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Global Synchronization and Anti-Synchronization of Fractional-Order Complex-Valued Gene Regulatory Networks With Time-Varying Delays

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ABSTRACT This paper presents the analytical and numerical investigation on the global synchronization and anti-synchronization for a class of drive-response systems of fractional-order complex-valued gene regulatory networks with time-varying delays (DFGRNs). In our design, two kinds of adaptive feedback controllers are used to synchronize and anti-synchronize the proposed drive-response systems, and some sufficient conditions on the global asymptotical synchronization and anti-synchronization are given with the methods of the fractional Lyapunov-like functions and the fractional-order inequalities. In the numerical simulations, two minimum "estimated time", T_1 and T_2, are computed to achieve the synchronization and anti-synchronization. We find that T_1 and T_2 increase with the decreasing of the fractional order of DFGRNs.

INDEX TERMS Complex-valued, feedback controller, fractional-order, gene regulatory networks (GRNs), synchronization and anti-synchronization, time-varying delays.

I. INTRODUCTION

Genetic regulatory networks (GRNs) are fundamental and important biological networks that describe the interaction functions in gene expressions between DNAs, RNAs, proteins and small molecules in an organism [1]–[5]. Various GRNs models, such as Boolean networks [2], Bayesian networks [5], Petri networks [6], differential equation models [7]–[9], have been proposed by researchers over a period of time. And the researches on GRNs not only provide a powerful tool for elucidating the gene regulation processes in living organisms, but also contribute to the diagnosis of cancers, diabetes and other complex diseases [10]–[16].

It is worthy to note that, the differential equation models involving integer-order type and fractional-order type, where the state variables usually denote concentrations of messenger ribonucleic acids (mRNAs), proteins and other small molecules, is one of important GRNs model and widely applied to describe the gene regulatory process [10], [11], [17]. Particularly, in [18]–[20], the authors pointed out that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various processes. Taking into account these facts, the incorporation of a memory term into a neural network model is an extremely important improvement [21]. Subsequently, the dynamics of the fractional-order GRNs becomes a hot topic and some remarkable results were reported in [7], [12], [16], [22]. Ji *et al.* [22] constructed fractional-order GRNs and demonstrated that the fractional-order model has stronger data approximation ability, which is more suitable for modeling gene regulatory mechanism.

Since the pioneer work of Perora and Carroll [23], synchronization and anti-synchronization as one of the most important dynamic behaviors has attracted increasing attentions and many significant results were derived [21], [24]–[35]. Different types of synchronization and anti-synchronization, such as adaptive synchronization [27], adaptive anti-synchronization [28], finite-time synchronization [33], [36],

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finite-time anti-synchronization [34], Mittag-Leffler synchronization [30], [31], [37] and projective synchronization [21], have been widely investigated. Song *et al.* [25] considered the synchronization of fractional-order Lorenz chaotic systems and Chen chaotic systems with time delay, and designed the compensation controllers and optimal controllers. Chen *et al.* [26] studied the synchronization of memristor-based BAM neural networks with delays and realized asymptotic synchronization and exponential synchronization by designing two kinds of adaptive feedback controllers, respectively. Bao *et al.* [27] investigated the synchronization for fractional-order complex-valued neural networks with constant time delay and obtained the sufficient conditions of synchronization by using linear delay feedback control. And the main methods analyzing the synchronization and anti-synchronization of fractional-order dynamic system include, but are not limited to, direct quaternion approach [37], fractional-order comparison theory [18], fractional-order inequality [27] and fractional Lyapunov function approach [11], [18], [31]. The fractional Lyapunov method is a powerful tool for analyzing the synchronization and anti-synchronization of fractional-order dynamic system, which can be verified easily without solving the system.

In recent years, the investigation on the synchronization of GRNs, which may be helpful to explore the biological rhythm and internal mechanisms at the molecular and cellular levels, has attracted attentions of researchers [11], [17]. Jiang *et al.* [17] considered the finite-time synchronization of integer-order GRNs without time delays, and established some sufficient conditions for finite-time stochastic synchronization by designing a continuous finite-time controller. Due to slow biochemical processes such as gene transcription, translation and transportation, time delays are omnipresent in GRNs [4], [9], [11], [14], [15], [38]–[40]. Qiao *et al.* [11] established some sufficient conditions of finite-time synchronization for fractional-order GRNs with constant time delay by designing respectively the state feedback controller and the adaptive controller. As time delays often change with time and its precise measurement is difficult in the real biological networks [41], it is therefore better practical significance to consider the time-varying delays, rather than the constant delays, for the GRNs [9], [40].

However, according to our knowledge, few researches have been given to the synchronization and anti-synchronization of fractional-order complex-valued GRNs with time-varying delays.

From above discussions, we will focus on the global asymptotical synchronization and anti-synchronization for a class of complex-valued FGRNs with time-varying delays. The rest of this paper is organized as follows. The preliminaries and the model description are provided in Section II. Section III proposes some sufficient criteria on global asymptotical synchronization and anti-synchronization for the DFGRNs. Section IV gives some numerical simulations to support our findings. And finally, Section V presents a brief discussion and the summary around the main results.

II. PROBLEM DESCRIPTION AND PRELIMINARIES

In general, three definitions of the fractional-order derivatives, the Grunwald-Letnikov derivative, the Riemann-Liouville derivative and the Caputo derivative, are mentioned. Rather than the other two definitions, the initial conditions for the Caputo fractional derivative can be determined only by the integer derivative, and the Laplace transform can be performed more concisely, it is therefore widely used in differential equation models. In particular, the Caputo fractional derivative is more suitable for the GRNs due to its more accurate description of the memory and hereditary characteristics of various materials and processes [37], [42]. Therefore, we adopt the Caputo fractional-order derivative.

Definition 1 [42]: The fractional integral of order q for a function $f(t)$ is defined as

$$ {}_aI_t^q f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} f(\tau) d\tau, $$

where $t \geq a, a \in R, q > 0$. The Gamma function $\Gamma(q)$ is defined by the integral $\Gamma(q) = \int_0^\infty t^{q-1}e^{-t}dt$.

Definition 2 [42]: The Caputo's fractional derivative of order q for a function $f(t)$ is defined by

$$ {}_a^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{1}{(t-\tau)^{q-n+1}} f^{(n)}(\tau) d\tau, $$

where $t \geq a$ and n is a positive integer such that $n-1 < q < n$.

The following continuous fractional-order complex-valued GRNs with time-varying delays are considered as the drive system:

$$ \begin{cases} D^q m(t) = -Am(t) + WF(p(t)) \\ \qquad +EG(p(t-\tau_1(t))) + B + J_1(t), \\ D^q p(t) = -Cp(t) + Dm(t) \\ \qquad +Hm(t-\tau_2(t)) + J_2(t), \end{cases} \tag{1} $$

where

$$ \begin{aligned} m(t) &= [m_1(t), \cdots, m_n(t)]^T \in C^n, \\ p(t) &= [p_1(t), \cdots, p_n(t)]^T \in C^n, t \geq 0, \\ A &= \mathrm{diag}\{a_1, \cdots, a_n\} \in R^{n\times n}, \\ C &= \mathrm{diag}\{c_1, \cdots, c_n\} \in R^{n\times n}, \\ D &= \mathrm{diag}\{d_1, \cdots, d_n\} \in C^{n\times n}, \\ H &= \mathrm{diag}\{h_1, \cdots, h_n\} \in C^{n\times n}, \\ W &= (w_{jk})_{n\times n} \in C^{n\times n}, \\ E &= (e_{jk})_{n\times n} \in C^{n\times n}, \\ J_1(t) &= (J_{11}(t), \cdots, J_{1n}(t))^T \in C^n, \\ J_2(t) &= (J_{21}(t), \cdots, J_{2n}(t))^T \in C^n, \\ F(p(t)) &= [F_1(p_1(t)), \cdots, F_n(p_n(t))]^T : C^n \to C^n, \\ B &= (B_1, \cdots, B_n)^T \in C^n, \\ G(p(t-\tau_1(t))) &= [G_1(p_1(t-\tau_1(t))), \\ &\quad \cdots, G_n(p_n(t-\tau_1(t)))]^T : C^n \to C^n, \end{aligned} $$

$D^q = {}_0^C D_t^q$ represents the Caputo's fractional derivative, and $q \in (0,1)$, $m(t), p(t)$ represent the state vectors, and the moduli of $m_i(t)$ and $p_i(t)$ represent respectively the mRNA concentrations and the protein concentrations of the ith node; The parameters $a_i > 0$ and $c_i > 0$ respectively represent the decay rates of mRNA and protein; The moduli of d_i and h_i mean the translation rates; Both $F_j(p_j(t))$ and $G_j(p_j(t-\tau_1(t)))$ represent the feedback regulation of the protein on the transcription; $B_j = \sum_{k\in I_j} b_{jk} + \sum_{k\in \bar{I}_j} \bar{b}_{jk}$, the moduli of b_{jk} and $\bar{b}_{jk}$ are bounded constants representing the dimensionless transcriptional rate of transcription factor k to j at time t and $t-\tau_1(t)$, and $I_j, \bar{I}_j$, respectively, represent the set of all the k where the transcription factor k is a repressor of gene j at time t and $t-\tau_1(t)$; The matrix $W = (w_{jk})_{n\times n}, E = (e_{jk})_{n\times n}$ mean the coupling matrix of the gene network, which are defined as follows:

$$w_{jk}(e_{jk}) = \begin{cases} b_{jk}(\bar{b}_{jk}), & \text{if } k \text{ is an activator of gene } j, \\ -b_{jk}(-\bar{b}_{jk}), & \text{if } k \text{ is a repressor of gene } j, \\ 0, & \text{if there is no link from } k \text{ to } j. \end{cases}$$

The transcriptional delay $\tau_1(t)$ and translational delay $\tau_2(t)$ are bounded continuous functions on R with $0 \le \tau_1(t) \le \tau_1^*$ and $0 \le \tau_2(t) \le \tau_2^*$, here τ_1^* and τ_2^* are positive constants. $J_1(t)$ and $J_2(t)$ represent the external input vectors.

The response system of the drive system (1) is as follows:

$$\begin{cases} D^q\widetilde{m}(t) = -A\widetilde{m}(t) + WF(\widetilde{p}(t)) + EG(\widetilde{p}(t-\tau_1(t))) \\ \qquad +B + J_1(t) + U_1(t), \\ D^q\widetilde{p}(t) = -C\widetilde{p}(t) + D\widetilde{m}(t) + H\widetilde{m}(t-\tau_2(t)) \\ \qquad +J_2(t) + U_2(t), \end{cases} \tag{2}$$

where $U_1(t) = (U_{11}(t), \cdots, U_{1n}(t))^T \in C^n$ and $U_2(t) = (U_{21}(t), \cdots, U_{2n}(t))^T \in C^n$ represent the control input vectors.

Let

$$\begin{aligned} m(t) &= r(t) + i\eta(t), p(t) = \lambda(t) + i\mu(t), \\ \widetilde{m}(t) &= \widetilde{r}(t) + i\widetilde{\eta}(t), \widetilde{p}(t) = \widetilde{\lambda}(t) + i\widetilde{\mu}(t), \\ W^R &= Re(W) = [w_{jk}^R]_{n\times n}, \\ W^I &= Im(W) = [w_{jk}^I]_{n\times n}, \\ E^R &= Re(E) = [e_{jk}^R]_{n\times n}, \\ E^I &= Im(E) = [e_{jk}^I]_{n\times n}, \\ D^R &= Re(D) = \text{diag}\{d_1^R, \cdots, d_n^R\}, \\ D^I &= Im(D) = \text{diag}\{d_1^I, \cdots, d_n^I\}, \\ H^R &= Re(H) = \text{diag}\{h_1^R, \cdots, h_n^R\}, \\ H^I &= Im(H) = \text{diag}\{h_1^I, \cdots, h_n^I\}, \\ J_1(t) &= J_1^R(t) + iJ_1^I(t), \\ J_2(t) &= J_2^R(t) + iJ_2^I(t), \\ U_1(t) &= u_1(t) + iv_1(t), \\ U_2(t) &= u_2(t) + iv_2(t), \\ B^R &= Re(B) = (B_1^R, \cdots, B_n^R)^T, \\ B^I &= Im(B) = (B_1^I, \cdots, B_n^I)^T, \end{aligned}$$

$$\begin{aligned} F_j(p_j(t)) &= F_j^R(\lambda_j(t), \mu_j(t)) + iF_j^I(\lambda_j(t), \mu_j(t)), \\ G_j(p_j(t-\tau_1(t))) &= G_j^R(\lambda_j(t-\tau_1(t)), \mu_j(t-\tau_1(t))) \\ &\quad + iG_j^I(\lambda_j(t-\tau_1(t)), \mu_j(t-\tau_1(t))) \end{aligned}$$

where

$$\begin{aligned} r(t) &= \big(r_1(t), \cdots, r_n(t)\big)^T, \\ \eta(t) &= \big(\eta_1(t), \cdots, \eta_n(t)\big)^T, \\ \lambda(t) &= \big(\lambda_1(t), \cdots, \lambda_n(t)\big)^T, \\ \mu(t) &= \big(\mu_1(t), \cdots, \mu_n(t)\big)^T, \\ \widetilde{r}(t) &= \big(\widetilde{r}_1(t), \cdots, \widetilde{r}_n(t)\big)^T, \\ \widetilde{\eta}(t) &= \big(\widetilde{\eta}_1(t), \cdots, \widetilde{\eta}_n(t)\big)^T, \\ \widetilde{\lambda}(t) &= \big(\widetilde{\lambda}_1(t), \cdots, \widetilde{\lambda}_n(t)\big)^T, \\ \widetilde{\mu}(t) &= \big(\widetilde{\mu}_1(t), \cdots, \widetilde{\mu}_n(t)\big)^T, \\ J_1^R(t) &= \big(J_{11}^R(t), \cdots, J_{1n}^R(t)\big)^T, \\ J_1^I(t) &= \big(J_{11}^I(t), \cdots, J_{1n}^I(t)\big)^T, \\ J_2^R(t) &= \big(J_{21}^R(t), \cdots, J_{2n}^R(t)\big)^T, \\ J_2^I(t) &= \big(J_{21}^I(t), \cdots, J_{2n}^I(t)\big)^T, \\ u_1(t) &= \big(u_{11}(t), \cdots, u_{1n}(t)\big)^T, \\ u_2(t) &= \big(u_{21}(t), \cdots, u_{2n}(t)\big)^T, \\ v_1(t) &= \big(v_{11}(t), \cdots, v_{1n}(t)\big)^T, \\ v_2(t) &= \big(v_{21}(t), \cdots, v_{2n}(t)\big)^T, \end{aligned}$$

$F_j^R(\cdot,\cdot)$, $F_j^I(\cdot,\cdot)$, $G_j^R(\cdot,\cdot)$, $G_j^I(\cdot,\cdot)$: $R^2 \to R$ ($j = 1, 2, \cdots, n$) and $r(t), \eta(t), \lambda(t), \mu(t), \widetilde{r}(t), \widetilde{\eta}(t), \widetilde{\lambda}(t), \widetilde{\mu}(t), J_1^R(t), J_1^I(t), J_2^R(t), J_2^I(t), u_1(t), u_2(t), v_1(t), v_2(t) \in R^n$.

Then the drive system (1) and the response system (2) can be expressed respectively by separating them into the real part and the imaginary part as

$$\begin{cases} D^q r(t) = -Ar(t) + W^R F^R(\lambda(t), \mu(t)) \\ \qquad -W^I F^I(\lambda(t), \mu(t)) \\ \qquad +E^R G^R(\lambda(t-\tau_1(t)), \mu(t-\tau_1(t))) \\ \qquad -E^I G^I(\lambda(t-\tau_1(t)), \mu(t-\tau_1(t))) \\ \qquad +B^R + J_1^R(t), \\ D^q\eta(t) = -A\eta(t) + W^R F^I(\lambda(t), \mu(t)) \\ \qquad +W^I F^R(\lambda(t), \mu(t)) \\ \qquad +E^I G^R(\lambda(t-\tau_1(t)), \mu(t-\tau_1(t))) \\ \qquad +E^R G^I(\lambda(t-\tau_1(t)), \mu(t-\tau_1(t))) \\ \qquad +B^I + J_1^I(t), \\ D^q\lambda(t) = -C\lambda(t) + D^R r(t) - D^I\eta(t) + J_2^R(t) \\ \qquad +H^R r(t-\tau_2(t)) - H^I\eta(t-\tau_2(t)), \\ D^q\mu(t) = -C\mu(t) + D^R\eta(t) + D^I r(t) + J_2^I(t) \\ \qquad +H^I r(t-\tau_2(t)) + H^R\eta(t-\tau_2(t)), \end{cases} \tag{3}$$

and

$$\begin{cases} D^q\widetilde{r}(t) = -A\widetilde{r}(t) + W^R F^R(\widetilde{\lambda}(t), \widetilde{\mu}(t)) \\ \qquad -W^I F^I(\widetilde{\lambda}(t), \widetilde{\mu}(t)) \\ \qquad +E^R G^R(\widetilde{\lambda}(t-\tau_1(t)), \widetilde{\mu}(t-\tau_1(t))) \\ \qquad -E^I G^I(\widetilde{\lambda}(t-\tau_1(t)), \widetilde{\mu}(t-\tau_1(t))) \\ \qquad +B^R + J_1^R(t) + u_1(t), \\ D^q\widetilde{\eta}(t) = -A\widetilde{\eta}(t) + W^R F^I(\widetilde{\lambda}(t), \widetilde{\mu}(t)) \\ \qquad +W^I F^R(\widetilde{\lambda}(t), \widetilde{\mu}(t)) \\ \qquad +E^I G^R(\widetilde{\lambda}(t-\tau_1(t)), \widetilde{\mu}(t-\tau_1(t))) \\ \qquad +E^R G^I(\widetilde{\lambda}(t-\tau_1(t)), \widetilde{\mu}(t-\tau_1(t))) \\ \qquad +B^I + J_1^I(t) + v_1(t), \\ D^q\widetilde{\lambda}(t) = -C\widetilde{\lambda}(t) + D^R\widetilde{r}(t) - D^I\widetilde{\eta}(t) \\ \qquad +H^R\widetilde{r}(t-\tau_2(t)) - H^I\widetilde{\eta}(t-\tau_2(t)) \\ \qquad +J_2^R(t) + u_2(t), \\ D^q\widetilde{\mu}(t) = -C\widetilde{\mu}(t) + D^R\widetilde{\eta}(t) + D^I\widetilde{r}(t) \\ \qquad +H^I\widetilde{r}(t-\tau_2(t)) + H^R\widetilde{\eta}(t-\tau_2(t)) \\ \qquad +J_2^I(t) + v_2(t). \end{cases} \tag{4}$$

System (3) and (4) can also be described respectively as follows:

$$\begin{cases} D^q r_j(t) = -a_j r_j(t) + \sum_{k=1}^{n} w_{jk}^R F_k^R(\lambda_k(t), \mu_k(t)) \\ \qquad - \sum_{k=1}^{n} w_{jk}^I F_k^I(\lambda_k(t), \mu_k(t)) + B_j^R + J_{1j}^R(t) \\ \qquad + \sum_{k=1}^{n} e_{jk}^R G_k^R(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t))) \\ \qquad - \sum_{k=1}^{n} e_{jk}^I G_k^I(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t))), \\ D^q \eta_j(t) = -a_j \eta_j(t) + \sum_{k=1}^{n} w_{jk}^R F_k^I(\lambda_k(t), \mu_k(t)) \\ \qquad + \sum_{k=1}^{n} w_{jk}^I F_k^R(\lambda_k(t), \mu_k(t)) + B_j^I + J_{1j}^I(t) \\ \qquad + \sum_{k=1}^{n} e_{jk}^I G_k^R(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t))) \\ \qquad + \sum_{k=1}^{n} e_{jk}^R G_k^I(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t))), \\ D^q \lambda_j(t) = -c_j\lambda_j(t) + d_j^R r_j(t) - d_j^I \eta_j(t) + J_{2j}^R(t) \\ \qquad + h_j^R r_j(t-\tau_2(t)) - h_j^I \eta_j(t-\tau_2(t)), \\ D^q \mu_j(t) = -c_j\mu_j(t) + d_j^R \eta_j(t) + d_j^I r_j(t) + J_{2j}^I(t) \\ \qquad + h_j^I r_j(t-\tau_2(t)) + h_j^R \eta_j(t-\tau_2(t)), \end{cases} \tag{5}$$

and

$$\begin{cases} D^q\widetilde{r}_j(t) = -a_j\widetilde{r}_j(t) + \sum_{k=1}^{n} w_{jk}^R F_k^R(\widetilde{\lambda}_k(t), \widetilde{\mu}_k(t)) \\ \qquad - \sum_{k=1}^{n} w_{jk}^I F_k^I(\widetilde{\lambda}_k(t), \widetilde{\mu}_k(t)) \\ \qquad + \sum_{k=1}^{n} e_{jk}^R G_k^R(\widetilde{\lambda}_k(t-\tau_1(t)), \widetilde{\mu}_k(t-\tau_1(t))) \\ \qquad - \sum_{k=1}^{n} e_{jk}^I G_k^I(\widetilde{\lambda}_k(t-\tau_1(t)), \widetilde{\mu}_k(t-\tau_1(t))) \\ \qquad +B_j^R + J_{1j}^R(t) + u_{1j}(t), \\ D^q\widetilde{\eta}_j(t) = -a_j\widetilde{\eta}_j(t) + \sum_{k=1}^{n} w_{jk}^R F_k^I(\widetilde{\lambda}_k(t), \widetilde{\mu}_k(t)) \\ \qquad + \sum_{k=1}^{n} w_{jk}^I F_k^R(\widetilde{\lambda}_k(t), \widetilde{\mu}_k(t)) \\ \qquad + \sum_{k=1}^{n} e_{jk}^I G_k^R(\widetilde{\lambda}_k(t-\tau_1(t)), \widetilde{\mu}_k(t-\tau_1(t))) \\ \qquad + \sum_{k=1}^{n} e_{jk}^R G_k^I(\widetilde{\lambda}_k(t-\tau_1(t)), \widetilde{\mu}_k(t-\tau_1(t))) \\ \qquad +B_j^I + J_{1j}^I(t) + v_{1j}(t), \\ D^q\widetilde{\lambda}_j(t) = -c_j\widetilde{\lambda}_j(t) + d_j^R\widetilde{r}_j(t) - d_j^I\widetilde{\eta}_j(t) \\ \qquad +h_j^R\widetilde{r}_j(t-\tau_2(t)) - h_j^I\widetilde{\eta}_j(t-\tau_2(t)) \\ \qquad +J_{2j}^R(t) + u_{2j}(t), \\ D^q\widetilde{\mu}_j(t) = -c_j\widetilde{\mu}_j(t) + d_j^R\widetilde{\eta}_j(t) + d_j^I\widetilde{r}_j(t) \\ \qquad +h_j^I\widetilde{r}_j(t-\tau_2(t)) + h_j^R\widetilde{\eta}_j(t-\tau_2(t)) \\ \qquad +J_{2j}^I(t) + v_{2j}(t). \end{cases} \tag{6}$$

The initial conditions of systems (1) and (2) are given respectively as follows:

$$\begin{cases} m_j(s) = \phi_j(s) + i\psi_j(s),\ s \in [-\tau_2^*, 0], \\ p_j(s) = \alpha_j(s) + i\beta_j(s),\ s \in [-\tau_1^*, 0], \end{cases} \tag{7}$$

and

$$\begin{cases} \widetilde{m}_j(s) = \widetilde{\phi}_j(s) + i\widetilde{\psi}_j(s),\ s \in [-\tau_2^*, 0], \\ \widetilde{p}_j(s) = \widetilde{\alpha}_j(s) + i\widetilde{\beta}_j(s),\ s \in [-\tau_1^*, 0], \end{cases} \tag{8}$$

where $\phi_j(s), \psi_j(s), \widetilde{\phi}_j(s), \widetilde{\psi}_j(s) \in C([-\tau_2^*, 0], R)$, $\alpha_j(s), \beta_j(s), \widetilde{\alpha}_j(s), \widetilde{\beta}_j(s) \in C([-\tau_1^*, 0], R)$, $j = 1, 2, \cdots, n$.

Assumption 1: Assume that the functions $F_j(p_j(t))$ and $G_j(p_j(t-\tau_1(t)))$ can be expressed by separating into their real and imaginary parts as $F_j(p_j(t)) = F_j^R(\lambda_j(t), \mu_j(t)) + iF_j^I(\lambda_j(t), \mu_j(t))$, $G_j(p_j(t-\tau_1(t))) = G_j^R(\lambda_j(t-\tau_1(t)), \mu_j(t-\tau_1(t))) + iG_j^I(\lambda_j(t-\tau_1(t)), \mu_j(t-\tau_1(t)))$, respectively, and the following inequalities hold:

$$\begin{aligned} &|F_j^R(\widetilde{\lambda}_j(t), \widetilde{\mu}_j(t)) - F_j^R(\lambda_j(t), \mu_j(t))| \\ &\quad \le \delta_j^1|\widetilde{\lambda}_j(t) - \lambda_j(t)| + \delta_j^2|\widetilde{\mu}_j(t) - \mu_j(t)|, \\ &|F_j^I(\widetilde{\lambda}_j(t), \widetilde{\mu}_j(t)) - F_j^I(\lambda_j(t), \mu_j(t))| \\ &\quad \le \delta_j^3|\widetilde{\lambda}_j(t) - \lambda_j(t)| + \delta_j^4|\widetilde{\mu}_j(t) - \mu_j(t)|, \\ &|G_j^R(\widetilde{\lambda}_j(t-\tau_1(t)), \widetilde{\mu}_j(t-\tau_1(t))) \\ &\quad -G_j^R(\lambda_j(t-\tau_1(t)), \mu_j(t-\tau_1(t)))| \\ &\quad \le \delta_j^5|\widetilde{\lambda}_j(t-\tau_1(t)) - \lambda_j(t-\tau_1(t))| \\ &\quad +\delta_j^6|\widetilde{\mu}_j(t-\tau_1(t)) - \mu_j(t-\tau_1(t))|, \\ &|G_j^I(\widetilde{\lambda}_j(t-\tau_1(t)), \widetilde{\mu}_j(t-\tau_1(t))) \end{aligned}$$

$$
\begin{aligned}
&-G_j^I(\lambda_j(t-\tau_1(t)),\mu_j(t-\tau_1(t)))| \\
&\le \delta_j^7|\widetilde{\lambda}_j(t-\tau_1(t))-\lambda_j(t-\tau_1(t))| \\
&\quad +\delta_j^8|\widetilde{\mu}_j(t-\tau_1(t))-\mu_j(t-\tau_1(t))|,
\end{aligned}
$$

where $\delta_j^k > 0$ are Lipschitz constants, $k = 1, 2, \cdots, 8, j = 1, 2, \cdots, n$.

Lemma 1: [43] If $x(t) \in C^1([0, +\infty), R)$ is a continuously differentiable function, then the following inequality holds almost everywhere:

$$D^q|x(t)| \le \mathrm{sgn}(x(t))D^q x(t), \quad 0 < q \le 1.$$

Lemma 2: [21] Suppose that $g(t)$ is a differential and non-decreasing function on $t \in [0, +\infty)$, then for any constant b and $t \in [0, +\infty)$, there exists

$$D^q(g(t)-b)^2 \le 2(g(t)-b)D^q g(t),$$

where $0 < q < 1$.

Lemma 3: Let $q \in (0, 1)$ and $\Psi(t)$, $\Phi(t)$ are differential and nonnegative functions on $t \in [0, +\infty)$. Suppose that for positive constants L^*, M^*, the following inequalities hold: for all $t \ge 0$, (i) $D^q\Psi(t) \le -L^*\Phi(t)$, (ii) $|D^q\Phi(t)| \le M^*$. Then $\lim_{t\to+\infty}\Phi(t) = 0$.

Proof: The proof of the Lemma 3 is similar to that of $\lim_{t\to\infty} U(t) = 0$ in Theorem 1 [44]. □

Remark 1: Since the above Lemma 3 is not given in [44], for the readers' convenience, we give the proof in the Appendix.

III. MAIN RESULTS

First, we consider the global synchronization of DFGRNs based on the adaptive feedback controller.

Let $x_j(t) = \widetilde{m}_j(t) - m_j(t) = x_j^R(t) + ix_j^I(t)$, $y_j(t) = \widetilde{p}_j(t) - p_j(t) = y_j^R(t) + iy_j^I(t)$ $(j = 1, 2, \cdots, n)$, where

$$
\begin{aligned}
&x_j^R(t) = \widetilde{r}_j(t) - r_j(t), x_j^I(t) = \widetilde{\eta}_j(t) - \eta_j(t), \\
&y_j^R(t) = \widetilde{\lambda}_j(t) - \lambda_j(t), y_j^I(t) = \widetilde{\mu}_j(t) - \mu_j(t).
\end{aligned}
$$

The adaptive feedback controllers are designed as follows:

$$
\begin{cases}
u_{1j}(t) = -\bar{a}_{1j}(t)x_j^R(t) \\
\qquad\quad -\mathrm{sgn}(x_j^R(t))\bar{c}_{1j}(t)|x_j^R(t-\tau_2(t))|, \\
v_{1j}(t) = -\bar{b}_{1j}(t)x_j^I(t) \\
\qquad\quad -\mathrm{sgn}(x_j^I(t))\bar{d}_{1j}(t)|x_j^I(t-\tau_2(t))|, \\
u_{2j}(t) = -\bar{a}_{2j}(t)y_j^R(t) \\
\qquad\quad -\mathrm{sgn}(y_j^R(t))\bar{c}_{2j}(t)|y_j^R(t-\tau_1(t))|, \\
v_{2j}(t) = -\bar{b}_{2j}(t)y_j^I(t) \\
\qquad\quad -\mathrm{sgn}(y_j^I(t))\bar{d}_{2j}(t)|y_j^I(t-\tau_1(t))|, \\
D^q\bar{a}_{1j}(t) = L_{1j}|x_j^R(t)|, \\
D^q\bar{c}_{1j}(t) = L_{2j}|x_j^R(t-\tau_2(t))|, \\
D^q\bar{b}_{1j}(t) = L_{3j}|x_j^I(t)|, \\
D^q\bar{d}_{1j}(t) = L_{4j}|x_j^I(t-\tau_2(t))|, \\
D^q\bar{a}_{2j}(t) = L_{5j}|y_j^R(t)|, \\
D^q\bar{c}_{2j}(t) = L_{6j}|y_j^R(t-\tau_1(t))|, \\
D^q\bar{b}_{2j}(t) = L_{7j}|y_j^I(t)|, \\
D^q\bar{d}_{2j}(t) = L_{8j}|y_j^I(t-\tau_1(t))|,
\end{cases}
\tag{9}
$$

where $L_{kj}(k = 1, 2, \cdots, 8, j = 1, 2, \cdots, n)$ are arbitrary positive constants, $\bar{a}_{kj}(t)$, $\bar{b}_{kj}(t)$, $\bar{c}_{kj}(t)$, $\bar{d}_{kj}(t)$ $(k = 1, 2, j = 1, 2, \cdots, n)$ are adaptive coupling strengths.

Remark 2: The state feedback control is designed in many dynamic systems such as GRNs. The adaptive feedback control built on adaptive techniques without knowing the values of the unknown parameters in advance is more flexible than the state feedback control, and can adjust the coupling weights adaptively to avoid high feedback gains [11].

It is well known that when $t \to \infty$, the facts $x_j^R(t) \to 0$, $x_j^I(t) \to 0$, $y_j^R(t) \to 0$, $y_j^I(t) \to 0$ $(i = 1, 2, \cdots, n)$ mean that the drive system (1) and the response system (2) are synchronized.

From systems (5) and (6), we can get the following error system:

$$
\begin{cases}
D^q x_j^R(t) = -a_j x_j^R(t) \\
\quad +\sum\limits_{k=1}^{n} w_{jk}^R[F_k^R(\widetilde{\lambda}_k(t),\widetilde{\mu}_k(t)) - F_k^R(\lambda_k(t),\mu_k(t))] \\
\quad -\sum\limits_{k=1}^{n} w_{jk}^I[F_k^I(\widetilde{\lambda}_k(t),\widetilde{\mu}_k(t)) - F_k^I(\lambda_k(t),\mu_k(t))] \\
\quad +\sum\limits_{k=1}^{n} e_{jk}^R[G_k^R(\widetilde{\lambda}_k(t-\tau_1(t)),\widetilde{\mu}_k(t-\tau_1(t))) \\
\quad -G_k^R(\lambda_k(t-\tau_1(t)),\mu_k(t-\tau_1(t)))] \\
\quad -\sum\limits_{k=1}^{n} e_{jk}^I[G_k^I(\widetilde{\lambda}_k(t-\tau_1(t)),\widetilde{\mu}_k(t-\tau_1(t))) \\
\quad -G_k^I(\lambda_k(t-\tau_1(t)),\mu_k(t-\tau_1(t)))] + u_{1j}(t), \\
D^q x_j^I(t) = -a_j x_j^I(t) \\
\quad +\sum\limits_{k=1}^{n} w_{jk}^R[F_k^I(\widetilde{\lambda}_k(t),\widetilde{\mu}_k(t)) - F_k^I(\lambda_k(t),\mu_k(t))] \\
\quad +\sum\limits_{k=1}^{n} w_{jk}^I[F_k^R(\widetilde{\lambda}_k(t),\widetilde{\mu}_k(t)) - F_k^R(\lambda_k(t),\mu_k(t))] \\
\quad +\sum\limits_{k=1}^{n} e_{jk}^I[G_k^R(\widetilde{\lambda}_k(t-\tau_1(t)),\widetilde{\mu}_k(t-\tau_1(t))) \\
\quad -G_k^R(\lambda_k(t-\tau_1(t)),\mu_k(t-\tau_1(t)))] \\
\quad +\sum\limits_{k=1}^{n} e_{jk}^R[G_k^I(\widetilde{\lambda}_k(t-\tau_1(t)),\widetilde{\mu}_k(t-\tau_1(t))) \\
\quad -G_k^I(\lambda_k(t-\tau_1(t)),\mu_k(t-\tau_1(t)))] + v_{1j}(t), \\
D^q y_j^R(t) = -c_j y_j^R(t) + d_j^R x_j^R(t) - d_j^I x_j^I(t) \\
\quad +h_j^R x_j^R(t-\tau_2(t)) - h_j^I x_j^I(t-\tau_2(t)) + u_{2j}(t), \\
D^q y_j^I(t) = -c_j y_j^I(t) + d_j^R x_j^I(t) + d_j^I x_j^R(t) \\
\quad +h_j^I x_j^R(t-\tau_2(t)) + h_j^R x_j^I(t-\tau_2(t)) + v_{2j}(t).
\end{cases}
\tag{10}
$$

From (7) and (8), we have the initial conditions of system (10) are as follows

$$
\begin{cases}
x_j^R(s) = \widetilde{\phi}_j(s) - \phi_j(s), s \in [-\tau_2^*, 0], \\
x_j^I(s) = \widetilde{\psi}_j(s) - \psi_j(s), s \in [-\tau_2^*, 0], \\
y_j^R(s) = \widetilde{\alpha}_j(s) - \alpha_j(s), s \in [-\tau_1^*, 0], \\
y_j^I(s) = \widetilde{\beta}_j(s) - \beta_j(s), s \in [-\tau_1^*, 0].
\end{cases}
\tag{11}
$$

Theorem 1: If assumption 1 holds, then the drive system (1) and the response system (2) are globally asymptotically synchronized based on adaptive feedback controller (9).

Proof: Suppose that $\left(x_j^R(t), x_j^I(t), y_j^R(t), y_j^I(t)\right)^T$ is an arbitrary solution of system (10) with any initial conditions (11).

Construct a Lyapunov-like function:

$$\begin{aligned}\Psi(t) = &\sum_{j=1}^{n}|x_j^R(t)| + \sum_{j=1}^{n}|x_j^I(t)| + \sum_{j=1}^{n}|y_j^R(t)| + \sum_{j=1}^{n}|y_j^I(t)|\\ &+\sum_{j=1}^{n}\frac{1}{2L_{1j}}(\bar{a}_{1j}(t)-\hat{a}_{1j})^2 + \sum_{j=1}^{n}\frac{1}{2L_{2j}}(\bar{c}_{1j}(t)-\hat{c}_{1j})^2\\ &+\sum_{j=1}^{n}\frac{1}{2L_{3j}}(\bar{b}_{1j}(t)-\hat{b}_{1j})^2 + \sum_{j=1}^{n}\frac{1}{2L_{4j}}(\bar{d}_{1j}(t)-\hat{d}_{1j})^2\\ &+\sum_{j=1}^{n}\frac{1}{2L_{5j}}(\bar{a}_{2j}(t)-\hat{a}_{2j})^2 + \sum_{j=1}^{n}\frac{1}{2L_{6j}}(\bar{c}_{2j}(t)-\hat{c}_{2j})^2\\ &+\sum_{j=1}^{n}\frac{1}{2L_{7j}}(\bar{b}_{2j}(t)-\hat{b}_{2j})^2 + \sum_{j=1}^{n}\frac{1}{2L_{8j}}(\bar{d}_{2j}(t)-\hat{d}_{2j})^2,\end{aligned} \tag{12}$$

where $\hat{a}_{kj}, \hat{c}_{kj}, \hat{b}_{kj}$ and $\hat{d}_{kj}$ $(k = 1, 2, j = 1, 2, \cdots, n)$ are constants which are determined later.

By applying assumption 1, Lemma 1, Lemma 2 and controllers (9), and calculating the fractional-order derivatives of $\Psi(t)$ along the solution of (10), we can get

$$\begin{aligned}D^q\Psi(t) \le &\sum_{j=1}^{n}\mathrm{sgn}(x_j^R(t))D^q x_j^R(t) + \sum_{j=1}^{n}\mathrm{sgn}(x_j^I(t))D^q x_j^I(t)\\ &+\sum_{j=1}^{n}\mathrm{sgn}(y_j^R(t))D^q y_j^R(t) + \sum_{j=1}^{n}\mathrm{sgn}(y_j^I(t))D^q y_j^I(t)\\ &+\sum_{j=1}^{n}\frac{1}{L_{1j}}(\bar{a}_{1j}(t)-\hat{a}_{1j})D^q\bar{a}_{1j}(t)\\ &+\sum_{j=1}^{n}\frac{1}{L_{2j}}(\bar{c}_{1j}(t)-\hat{c}_{1j})D^q\bar{c}_{1j}(t)\\ &+\sum_{j=1}^{n}\frac{1}{L_{3j}}(\bar{b}_{1j}(t)-\hat{b}_{1j})D^q\bar{b}_{1j}(t)\\ &+\sum_{j=1}^{n}\frac{1}{L_{4j}}(\bar{d}_{1j}(t)-\hat{d}_{1j})D^q\bar{d}_{1j}(t)\\ &+\sum_{j=1}^{n}\frac{1}{L_{5j}}(\bar{a}_{2j}(t)-\hat{a}_{2j})D^q\bar{a}_{2j}(t)\\ &+\sum_{j=1}^{n}\frac{1}{L_{6j}}(\bar{c}_{2j}(t)-\hat{c}_{2j})D^q\bar{c}_{2j}(t)\\ &+\sum_{j=1}^{n}\frac{1}{L_{7j}}(\bar{b}_{2j}(t)-\hat{b}_{2j})D^q\bar{b}_{2j}(t)\\ &+\sum_{j=1}^{n}\frac{1}{L_{8j}}(\bar{d}_{2j}(t)-\hat{d}_{2j})D^q\bar{d}_{2j}(t)\\ = &\sum_{j=1}^{n}\mathrm{sgn}(x_j^R(t))\Big\{-a_j x_j^R(t)\\ &+\sum_{k=1}^{n}w_{jk}^R[F_k^R(\widetilde{\lambda}_k(t),\widetilde{\mu}_k(t)) - F_k^R(\lambda_k(t),\mu_k(t))]\\ &-\sum_{k=1}^{n}w_{jk}^I[F_k^I(\widetilde{\lambda}_k(t),\widetilde{\mu}_k(t)) - F_k^I(\lambda_k(t),\mu_k(t))]\\ &+\sum_{k=1}^{n}e_{jk}^R[G_k^R(\widetilde{\lambda}_k(t-\tau_1(t)),\widetilde{\mu}_k(t-\tau_1(t)))\\ &-G_k^R(\lambda_k(t-\tau_1(t)),\mu_k(t-\tau_1(t)))]\\ &-\sum_{k=1}^{n}e_{jk}^I[G_k^I(\widetilde{\lambda}_k(t-\tau_1(t)),\widetilde{\mu}_k(t-\tau_1(t)))\\ &-G_k^I(\lambda_k(t-\tau_1(t)),\mu_k(t-\tau_1(t)))]\\ &-\bar{a}_{1j}(t)x_j^R(t) - \mathrm{sgn}(x_j^R(t))\bar{c}_{1j}(t)|x_j^R(t-\tau_2(t))|\Big\}\\ &+\sum_{j=1}^{n}\mathrm{sgn}(x_j^I(t))\Big\{-a_j x_j^I(t)\\ &+\sum_{k=1}^{n}w_{jk}^R[F_k^I(\widetilde{\lambda}_k(t),\widetilde{\mu}_k(t)) - F_k^I(\lambda_k(t),\mu_k(t))]\\ &+\sum_{k=1}^{n}w_{jk}^I[F_k^R(\widetilde{\lambda}_k(t),\widetilde{\mu}_k(t)) - F_k^R(\lambda_k(t),\mu_k(t))]\\ &+\sum_{k=1}^{n}e_{jk}^I[G_k^R(\widetilde{\lambda}_k(t-\tau_1(t)),\widetilde{\mu}_k(t-\tau_1(t)))\\ &-G_k^R(\lambda_k(t-\tau_1(t)),\mu_k(t-\tau_1(t)))]\\ &+\sum_{k=1}^{n}e_{jk}^R[G_k^I(\widetilde{\lambda}_k(t-\tau_1(t)),\widetilde{\mu}_k(t-\tau_1(t)))\\ &-G_k^I(\lambda_k(t-\tau_1(t)),\mu_k(t-\tau_1(t)))]\\ &-\bar{b}_{1j}(t)x_j^I(t) - \mathrm{sgn}(x_j^I(t))\bar{d}_{1j}(t)|x_j^I(t-\tau_2(t))|\Big\}\\ &+\sum_{j=1}^{n}\mathrm{sgn}(y_j^R(t))\Big\{-c_j y_j^R(t) + d_j^R x_j^R(t) - d_j^I x_j^I(t)\\ &+h_j^R x_j^R(t-\tau_2(t)) - h_j^I x_j^I(t-\tau_2(t))\\ &-\bar{a}_{2j}(t)y_j^R(t) - \mathrm{sgn}(y_j^R(t))\bar{c}_{2j}(t)|y_j^R(t-\tau_1(t))|\Big\}\\ &+\sum_{j=1}^{n}\mathrm{sgn}(y_j^I(t))\Big\{-c_j y_j^I(t) + d_j^R x_j^I(t) + d_j^I x_j^R(t)\\ &+h_j^I x_j^R(t-\tau_2(t)) + h_j^R x_j^I(t-\tau_2(t)) - \bar{b}_{2j}(t)y_j^I(t)\end{aligned}$$

$$
\begin{aligned}
&-\operatorname{sgn}(y_j^I(t))\bar{d}_{2j}(t)|y_j^I(t-\tau_1(t))|\Big\}\\
&+\sum_{j=1}^{n}[\bar{a}_{1j}(t)-\hat{a}_{1j}]|x_j^R(t)|\\
&+\sum_{j=1}^{n}[\bar{c}_{1j}(t)-\hat{c}_{1j}]|x_j^R(t-\tau_2(t))|\\
&+\sum_{j=1}^{n}[\bar{b}_{1j}(t)-\hat{b}_{1j}]|x_j^I(t)|\\
&+\sum_{j=1}^{n}[\bar{d}_{1j}(t)-\hat{d}_{1j}]|x_j^I(t-\tau_2(t))|\\
&+\sum_{j=1}^{n}[\bar{a}_{2j}(t)-\hat{a}_{2j}]|y_j^R(t)|\\
&+\sum_{j=1}^{n}[\bar{c}_{2j}(t)-\hat{c}_{2j}]|y_j^R(t-\tau_1(t))|\\
&+\sum_{j=1}^{n}[\bar{b}_{2j}(t)-\hat{b}_{2j}]|y_j^I(t)|\\
&+\sum_{j=1}^{n}[\bar{d}_{2j}(t)-\hat{d}_{2j}]|y_j^I(t-\tau_1(t))|\\
\leq&\sum_{j=1}^{n}\Big\{-a_j|x_j^R(t)|-\hat{a}_{1j}|x_j^R(t)|-\hat{c}_{1j}|x_j^R(t-\tau_2(t))|\\
&+\sum_{k=1}^{n}|w_{jk}^R|\left(\delta_k^1|y_k^R(t)|+\delta_k^2|y_k^I(t)|\right)\\
&+\sum_{k=1}^{n}|w_{jk}^I|\left(\delta_k^3|y_k^R(t)|+\delta_k^4|y_k^I(t)|\right)\\
&+\sum_{k=1}^{n}|e_{jk}^R|\left(\delta_k^5|y_k^R(t-\tau_1(t))|+\delta_k^6|y_k^I(t-\tau_1(t))|\right)\\
&+\sum_{k=1}^{n}|e_{jk}^I|\left(\delta_k^7|y_k^R(t-\tau_1(t))|+\delta_k^8|y_k^I(t-\tau_1(t))|\right)\Big\}\\
&+\sum_{j=1}^{n}\Big\{-a_j|x_j^I(t)|-\hat{b}_{1j}|x_j^I(t)|-\hat{d}_{1j}|x_j^I(t-\tau_2(t))|\\
&+\sum_{k=1}^{n}|w_{jk}^R|\left(\delta_k^3|y_k^R(t)|+\delta_k^4|y_k^I(t)|\right)\\
&+\sum_{k=1}^{n}|w_{jk}^I|\left(\delta_k^1|y_k^R(t)|+\delta_k^2|y_k^I(t)|\right)\\
&+\sum_{k=1}^{n}|e_{jk}^I|\left(\delta_k^5|y_k^R(t-\tau_1(t))|+\delta_k^6|y_k^I(t-\tau_1(t))|\right)\\
&+\sum_{k=1}^{n}|e_{jk}^R|\left(\delta_k^7|y_k^R(t-\tau_1(t))|+\delta_k^8|y_k^I(t-\tau_1(t))|\right)\Big\}\\
&+\sum_{j=1}^{n}\Big\{-c_j|y_j^R(t)|-\hat{a}_{2j}|y_j^R(t)|-\hat{c}_{2j}|y_j^R(t-\tau_1(t))|\\
&+|d_j^R||x_j^R(t)|+|d_j^I||x_j^I(t)|\\
&+|h_j^R||x_j^R(t-\tau_2(t))|+|h_j^I||x_j^I(t-\tau_2(t))|\Big\}\\
&+\sum_{j=1}^{n}\Big\{-c_j|y_j^I(t)|-\hat{b}_{2j}|y_j^I(t)|-\hat{d}_{2j}|y_j^I(t-\tau_1(t))|\\
&+|d_j^R||x_j^I(t)|+|d_j^I||x_j^R(t)|\\
&+|h_j^I||x_j^R(t-\tau_2(t))|+|h_j^R||x_j^I(t-\tau_2(t))|\Big\}\\
=&\sum_{j=1}^{n}\left(-a_j-\hat{a}_{1j}+|d_j^R|+|d_j^I|\right)|x_j^R(t)|\\
&+\sum_{j=1}^{n}\left(-\hat{c}_{1j}+|h_j^R|+|h_j^I|\right)|x_j^R(t-\tau_2(t))|\\
&+\sum_{j=1}^{n}\left(-a_j-\hat{b}_{1j}+|d_j^R|+|d_j^I|\right)|x_j^I(t)|\\
&+\sum_{j=1}^{n}\left(-\hat{d}_{1j}+|h_j^R|+|h_j^I|\right)|x_j^I(t-\tau_2(t))|\\
&+\sum_{j=1}^{n}\Big\{-c_j-\hat{a}_{2j}+\sum_{k=1}^{n}|w_{kj}^R|\delta_j^1+\sum_{k=1}^{n}|w_{kj}^I|\delta_j^3\\
&+\sum_{k=1}^{n}|w_{kj}^R|\delta_j^3+\sum_{k=1}^{n}|w_{kj}^I|\delta_j^1\Big\}|y_j^R(t)|\\
&+\sum_{j=1}^{n}\Big\{-\hat{c}_{2j}+\sum_{k=1}^{n}|e_{kj}^R|\delta_j^5+\sum_{k=1}^{n}|e_{kj}^I|\delta_j^7\\
&+\sum_{k=1}^{n}|e_{kj}^I|\delta_j^5+\sum_{k=1}^{n}|e_{kj}^R|\delta_j^7\Big\}|y_j^R(t-\tau_1(t))|\\
&+\sum_{j=1}^{n}\Big\{-c_j-\hat{b}_{2j}+\sum_{k=1}^{n}|w_{kj}^R|\delta_j^2+\sum_{k=1}^{n}|w_{kj}^I|\delta_j^4\\
&+\sum_{k=1}^{n}|w_{kj}^R|\delta_j^4+\sum_{k=1}^{n}|w_{kj}^I|\delta_j^2\Big\}|y_j^I(t)|\\
&+\sum_{j=1}^{n}\Big\{-\hat{d}_{2j}+\sum_{k=1}^{n}|e_{kj}^R|\delta_j^6+\sum_{k=1}^{n}|e_{kj}^I|\delta_j^8\\
&+\sum_{k=1}^{n}|e_{kj}^I|\delta_j^6+\sum_{k=1}^{n}|e_{kj}^R|\delta_j^8\Big\}|y_j^I(t-\tau_1(t))|.
\end{aligned}
$$

Then we can properly choose $\hat{a}_{kj}, \hat{c}_{kj}, \hat{b}_{kj}$ and $\hat{d}_{kj}$ satisfying the following inequalities:

$$\begin{cases} a_j + \hat{a}_{1j} - |d_j^R| - |d_j^I| > 0, \\ \hat{c}_{1j} - |h_j^R| - |h_j^I| > 0, \\ a_j + \hat{b}_{1j} - |d_j^I| - |d_j^R| > 0, \\ \hat{d}_{1j} - |h_j^R| - |h_j^I| > 0, \\ c_j + \hat{a}_{2j} - \sum_{k=1}^n |w_{kj}^R|\delta_j^1 - \sum_{k=1}^n |w_{kj}^I|\delta_j^3 \\ -\sum_{k=1}^n |w_{kj}^R|\delta_j^3 - \sum_{k=1}^n |w_{kj}^I|\delta_j^1 > 0, \\ \hat{c}_{2j} - \sum_{k=1}^n |e_{kj}^R|\delta_j^5 \\ -\sum_{k=1}^n |e_{kj}^I|\delta_j^7 - \sum_{k=1}^n |e_{kj}^I|\delta_j^5 \\ \qquad -\sum_{k=1}^n |e_{kj}^R|\delta_j^7 > 0, \\ c_j + \hat{b}_{2j} - \sum_{k=1}^n |w_{kj}^R|\delta_j^2 - \sum_{k=1}^n |w_{kj}^I|\delta_j^4 \\ -\sum_{k=1}^n |w_{kj}^R|\delta_j^4 - \sum_{k=1}^n |w_{kj}^I|\delta_j^2 > 0, \\ \hat{d}_{2j} - \sum_{k=1}^n |e_{kj}^R|\delta_j^6 - \sum_{k=1}^n |e_{kj}^I|\delta_j^8 \\ -\sum_{k=1}^n |e_{kj}^I|\delta_j^6 - \sum_{k=1}^n |e_{kj}^R|\delta_j^8 > 0. \end{cases} \tag{13}$$

Let

$$\begin{aligned} L_1 &= \min_{1\le j\le n} \{a_j + \hat{a}_{1j} - |d_j^R| - |d_j^I|\} > 0, \\ L_2 &= \min_{1\le j\le n} \{\hat{c}_{1j} - |h_j^R| - |h_j^I|\} > 0, \\ L_3 &= \min_{1\le j\le n} \{a_j + \hat{b}_{1j} - |d_j^I| - |d_j^R|\} > 0, \\ L_4 &= \min_{1\le j\le n} \{\hat{d}_{1j} - |h_j^R| - |h_j^I|\} > 0, \\ L_5 &= \min_{1\le j\le n} \Big\{c_j + \hat{a}_{2j} - \sum_{k=1}^n |w_{kj}^R|\delta_j^1 - \sum_{k=1}^n |w_{kj}^I|\delta_j^3 \\ &\quad - \sum_{k=1}^n |w_{kj}^R|\delta_j^3 - \sum_{k=1}^n |w_{kj}^I|\delta_j^1\Big\} > 0, \\ L_6 &= \min_{1\le j\le n} \Big\{\hat{c}_{2j} - \sum_{k=1}^n |e_{kj}^R|\delta_j^5 - \sum_{k=1}^n |e_{kj}^I|\delta_j^7 \\ &\quad - \sum_{k=1}^n |e_{kj}^I|\delta_j^5 - \sum_{k=1}^n |e_{kj}^R|\delta_j^7\Big\} > 0, \\ L_7 &= \min_{1\le j\le n} \Big\{c_j + \hat{b}_{2j} - \sum_{k=1}^n |w_{kj}^R|\delta_j^2 - \sum_{k=1}^n |w_{kj}^I|\delta_j^4 \\ &\quad - \sum_{k=1}^n |w_{kj}^R|\delta_j^4 - \sum_{k=1}^n |w_{kj}^I|\delta_j^2\Big\} > 0, \\ L_8 &= \min_{1\le j\le n} \Big\{\hat{d}_{2j} - \sum_{k=1}^n |e_{kj}^R|\delta_j^6 \\ &\quad - \sum_{k=1}^n |e_{kj}^I|\delta_j^8 - \sum_{k=1}^n |e_{kj}^I|\delta_j^6 - \sum_{k=1}^n |e_{kj}^R|\delta_j^8\Big\} > 0. \end{aligned}$$

Suppose that $L = \min\{L_1, L_3, L_5, L_7\}$ and

$$\Phi(t) = \sum_{j=1}^n |x_j^R(t)| + \sum_{j=1}^n |x_j^I(t)| + \sum_{j=1}^n |y_j^R(t)| + \sum_{j=1}^n |y_j^I(t)|, \tag{14}$$

then we can get

$$\begin{aligned} D^q\Psi(t) &\le -L_1 \sum_{j=1}^n |x_j^R(t)| - L_2 \sum_{j=1}^n |x_j^R(t-\tau_2(t))| \\ &\quad -L_3 \sum_{j=1}^n |x_j^I(t)| - L_4 \sum_{j=1}^n |x_j^I(t-\tau_2(t))| \\ &\quad -L_5 \sum_{j=1}^n |y_j^R(t)| - L_6 \sum_{j=1}^n |y_j^R(t-\tau_1(t))| \\ &\quad -L_7 \sum_{j=1}^n |y_j^I(t)| - L_8 \sum_{j=1}^n |y_j^I(t-\tau_1(t))| \\ &\le -L_1 \sum_{j=1}^n |x_j^R(t)| - L_3 \sum_{j=1}^n |x_j^I(t)| \\ &\quad -L_5 \sum_{j=1}^n |y_j^R(t)| - L_7 \sum_{j=1}^n |y_j^I(t)| \\ &\le -L\Phi(t) \le 0, \quad t \ge 0. \end{aligned} \tag{15}$$

From the Definition 1, we can obtain

$$\Psi(t) - \Psi(0) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} D^q\Psi(s)ds \le 0.$$

Hence

$$\Psi(t) \le \Psi(0), \quad t \ge 0.$$

Combined with (12), we know that $x_j^R(t), x_j^I(t), y_j^R(t), y_j^I(t)$, $\bar{a}_{kj}(t)$, $\bar{c}_{kj}(t)$, $\bar{b}_{kj}(t)$ and $\bar{d}_{kj}(t)$ $(k = 1, 2, j = 1, 2, \cdots, n)$ are bounded on $t \ge 0$.

So, there exists a constant $M > 0$ satisfying

$$|D^q\Phi(t)| \le M, t \ge 0. \tag{16}$$

According to Lemma 3, we have $\lim_{t\to\infty} \Phi(t) = 0$, that is, the drive system (1) and the response system (2) are globally asymptotically synchronized based on the controller (9). The proof is completed. □

Remark 3: Since the well-known Leibniz chain rule is invalid for fractional-order derivative [45], combining

Lemma 2, Lyapunov-like function $\Psi(t)$, which contains some square terms of the adaptive coupling strengths, become difficult to appear on the right side of $D^q\Psi(t)$. Therefore, compared with the classical Lyapunov method [45], Lemma 3 is added in the proof of Theorem 1.

Remark 4: In systems (1) *and* (2), *if* $m(t), p(t), \widetilde{m}(t), \widetilde{p}(t), B$ *are real-valued vectors,* W, E, D, H *are real-valued matrices,* $F(p(t)), G(p(t-\tau_1(t))), J_1(t), J_2(t), U_1(t), U_2(t)$ *are real-valued vector-valued functions and the controller* (9) *becomes*

$$\begin{cases} u_{1j}(t)=-\bar{a}_{1j}(t)x_j(t)-\operatorname{sgn}(x_j(t))\bar{c}_{1j}(t)|x_j(t-\tau_2(t))|,\\ u_{2j}(t)=-\bar{a}_{2j}(t)y_j(t)-\operatorname{sgn}(y_j(t))\bar{c}_{2j}(t)|y_j(t-\tau_1(t))|,\\ D^q\bar{a}_{1j}(t)=L_{1j}|x_j(t)|,\\ D^q\bar{c}_{1j}(t)=L_{2j}|x_j(t-\tau_2(t))|,\\ D^q\bar{a}_{2j}(t)=L_{5j}|y_j(t)|,\\ D^q\bar{c}_{2j}(t)=L_{6j}|y_j(t-\tau_1(t))|. \end{cases} \tag{17}$$

In this case, assumption 1 becomes:

Assumption 2: The functions $F_j(p_j(t)), G_j(p_j(t-\tau_1(t)))$ satisfy the following inequalities

$$\begin{aligned} &|F_j(\widetilde{p}_j(t))-F_j(p_j(t))|\le\delta_j^1|\widetilde{p}_j(t)-p_j(t)|,\\ &|G_j(\widetilde{p}_j(t-\tau_1(t)))-G_j(p_j(t-\tau_1(t)))|\\ &\le\delta_j^5|\widetilde{p}_j(t-\tau_1(t))-p_j(t-\tau_1(t))|, \end{aligned}$$

where $\delta_j^1>0, \delta_j^5>0$ are Lipschitz constants, $j=1,\cdots,n$.

Then we can obtain the following result:

Corollary 1: If assumption 2 holds, then the drive system (1) *and the response system* (2) *are globally asymptotically synchronized based on the controller* (17).

Remark 5: In Remark 4, if we take matrix $W=D=0$, *vector* $J_1(t)=J_2(t)=0$, *time-varying delays* $\tau_1(t)=\tau$*(constant),* $\tau_2(t)=0$, *then systems* (1) *and* (2) *convert to systems (1) and (2) in [11], respectively. The drive system (1) and the response system (2) reach finite-time synchronization based on feedback controller (7) in [11].*

Remark 6: In Remark 4, if we adopt $E=H=0$ *in systems* (1) *and* (2), *then the controller* (17) *becomes*

$$\begin{cases} u_{1j}(t)=-\bar{a}_{1j}(t)x_j(t),\\ u_{2j}(t)=-\bar{a}_{2j}(t)y_j(t),\\ D^q\bar{a}_{1j}(t)=L_{1j}|x_j(t)|,\\ D^q\bar{a}_{2j}(t)=L_{5j}|y_j(t)|, \end{cases} \tag{18}$$

and assumption 2 becomes:

Assumption 3: The functions $F_j(p_j(t))$ *satisfy the following inequalities*

$$|F_j(\widetilde{p}_j(t))-F_j(p_j(t))|\le\delta_j^1|\widetilde{p}_j(t)-p_j(t)|,\quad j=1,2,\cdots,n,$$

where $\delta_j^1>0$ *are Lipschitz constants.*

Then we have the following conclusion:

Corollary 2: If assumption 3 holds, then the drive system (1) *without time delay and the response system* (2) *are globally asymptotically synchronized based on feedback controller* (18).

Remark 7: In Remark 6, if we take $q=1$, $J_1(t)=J_2(t)=0$, $\tau_1(t)=\tau_2(t)=0$, *then system* (1) *without time delay convert into drive system (1) in [17], which exhibits the stochastic finite-time synchronization for a class of integer-order GRNs. For more details, see [17].*

Next, we investigate the global anti-synchronization of DFGRNs based on the other adaptive feedback controller.

Let $x_j(t)=\widetilde{m}_j(t)+m_j(t)=x_j^R(t)+ix_j^I(t)$, $y_j(t)=\widetilde{p}_j(t)+p_j(t)=y_j^R(t)+iy_j^I(t)$ $(j=1,2,\cdots,n)$, where

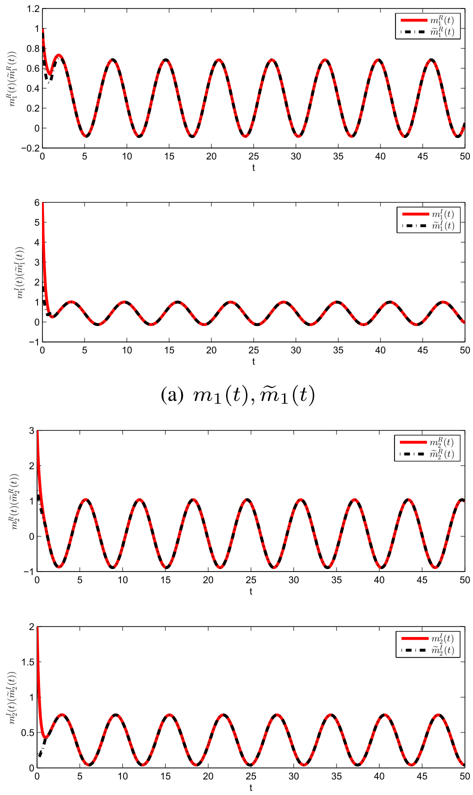
$$\begin{aligned} x_j^R(t)&=\widetilde{r}_j(t)+r_j(t), x_j^I(t)=\widetilde{\eta}_j(t)+\eta_j(t),\\ y_j^R(t)&=\widetilde{\lambda}_j(t)+\lambda_j(t), y_j^I(t)=\widetilde{\mu}_j(t)+\mu_j(t). \end{aligned}$$

We choose the other adaptive feedback controllers as follows:

$$\begin{cases} u_{1j}(t)=-\bar{a}_{1j}(t)x_j^R(t)-2B_j^R-2J_{1j}^R(t)+S_1(t)\\ \qquad -\operatorname{sgn}(x_j^R(t))\bar{c}_{1j}(t)|x_j^R(t-\tau_2(t))|,\\ v_{1j}(t)=-\bar{b}_{1j}(t)x_j^I(t)-2B_j^I-2J_{1j}^I(t)+S_2(t)\\ \qquad -\operatorname{sgn}(x_j^I(t))\bar{d}_{1j}(t)|x_j^I(t-\tau_2(t))|,\\ u_{2j}(t)=-\bar{a}_{2j}(t)y_j^R(t)+2h_j^Ix_j^I(t-\tau_2(t))\\ \qquad -\operatorname{sgn}(y_j^R(t))\bar{c}_{2j}(t)|y_j^R(t-\tau_1(t))|-2J_{2j}^R(t)\\ \qquad -2d_j^Rx_j^R(t)+2d_j^Ix_j^I(t)-2h_j^Rx_j^R(t-\tau_2(t)),\\ v_{2j}(t)=-\bar{b}_{2j}(t)y_j^I(t)-2d_j^Rx_j^I(t)-2d_j^Ix_j^R(t)\\ \qquad -\operatorname{sgn}(y_j^I(t))\bar{d}_{2j}(t)|y_j^I(t-\tau_1(t))|-2J_{2j}^I(t)\\ \qquad -2h_j^Ix_j^R(t-\tau_2(t))-2h_j^Rx_j^I(t-\tau_2(t)),\\ D^q\bar{a}_{1j}(t)=L_{1j}|x_j^R(t)|,\\ D^q\bar{c}_{1j}(t)=L_{2j}|x_j^R(t-\tau_2(t))|,\\ D^q\bar{b}_{1j}(t)=L_{3j}|x_j^I(t)|,\\ D^q\bar{d}_{1j}(t)=L_{4j}|x_j^I(t-\tau_2(t))|,\\ D^q\bar{a}_{2j}(t)=L_{5j}|y_j^R(t)|,\\ D^q\bar{c}_{2j}(t)=L_{6j}|y_j^R(t-\tau_1(t))|,\\ D^q\bar{b}_{2j}(t)=L_{7j}|y_j^I(t)|,\\ D^q\bar{d}_{2j}(t)=L_{8j}|y_j^I(t-\tau_1(t))|, \end{cases} \tag{19}$$

where $L_{kj}(k=1,2,\cdots,8, j=1,2,\cdots,n)$ are arbitrary positive constants, $\bar{a}_{kj}(t)$, $\bar{b}_{kj}(t)$, $\bar{c}_{kj}(t)$, $\bar{d}_{kj}(t)$ $(k=1,2$, $j=1,2,\cdots,n)$ are adaptive coupling strengths, and

$$\begin{aligned} S_1(t)=&\sum_{k=1}^n w_{jk}^R(-F_k^R(\lambda_k(t),\mu_k(t))\\ &-F_k^R(-\lambda_k(t),-\mu_k(t)))\\ &-\sum_{k=1}^n w_{jk}^I(-F_k^I(\lambda_k(t),\mu_k(t))\\ &-F_k^I(-\lambda_k(t),-\mu_k(t)))\\ &+\sum_{k=1}^n e_{jk}^R(-G_k^R(\lambda_k(t-\tau_1(t)),\mu_k(t-\tau_1(t))) \end{aligned}$$



(a) $m_1(t), \widetilde{m}_1(t)$

(b) $m_2(t), \widetilde{m}_2(t)$

FIGURE 1. **Curves of coupled DFGRNs (23) and (24) in 2-dimensional space based on controller (9) with $q = 0.98$ and (a) $m_1(t), \widetilde{m}_1(t)$; (b) $m_2(t), \widetilde{m}_2(t)$.**

$$
\begin{aligned}
&-G_k^R(-\lambda_k(t-\tau_1(t)), -\mu_k(t-\tau_1(t))))\\
&-\sum_{k=1}^{n} e_{jk}^I(-G_k^I(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t)))\\
&-G_k^I(-\lambda_k(t-\tau_1(t)), -\mu_k(t-\tau_1(t)))),\\
S_2(t) = &\sum_{k=1}^{n} w_{jk}^I(-F_k^R(\lambda_k(t), \mu_k(t))\\
&-F_k^R(-\lambda_k(t), -\mu_k(t)))\\
&+\sum_{k=1}^{n} w_{jk}^R(-F_k^I(\lambda_k(t), \mu_k(t))\\
&-F_k^I(-\lambda_k(t), -\mu_k(t)))\\
&+\sum_{k=1}^{n} e_{jk}^I(-G_k^R(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t)))\\
&-G_k^R(-\lambda_k(t-\tau_1(t)), -\mu_k(t-\tau_1(t))))\\
&+\sum_{k=1}^{n} e_{jk}^R(-G_k^I(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t)))\\
&-G_k^I(-\lambda_k(t-\tau_1(t)), -\mu_k(t-\tau_1(t)))).
\end{aligned}
$$

It is well known that when $t \to \infty$, the facts $x_j^R(t) \to 0$, $x_j^I(t) \to 0$, $y_j^R(t) \to 0$, $y_j^I(t) \to 0$ $(i = 1, 2, \cdots, n)$ mean that the drive system (1) and the response system (2) are anti-synchronized.

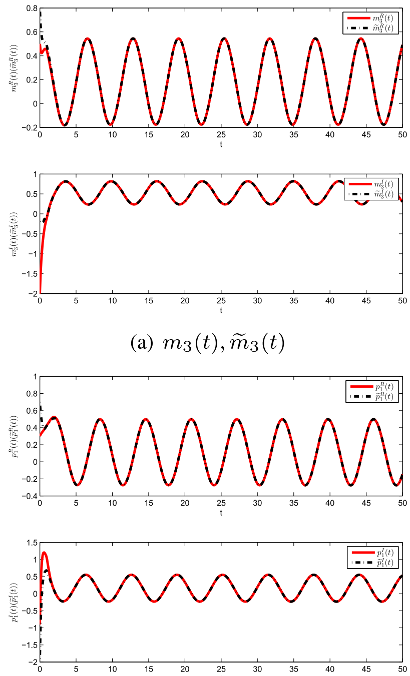
From systems (5) and (6), we can get the following error system:

$$
\begin{cases}
D^q x_j^R(t) = -a_j x_j^R(t) + \sum_{k=1}^{n} w_{jk}^R[F_k^R(\widetilde{\lambda}_k(t), \widetilde{\mu}_k(t))\\
\qquad +F_k^R(\lambda_k(t), \mu_k(t))]\\
\qquad -\sum_{k=1}^{n} w_{jk}^I[F_k^I(\widetilde{\lambda}_k(t), \widetilde{\mu}_k(t))\\
\qquad +F_k^I(\lambda_k(t), \mu_k(t))]\\
\qquad +\sum_{k=1}^{n} e_{jk}^R[G_k^R(\widetilde{\lambda}_k(t-\tau_1(t)), \widetilde{\mu}_k(t-\tau_1(t)))\\
\qquad +G_k^R(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t)))]\\
\qquad -\sum_{k=1}^{n} e_{jk}^I[G_k^I(\widetilde{\lambda}_k(t-\tau_1(t)), \widetilde{\mu}_k(t-\tau_1(t)))\\
\qquad +G_k^I(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t)))]\\
\qquad +u_{1j}(t) + 2J_{1j}^R(t) + 2B_j^R,\\
D^q x_j^I(t) = -a_j x_j^I(t) + \sum_{k=1}^{n} w_{jk}^R[F_k^I(\widetilde{\lambda}_k(t), \widetilde{\mu}_k(t))\\
\qquad +F_k^I(\lambda_k(t), \mu_k(t))]\\
\qquad +\sum_{k=1}^{n} w_{jk}^I[F_k^R(\widetilde{\lambda}_k(t), \widetilde{\mu}_k(t))\\
\qquad +F_k^R(\lambda_k(t), \mu_k(t))]\\
\qquad +\sum_{k=1}^{n} e_{jk}^I[G_k^R(\widetilde{\lambda}_k(t-\tau_1(t)), \widetilde{\mu}_k(t-\tau_1(t)))\\
\qquad +G_k^R(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t)))]\\
\qquad +\sum_{k=1}^{n} e_{jk}^R[G_k^I(\widetilde{\lambda}_k(t-\tau_1(t)), \widetilde{\mu}_k(t-\tau_1(t)))\\
\qquad +G_k^I(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t)))]\\
\qquad +v_{1j}(t) + 2J_{1j}^I(t) + 2B_j^I,\\
D^q y_j^R(t) = -c_j y_j^R(t) + d_j^R x_j^R(t) - d_j^I x_j^I(t)\\
\qquad +h_j^R x_j^R(t-\tau_2(t)) - h_j^I x_j^I(t-\tau_2(t))\\
\qquad +u_{2j}(t) + 2J_{2j}^R(t),\\
D^q y_j^I(t) = -c_j y_j^I(t) + d_j^R x_j^I(t) + d_j^I x_j^R(t)\\
\qquad +h_j^I x_j^R(t-\tau_2(t)) + h_j^R x_j^I(t-\tau_2(t))\\
\qquad +v_{2j}(t) + 2J_{2j}^I(t).
\end{cases}
\tag{20}
$$

Theorem 2: If assumption 1 holds, then the drive system (1) *and the response system* (2) *are globally asymptotically anti-synchronized based on feedback controller* (19).

Proof: The proof of Theorem 2 is similar to that of Theorem 1. □

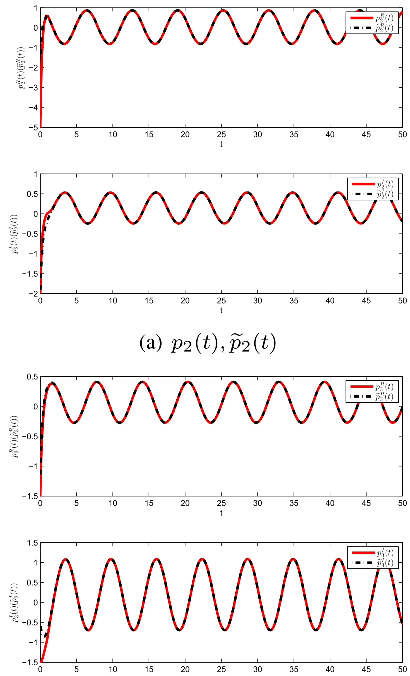
Remark 8: In systems (1) *and* (2)*, if* $m(t), p(t), \widetilde{m}(t), \widetilde{p}(t), B$ *are real-valued vectors,* W, E, D, H *are real-valued*



(a) $m_3(t), \widetilde{m}_3(t)$

(b) $p_1(t), \widetilde{p}_1(t)$

FIGURE 2. **Curves of coupled DFGRNs (23) and (24) in 2-dimensional space based on controller (9) with $q = 0.98$ and (a) $m_3(t), \widetilde{m}_3(t)$; (b) $p_1(t), \widetilde{p}_1(t)$.**



(a) $p_2(t), \widetilde{p}_2(t)$

(b) $p_3(t), \widetilde{p}_3(t)$

FIGURE 3. **Curves of coupled DFGRNs (23) and (24) in 2-dimensional space based on controller (9) with $q = 0.98$ and (a) $p_2(t), \widetilde{p}_2(t)$; (b) $p_3(t), \widetilde{p}_3(t)$.**

matrices, $F(p(t)), G(p(t - \tau_1(t))), J_1(t), J_2(t), U_1(t), U_2(t)$ *are real-valued vector-valued functions and the controller* (19) *becomes*

$$\begin{cases} u_{1j}(t) = -\bar{a}_{1j}(t)x_j(t) - 2B_j - 2J_{1j}(t) \\ \qquad -\mathrm{sgn}(x_j(t))\bar{c}_{1j}(t)|x_j(t-\tau_2(t))| \\ \qquad +\sum_{k=1}^{n} w_{jk}(-F_k(\lambda_k(t), \mu_k(t)) \\ \qquad -F_k(-\lambda_k(t), -\mu_k(t))) \\ \qquad +\sum_{k=1}^{n} e_{jk}(-G_k(\lambda_k(t-\tau_1(t)), \mu_k(t-\tau_1(t))) \\ \qquad -G_k(-\lambda_k(t-\tau_1(t)), -\mu_k(t-\tau_1(t)))), \\ u_{2j}(t) = -\bar{a}_{2j}(t)y_j(t) - 2J_{2j}(t) \\ \qquad -\mathrm{sgn}(y_j(t))\bar{c}_{2j}(t)|y_j(t-\tau_1(t))| \\ \qquad -2d_jx_j(t) - 2h_jx_j(t-\tau_2(t)), \\ D^q\bar{a}_{1j}(t) = L_{1j}|x_j(t)|, \\ D^q\bar{c}_{1j}(t) = L_{2j}|x_j(t-\tau_2(t))|, \\ D^q\bar{a}_{2j}(t) = L_{5j}|y_j(t)|, \\ D^q\bar{c}_{2j}(t) = L_{6j}|y_j(t-\tau_1(t))|. \end{cases} \tag{21}$$

Then we can obtain the following result:

Corollary 3: If assumption 2 holds, then the drive system (1) *and the response system* (2) *are globally asymptotically anti-synchronized based on controller* (21).

Remark 9: In Remark 8, if we adopt $E = H = 0$ *in systems* (1) *and* (2), *then the controller* (21) *becomes*

$$\begin{cases} u_{1j}(t) = -\bar{a}_{1j}(t)x_j(t) - 2B_j - 2J_{1j}(t) \\ \qquad +\sum_{k=1}^{n} w_{jk}(-F_k(\lambda_k(t), \mu_k(t)) \\ \qquad -F_k(-\lambda_k(t), -\mu_k(t))), \\ u_{2j}(t) = -\bar{a}_{2j}(t)y_j(t) - 2J_{2j}(t) - 2d_jx_j(t), \\ D^q\bar{a}_{1j}(t) = L_{1j}|x_j(t)|, \\ D^q\bar{a}_{2j}(t) = L_{5j}|y_j(t)|. \end{cases} \tag{22}$$

Then we have the following conclusion:

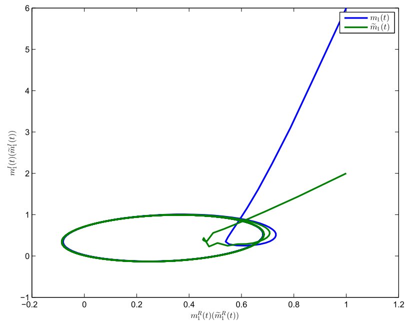
Corollary 4: If assumption 3 holds, then the drive system (1) *without time delay and the response system* (2) *are globally asymptotically anti-synchronized based on feedback controller* (22).

Remark 10: All the results of our corollaries are still new.

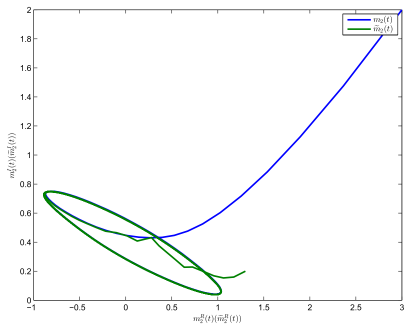
IV. NUMERICAL EXAMPLE

In this section, we give some numerical examples to illustrate the effectiveness of above theoretical results. We take the step-length $h = 0.1$ for the improved Adams-Bashforth-Moulton predictor-corrector scheme [46], which is available on the fractional-order differential equations with time-varying delays.

The following fractional-order complex-valued GRNs of three mRNA and protein nodes with time-varying delays



(a) $m_1(t), \widetilde{m}_1(t)$



(b) $m_2(t)$ $\widetilde{m}_2(t)$

FIGURE 4. Phase plot of coupled DFGRNs (23) and (24) based on controller (9) with $q = 0.98$ and (a) $m_1(t), \widetilde{m}_1(t)$; (b) $m_2(t), \widetilde{m}_2(t)$.

are considered:

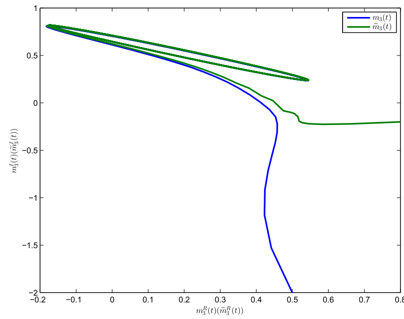
$$\begin{cases} D^q m(t) = -Am(t) + WF(p(t)) \\ \qquad\qquad +EG(p(t-\tau_1(t))) + B + J_1(t), \\ D^q p(t) = -Cp(t) + Dm(t) \\ \qquad\qquad +Hm(t-\tau_2(t)) + J_2(t), \end{cases} \tag{23}$$

where $m(t) = (m_1(t), m_2(t), m_3(t))^T$, $p(t) = (p_1(t), p_2(t), p_3(t))^T$ and, A, C, D, H, W, E, $F_j(p_j(t))$, $G_j(p_j(t))$, $J_1(t)$, and $J_2(t)$, as shown at the bottom of the next page. The response system of the drive system (23) is as follows:

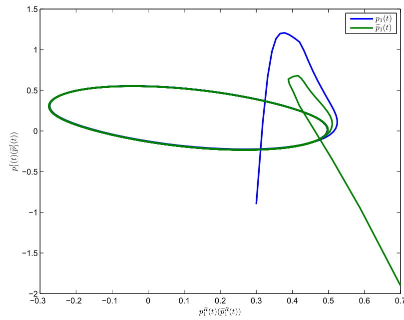
$$\begin{cases} D^q \widetilde{m}(t) = -A\widetilde{m}(t) + WF(\widetilde{p}(t)) + B + J_1(t) \\ \qquad\qquad +EG(\widetilde{p}(t-\tau_1(t))) + U_1(t), \\ D^q \widetilde{p}(t) = -C\widetilde{p}(t) + D\widetilde{m}(t) + J_2(t) \\ \qquad\qquad +H\widetilde{m}(t-\tau_2(t)) + U_2(t), \end{cases} \tag{24}$$

where $\widetilde{m}(t) = (\widetilde{m}_1(t), \widetilde{m}_2(t), \widetilde{m}_3(t))^T$, $\widetilde{p}(t) = (\widetilde{p}_1(t), \widetilde{p}_2(t), \widetilde{p}_3(t))^T$, $U_1(t) = (u_{11}(t), u_{12}(t), u_{13}(t))^T$, $U_2(t) = (u_{21}(t), u_{22}(t), u_{23}(t))^T$.

Let $q = 0.98, \tau_1(t) = \tau_2(t) = \frac{|\cos t|+1}{2}$, the initial conditions $(m_1(s), m_2(s), m_3(s), p_1(s), p_2(s), p_3(s))^T = [1-6i, 3-2i, 0.5+2i, 0.3+0.9i, -5+2i, -1.5+1.5i]^T$ and $(\widetilde{m}_1(s), \widetilde{m}_2(s), \widetilde{m}_3(s), \widetilde{p}_1(s), \widetilde{p}_2(s), \widetilde{p}_3(s))^T = [1-2i, 1.3-0.2i, 0.8+0.2i, 0.7+1.9i, -1.5+2i, -1.2+0.5i]^T, s \in [-1, 0]$. From the selected $F_j(\cdot), G_j(\cdot)$ functions, we have $\delta_1 = \delta_5 = 0.5, \delta_2 = \delta_3 = \delta_6 = \delta_7 = 0, \delta_4 = \delta_8 = 0.25$. Hence assumption 1 is satisfied. Now we will discuss system (24) in two cases:

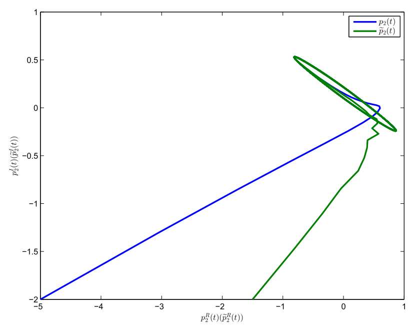


(a) $m_3(t), \widetilde{m}_3(t)$

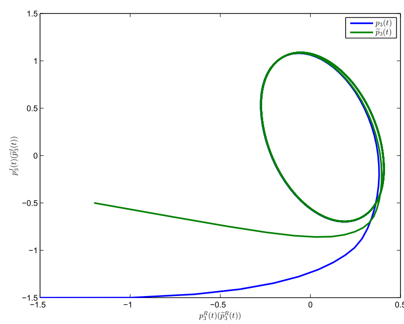


(b) $p_1(t), \widetilde{p}_1(t)$

FIGURE 5. Phase plot of coupled DFGRNs (23) and (24) based on controller (9) with $q = 0.98$ and (a) $m_3(t), \widetilde{m}_3(t)$; (b) $p_1(t), \widetilde{p}_1(t)$.



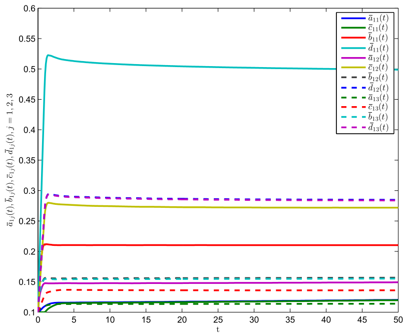
(a) $p_2(t), \widetilde{p}_2(t)$



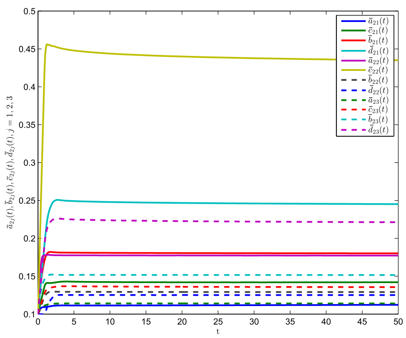
(b) $p_3(t), \widetilde{p}_3(t)$

FIGURE 6. Phase plot of coupled DFGRNs (23) and (24) based on controller (9) with $q = 0.98$ and (a) $p_2(t), \widetilde{p}_2(t)$; (b) $p_3(t), \widetilde{p}_3(t)$.

(i) **Use the synchronization controller in the response system** (24). Let $\bar{a}_{kj}(s) = \bar{c}_{kj}(s) = \bar{b}_{kj}(s) = \bar{d}_{kj}(s) = 0.1$ $(k = 1, 2, j = 1, 2, 3), s \in [-1, 0]$, $L_{kj} = 0.1$



(a) $\bar{a}_{1j}(t), \bar{b}_{1j}(t), \bar{c}_{1j}(t), \bar{d}_{1j}(t), j = 1, 2, 3$



(b) $\bar{a}_{2j}(t), \bar{b}_{2j}(t), \bar{c}_{2j}(t), \bar{d}_{2j}(t), j = 1, 2, 3$

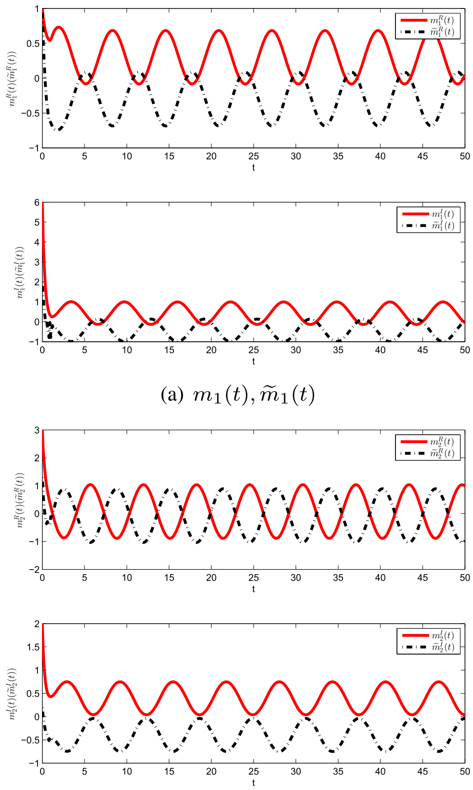
FIGURE 7. Time response of the control gains in controller (9) with $q = 0.98$ and (a) $\bar{a}_{1j}(t), \bar{b}_{1j}(t), \bar{c}_{1j}(t), \bar{d}_{1j}(t), j = 1, 2, 3$; (b) $\bar{a}_{2j}(t), \bar{b}_{2j}(t), \bar{c}_{2j}(t), \bar{d}_{2j}(t), j = 1, 2, 3$.

$(k = 1, 2, \cdots, 8, j = 1, 2, 3)$ in controller (9). We take $\hat{a}_{kj} = \hat{c}_{kj} = \hat{b}_{kj} = \hat{d}_{kj} = 1\,(k = 1, 2, j = 1, 2, 3)$, then we have $L_1 = 3, L_2 = 0.7, L_3 = 3, L_4 = 0.7, L_5 = 2.5952, L_6 = 0.4025, L_7 = 3.0857, L_8 = 0.7013$. So the inequalities (13) hold.

According to Theorem 1, the response system (24) will synchronize with the drive system (23) under controller (9). The 2-dimensional synchronization diagrams are shown in Figures 1 – 3, where the X-axis represents the time t and the Y-axis represents the real(imaginary) part of the state variables. The phase trajectories are shown in Figures 4 – 6, where the X and Y-axes represent the real and imaginary parts of the state variables, respectively. In Figures 1 – 6, the drive-response systems (23) and (24) achieve synchronization under controller (9). Figure 7, where the X-axis represents the time t and the Y-axis represents the adaptive gains, shows the adaptive gains $\bar{a}_{kj}(t)$, $\bar{c}_{kj}(t)$, $\bar{b}_{kj}(t)$ and $\bar{d}_{kj}(t)$ in controller (9) respectively converge to the corresponding positive constants $\hat{a}_{kj}$, $\hat{c}_{kj}$, $\hat{b}_{kj}$ and $\hat{d}_{kj}$ $(k = 1, 2, j = 1, 2, 3)$. Also, the synchronization phenomena exists for arbitrary initial conditions and $q \in (0, 1)$.

(ii) **Use the anti-synchronization controller in the response system** (24). In controller (19), we take the same parameters and initial conditions with case (i). According to Theorem 2, the response system (24) will anti-synchronize with the drive system (23) under controller (19). The 2-dimensional anti-synchronization diagrams are shown in Figures 8 – 10, where the X-axis represents the time t and the Y-axis represents the real(imaginary) part of the state variables. The phase trajectories are shown in Figures 11 – 13, where the X and Y-axes represent the real and imaginary parts of the state variables, respectively. In Figures 8 – 13, the drive-response systems (23) and (24) achieve anti-synchronization under controller (19). Figure 14, where the X-axis represents the time t and the Y-axis represents the adaptive gains, shows the adaptive gains $\bar{a}_{kj}(t), \bar{c}_{kj}(t), \bar{b}_{kj}(t)$ and $\bar{d}_{kj}(t)$ in controller (19) respectively converge to the corresponding positive constants $\hat{a}_{kj}$, $\hat{c}_{kj}$, $\hat{b}_{kj}$

$$
\begin{aligned}
A &= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, C = \begin{pmatrix} 3.5 & 0 & 0 \\ 0 & 3.5 & 0 \\ 0 & 0 & 3.5 \end{pmatrix}, \\
D &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, H = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{pmatrix}, \\
W &= \begin{pmatrix} 0.8147 - 0.6948i & -0.9134 + 0.0344i & 0.2785 + 0.7655i \\ 0.9058 + 0.3171i & 0.6324 - 0.4387i & -0.5469 + 0.7952i \\ -0.1270 - 0.9502i & 0.0975 + 0.3816i & 0.9575 + 0.1869i \end{pmatrix}, \\
E &= \begin{pmatrix} 0.2895 + 0.1469i & 0.2872 - 0.2128i & -0.0426 + 0.2039i \\ 0.0473 - 0.1337i & -0.1456 + 0.2264i & 0.1265 + 0.1965i \\ -0.2912 + 0.1939i & 0.2401 + 0.0828i & 0.2747 - 0.0488i \end{pmatrix}, \\
F_j(p_j(t)) &= \frac{1 - e^{-\lambda_j(t)}}{1 + e^{-\lambda_j(t)}} + i\frac{1}{1 + e^{-\mu_j(t)}} \; (j = 1, 2, 3), \\
G_j(p_j(t)) &= \frac{1 - e^{-\mu_j(t)}}{1 + e^{-\mu_j(t)}} + i\frac{1}{1 + e^{-\lambda_j(t)}} \; (j = 1, 2, 3), \\
J_1(t) &= [\sin(t) - 2i\cos(t), 3\cos(t+1) + i\sin(t-1), \cos(t) - i\cos(t)]^T, \\
J_2(t) &= [\sin(t) + 2i\cos(t), 2\cos(t) - i\cos(t), \sin(t) - 3i\cos(t)]^T.
\end{aligned}
$$



(a) $m_1(t), \widetilde{m}_1(t)$

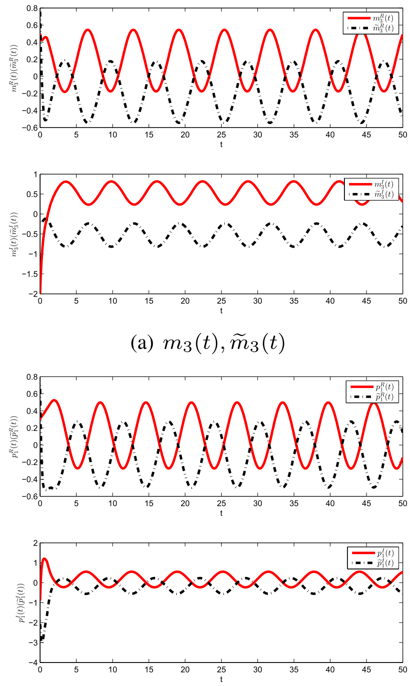
(b) $m_2(t), \widetilde{m}_2(t)$

FIGURE 8. **Curves of coupled DFGRNs (23) and (24) in 2-dimensional space based on controller (19) with $q = 0.98$ and (a) $m_1(t), \widetilde{m}_1(t)$; (b) $m_2(t), \widetilde{m}_2(t)$.**

and $\hat{d}_{kj}$ ($k = 1, 2, j = 1, 2, 3$). Also, the anti-synchronization phenomena exists for arbitrary initial conditions and $q \in (0, 1)$.

In order to investigate the effects of the fractional-order q on synchronization and anti-synchronization of systems (23) and (24), let error $e^*(t) = \frac{1}{12}\left(\sum_{i=1}^{3}|x_i^R(t)| + \sum_{i=1}^{3}|x_i^I(t)| + \sum_{i=1}^{3}|y_i^R(t)| + \sum_{i=1}^{3}|y_i^I(t)|\right)$, $T = \min\{t : e^*(t) \leq \bar{\varepsilon}\}$, where $\bar{\varepsilon}$ is the error limit. We get "estimated time" T_1 and T_2 to achieve synchronization and anti-synchronization by calculating T, respectively. For different fractional-order q, the values T_1 and T_2 are seen in Table 1 with $\bar{\varepsilon} = 0.1$, and the trajectories of the errors for synchronization and anti-synchronization are shown in Figure 15. In Figure 15, the X-axis of represents the time t and the Y-axis represents the errors of synchronization or anti-synchronization.

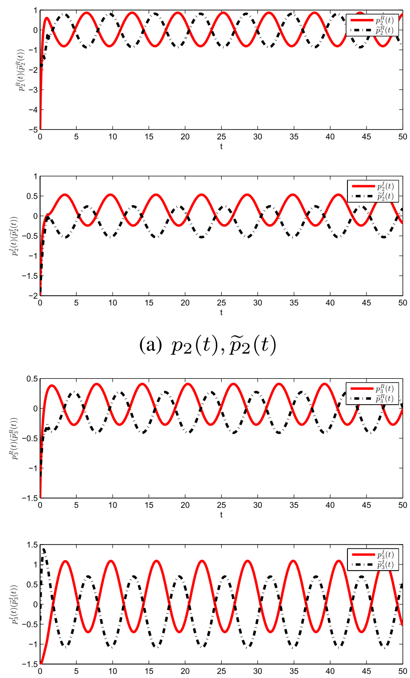
From Table 1, we find that the minimum "estimated time" T_1 and T_2 increase with the decreasing of the fractional-order q, which means that the fractional-order q can effect the synchronization and anti-synchronization of DFGRNs.



(a) $m_3(t), \widetilde{m}_3(t)$

(b) $p_1(t), \widetilde{p}_1(t)$

FIGURE 9. **Curves of coupled DFGRNs (23) and (24) in 2-dimensional space based on controller (19) with $q = 0.98$ and (a) $m_3(t), \widetilde{m}_3(t)$; (b) $p_1(t), \widetilde{p}_1(t)$.**

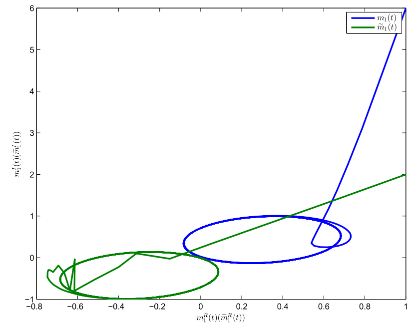


(a) $p_2(t), \widetilde{p}_2(t)$

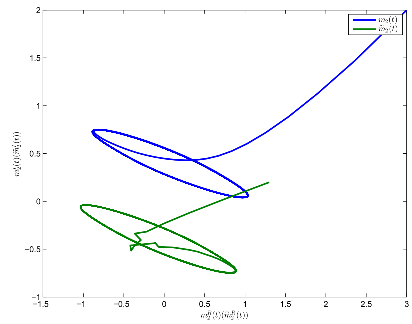
(b) $p_3(t), \widetilde{p}_3(t)$

FIGURE 10. **Curves of coupled DFGRNs (23) and (24) in 2-dimensional space based on controller (19) with $q = 0.98$ and (a) $p_2(t), \widetilde{p}_2(t)$; (b) $p_3(t), \widetilde{p}_3(t)$.**

Remark 11: From (1), (2) *and synchronization controller* (9) *(or anti-synchronization controller* (19)*), when the GRNs consists of n mRNAs and n proteins, we need to solve*

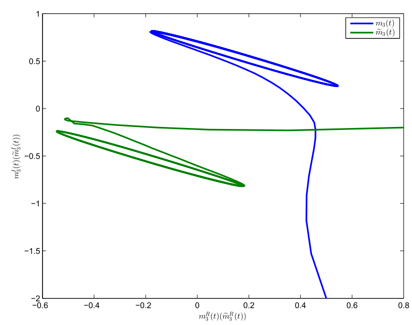


(a) $m_1(t), \widetilde{m}_1(t)$

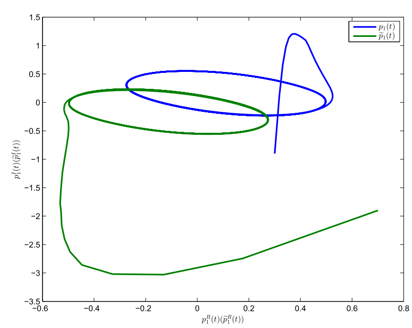


(b) $m_2(t), \widetilde{m}_2(t)$

FIGURE 11. **Phase plot of coupled DFGRNs (23) and (24) based on controller (19) with $q = 0.98$ and (a) $m_1(t), \widetilde{m}_1(t)$; (b) $m_2(t), \widetilde{m}_2(t)$.**

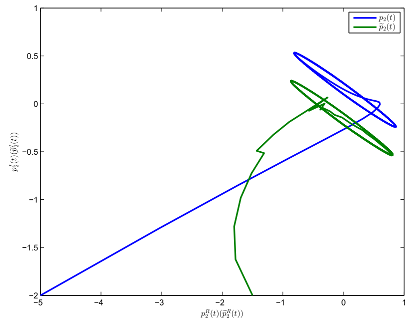


(a) $m_3(t), \widetilde{m}_3(t)$

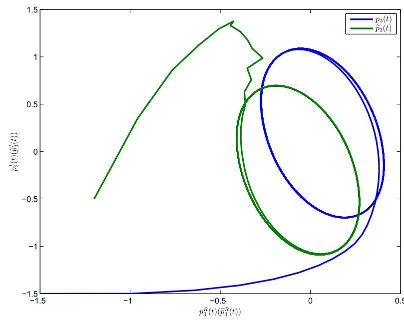


(b) $p_1(t), \widetilde{p}_1(t)$

FIGURE 12. **Phase plot of coupled DFGRNs (23) and (24) based on controller (19) with $q = 0.98$ and (a) $m_3(t), \widetilde{m}_3(t)$; (b) $p_1(t), \widetilde{p}_1(t)$.**

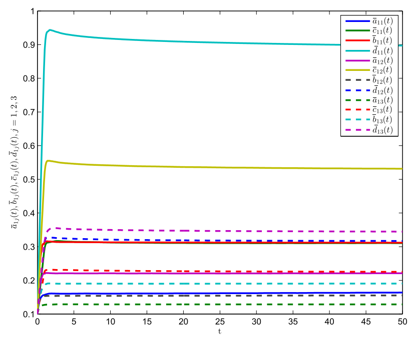


(a) $p_2(t), \widetilde{p}_2(t)$

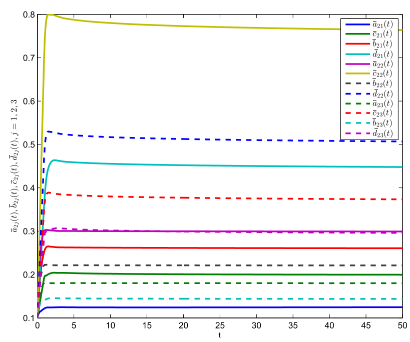


(b) $p_3(t), \widetilde{p}_3(t)$

FIGURE 13. **Phase plot of coupled DFGRNs (23) and (24) based on controller (19) with $q = 0.98$ and (a) $p_2(t), \widetilde{p}_2(t)$; (b) $p_3(t), \widetilde{p}_3(t)$.**



(a) $\bar{a}_{1j}(t), \bar{b}_{1j}(t), \bar{c}_{1j}(t), \bar{d}_{1j}(t), j = 1, 2, 3$



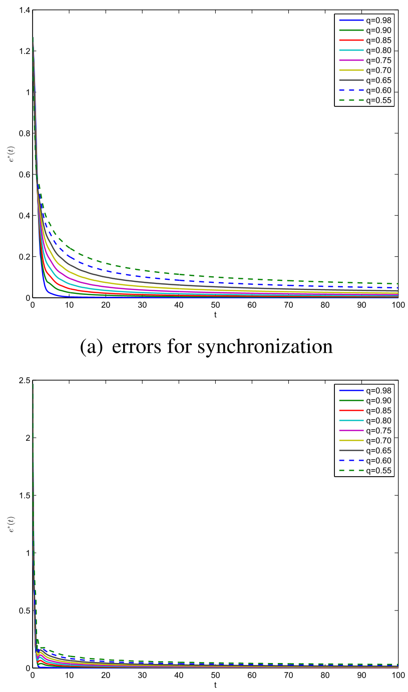
(b) $\bar{a}_{2j}(t), \bar{b}_{2j}(t), \bar{c}_{2j}(t), \bar{d}_{2j}(t), j = 1, 2, 3$

FIGURE 14. **Time response of the control gains in controller (19) with $q = 0.98$ and (a) $\bar{a}_{1j}(t), \bar{b}_{1j}(t), \bar{c}_{1j}(t), \bar{d}_{1j}(t), j = 1, 2, 3$; (b) $\bar{a}_{2j}(t), \bar{b}_{2j}(t), \bar{c}_{2j}(t), \bar{d}_{2j}(t), j = 1, 2, 3$.**

a system that contains $12n$ *fractional-order differential equations. When the time range is* $[0, 50]$ *and step-length* $h = 0.1$*, the numerical result is a* $12n \times 500$ *matrix. This means that the time complexity and space complexity of our calculation are* $O(n)$*, that is, the running time and storage space are linear functions of n.*

TABLE 1. The minimum "estimated time" T_1 and T_2 with different fractional order q.

q	0.98	0.90	0.85	0.80	0.75	0.70	0.65	0.60	0.55
T_1	2.90	3.50	5.10	6.80	9.50	13.6	20.1	31.0	50.3
T_2	1.10	1.10	1.10	1.10	1.30	3.70	5.30	7.40	10.9



(a) errors for synchronization

(b) errors for anti-synchronization

FIGURE 15. The errors of synchronization and anti-synchronization with different fractional order q and (a) errors for synchronization; (b) errors for anti-synchronization.

V. CONCLUSION

By designing two kinds of adaptive feedback controllers respectively, this paper deals with the global asymptotical synchronization and anti-synchronization for a class of drive-response system of fractional-order complex-valued GRNs with time-varying delays. And by combining fractional-order Lyapunov-like function method with the fractional-order inequality techniques, some sufficient criteria for global asymptotical synchronization and anti-synchronization are derived.

Compared with the other synchronization conditions for GRNs, which expressed via linear matrix inequality (LMI) [17], our sufficient conditions of synchronization and anti-synchronization are simpler and need not to be calculated by the MATLAB LMI toolbox in the simulation part, which avoids the computational complexity caused by high dimension matrix.

In Table 1, we give the minimum "estimated time" T_1 and T_2 to achieve synchronization and anti-synchronization, respectively. And we find that the values of T_1 and T_2 increase with the decreasing of the fractional-order q of DFGRNs.

In the future, we will pay more attention to dynamic behaviors of FGRNs with leakage delay, structured uncertainties or stochastic disturbance, such as finite-time synchronization, finite-time stability, Hopf bifurcation, and explore their corresponding practical application.

APPENDIX

Proof of Lemma 3

Proof: Using the contradiction method. Otherwise, there is a constant $\varepsilon > 0$ and the time series $\{s_k\}$ satisfying $0 < s_1 < s_2 < \cdots < s_k < s_{k+1} < \cdots$ and $\lim_{k\to\infty} s_k = \infty$ such that

$$\Phi(s_k) \geq \varepsilon, \quad k = 1, 2, \cdots. \tag{25}$$

Denote $T = \left(\frac{\Gamma(q+1)\varepsilon}{2M^*}\right)^{\frac{1}{q}} > 0$. When $s_k \leq t \leq s_k + T$, $k = 1, 2, \cdots$, according to inequality (ii) and (25), we have

$$\begin{aligned}\Phi(t) - \Phi(s_k) &= \frac{1}{\Gamma(q)}\int_{s_k}^{t}(t-s)^{q-1}D^q\Phi(s)ds\\ &\geq \frac{-M^*}{\Gamma(q)}\int_{s_k}^{t}(t-s)^{q-1}ds\\ &= \frac{-M^*}{\Gamma(q+1)}(t-s_k)^q \geq \frac{-M^*}{\Gamma(q+1)}T^q = -\frac{\varepsilon}{2},\end{aligned}$$

which shows that $\Phi(t) \geq \Phi(s_k) - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}, s_k \leq t \leq s_k + T, k = 1, 2, \cdots$.

When $s_k - T \leq t \leq s_k, k = 1, 2, \cdots$, from inequality (ii) and (25), we get

$$\begin{aligned}\Phi(s_k) - \Phi(t) &= \frac{1}{\Gamma(q)}\int_{t}^{s_k}(s_k-s)^{q-1}D^q\Phi(s)ds\\ &\leq \frac{M^*}{\Gamma(q)}\int_{t}^{s_k}(s_k-s)^{q-1}ds\\ &= \frac{M^*}{\Gamma(q+1)}(s_k-t)^q \leq \frac{M^*}{\Gamma(q+1)}T^q = \frac{\varepsilon}{2},\end{aligned}$$

which implies that $\Phi(t) \geq \Phi(s_k) - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}, s_k - T \leq t \leq s_k, k = 1, 2, \cdots$.

Therefore,

$$\Phi(t) \geq \frac{\varepsilon}{2}, s_k - T \leq t \leq s_k + T, \quad k = 1, 2, \cdots. \tag{26}$$

Without losing generality, we assume that these interval disjoint and $s_1 - T > 0$, then for any $k = 1, 2, \cdots$, we get

$$s_{k-1} + T < s_k - T < s_k + T < s_{k+1} - T. \tag{27}$$

When $s_k - T \leq t \leq s_k + T$, from inequality (i) and (26), we get

$$D^q\Psi(t) \leq -\frac{L^*\varepsilon}{2}.$$

Then for any $k = 1, 2, \cdots$, we can obtain

$$\begin{aligned}&\Psi(s_k+T) - \Psi(s_k-T)\\ &\quad = \frac{1}{\Gamma(q)}\int_{s_k-T}^{s_k+T}(s_k+T-s)^{q-1}D^q\Psi(s)ds\end{aligned}$$

$$\leq \frac{-L^*\varepsilon}{2\Gamma(q)}\int_{s_k-T}^{s_k+T}(s_k+T-s)^{q-1}ds$$
$$= -\frac{2^{q-1}L^*\varepsilon}{\Gamma(q+1)}T^q. \tag{28}$$

From the Definition 1, we can get

$$\Psi(s_k-T)-\Psi(s_{k-1}+T) = \frac{1}{\Gamma(q)}\int_{s_{k-1}+T}^{s_k-T}(s_k-T-s)^{q-1}D^q\Psi(s)ds, \tag{29}$$

and

$$\Psi(s_1-T)-\Psi(0)=\frac{1}{\Gamma(q)}\int_0^{s_1-T}(s_1-T-s)^{q-1}D^q\Psi(s)ds. \tag{30}$$

From inequality (i), (27), (29) and (30), we can obtain

$$\Psi(s_{k-1}+T)\geq\Psi(s_k-T),\quad k=1,2,\cdots, \tag{31}$$

and $\Psi(0)\geq\Psi(s_1-T)$.

From (28) and (31), we have

$$\begin{aligned}\Psi(s_k+T)-\Psi(0) &= [\Psi(s_k+T)-\Psi(s_{k-1}+T)]\\ &\quad+[\Psi(s_{k-1}+T)-\Psi(s_{k-2}+T)]\\ &\quad+\cdots+[\Psi(s_2+T)-\Psi(s_1+T)]\\ &\quad+[\Psi(s_1+T)-\Psi(0)]\\ &\leq [\Psi(s_k+T)-\Psi(s_k-T)]\\ &\quad+[\Psi(s_{k-1}+T)-\Psi(s_{k-1}-T)]\\ &\quad+\cdots+[\Psi(s_2+T)-\Psi(s_2-T)]\\ &\quad+[\Psi(s_1+T)-\Psi(s_1-T)]\\ &\leq -\frac{2^{q-1}L^*\varepsilon}{\Gamma(q+1)}T^qk. \end{aligned} \tag{32}$$

From (32), we have $\Psi(s_k+T)\leq\Psi(0)-\frac{2^{q-1}L^*\varepsilon}{\Gamma(q+1)}T^qk$. It reveals that $\Psi(s_k+T)\to-\infty$ when $k\to+\infty$, which contradict with $\Psi(t)\geq 0$. So $\lim_{t\to\infty}\Phi(t)=0$. □

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