IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received July 20, 2020, accepted August 4, 2020, date of publication August 7, 2020, date of current version August 19, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3014883

Fault-Tolerant Metric Dimension of Interconnection Networks

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The work of Sakander Hayat and Muhammad Imran was supported by the University Programme for Advance Research (UPAR) grant of United Arab Emirates University under Grant G00003271.

ABSTRACT A fixed interconnection parallel architecture is characterized by a graph, with vertices corresponding to processing nodes and edges representing communication links. An ordered set R of nodes in a graph G is said to be a resolving set of G if every node in G is uniquely determined by its vector of distances to the nodes in R. Each node in R can be thought of as the site for a sonar or loran station, and each node location must be uniquely determined by its distances to the sites in R. A fault-tolerant resolving set R for which the failure of any single station at node location v in R leaves us with a set that still is a resolving set. The metric dimension (resp. fault-tolerant metric dimension) is the minimum cardinality of a resolving set (resp. fault-tolerant resolving set). In this article, we study the metric and fault-tolerant dimension of certain families of interconnection networks. In particular, we focus on the fault-tolerant metric dimension of the butterfly, the Benes and a family of honeycomb derived networks called the silicate networks. Our main results assert that three aforementioned families of interconnection have an unbounded fault-tolerant resolvability structures. We achieve that by determining certain maximal and minimal results on their fault-tolerant metric dimension.

INDEX TERMS Graph theory, metric dimension, fault-tolerant metric dimension, NP-complete problems, interconnection networks.

I. INTRODUCTION

A. LITERATURE BACKGROUND

The concept of the metric dimension was put forward independently by Slater [34] and Harary and Melter [12] in 1975 and 1976 respectively. Since then, this graph parameter has found potential applications in many scientific areas such as chemistry [8], the robot navigation [19], network discovery and verification [3] and geographical routing protocols [25], among others.

Note that finding metric dimension of a graph is an NP-complete problem [11]. Therefore, it is interesting to study the minimum metric dimension (MMD) problem for infinite families of graph-theoretic interest. Bailey and Cameron [1] studied the MMD problem for the Kneser and Johnson graphs. Bailey and Meagher [2] determined the metric dimension of Grassmann graphs. Fehr *et al.* [10] studied the MMD problem for Cayley digraphs. Cáceres *et al.* [6] (resp. Cáceres *et al.* [5]) studied the problem for Cartesian product graphs (resp. infinite graphs). It is interesting to know that the metric dimension of an infinite graph can both be finite or infinite. The metric dimension of wheel-related graphs was studied by Siddiqui and Imran [33] and of convex polytopes by Kratica *et al.* [20]. The MMD problem of convex polytopes which could be generated from wheel-related graphs was studied by Imran *et al.* [17]. Hsieh and Hsiao [15] studied certain topological properties of the k-degree Caley networks.

Chartrand and Zhang [7] suggested to use the members of metric basis as censors. Given that, a faulty censor will lead to failure in recognizing the thief (intruder, fire etc.) in the system. The concept of a fault-tolerant resolving set resolves

The associate editor coordinating the review of this manuscript and approving it for publication was Ton Duc Do.

this issue by adding the assumption that a faulty censor will not lead to the system failure as the remaining censors will still be able to deal with the intruder. This suggest that applications of fault-tolerant metric dimension are as diverse as they are for the classical metric dimension. For more detailed information regarding application and mathematical properties of the fault-tolerance in resolvability, we refer the reader to [18], [32].

Fault-tolerant metric dimension has been studied in various fields. Raza *et al.* [30] studied applications of the fault-tolerant metric dimension in certain direct interconnection architectures. Somasundari and Raj [35] studied the fault-tolerant metric dimension of oxide interconnection networks. Krishnan and Rajan [21] studied applications of the fault-tolerant metric dimension in crystalline structures. Raza *et al.* [31] studied applications of fault-tolerant metric dimension in convex polytopes. Liu *et al.* [23] studied the fault-tolerant metric dimension of wheel related graphs. Raza *et al.* [29] studied the extremal structure of graphs with respect to the fault-tolerant metric dimension. For more applications, we refer the reader to [37].

In view of this study, it is important to notice that the MMD problem for interconnection networks was first studied by Manuel *et al.* [26]. We continue this study by investigating the fault-tolerant metric dimension problem for three important classes of interconnection networks known as butterfly networks, Benes networks and silicate networks.

B. MATHEMATICAL PRELIMINARIES

A graph G is an ordered pair (V, E), where the set V comprising the nodes called the vertex-set and E is the set of the links between those nodes call the edge-set. The distance between two vertices say x and y is denoted by $d(x, y)$ and is defined to be the length of a shortest path between x and y. A node $x \in G$ is said to *resolve* two nodes u and v, if $d(x, u) \neq d(x, v)$. We refer the reader to the book by Bondy and Murty [4] for standard terminologies on graph theory.

For any $W \subset V(G)$ if there exist a vertex $w \in W$ such that every pair of vertices $v_1, v_2 \in V$ is resolved by w, then we say that W is a resolving set of G. For a resolving set W and $w \in W$, if $W \setminus \{w\}$ is still a resolving set we say that W is a fault-tolerant resolving set. The smallest possible cardinality of a resolving set (resp. fault-tolerant resolving set) in G is called the metric dimension $\beta(G)$ (resp. fault-tolerant metric dimension $\beta'(G)$). A resolving (resp. fault-tolerant resolving) set of cardinality $\beta(G)$ (resp. $\beta'(G)$) is said to be the metric basis (resp. fault-tolerant metric basis) of G.

Note that the definition of the fault-tolerant metric dimension naturally suggests the following inequality. For any graph G,

$$\beta(G) + 1 \leq \beta'(G). \qquad (1)$$

Note that the equality in G holds for cycles.

In order to understand the concept of metric and fault-tolerant metric dimension better, we provide an example of a tree $\mathcal{T}$ with matric dimension 10 and fault-tolerant metric dimension 14. Figure 1 exhibits the tree $\mathcal{T}$. Note that the vertices 1 to 10 form a metric basis whereas adding y, v, r, s to these 10 vertices generate a fault-tolerant metric basis of $\mathcal{T}$.

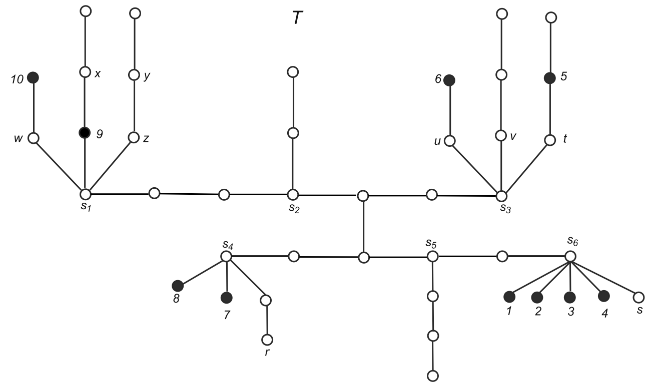

FIGURE 1. The tree $\mathcal{T}$ with $\beta'(\mathcal{T}) = 14$ and $\beta(\mathcal{T}) = 10$.

Javaid *et al.* proved the following lemma which shows an alternative way to trace a fault-tolerant resolving set in a graph.

Lemma 1.1 [18]: A resolving set R of graph G is fault-tolerant if and only if every pair of nodes of G is resolved by at least two nodes of R.

In view of the relation between a resolving and fault-tolerant resolving set, Hernando *et al.* proved the following relation between the parameters $\beta(G)$ and $\beta'(G)$.

Theorem 1.2 [14]: Fault-tolerant metric dimension is bounded by a function of the metric dimension (independent of the graph). In particular, $\beta'(G) \leq \beta(G)\left(1+2.5^{\beta(G)-1}\right)$ *for every graph G.*

Javaid *et al.* proved the following interesting result which shows that the difference between metric dimension and fault-tolerant metric dimension of a graph can be arbitrary large.

Theorem 1.3 [18]: For every natural number ν*, there exist a graph such that* $\beta'(G) \geq \beta(G) + \nu$.

For any set $W \subset V(G)$, let $\gamma(W)$ be the set of all the common neighbors of vertices in W. Based on this concept, Raza *et al.* [31] determined the following interesting relation between a resolving and a fault-tolerant resolving set.

Lemma 1.4 [31]: Let W be a resolving set of graph G. Then $W' := \cup_{v \in W}\left(N[v] \cup \gamma(N(v))\right)$ *is a fault-tolerant resolving set of G.*

II. A TOPOLOGICAL REPRESENTATION OF BUTTERFLY AND BENES NETWORKS

In this section, we present a topological representation of butterfly and Benes and networks introduced by Manuel *et al.* [26]. They have called this new representation and the usual structural drawing, a diamond representation and a normal representation respectively. The diamond representation plays a key role in locating the resolving and fault-tolerant resolving sets in these networks. Moreover, this representation is very useful and efficient in studying certain cyclic and other geometric properties of these interconnection networks. Kruskal and Snir [22] studied the structural properties of interconnection networks and presented a unified theory for them. They also studied the flow mechanisms of interconnection networks and its role in certain geometric properties of these multistage interconnection structures.

In order to visualize the normal and diamond representations, it is required to properly define these networks. The representation of an interprocessor communication model as an undirected graph takes processors as its nodes and links between processors as the edges. The butterfly networks of dimension r has the node-set denoted by $BF(r)$ which are of the form of pairs $[s, i]$, in which i is the dimension/level of a node which varies from 0 to r and s is an r-bit binary string which denoted the row of the corresponding node. The nodes $[s, i]$ and $[s', i']$ are connected by an edge if and only if $i' = i + 1$ and either s and s' are identical, or both s and s' differ in precisely the ith bit. The number of nodes and edges in $BF(r)$ are $2^r(r + 1)$ and $r2^{r+1}$ respectively.

An *r-dimensional Benes network* denoted by $B(r)$ is defined in a similar fashion, in fact the number of levels are $2r + 1$ while having 2^r nodes in each level. The nodes are of the form $[s, i]$, $(0 \leq i \leq 2r)$ and linked to each other by the same fashion we have in $BF(r)$. Thus, in other words $B(r)$ is back-to-back butterflies as one can obtain a copy of $BF(r)$ from level 0 to level r. The number of nodes and edges in $B(r)$ are $2^r(2r + 1)$ and $r2^{r+2}$ respectively. It is necessary to mention that the edges in both $BF(r)$ and $B(r)$ are undirected. The representation of butterfly and Benes networks, which we obtain by arranging the nodes level-wise and link them as defined above, is called their *normal representation*.

The description of the diamond representation of $BF(r)$ is explained as follows: Corresponding to any array of level 0 nodes, two butterfly networks of dimension $r - 1$ generate reflexive images. Bridging the two $BF(r - 1)$, the nodes on level 0 are the nodes of chord-less quadrangles in the diamond representation. So every quadrangle is considered a diamond in the later representation. Both normal and diamond representation of $BF(3)$ i.e. the 3-dimensional butterfly networks are exhibited in Fig. 2. Given this sceneries, the diamond representation Benes network comprises back-to-back butterfly networks, Fig. 3. This new representation assist in understanding the spanning trees and cyclic properties better in these networks, since this representation has a better structure visualization.

Although an r-dimensional Benes network comprises back-to-back butterflies, a considerable topological and structural difference can be observed between butterfly and Benes networks. The structural similarity and dissimilarity can be observed as follows. The deletion of nodes of level 0 generates two disjoint copies of $(r - 1)$-dimensional butterflies, and in the same spirit, the removal of level r of also leaves two disjoint copies of $BF(r - 1)$. An alternative way to view this recursive structure is as follows: the deletion of nodes of levels both 0 and r of an r-dimensional butterfly network generates 4 disjoint copies of of an $(r - 2)$-dimensional

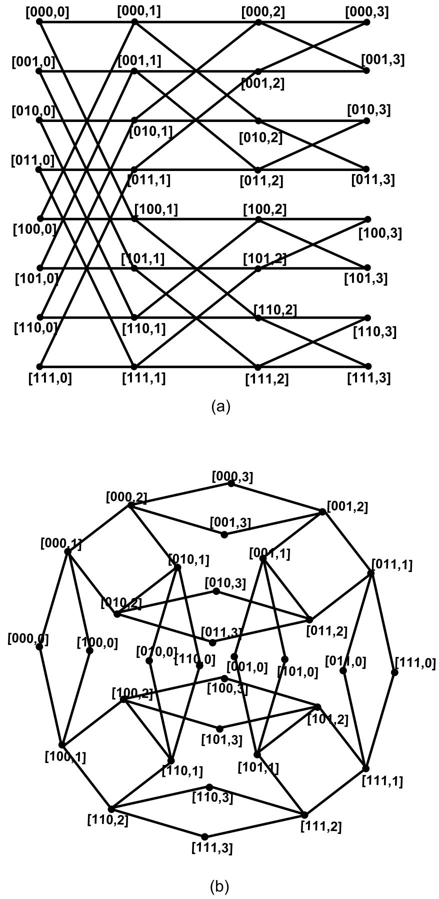

FIGURE 2. (a): Normal representation of $BF(3)$, (b): diamond representation of $BF(3)$.

butterfly network. Whereas, the removal of level 0 and $2r$ nodes in $B(r)$ leaves 2 disjoint copies of a $B(r-1)$. In other words, the butterfly has dual symmetry, which the Benes does not have. Imran *et al.* [16] studied certain topological properties of the Benes and butterfly networks.

Lemma 3.1 [26]: The normal and diamond representations of butterfly and Benes networks are isomorphic.

III. FAULT-TOLERANT METRIC DIMENSION OF BENES NETWORKS

We study the fault-tolerant metric dimension problem in this section. Note that the diamond representation of an r-dimensional Benes network $B(r)$ is embeddable on $2D$ grid. An embedding of this type of $B(3)$ on grid is depicted in Fig. 4. We call two nodes x and y the *horizontal*, if they lie in the same row of the underlying grid. In a similar fashion, they are called *vertical*, if they are drawn in the same column of the grid. For example, the nodes [000, 2] and [110, 2] are vertical nodes in Fig. 3 whereas, in the same Fig, the nodes [000, 1] and [000, 5] are horizontal nodes.

There is an important structural observation for Benes network $B(r)$. By deleting all the nodes of level r, we can partition $B(r)$ into $(r-1)$-dimensional butterflies, say, $BF_1(r-1)$, $BF_2(r - 1)$, $BF_3(r - 1)$, and $BF_4(r - 1)$. This structural

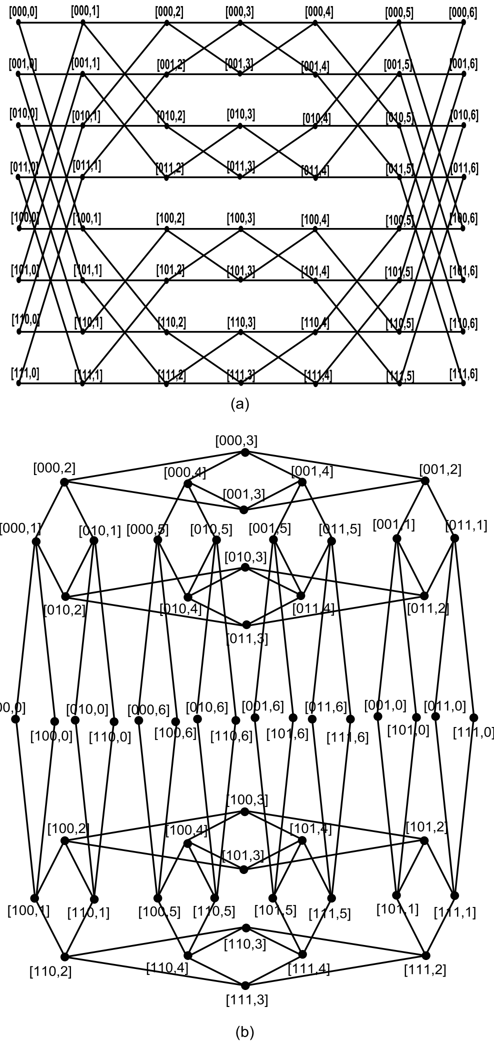

FIGURE 3. (a): Normal representation of $B(3)$, (b): diamond representation of $B(3)$.

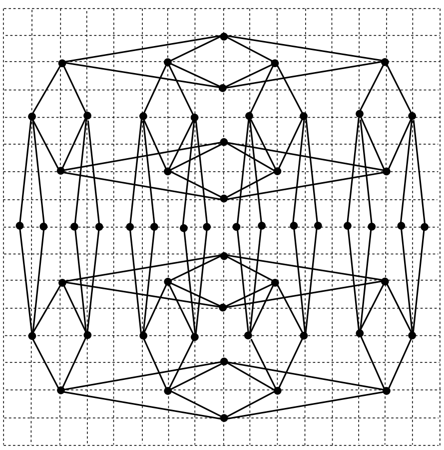

FIGURE 4. An embedding of 3-dimensional Benes network on grid.

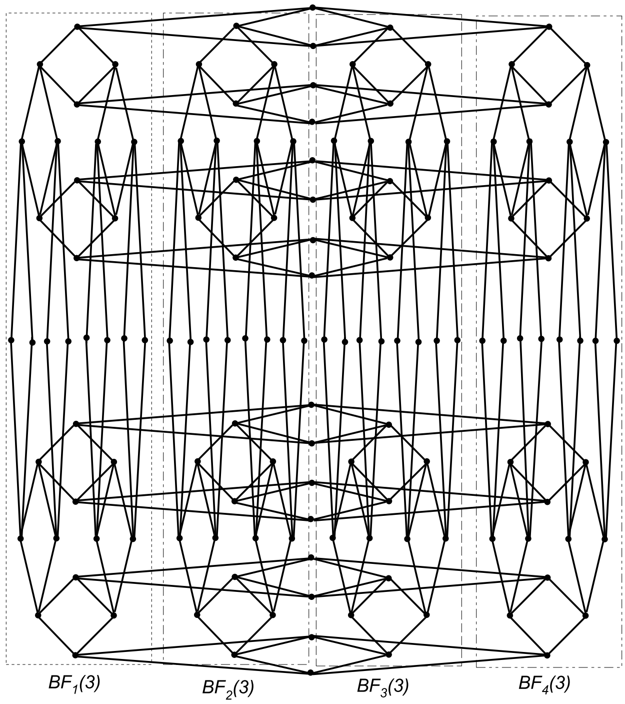

FIGURE 5. A structural decomposition of $B(4)$ into $BF_1(3)$, $BF_2(3)$, $BF_3(3)$, and $BF_4(3)$. They are marked with rectangles in different styles.

decomposition of in a 3-dimensional Benes network can be viewed in Fig. 5. The butterfly network $BF_1(r-1)$ consists of nodes $\{[t_1t_2\dots t_{r-1}0, l] : t_1t_2\dots t_{r-1}$ *is any binary string and* $0 \le l \le (r-1)\}$ and similarly $BF_2(r-1)$ comprises nodes $\{[t_1t_2\dots t_{r-1}0, l] : t_1t_2\dots t_{r-1}$ *is any binary string and* $(r+1) \le l \le 2r\}$. On the other hand, the butterfly network $BF_3(r-1)$ contains nodes $\{[t_1t_2\dots t_{r-1}1, l] : t_1t_2\dots t_{r-1}$ *is any binary string and* $(r+1) \le l \le 2r\}$ and, in the similar fashion, $BF_4(r-1)$ consists of nodes $\{[t_1t_2\dots t_{r-1}1, l] : t_1t_2\dots t_{r-1}$ *is any binary string and* $0 \le l \le (r-1)\}$. We will notice in Section IV that the butterfly networks do not possess this type of structural decomposition.

We need the following lemma before we present our main theorem of this section.

Lemma 4.1: Let $B(r)$ be an r-dimensional Benes network. If $F_0 = \{[0s_2\dots s_r, 0], [0s_2\dots s_r, 2r] : s_2\dots s_r$ is any binary string$\}$ and $F_1 = \{[s_1s_2\dots s_{r-1}0, r] : s_1\dots s_{r-1}$ is any binary string$\}$, then $\gamma(N(F_0)) = \{[1s_2\dots s_r, 0], [1s_2\dots s_r, 2r] : s_2\dots s_r$ is any binary string$\}$ and $\gamma(N(F_1)) = \varnothing$.

Proof: Let x be a node of r-dimensional Benes network $B(r)$ such that $x \in F_0$. Then x is either of the form $[0s_2\dots s_r, 0]$ or $[0s_2\dots s_r, 2r]$. If x is a node of the form $[0s_2\dots s_r, 0]$, then $N(x) = \{[s_1s_2\dots s_r, 1] : s_1s_2\dots s_r$ *is any binary string*$\}$. Note that, in this case, the only common neighbors of $[0s_2\dots s_r, 1]$ and $[1s_2\dots s_r, 1]$ are $[1s_2\dots s_r, 0]$ where $s_2\dots s_r$ is any binary string. A similar kind of argument holds for x to be the other possible case. Thus we obtain $\gamma(N(F_0)) = \{[1s_2\dots s_r, 0], [1s_2\dots s_r, 2r] : s_2\dots s_r$ is any binary string$\}$. Now assume x to be of the form $[s_1s_2\dots s_{r-1}0, r]$, then $N(x) = \{[s_1s_2\dots s_{r-1}0, r-1], [s_1s_2\dots s_{r-1}1, r-1], [s_1s_2\dots s_{r-1}0, r+1], [s_1s_2\dots s_{r-1}1, r+1] : s_1s_2\dots s_{r-1}$

is any binary string}. By the definition of Benes network, there can not be any common neighbor of the nodes of these four types. Thus in this case we obtain $\gamma(N(F_1)) = \varnothing$. □

Now we present our main result of this section. Claims 1 & 2 were essentially shown by Manuel *et al.* [26].

Theorem 4.2: Let $B(r)$ be the r-dimensional Benes network with $r > 2$. Let $F_0 = \{[s_1s_2\dots s_r, \ell] : s_1\dots s_r$ is any binary string and $\ell = 0, 1, r-1, r+1, 2r-1, 2r\}$ and $F_1 = \{[s_1s_2\dots s_{r-1}0, r] : s_1\dots s_{r-1}$ is any binary string$\}$. Then $F_0 \cup F_1$ is a fault-tolerant resolving set of $B(r)$.

Proof: Let G be r-dimensional Benes network. Let c_1, c_2, c_3, c_4, c_5 and c_6 are the nodes of types $[0t_2\dots t_k, 0]$, $[1t_2\dots t_k, 0]$, $[0t_2\dots t_k, 2r]$, $[1t_2\dots t_k, 2r]$, $[t_1t_2\dots t_{k-1}0, r]$ and $[t_1t_2\dots t_{k-1}1, r]$ respectively.

Claim 1: Any resolving set R of G comprises either c_1 or c_2, either c_3 or c_4, and either c_5 or c_6.

Note that the nodes $[0t_2\dots t_k, 0]$, $[0t_2\dots t_k, 1]$, $[1t_2\dots t_k, 0]$, and $[1t_2\dots t_k, 1]$ lie on a cycle of length 4, say, C in G. Furthermore, the nodes $[0t_2\dots t_k, 0]$ and $[1t_2\dots t_k, 0]$ are having degree 2 in C. Let u be a node of G. If a geodesic between u and $[0t_2\dots t_k, 0]$ traverses $[0t_2\dots t_k, 1]$, then a geodesic between u and $[1t_2\dots t_k, 0]$ also traverses the same node $[0t_2\dots t_k, 1]$. Thus $d_G(u, [0t_2\dots t_k, 0]) = d_G(u, [1t_2\dots t_k, 0])$ for any node u of G. Thus we deduce that any resolving set R of G consists of either c_1 or c_2. A similar argument also works for other two possible cases. Thus the claim follows.

Let $S_0 = \{[0s_2\dots s_r, 0], [0s_2\dots s_r, 2r] : s_2\dots s_r$ *is any binary string*$\}$ and $S_1 = \{[s_1s_2\dots s_{r-1}0, r] : s_1\dots s_{r-1}$ *is any binary string*$\}$.

Claim 2: $S_0 \cup S_1$ is a resolving set of G.

Let $R = S_0 \cup S_1$, and let x and y be the arbitrary nodes of $V \setminus R$ of G. Assume that $x = [x_1x_2\dots x_r, j]$ and $y = [y_1y_2\dots y_r, k]$. Note that it is possible to locate a node $u \in R$ such that $d(x, u) \neq d(y, u)$, if one of x and y lies at level $2r$, level r, or level 0. Thus it is enough to assume that the nodes x and y are not at level $2r$, level r, and level 0. In that case, we are left with three possible cases for the nodes x and y.

As case 1, we assume that both x and y are vertical. Note that any two nodes lie at same level, whenever they are vertical. Thus $k = j$ in the nodes x and y. By considering a subcase where both x and y are the nodes of $BF_1(r-1)$, we obtain that $j < r$ and $x_r = y_r = 0$. Recall that a landmark is an element of a resolving set in a graph. Now we consider the landmark $u = [x_1x_2\dots x_{r-1}0, r]$. We may assume without loss of generality that $x_1x_2\dots x_r < y_1y_2\dots y_r$. Let p be the smallest index such that $x_1x_2\dots x_p = y_1y_2\dots y_p$ and $x_{p+1} \neq y_{p+1}$. Also note that a geodesic between $y = [y_1y_2\dots y_r, j]$ and $u = [x_1x_2\dots x_{r-1}0, r]$ is $[y_1y_2\dots y_py_{p+1}\dots y_r, j], [y_1y_2\dots y_py_{p+1}\dots y_r, j-1], \dots, [y_1y_2\dots y_py_{p+1}\dots y_r, p], [y_1y_2\dots y_py_{p+1}\dots y_r, p+1], \dots, [y_1y_2\dots y_px_{p+1}\dots x_r, j], [y_1y_2\dots y_px_{p+1}\dots x_r, j+1], \dots, [y_1y_2\dots y_px_{p+1}\dots x_r, r]$. By assumption $x_1x_2\dots x_p = y_1y_2\dots y_p$ and $x_r = 0$. Thus it follows that $[y_1y_2\dots y_px_{p+1}\dots x_r, r] = [x_1x_2\dots x_{r-1}0, r]$. We conclude that this geodesic between y and u traverses x. And thus $d(y, u) \neq d(x, u)$. The remaining subcases are of similar argument.

Now we consider as a case 2, that both x and y are horizontal. Note that this case is similar to the case 1. However, the corresponding element of R is $[0x_2\dots x_r, 0]$ or $[0x_2\dots x_r, 2r]$.

Finally we assume the case when x and y are neither horizontal nor vertical. Let us assume that $x = [x_1x_2\dots x_r, j]$ and $y = [y_1y_2\dots y_r, k]$. Among possible subcases, first we consider the subcase when both x and y belong to $BF_1(r-1)$. This implies that $j < r$, $k < r$, and $u_r = v_r = 0$. We first consider the case when $j \neq k$ as the case $j = k$ is essentially similar. Let us assume that $j > k$. Consider an element of $u \in R$ such that $u = [x_1x_2\dots x_{r-1}0, r]$. It can possibly be verified that $d(u, x) = r - j$. It is important to notice that all the nodes of level p in $B(r)$ form a co-clique (i.e. an independent set) for every value of p. This implies that, a geodesic from u at level r to y at level k passes through some node at level l, where $k \leq l \leq r$. Thus $d(u, y) \geq r - k$. Since $j > k$, we have $r - k > r - j$ and thus $d(y, u) \neq d(x, u)$. The other possible subcases are similar, so we skip them.

Claim 3: $F_0 \cup F_1$ is a fault-tolerant resolving set of G.

By Claim 1, any resolving set of G comprises at least $3(2^{r-1})$ nodes. Thus the resolving set $S := S_0 \cup S_1$ is a metric basis of G. Even though S is a metric basis of G, Lemma 1.1 does not assure us to generate a corresponding fault-tolerant metric basis of G. In fact we will obtain an upper bound by using the Lemma 1.1. Note that by Lemma 1.1, $\cup_{0\leq i\leq 1}\left(N[S_i] \cup \gamma(N(S_i))\right)$ is a fault-tolerant resolving set of G. By definition of Benes networks, we obtain that $N[S_0] = S_0 \cup A$ such that $A = \{[s_1s_2\dots s_r, 1], [s_1s_2\dots s_r, 2r-1] : s_1s_2\dots s_r$ *is any binary string*$\}$. In a similar manner, we obtain that $N[S_1] = S_1 \cup B$ where $B = \{[s_1s_2\dots s_r, r-1], [s_1s_2\dots s_r, r+1] : s_1s_2\dots s_r$ *is any binary string*$\}$. By simultaneously using these facts, Lemma 4.1 and the Lemma 1.1 we obtain that $F_0 \cup F_1$ is a fault-tolerant resolving set of G. This completes the proof. □

As a corollary to Theorem 4.2, we present the following result.

Corollary 4.3: Let G be an r-dimensional Benes network, then $\beta'(G) \leq 13(2^{r-1})$.

Proof: By Theorem 4.2, there exists a fault-tolerant resolving set of order $|F_0 \cup F_1| = 13(2^{r-1})$ in G. Thus the fault-tolerant metric dimension $\beta'(G)$ can be at most $13(2^{r-1})$. □

As the difference between the metric dimension and the fault-tolerant metric dimension can be arbitrary large, it is not surprising that we obtain the upper bound of $\beta'(B(r))$ is comparatively larger than $\beta(B(r))$. Note that, in the proof of Theorem 4.2, $S_0 \cup S_1$ is a metric basis of $B(r)$. Thus we record our calculations by proposing the following conjecture.

Conjecture 4.4: Let G be an r-dimensional Benes network with $r > 2$. Then $\beta'(G) \geq 13(2^{r-1})$ and thus $\beta'(G) = 13(2^{r-1})$.

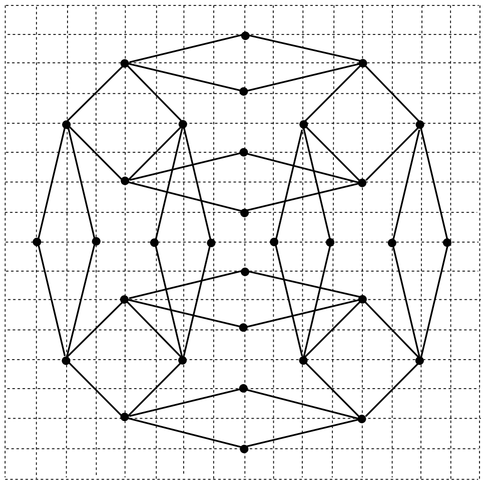

FIGURE 6. ***BF*(3) on grid.**

Proposition 4.5: Let G be an r-dimensional Benes network with $r > 2$. Then G is a family of interconnection networks with an unbounded fault-tolerant resolvability structures.

IV. FAULT-TOLERANT METRIC DIMENSION OF BUTTERFLY NETWORKS

In this section we study the fault-tolerant metric dimension of butterfly networks. As we have noticed in Section II that a Benes network is a back-to-back butterflies. This recursive structural property of Benes network leads us to the fact that it shares some topological properties with butterfly networks. This suggests a considerable similarity in the proof of our main result in this section. Even though we will skip the similar parts of the proof, we will properly state the assumption and results. In a similar spirit to the embedding of Benes network on grid, the butterfly network can be embedded to the grid as well. Fig. 6 shows an embedding of $BF(3)$ on grid.

As we described in Section III that the removal of level r nodes in $B(r)$ yields four disjoint copies of $BF(r-1)$. In the similar manner, the removal of nodes either of level 0 or of level r in $BF(r)$ leaves two disjoint copies of $BF(r-1)$. Let us denote these two copies of $(r-1)$-dimensional butterfly networks by $BF_1(r-1)$ and $BF_2(r-1)$, respectively. This phenomenon for $BF(4)$ can be observed in Fig. 7. As we have observed already in the proof of Theorem 4.2 that this sort of structural decomposition of Benes network is important. In case of butterfly networks, we will skip those part of proof of main theorem in this section which have structural resemblance with those corresponding parts of Theorem 4.2.

The following lemma gives us the necessary information to prove our main theorem of this section.

Lemma 5.1: Let $BF(r)$ be an r-dimensional butterfly network. If $F_0 = \{[0s_2\ldots s_r, 0] : s_2\ldots s_r$ is any binary string$\}$ and $F_1 = \{[s_1s_2\ldots s_{r-1}0, r] : s_1\ldots s_{r-1}$ is any binary string$\}$, then $\gamma(N(F_0)) = \{[1s_2\ldots s_r, 0] : s_2\ldots s_r$ is any binary string$\}$ and $\gamma(N(F_1)) = \{[s_1s_2\ldots s_{r-1}1, r] : s_1\ldots s_{r-1}$ is any binary string$\}$.

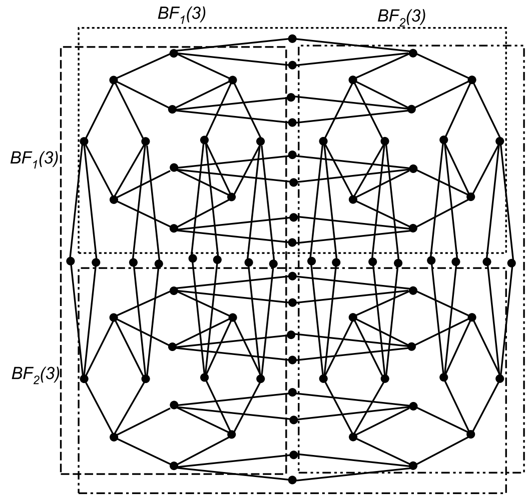

FIGURE 7. A structural decomposition of *BF*(4) in to *BF*$_1$(3) and *BF*$_2$(3). They are marked with rectangles in different styles.

Proof: Let $x \in F_0$. Then $x = [s_1s_2\ldots s_{r-1}0, r]$ and $N(x) = \{[s_1s_2\ldots s_r, 1] : s_1s_2\ldots s_r$ *is any binary string*$\}$. The common neighbors of $[0s_2\ldots s_r, 1]$ and $[1s_2\ldots s_r, 1]$ are $[1s_2\ldots s_r, 0]$ where $s_2\ldots s_r$ is any binary string. Then we obtain $\gamma(N(F_0)) = \{[1s_2\ldots s_r, 0] : s_2\ldots s_r$ is any binary string$\}$. The other case is similar. □

The following result gives us a fault-tolerant resolving set of $BF(r)$ for $r > 2$. Claims 1 & 2 were essentially shown by Manual *et al.* [26].

Theorem 5.2: Let $BF(r)$ be the r-dimensional butterfly network with $r > 2$. Then $F = \{[s_1s_2\ldots s_r, \ell] : s_1\ldots s_r$ is any binary string and $\ell = 0, 1, r-1, r\}$ is a fault-tolerant resolving set of $BF(r)$.

Proof: Let G be an r-dimensional butterfly network. Let c_1, c_2, c_3, and c_4 be the nodes of the types $[0t_2\ldots t_k, 0]$, $[1t_2\ldots t_k, 0]$, $[t_1t_2\ldots t_{k-1}0, r]$ and $[t_1t_2\ldots t_{k-1}1, r]$, respectively.

Claim 1: Any resolving set R of G consists of either c_1 or c_2 and either c_3 or c_4.

The proof of Claim 1 is similar to the proof of Claim 1 in Theorem 4.2. Thus, we skip it. Let $S_0 = \{[0s_2\ldots s_r, 0] : s_2\ldots s_r$ *is any binary string*$\}$ and $S_1 = \{[s_1s_2\ldots s_{r-1}0, r] : s_1\ldots s_{r-1}$ *is any binary string*$\}$.

Claim 2: $S_0 \cup S_1$ is a resolving set of G.

We skip the proof of Claim 2 due to similarity with the proof of Claim 2 in Theorem 4.2. Note that the vertical and horizontal nodes of G are the nodes in the same column and same row in the grid, see also Fig. 6. On the other hand, the butterflies $BF_1(r-1)$ and $BF_2(r-2)$ correspond to the $(r-1)$-dimensional butterflies in the Fig. 7.

Claim 3: F is a fault-tolerant resolving set of G.

Note that $N(S_0) = \{[s_1s_2\ldots s_r, 1] : s_1s_2\ldots s_r$ *is any binary string*$\}$ and $N(S_1) = \{[s_1s_2\ldots s_r, r-1] : s_1s_2\ldots s_r$ *is any binary string*$\}$. Now we use the Lemma 1.1 and Lemma 5.1 simultaneously to deduce that F is a fault-tolerant resolving set of G. This finishes the proof. □

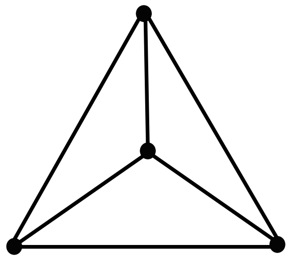

FIGURE 8. The tetrahedron. It acts as the unit for silicate sheets and networks.

As a corollary to Theorem 5.2, we present the following result.

Corollary 5.3: Let G be an r-dimensional butterfly network, then $\beta'(G) \leq 4(2^r)$.

Proof: By Theorem 5.2, there exists a fault-tolerant resolving set of order $|F| = 4(2^r)$ in G. Thus the fault-tolerant metric dimension $\beta'(G)$ can be at most $4(2^r)$. □

Note that, in the proof of Theorem 5.2, $S_0 \cup S_1$ is a metric basis of $BF(r)$. Thus we propose the following conjecture.

Conjecture 5.4: Let G be an r-dimensional butterfly network with $r > 2$. *Then* $\beta'(G) \geq 4(2^r)$ *and thus* $\beta'(G) = 4(2^r)$.

Proposition 5.5: Let G be an r-dimensional buterfly network with $r > 2$. *Then G is a family of interconnection networks with an unbounded fault-tolerant resolvability structures.*

V. FAULT-TOLERANT METRIC DIMENSION OF HONEYCOMB DERIVED NETWORKS

Fusion process in metallic carbonates or oxides with sand generate silicates. The key unit is the SiO_4 tetrahedron which exists in essentially-all silicates. From chemical perspectives, the central node of the tetrahedron symbolizes the silicon ion, whereas, the boundary atoms represent the oxygen ions. By following a similar graph-theoretic terminology, the corner/boundary vertices are called the oxygen nodes and the central vertex is called the silicon node. See Figure 8.

By recursively fusing oxygen nodes from two tetrahedra of distinct silicates, certain minerals are obtained. Different tetrahedra arrangements generate different silicate structures are obtained. These different arrangements include linked & unlinked distinctive entities, 1D chains, 2D sheets and 3D framework/structures. Some of these silicates are exhibited in Figure 9. Based on the tetrahedra arrangements in these silicates, they are named as ortho-, pyro-, chain and cyclic silicates.

Note that simple structures of orthosilicates comprises discrete SiO_4 tetrahedron as their units. Pyrosilicate are formed by joining an oxygen node of two tetrahedra. On the other way, a linear arrangement of tetrahedra gives rise to the chain silicates. Some sheet and cyclic silicates are shown in Figure 10.

A honeycomb network $HC(n)$ is constructed by attaching $n-1$ layers of hexagons on a central hexagons called its

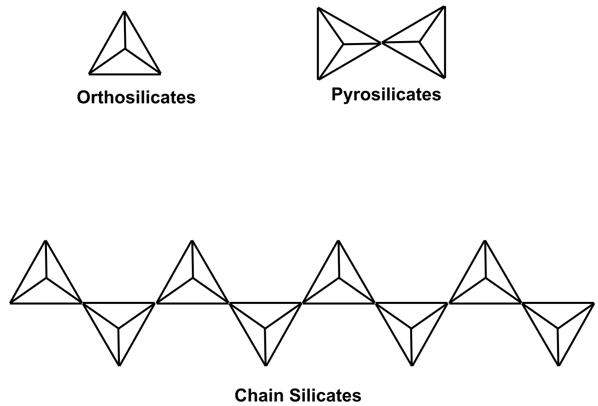

FIGURE 9. Types of linear silicates.

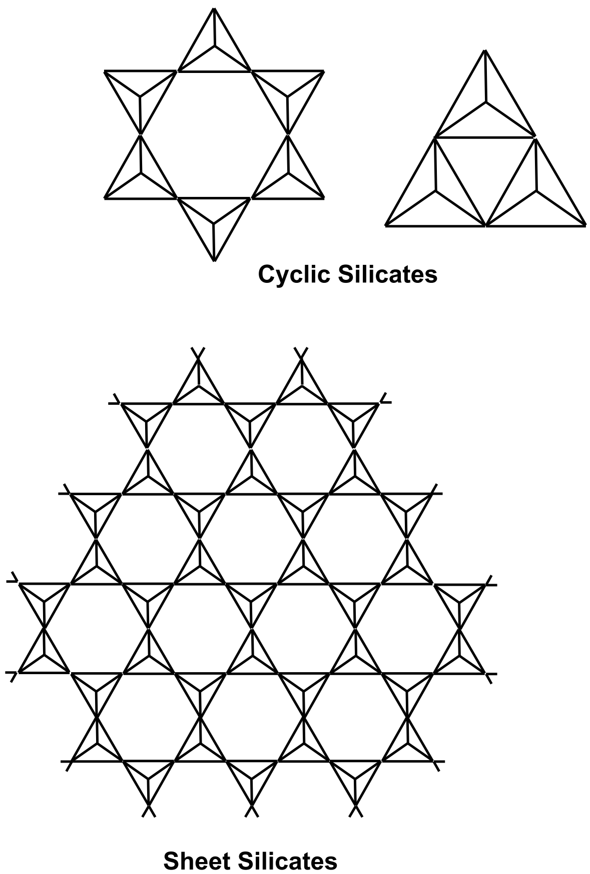

FIGURE 10. Some types of cyclic and sheet silicates.

core. A silicate network of dimension n symbolized as $SL(n)$, where n is the number of hexagons between the center and boundary of $SL(n)$. Addressing, routing and broadcasting in honeycomb networks is discuss in [9]. A construction of $SL(2)$ from $HC(3)$ is given in Figure 11. By deleting all the central nodes, which are called the silicon nodes, from the silicate network we obtain the Oxide network. We symbolize the oxide network of dimension n by $OX(n)$. It is important to notice that $SL(n)$ has both $OX(n)$ and $HC(n)$ as its subgraphs.

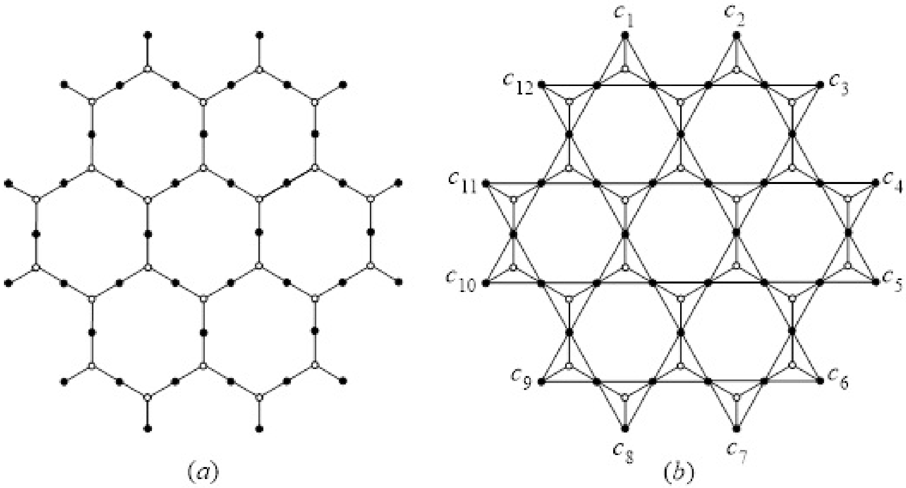

FIGURE 11. Silicate network construction and boundary nodes.

However, $OX(n)$ plays an important role in studying the structural and metric properties of $SL(n)$. First observation in this regard is that both $SL(n)$ and $OX(n)$ has the same diameter [27]. Some topological properties of the silicate and other related networks are derived in [13], [24].

Now we consider the aforementioned family of silicate networks as a multistage interconnection network. This is achieved by suggesting a coordinate system to this network and the first step in doing that is to allot an *id* to the nodes of the oxide network. And in the next step, the proposed coordinate system is extended [27] to the family of silicate networks. Note that a similar coordinate system has been proposed by Nocetti *et al.* [28] (resp. Stojmenovic [36]) for the hexagonal (resp. honeycomb) networks. Parallel to the three edge orientations, three coordinate axis namely α, β and γ having a mutual angle of 120 between any two of them, are proposed. The corresponding coordinate axes for β, α and γ axes are $\beta = 0$, $\alpha = 0$ and $\gamma = 0$ respectively. The lines parallel to each coordinate axis are called α-, β- and γ-lines. On the other hand, $\alpha = -k$ and $\alpha = h$ are the lines lie on on the two sides of α-axis. Any arbitrary node in an n-dimensional oxide network $OX(n)$ is assigned a triplet (a, b, c) and a, b and c are the coordinate position for the intersection of lines $\alpha = a$, $\beta = b$, and $\gamma = c$. Since each silicon node is the central node in the tetrahedron. One may assign *ids* to a silicon node by applying the formula of centroid of an equilateral triangle. See Figure 12.

Theorem 6.1: Let $SL(n)$ be the n-dimensional silicate network with $n > 2$. Then $\beta(SL(n)) \geq 6n$.

Proof: Let c be a boundary node and x be a central/silion node adjacent to c. Then note that any other node of the silicate network, say y, has the distance property $d(c, y) = d(x, y)$ holds. This shows that neither c nor x resolve the vertex y. This simply suggest to include either c or x in any metric basis of the silicate network. Note that the number of boundary nodes are exactly $6n$. This suggests that $\beta(SL(n)) \geq 6n$. □

Theorem 6.2 [26]: Let $SL(n)$ be the n-dimensional silicate network with $n > 2$. Then $\beta(SL(n)) = 6n$.

Proof: We claim that the set of boundary nodes is a metric basis. In view of symmetry of the network α-lines is the starting point of our discussion. The odd/even property of the parameter k determines the line $\alpha = k$ to be odd or even. α-channel is the region between two consecutive or alternative α-lines. One of the structural property suggests that all the boundary nodes of $SL(n)$ lie only on the α-lines. Figure 12 depicts this fact.

The dotted lines in Figure 12 are even axes and black lines are odd axes. For any oxygen node (a, b, c), $\alpha = a$, $\beta = b$, and $\gamma = c$ are the axis lines passing through (a, b, c). Among these three axis lines, two are odd and one is even. For the oxygen node X in Figure 12, α and γ lines are odd axis lines and γ line is an even axis line. Similarly for node N, γ and β are odd axis lines and α is an even axis line. Further if we call an edge joining two oxygen nodes as an oxide edge then each such oxide edge is on some odd axis line.

Now consider any two oxygen nodes A and B. Let $AXBY$ be the parallelogram with nodes A and B as corner vertices. Let $P(A, B)$ be a shortest path between A and B. Since every shortest path between A and B lies inside the parallelogram $AXBY$, the path $P(A, B)$ also lies inside parallelogram $AXBY$. Let $P(A, B)$ be an (A, L)-path followed by an (L, N)-path followed by an (N, B)-path. Let e be the last edge of $P(A, B)$ which is incident on node B. This edge e lies on one of the three odd axis lines. In Figure 14, e lies on an odd β-line. The node B divides this β-line into two segments one of which does not contain e. Let C denote the boundary node on this segment. Now consider the parallelogram $AXCZ$. Define a path $P(A, C)$ as (A, L)-path followed by an (L, N)-path followed by an (N, C)-path which is a shortest path between A and C passing through B. Thus $d(A, C) \neq d(B, C)$.

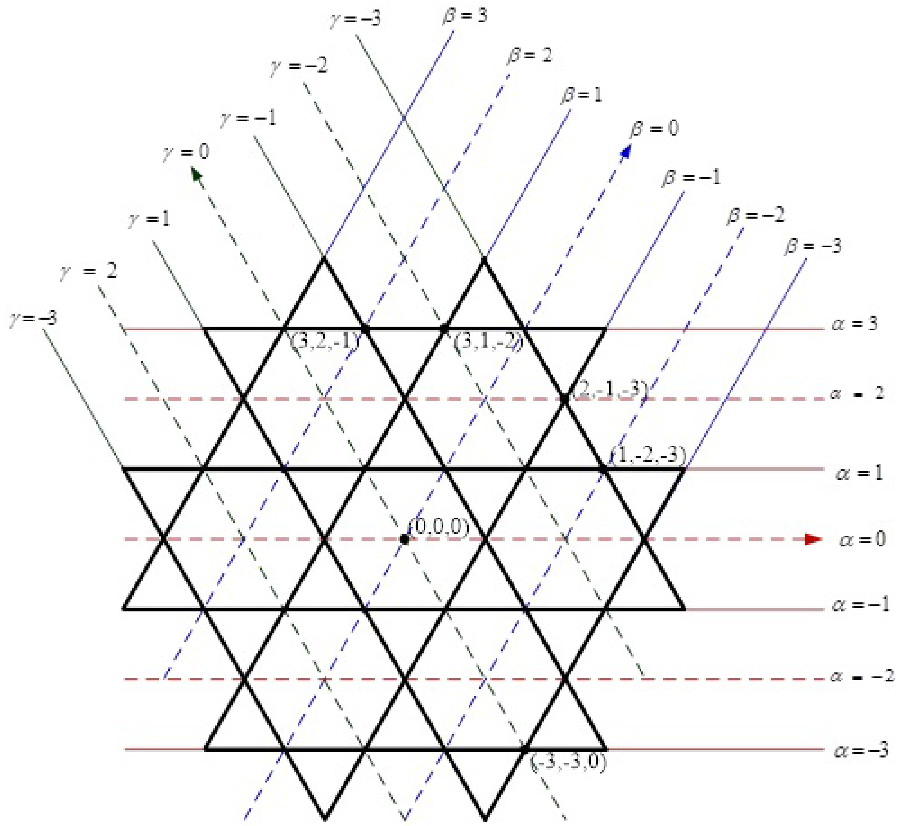

FIGURE 12. Coordinate system in oxide networks.

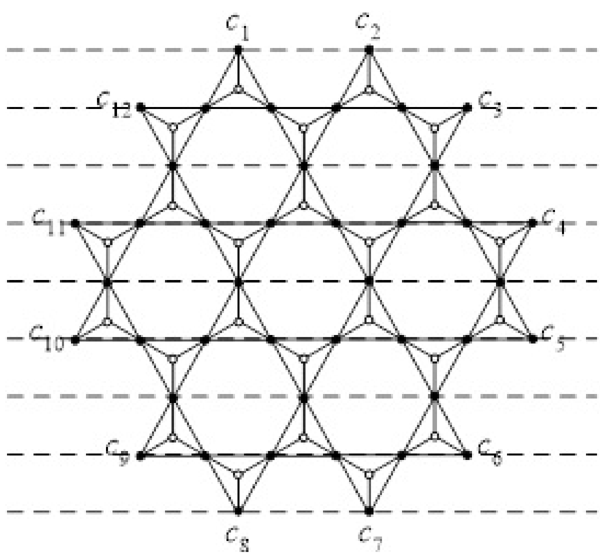

FIGURE 13. Channels in silicate networks.

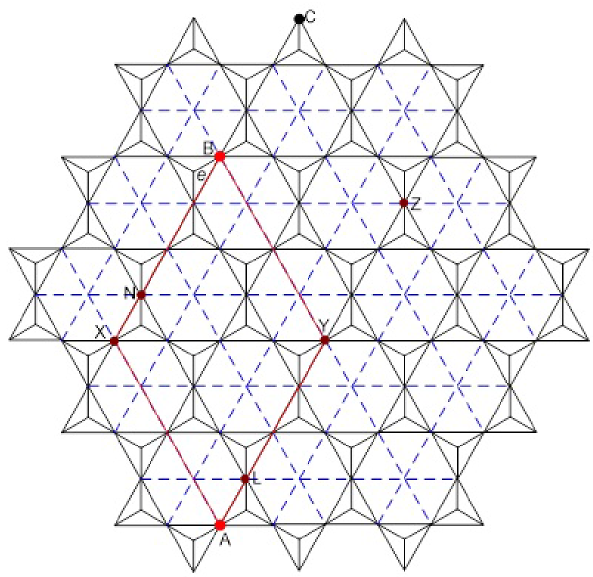

FIGURE 14. Proof cases in Theorem 6.2.

Let A and B be two silicon nodes of $SL(n)$ lying in some α-channel. In this case $d(A, C) \neq d(B, C)$ where C is a boundary node of $SL(n)$ belonging to the odd α-line bounding the channel containing A and B.

If A and B are two silicon nodes not lying in the same channel, choose the odd axis line at distance one from one of A or B such that A and B lie on the same side of the axis line. A boundary node C belonging to this axis line is such that $d(A, C) \neq d(B, C)$.

The case when A is an oxygen node and B a silicon node can be treated similarly. □

Next we locate a fault-tolerant resolving set in the silicate networks $SL(n)$.

Lemma 6.3: Let c be a boundary vertex in the silicate network $SL(n)$ with $n > 2$. Then $\gamma(N(c)) = \{c\}$.

Proof: Note that the boundary vertex c is the only vertex which is the common neighbor of all the three vertices in the open-neighborhood $N(c)$. □

The next proposition locates a fault-tolerant resolving set in $SL(n)$.

Proposition 6.4: Let $SL(n)$ be the n-dimensional silicate network with $n > 2$. Then $6n + 1 \leq \beta'(SL(n)) \leq 21n$.

Proof: By Theorem 6.1 and 6.2, we know that the $6n$ boundary vertices in $SL(n)$ provides a metric basis. By using Lemma 1.1, we know that $R' = \cup_{v\in R}\big(N[v] \cup \gamma(N(v))\big)$ is a fault-tolerant resolving set corresponding to any resolving set R. Let R be the set of boundary vertices in $SL(n)$. Then by Lemma 6.3, for any vertex $c \in R$ $\gamma(N(c)) \subseteq N[c]$. Moreover, there are $15n$ vertices in the neighborhood of the $6n$ boundary vertices. by combining these facts with 1, we obtain the result. □

Proposition 6.4 gives rise the following immediate corollary.

Corollary 6.5: The family of silicate networks is a family of interconnection networks with an unbounded fault-tolerant metric dimension.

VI. COMPUTATIONAL AND ALGORITHMIC COMPLEXITY

It is proved in [11], [19] that the metric dimension problem in *NP*-complete for general graphs. It was further shown by Manuel *et al.* [26] that the problem of find metric dimension of an arbitrary bipartite graph is *NP*-complete. The computational complexity of fault-tolerant metric dimension is not explicitly studied so far, and it is believed that the complexity status of this problem is *NP*-complexity as well due to the fact that, to develop an algorithm for fault-tolerant metric dimension problem one will have to go through the metric dimension problem. Even though the fault-tolerant metric dimension problem might be polynomially solvable for Benes, butterfly and mesh networks, we believe that is *NP*-complete for bipartite graphs. Thus our study fills up the gap between *NP*-complete and *P*-type structure for fault-tolerant metric dimension problem for bipartite graphs. We record our computational work here by proposing the following conjecture.

Conjecture 7.1: The problem of finding the fault-tolerant metric dimension is polynomially solvable for Benes and butterfly networks.

VII. CONCLUSION AND FUTURE WORK

We have studied the fault-tolerant metric dimension problem for three infinite families of bipartite interconnection networks. Although our results in this article produce upper bounds on this problem for those families of networks, we firmly believe that the lower bounds for fault-tolerant metric dimension for Benes and butterfly network are also the same and thus these two families have a bounded fault-tolerant metric dimension. On the other hand, mesh network share a constant fault-tolerant metric dimension for any values of defining parameters m and n.

The study of problems of finding the metric dimension and the fault-tolerant metric dimension of interconnection networks is at its early stage. It is open to investigate these problems for various interconnection networks such as hypercube, shuffle exchange, star, pancake, De Bruijn, banyan, delta, omega, bidelta, baseline and torus architectures. Moreover, the *NP*-complete problems such as achromatic number problem and minimum crossing number problem [11] are open for Benes and butterfly networks.

ACKNOWLEDGMENT

This work is a part of the Ph.D. thesis of Muhammad Yasir Hayat Malik.

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