2| + |(v_{ij}^1)^2 - (v_{ij}^2)^2| + |(\pi_{ij}^1)^2 - (\pi_{ij}^2)^2|) \quad (21)$$

is called the distance between P^1 and P^2.

Definition 3.8: Assume a PFPR $P = (p_{ij})_{n \times n}$ with $p_{ij} = (\mu_{ij}, v_{ij})$, and its additive consistent PFPR $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ with $\tilde{p}_{ij} = (\tilde{\mu}_{ij}, \tilde{v}_{ij})$, to make P approximate $\tilde{P}$ as much as possible, we define $CI(P)$ as a consistency index (CI) of the PFPR P as follows.

$$CI(P) = \frac{1}{n(n-1)/2} \sum_{i<j} \frac{1}{2} (|(\mu_{ij})^2 - (\tilde{\mu}_{ij})^2| + |(v_{ij})^2 - (\tilde{v}_{ij})^2| + |(\pi_{ij})^2 - (\tilde{\pi}_{ij})^2|). \quad (22)$$

According to **Definition 3.7**, the $CI(P)$ can be used to measure the distance between P and $\tilde{P}$.

Theorem 3.7: The consistency index $CI(P)$ between two PFPRs P and its additive consistent PFPR $\tilde{P}$ satisfies the following properties:
1. $0 \leq CI(P) \leq 1$;
2. $CI(P) = 0$ if and only if $P = \tilde{P}$.

Additionally, according to **Theorem 3.7**, the smaller the $CI(P)$, the more consistent the PFPR P. Especially, $CI(P) = 0$ if and only if P is an additive consistent PFPR.

In most cases, it is unrealistic to construct an additive consistent PFPR due to the reason that decision makers must be affected by many factors in the decision-making process. Based on this, a definition of acceptably additive consistent PFPR will be further developed to allow a certain of level of acceptable deviation.

Definition 3.9: Let $P = (p_{ij})_{n \times n}$ be a PFPR. Given a threshold value $\overline{CI}$, if the additive consistency index satisfies the following,

$$CI(P) \leq \overline{CI} \quad (23)$$

then we call a PFPR P with acceptably additive consistency.

The value of $\overline{CI}$ can be determined according to the preferences of the decision maker or the actual situation of the problem. Which is a question worthy of further discussion in the future.

Due to the complexity of objective things and the limitations of human cognition, the PFPR P constructed by DMs often has unacceptable additive consistency, i.e., $CI(P) \geq \overline{CI}$. In order to obtain more reasonable results, DMs need to construct a new PFPR based on additive consistency. To help

the DMs to obtain an additive consistent PFPR, we provide the following formula to adjust or repair the inconsistent PFPR $P^{(t)} = (p_{ij}^{(t)})_{n \times n}$ until it has acceptably additive consistency.

Theorem 3.8: Let $P^{(t+1)} = (p_{ij}^{(t+1)})_{n \times n}$ be a PFPR defined by **Eq. (24)**, as shown at the bottom of the next page. Then, we have $CI(P^{(t+1)}) \leq CI(P^{(t)})$.

Proof: $CI(P^{(t+1)})$, as shown at the bottom of the next page, where $\tilde{P}^{(0)} = \tilde{P}^{(i)}$, $i = 1, 2, \ldots, t+1$, which completes the proof. Moreover, $CI(P^{(t)}) \geq 0$, for each t. Thus, the sequence $\{CI(P^{(t)})\}$ is monotonically decreasing and has lower bounds.

Consistency means that decision makers do not have conflicts when expressing their preferences. If PFPR $P = (p_{ij})_{n \times n}$ does not have acceptable consistency, we should adjust it to achieve acceptable consistency before using it to resolve decision problems. We propose the following algorithm to modify $P = (p_{ij})_{n \times n}$ that do not have acceptable consistency to meet the consistency requirements.

Algorithm 1

Input: The original PFPR $P = (p_{ij})_{n \times n}$ with $p_{ij} = (\mu_{ij}, v_{ij})$, the parameter $\sigma \in (0, 1)$ that is the trade-off parameter between the inconsistent preference relation and the corresponding consistent preference relation, the maximum number of iterations t^*, and the threshold value $\overline{CI} \in (0, 1]$.

Output: The adjusted PFPR $\bar{P} = (\bar{p}_{ij})_{n \times n}$ with $\bar{p}_{ij} = (\bar{\mu}_{ij}, \bar{v}_{ij})$, and the consistency index $CI(\bar{P})$.

Step 1: Let $P^{(0)} = (\mu_{ij}^{(0)}, v_{ij}^{(0)})_{n \times n} = P = (\mu_{ij}, v_{ij})_{n \times n}$, $t = 0$. By calculating **M3**, construct the additive consistent PFPR $\tilde{P}^{(0)} = (\tilde{\mu}_{ij}^{(0)}, \tilde{v}_{ij}^{(0)})_{n \times n}$ with respect to $P^{(0)}$ based on **Eq. (17)**, where $\tilde{P}^{(0)} = \tilde{P}^{(i)}$, $i = 1, 2, \ldots, t+1$.

Step 2: Compute the consistency index $CI(P^{(t)})$ by **Eq. (22)**, i.e.,

$$CI(P^{(t)}) = \frac{1}{n(n-1)/2} \sum_{i<j} \frac{1}{2} (|(\mu_{ij}^{(t)})^2 - (\tilde{\mu}_{ij}^{(t)})^2|$$
$$+ |(v_{ij}^{(t)})^2 - (\tilde{v}_{ij}^{(t)})^2| + |(\pi_{ij}^{(t)})^2 - (\tilde{\pi}_{ij}^{(t)})^2|).$$

Step 3: If $CI(P^{(t)}) \leq \overline{CI}$ or $t \geq t^*$, then go to **Step 5**; otherwise, go to **Step 4**.

Step 4: Let $P^{(t+1)} = (p_{ij}^{(t+1)})_{n \times n} = (\mu_{ij}^{(t+1)}, v_{ij}^{(t+1)})_{n \times n}$, where

$$\begin{cases} \mu_{ij}^{(t+1)} = \sqrt{(1-\sigma)(\mu_{ij}^{(t)})^2 + \sigma(\tilde{\mu}_{ij}^{(t)})^2} \\ v_{ij}^{(t+1)} = \sqrt{(1-\sigma)(v_{ij}^{(t)})^2 + \sigma(\tilde{v}_{ij}^{(t)})^2} \\ \pi_{ij}^{(t+1)} = \sqrt{1 - (\mu_{ij}^{(t+1)})^2 - (v_{ij}^{(t+1)})^2} \\ \quad = \sqrt{(1-\sigma)(\pi_{ij}^{(t)})^2 + \sigma(\tilde{\pi}_{ij}^{(t)})^2} \end{cases}$$

Set $t = t + 1$ and go to **Step 2**.

Step 5: Let $\bar{P} = P^{(t)}$. Output the modified PFPR $\bar{P}$ and its consistency index $CI(\bar{P})$.
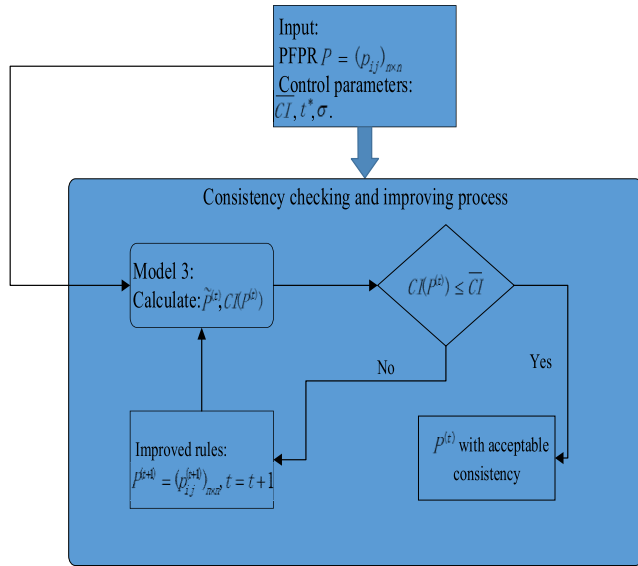
Step 6: End.

FIGURE 1. Algorithm 1: Consistency checking and improving process.

The proposed algorithm can be described by using **FIGURE. 1.**

IV. METHOD FOR DECISION MAKING WITH PFPR

In this section, we develop an approach to decision making based on Pythagorean fuzzy preference relation (PFPR) and its additive consistency, which can be described as Algorithm 2.

The proposed algorithm can also be described by using **FIGURE. 2.**

V. NUMERICAL EXAMPLES

This section presents a numerical example to validate the proposed models. In this paper, since $\frac{\sqrt{2}}{2}$ is irrational number, we make $\frac{\sqrt{2}}{2} \approx 0.7071$.
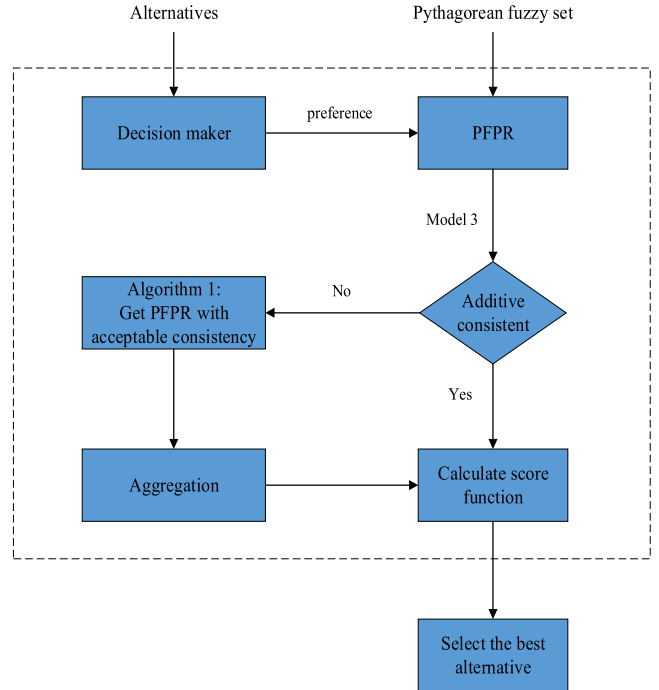

FIGURE 2. Algorithm 2: an approach to decision making based on PFPR.

A. EXAMPLE AND COMPARISON ANALYSIS

As the big data, cloud computing, AI and other fields developing at a high speed, the contradiction of the increasing data calculation volume, flexibility demand and the execution is gradually sharp. Computing-in-memory (CIM) based on DRAM integrates Computing and storage closely, which can partially relieve the "von neumann bottleneck", reduce data handling between on-chip cache and Memory, and greatly improve Memory access efficiency.

During the research of the CIM in recent years, r&d institutions have all taken the following four factors,

$$
\begin{cases}
\mu_{ij}^{(t+1)} = \sqrt{(1-\sigma)(\mu_{ij}^{(t)})^2 + \sigma(\tilde{\mu}_{ij}^{(t)})^2} \\
v_{ij}^{(t+1)} = \sqrt{(1-\sigma)(v_{ij}^{(t)})^2 + \sigma(\tilde{v}_{ij}^{(t)})^2} \\
\pi_{ij}^{(t+1)} = \sqrt{1 - (\mu_{ij}^{(t+1)})^2 - (v_{ij}^{(t+1)})^2} = \sqrt{(1-\sigma)(\pi_{ij}^{(t)})^2 + \sigma(\tilde{\pi}_{ij}^{(t)})^2}
\end{cases}
. \tag{24}
$$

$$
\begin{aligned}
& CI(P^{(t+1)}) \\
&= d(P^{(t+1)}, \tilde{P}^{(t+1)}) = d(P^{(t+1)}, \tilde{P}^{(t)}) \\
&= \frac{1}{n(n-1)/2} \sum_{i<j} \frac{1}{2}(|(\mu_{ij}^{(t+1)})^2 - (\tilde{\mu}_{ij}^{(t)})^2| + |(v_{ij}^{(t+1)})^2 - (\tilde{v}_{ij}^{(t)})^2| + |(\pi_{ij}^{(t+1)})^2 - (\tilde{\pi}_{ij}^{(t)})^2|) \\
&= \frac{1}{n(n-1)/2} \sum_{i<j} \frac{1}{2}(|(1-\sigma)(\mu_{ij}^{(t)})^2 + \sigma(\tilde{\mu}_{ij}^{(t)})^2 - (\tilde{\mu}_{ij}^{(t)})^2| + |(1-\sigma)(v_{ij}^{(t)})^2 + \sigma(\tilde{v}_{ij}^{(t)})^2 - (\tilde{v}_{ij}^{(t)})^2| \\
&\quad + |(1-\sigma)(\pi_{ij}^{(t)})^2 + \sigma(\tilde{\pi}_{ij}^{(t)})^2 - (\tilde{\pi}_{ij}^{(t)})^2|) \\
&= \frac{1-\sigma}{n(n-1)/2} \sum_{i<j} \frac{1}{2}(|(\mu_{ij}^{(t+1)})^2 - (\tilde{\mu}_{ij}^{(t)})^2| + |(v_{ij}^{(t+1)})^2 - (\tilde{v}_{ij}^{(t)})^2| + |(\pi_{ij}^{(t+1)})^2 - (\tilde{\pi}_{ij}^{(t)})^2|) \\
&= (1-\sigma)d(P^{(t)}, \tilde{P}^{(t)}) \leq CI(P^{(t)}).
\end{aligned}
$$

Algorithm 2

Step 1: Construct the PFPR matrix $P = (\mu_{ij}, v_{ij})_{n \times n}$ based on the decision-making information; and set the values of the predefined consistency threshold $\overline{CI}$.

Step 2: If P satisfies the order consistency, i.e., P satisfies that $p_{is} \geq p_{it}$ for all $i \in \{1, 2, \ldots, n\}$, where $s \in \{1, 2, \ldots, n\}$ and $t \in \{1, 2, \ldots, n\}$, then the ranking of the alternatives can be obtained; Otherwise, go to **Step 3**.

Step 3: According to **M3**, a goal programming model can be developed, we get the optimal normalized Pythagorean fuzzy weight vector:

$$\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)^T = ((\tilde{\omega}_{1\mu}, \tilde{\omega}_{1v}),$$
$$(\tilde{\omega}_{2\mu}, \tilde{\omega}_{2v}), \ldots, (\tilde{\omega}_{n\mu}, \tilde{\omega}_{nv}))^T$$

and the optimal deviation values $\varepsilon_{ij}^+ \geq 0, \varepsilon_{ij}^- \geq 0, \eta_{ij}^+ \geq 0, \eta_{ij}^- \geq 0$, for $i, j = 1, 2, \ldots, n$. If $J^* = 0$, the given PFPR $P = (p_{ij})_{n \times n}$ is additive consistent, then go to **Step 8**; If $J^* \neq 0$, the given PFPR $P = (p_{ij})_{n \times n}$ is not consistent, then go to **Step 4**.

Step 4: Construct the additive consistent PFPR $\tilde{P} = (\tilde{\mu}_{ij}, \tilde{v}_{ij})_{n \times n}$ with respect to P based on **Eq. (17)**.

Step 5: Utilize **Eq. (22)** to calculate the consistency index $CI(P)$. If $CI(P) \leq \overline{CI}$, go to **Step 7**; Otherwise, go to **Step 6**.

Step 6: Utilize **Algorithm 1** to modify the PFPR that do not achieve acceptable consistency degree. After implementation of **Algorithm 1**, we can get PFPRs $P' = (p'_{ij})_{n \times n} = (\mu'_{ij}, v'_{ij})_{n \times n}$ with acceptable consistency; then, go to **Step 7**.

Step 7: Utilize the Pythagorean fuzzy arithmetic averaging operators **Eq. (13)** to aggregate all $p'_{ij}(i, j = 1, 2, \ldots, n)$ into a collective Pythagorean fuzzy value $p'_i(i = 1, 2, \ldots, n)$ of the alternative $x_i(i = 1, 2, \ldots, n)$ over all the other alternatives.

Step 8: Rank all $p'_i(i = 1, 2, \ldots, n)$ by means of the score function **Eq. (9)** and the accuracy function **Eq. (10)**, and then rank all the alternatives $x_i(i = 1, 2, \ldots, n)$ and select the best one in accordance with the values of $p'_i(i = 1, 2, \ldots, n)$.

access speed, storage capacity, computing speed,and delay time(CAS),into consideration in the design of hardware structure optimization and software algorithm improvement,such as 3D stacked DRAM with CIM [62], approxPIM(A new CIM architecture for solving the problem of bandwidth and power mismatch between processor and memory) [63] and a mathematical framework integrated with 3D stacked DRAM accelerators [64].

So,An excellent CIM system is affected by many factors, including

x_1 : access speed

x_2 : storage capacity

x_3 : computing speed

x_4 : delay time(CAS)

Consider the case that a company prepare to know the importance of these factors for the design of CIM, and because decision makers lack the corresponding expertise, many uncertain messages are generated. Then by pairwise

comparison of x_i and x_j $(i, j = 1, 2, 3, 4)$, the decision maker construct a Pythagorean fuzzy preference relation (PFPR) $P^{(0)}$ as follows $P^{(0)}$, as shown at the bottom of the next page.

Step 1: Set $\sigma = 0.5, \overline{CI} = 0.1$ [58], [59] and the maximum number of iterations $t^* = 10$.

Step 2: By calculating **M3**, the following linear goal program is established Min J, as shown at the bottom of the next page.

Solving this model by an appropriate optimization computer package, it follows that the optimal objective value $J = 1.68$, which implies that PFPR $\tilde{P}$ is non-additive consistent, and we get the optimal Pythagorean fuzzy weights vector as:

$$\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4)^T$$
$$= ((0, 0.9024), (0.1754, 0.4950),$$
$$\times (0.1311, 0.5657), (0, 1.0000))^T.$$

Utilize **Eq. (17)**, a PFPR $\tilde{P}^{(0)}$ with complete additive consistency is constructed as follows $\tilde{P}^{(0)}$, as shown at the bottom of the next page.

Step 3: By **Eq. (22)**, calculate the additive consistent index $CI(P^{(0)}) = 0.2121$. Since $CI(P^{(0)}) > \overline{CI}$, PFPR $P^{(0)}$ is unacceptable additive consistent, then, go to **Step 4**.

Step 4: Utilize **Algorithm 1** to modify the PFPR $P^{(0)}$. After implementation of Algorithm 1, we can get $CI(P^{(3)}) = 0.0530$. Since $CI(P^{(3)}) < \overline{CI}$, we can get PFPRs $P^{(3)}$ with acceptable consistency is shown as follows $P^{(3)}$, as shown at the bottom of the next page.

Step 5: Utilize the Pythagorean fuzzy arithmetic averaging operators **Eq. (13)** to aggregate all $p_{ij}^{(3)}(i, j = 1, 2, \ldots, n)$ into collective Pythagorean fuzzy value $P_1^{(3)} = (0.5471, 0.6472)$, $P_2^{(3)} = (0.6494, 0.4578)$, $P_3^{(3)} = (0.6195, 0.4929)$, $P_4^{(3)} = (0.5167, 0.7182)$.

Step 6: Rank all $p_i^{(3)}(i = 1, 2, 3, 4)$ by means of the score function $s(p_i^{(3)})(i = 1, 2, 3, 4)$, and we can obtain $s(p_1^{(3)}) = -0.1540, s(p_2^{(3)}) = 0.2121, s(p_3^{(3)}) = 0.1408, s(p_4^{(3)}) = -0.2488$.

Step 7: The ranking of the score function of four alternatives is $s(p_2^{(3)}) > s(p_3^{(3)}) > s(p_1^{(3)}) > s(p_4^{(3)})$.

Step 8: The ranking of the four alternatives is $x_2 \succ x_3 \succ x_1 \succ x_4$.

B. COMPARISON WITH OTHER METHOD

Because of the value ranges of PFN and IFN are different, in the comparative analysis, we first convert PFN into IFN, and the converted preference relation matrix is denoted as P_*, as shown at the bottom of the next page.

Wang [48] proposed linear programming models for deriving intuitionistic fuzzy weights from intuitionistic fuzzy preference relations. Using the above method to deal with P^*, it follows that the optimal objective value $J^* = 1.8508$, and the optimal intuitionistic fuzzy weight vector is:

$$\omega^* = (\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*)$$
$$= ((0.1033, 0.6417), (0, 0.1417), (0.0383, 0.2167), (0, 1)),$$

$$P^{(0)} = \begin{pmatrix} (0.7071, 0.7071) & (0.3500, 0.6500) & (0.4000, 0.7000) & (0.8000, 0.3000) \\ (0.6500, 0.3500) & (0.7071, 0.7071) & (0.7000, 0.3000) & (0.5000, 0.6000) \\ (0.7000, 0.4000) & (0.3000, 0.7000) & (0.7071, 0.7071) & (0.9000, 0.2000) \\ (0.3000, 0.8000) & (0.6000, 0.5000) & (0.2000, 0.9000) & (0.7071, 0.7071) \end{pmatrix}$$

$$\text{Min } J = \varepsilon_{12}^+ + \varepsilon_{12}^- + \eta_{12}^+ + \eta_{12}^- + \varepsilon_{13}^+ + \varepsilon_{13}^- + \eta_{13}^+ + \eta_{13}^- + \varepsilon_{14}^+ + \varepsilon_{14}^- + \eta_{14}^+ + \eta_{14}^-$$
$$+ \varepsilon_{23}^+ + \varepsilon_{23}^- + \eta_{23}^+ + \eta_{23}^- + \varepsilon_{24}^+ + \varepsilon_{24}^- + \eta_{24}^+ + \eta_{24}^- + \varepsilon_{34}^+ + \varepsilon_{34}^- + \eta_{34}^+ + \eta_{34}^-$$

$$s.t. \begin{cases} \sqrt{\dfrac{(\omega_1^\mu)^2 + (\omega_2^\nu)^2}{2}} - 0.35 - \varepsilon_{12}^+ + \varepsilon_{12}^- = 0, \quad \sqrt{\dfrac{(\omega_1^\mu)^2 + (\omega_3^\nu)^2}{2}} - 0.40 - \varepsilon_{13}^+ + \varepsilon_{13}^- = 0, \\[2mm]
\sqrt{\dfrac{(\omega_1^\mu)^2 + (\omega_4^\nu)^2}{2}} - 0.80 - \varepsilon_{14}^+ + \varepsilon_{14}^- = 0, \quad \sqrt{\dfrac{(\omega_2^\mu)^2 + (\omega_3^\nu)^2}{2}} - 0.70 - \varepsilon_{23}^+ + \varepsilon_{23}^- = 0, \\[2mm]
\sqrt{\dfrac{(\omega_2^\mu)^2 + (\omega_4^\nu)^2}{2}} - 0.50 - \varepsilon_{24}^+ + \varepsilon_{24}^- = 0, \quad \sqrt{\dfrac{(\omega_3^\mu)^2 + (\omega_4^\nu)^2}{2}} - 0.90 - \varepsilon_{34}^+ + \varepsilon_{34}^- = 0, \\[2mm]
\sqrt{\dfrac{(\omega_1^\nu)^2 + (\omega_2^\mu)^2}{2}} - 0.65 - \eta_{12}^+ + \eta_{12}^- = 0, \quad \sqrt{\dfrac{(\omega_1^\nu)^2 + (\omega_3^\mu)^2}{2}} - 0.70 - \eta_{13}^+ + \eta_{13}^- = 0, \\[2mm]
\sqrt{\dfrac{(\omega_1^\nu)^2 + (\omega_4^\mu)^2}{2}} - 0.30 - \eta_{14}^+ + \eta_{14}^- = 0, \quad \sqrt{\dfrac{(\omega_2^\nu)^2 + (\omega_3^\mu)^2}{2}} - 0.30 - \eta_{23}^+ + \eta_{23}^- = 0, \\[2mm]
\sqrt{\dfrac{(\omega_2^\nu)^2 + (\omega_4^\mu)^2}{2}} - 0.60 - \eta_{24}^+ + \eta_{24}^- = 0, \quad \sqrt{\dfrac{(\omega_3^\nu)^2 + (\omega_4^\mu)^2}{2}} - 0.20 - \eta_{34}^+ + \eta_{34}^- = 0, \\[2mm]
\omega_2^\mu + \omega_3^\mu + \omega_4^\mu + \sqrt{1 - (\omega_1^\nu)^2} \le 1, \quad \omega_1^\mu + \omega_3^\mu + \omega_4^\mu + \sqrt{1 - (\omega_2^\nu)^2} \le 1, \\[2mm]
\omega_1^\mu + \omega_2^\mu + \omega_4^\mu + \sqrt{1 - (\omega_3^\nu)^2} \le 1, \quad \omega_1^\mu + \omega_2^\mu + \omega_3^\mu + \sqrt{1 - (\omega_4^\nu)^2} \le 1, \\[2mm]
\sqrt{1 - (\omega_2^\nu)^2} + \sqrt{1 - (\omega_3^\nu)^2} + \sqrt{1 - (\omega_4^\nu)^2} + \omega_1^\mu \ge 1, \\[2mm]
\sqrt{1 - (\omega_1^\nu)^2} + \sqrt{1 - (\omega_3^\nu)^2} + \sqrt{1 - (\omega_4^\nu)^2} + \omega_2^\mu \ge 1, \\[2mm]
\sqrt{1 - (\omega_1^\nu)^2} + \sqrt{1 - (\omega_2^\nu)^2} + \sqrt{1 - (\omega_4^\nu)^2} + \omega_3^\mu \ge 1, \\[2mm]
\sqrt{1 - (\omega_1^\nu)^2} + \sqrt{1 - (\omega_2^\nu)^2} + \sqrt{1 - (\omega_3^\nu)^2} + \omega_4^\mu \ge 1, \\[2mm]
0 \le \omega_1^\mu \le 1, 0 \le \omega_1^\nu \le 1, (\omega_1^\mu)^2 + (\omega_1^\nu)^2 \le 1, \\[1mm]
0 \le \omega_2^\mu \le 1, 0 \le \omega_2^\nu \le 1, (\omega_2^\mu)^2 + (\omega_2^\nu)^2 \le 1, \\[1mm]
0 \le \omega_3^\mu \le 1, 0 \le \omega_3^\nu \le 1, (\omega_3^\mu)^2 + (\omega_3^\nu)^2 \le 1, \\[1mm]
0 \le \omega_4^\mu \le 1, 0 \le \omega_4^\nu \le 1, (\omega_4^\mu)^2 + (\omega_4^\nu)^2 \le 1, \\[1mm]
\varepsilon_{12}^+ \ge 0, \varepsilon_{12}^- \ge 0, \eta_{12}^+ \ge 0, \eta_{12}^- \ge 0, \varepsilon_{13}^+ \ge 0, \varepsilon_{13}^- \ge 0, \eta_{13}^+ \ge 0, \eta_{13}^- \ge 0, \\[1mm]
\varepsilon_{14}^+ \ge 0, \varepsilon_{14}^- \ge 0, \eta_{14}^+ \ge 0, \eta_{14}^- \ge 0, \varepsilon_{23}^+ \ge 0, \varepsilon_{23}^- \ge 0, \eta_{23}^+ \ge 0, \eta_{23}^- \ge 0, \\[1mm]
\varepsilon_{24}^+ \ge 0, \varepsilon_{24}^- \ge 0, \eta_{24}^+ \ge 0, \eta_{24}^- \ge 0, \varepsilon_{34}^+ \ge 0, \varepsilon_{34}^- \ge 0, \eta_{34}^+ \ge 0, \eta_{34}^- \ge 0, \end{cases}$$

$$\tilde{P}^{(0)} = \begin{pmatrix} (0.7071, 0.7071) & (0.3500, 0.6500) & (0.4000, 0.6448) & (0.7071, 0.6381) \\ (0.6500, 0.3500) & (0.7071, 0.7071) & (0.4188, 0.3621) & (0.7179, 0.3500) \\ (0.6448, 0.4000) & (0.3621, 0.4188) & (0.7071, 0.7071) & (0.7132, 0.4000) \\ (0.6381, 0.7071) & (0.3500, 0.7179) & (0.4000, 0.7132) & (0.7071, 0.7071) \end{pmatrix}$$

$$P^{(3)} = \begin{pmatrix} (0.7071, 0.7071) & (0.3500, 0.6500) & (0.4000, 0.6590) & (0.7314, 0.5726) \\ (0.6500, 0.3500) & (0.7071, 0.7071) & (0.5040, 0.3476) & (0.6701, 0.4265) \\ (0.6590, 0.4000) & (0.3476, 0.5040) & (0.7071, 0.7071) & (0.7642, 0.3606) \\ (0.5726, 0.7314) & (0.4265, 0.6701) & (0.3606, 0.7642) & (0.7071, 0.7071) \end{pmatrix}$$

$$P^* = \begin{pmatrix} (0.5000, 0.5000) & (0.1225, 0.4225) & (0.1600, 0.4900) & (0.6400, 0.0900) \\ (0.4225, 0.1225) & (0.5000, 0.5000) & (0.4900, 0.0900) & (0.2500, 0.3600) \\ (0.4900, 0.1600) & (0.0900, 0.4900) & (0.5000, 0.5000) & (0.8100, 0.0400) \\ (0.0900, 0.6400) & (0.3600, 0.2500) & (0.0400, 0.8100) & (0.5000, 0.5000) \end{pmatrix}$$

Then the score functions are computed as:

$$s(\omega_1^*) = -0.5384, s(\omega_2^*) = -0.1417, s(\omega_3^*)$$
$$= -0.1784, s(\omega_4^*) = -1.$$

Since $s(\omega_2^*) > s(\omega_3^*) > s(\omega_1^*) > s(\omega_4^*)$, the four alternatives can be ranked as $x_2 \succ x_3 \succ x_1 \succ x_4$.

VI. DISCUSSIONS
Based on the numerical examples and comparative study, the characteristic of the proposed method is summarized as follows.

1) As the proposed model aim to derive the Pythagorean fuzzy priority weights by minimizing derivation of via a linear programming model, it can be observed that the model is built on PFN. As mentioned above, PFN have more powerful abilities than IFN do in modeling the uncertainty of practical decision-making problems. From this perspective, it can be concluded the proposed model has a wider range of applications.

(2) The specific implementation steps of the two methods are different. Wang [48]'s method developed a linear goal programming model to obtain its intuitionistic fuzzy weights, and then the best alternatives is selected. In the proposed models, we develop a linear goal programming model to obtain its Pythagorean fuzzy weights, and for the PFPR that does not satisfy the consistency, the consistency index is defined to measure the degree of consistency, and a consistency adjustment algorithm is proposed, this method makes our results more accurate.

(3) Finally, the proposed adjustment algorithm can be used to improve the additive consistency of a PFPR. By solving this algorithm, not only can the additive consistency of a PFPR be improved, but also can make reference for decision makers before making decisions.

However, the proposed methods still have some limitations. First, it needed to solve a linear programming model to obtain the Pythagorean fuzzy weights with additive consistency, and for PFPR that does not meet consistency, it needs to be adjusted, it may be a bit complex compared with other methods. However, these models can be easily solved by using some optimization packages, such as Lingo, MATLAB and CPLEX. In addition, the threshold of the additive consistency index is assumed to be given by decision makers. We argue that it will be more convincing if the threshold can be determined by using some automatic methods.

VII. CONCLUSION
As a new type of preference relation, PFPR not only expands the application scope of preference relations, but also more fully expresses the views of decision makers. In this paper, we mainly discussed the application of PFPR in decision making. Firstly, we defined PFPR and its additive consistency based on IFPR, and discussed some properties that satisfy the additive consistency of PFPR. Secondly, we developed a linear goal programming model for generating Pythagorean fuzzy weight based on the additive consistency of PFPR.

Then, for PFPRs that do not satisfy additive consistency, we defined the consistency index of PFPR and develop a feasible algorithm for improving consistency degree of PFPR. In the next section, we developed an approach to decision-making based on PFPR and its additive consistency. The proposed decision-making process and models may be used in many real-world applications in which the DMs may be able to provide his/her preferences for alternatives to a PFN, this was confirmed in the final section of this paper.

During the PPFR research process, we can find that there are some new directions that should be considered in future research, such as the application of PFPR in group decision making, the multiplicative consistency of PFPR and the consensus reaching process of groups in Pythagorean fuzzy environment.

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LI ZHANG is currently pursuing the Ph.D. degree in microelectronics and solid-state electronics with Anhui University. He is also a Lecturer with the School of Electronics and Information Engineering, Anhui University. His main research interests include computing in memory, control engineering theory, and decision making.

LIGANG ZHOU received the Ph.D. degree in operational research from Anhui University, Hefei, China, in 2013. He is currently a Professor with the School of Mathematical Sciences, Anhui University. He has contributed over 100 journal articles to professional journals, such as the IEEE Transactions on Fuzzy Systems, *Fuzzy Sets and Systems*, *Information Sciences*, *Applied Mathematical Modelling*, and *Group Decision and Negotiation*. He is also a Paper Reviewer of many international journals. His current research interests include information fusion, group decision making, aggregation operators, and combined forecasting.

KAIFANG YANG is currently the Graduate Student of electronic information with Anhui University. His main research interests include control engineering and sensing technology.

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