A Discrete-Time State Estimation for Nonlinear Systems With Noises

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ABSTRACT This paper proposes a discrete-time observer to estimate states of continuous-time nonlinear systems with presences of noises. Most conventional discrete-time state observers are designed using discrete-time models, which are derived by applying the forward-difference method to continuous-time models without considering their poor accuracy. This poor accuracy of the discretization may affect the accuracy of the observer-based on it and lead to a misevaluation. The proposed observer is based on an application of the extended Kalman filter (EKF) for a discrete-time model derived by a discretization method called continualized discretization. The proposed observer was applied to estimate the states of the Lorenz and van der Pol oscillators, which have complex dynamics such as limit cycle and chaos. The simulation results showed that the proposed observer gives better performances in estimating the real state and retaining the system dynamics of the original continuous-time model than the conventional method.

INDEX TERMS Discrete-time model, state estimation, nonlinear, continualized discretization, van der Pol oscillator, Lorenz system.

NOMENCLATURE

$\bar{\mathbf{x}}(t)$:	continuous-time state
$\mathbf{x}_k$:	discrete-time state
$\hat{\mathbf{x}}_k$:	estimated state
$\bar{\mathbf{y}}(t)$:	system output
$\bar{\mathbf{v}}(t)$:	system noise
$\bar{\mathbf{w}}(t)$:	measurement noise
$\mathbf{f}(\mathbf{x})$:	system function
$\mathbf{h}(\mathbf{x})$:	output function
T :	sampling interval
$\Gamma(\mathbf{x}_k, T)$:	discrete-time integration gain
$D\mathbf{f}$:	Jacobian matrix of function $\mathbf{f}$

I. INTRODUCTION

System state variables are required for most feedback control design and system analysis methods [1]. However, the entire states are usually too expensive or impossible to measure in most applications. Furthermore, even when the states are measurable, they may include noises, which come from measurements or disturbances. In these cases, some forms of state estimation (observer) are necessary to estimate the real state of the system. If the stochastic properties of the uncertainties are available, the Bayesian state estimation methods, such as Kalman filter, are employed. In the cases, when only the bounds of the uncertainties are available, the deterministic

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approach using set valued state estimation methods are applied [2]–[4]. While there are many observer techniques have been studied for linear systems successfully, the theory of observer for nonlinear systems is still a significant challenge [5]–[7].

Since the state observer and digital controller are implemented on computers, many studies on the state observer are based on an assumption that there exists a discrete-time model of the given system [8]–[13]. However, most physical systems are modelled by continuous-time differential equations. Thus, a discrete-time versions, which are usually derived by discretizing the continuous-time model, are needed for the implementation of the observer [14]. While there are some accurate discretization methods, such as Runge-Kutta families [15]–[17], usable for offline computation systems, those are available for online observers or controls design for nonlinear systems, are still relatively rare. In most conventional applications of the state observer, forward difference method (is also called Euler's method) [18] is usually used for the discretization due to its versatility and simplicity. However, the accuracy of this method is poor unless an efficient high sampling frequency is used. However, this high sampling frequency may be unachievable in many practical cases due to constrains on hardware and system cost [19], [20]. This poor accuracy of the discretization may affect the accuracy of the observer-based on it and lead to a misevaluation [18]. Unfortunately, most studies on state observer have not considered this issue. There is still a gap needed to be

filled to design the discrete-time observers for the continuous-time systems.

Among some rarely proposed discretization methods for nonlinear system, a discretization method called continualized discretization, which is based on discrete-time integration gain and continualization concepts, has been developed [22], [23]. This method has been used to derive highly accurate discrete-time models, which can retain complex dynamical properties for various continuous-time nonlinear systems, for example, the chaotic behavior in the Lorenz system and the limit cycle in the van der Pol oscillator [22]. A sampled-data feedback control based on this discretization method for a scalar Riccati system has been shown to give better performances than that one using the conventional forward-difference method [24]. This continualized discretization method is expected to fill the above-mentioned gap between the discrete-time state observer and the continuous-time system. The contributions of this paper are as below:

- Presents a continualized discretization method, which available for online observers or controls design for nonlinear systems, to derive discrete-time model from a continuous-time model.
- Proposes a discrete-time observer for autonomous nonlinear continuous-time systems with presences of noises by using this continualized discretization method and the dominated Extended Kalman Filter (EKF).
- Applies the proposed observer to estimate the states of the Lorenz and van der Pol oscillators, which have complex dynamics such as limit cycle and chaos.

The organization of this paper is as follows: Section II summarizes the continualized discretization method and the discrete-time model used in this paper. In section III, the discrete-time state observer using the continualized discretization method and EKF is proposed for general nonlinear systems. Section IV presents simulation results for van der Pol and Lorenz oscillators to evaluate the proposed method comparing with the conventional one using the forward-difference method. Some conclusions are given in Section V.

II. CONTINUALIZED DISCRETIZATION METHOD

Consider a continuous-time dynamical system given by the following differential equation

$$\frac{d\bar{\mathbf{x}}(t)}{dt} = \mathbf{f}(\bar{\mathbf{x}}(t)), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0, \tag{1}$$

where $\bar{\mathbf{x}} \in R^n$ is a system state vector and $\mathbf{f} : R^n \to R^n$ is a system function, which is nonlinear. The function $\mathbf{f}$ is assumed to satisfy the Lipschitz condition. Thus, there exists a solution of the differential equation (1) and this solution is unique.

A discrete-time state $\mathbf{x}_k$ is said to be a discretization of the continuous-time state $\bar{\mathbf{x}}(t)$ if the following condition is satisfied for any fixed instant τ

$$\lim_{\substack{T \to 0 \\ kT \leq \tau < (k+1)T}} \mathbf{x}_k = \bar{\mathbf{x}}(\tau). \tag{2}$$

When $\mathbf{x}_k$ is a discretization of $\bar{\mathbf{x}}(t)$ and $\mathbf{x}_k = \bar{\mathbf{x}}(kT)$ for any discrete-time instant k and sampling period T, it is said to be an exact discretization of $\bar{\mathbf{x}}(t)$ [26].

Following the above definition for discretization, the discrete-time model for the system in eq. (1) using continualized discretization method [24] is given by [19]

$$\delta \mathbf{x}_k = \mathbf{\Gamma}(\mathbf{x}_k, T) \mathbf{f}(\mathbf{x}_k), \quad \mathbf{x}_0 = \bar{\mathbf{x}}_0, \tag{3}$$

where δ is the delta operator, satisfying that

$$\delta \mathbf{x}_k = \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{T}. \tag{4}$$

The discrete-time model given by eq. (3) uses the same function $\mathbf{f}$ that appears in eq. (1). Function $\mathbf{\Gamma} \in R^{n \times n}$ in eq. (3) is the discrete-time integration gain. It is defined as

$$\mathbf{\Gamma}(\mathbf{x}_k, T) = \frac{1}{T} \int_0^T e^{[D\mathbf{f}(\mathbf{x}_k)]\tau} d\tau, \tag{5}$$

where $D\mathbf{f}(\mathbf{x}_k)$ is the Jacobian matrix of the function $\mathbf{f}$ at $\mathbf{x}_k$. When the Jacobian matrix $D\mathbf{f}(\mathbf{x}_k)$ is invertible, the discrete-time integration gain given by eq. (5) can be written by the following form

$$\mathbf{\Gamma}(\mathbf{x}_k, T) = \frac{e^{T[D\mathbf{f}(\mathbf{x}_k)]} - \mathbf{I}}{T} [D\mathbf{f}(\mathbf{x}_k)]^{-1}. \tag{6}$$

Remark 1: For the linear continuous-time system, i.e., the function $\mathbf{f}$ in eq. (1) is given by

$$\mathbf{f}(\bar{\mathbf{x}}(t)) = \mathbf{A}\bar{\mathbf{x}}(t) + \mathbf{b}, \tag{7}$$

where $\mathbf{A} \in R^{n \times n}$ and $\mathbf{b} \in R^n$ are system parameters, the discrete-time integration gain can be written as

$$\mathbf{\Gamma}(\mathbf{x}_k, T) = \frac{1}{T} \int_0^T e^{\mathbf{A}\tau} d\tau. \tag{8}$$

The discrete-time model given by eq. (3) with the above discrete-time integration gain is an exact discrete-time model.

Remark 2: When the integration gain $\mathbf{\Gamma}$ is defined by a unity matrix, eq. (3) represents the conventional forward difference method.

Remark 3 ([24]): When the Jacobian matric $D\mathbf{f}(\mathbf{x}_k)$ is invertible, the equilibrium points of the continualized discrete-time model given by eq. (3) and their asymptotical stabilities are identical to those of the continuous-time model (1).

III. DISCRETE-TIME STATE ESTIMATION FOR NONLINEAR SYSTEM WITH NOISES

Consider a nonlinear continuous-time system with the presence of noises given by

$$\frac{d\bar{\mathbf{x}}(t)}{dt} = \mathbf{f}(\bar{\mathbf{x}}(t)) + \bar{\mathbf{v}}(t), \tag{9}$$

$$\bar{\mathbf{y}}(t) = \mathbf{h}(\bar{\mathbf{x}}(t)) + \bar{\mathbf{w}}(t), \tag{10}$$

where $\bar{\mathbf{x}} \in R^n$ is the system state and $\bar{\mathbf{y}} \in R^m$ is a measurable system output. The functions $\mathbf{f} : R^n \to R^n$ and $\mathbf{h} : R^n \to R^m$ are nonlinear. Vectors $\bar{\mathbf{v}}(t) \in R^n$ is a system noise and $\bar{\mathbf{w}}(t) \in R^m$ is a measurement noise. Assume that the

sampled-data of $\bar{\mathbf{v}}(t)$ and $\bar{\mathbf{w}}(t)$ are zero-mean white Gaussian and have covariance matrices of $\mathbf{Q}$ and $\mathbf{R}$, respectively.

In the sampled-data systems, the system noise $\bar{\mathbf{v}}(t)$ and measurement noise $\bar{\mathbf{w}}(t)$ are assumed to affect the system through zero-order-hold and be constant during each sampling interval [21]. The discrete-time model of the continuous-time system given by eqs. (9) and (10) can be derived by using the contiualized discretization method presented in Section II, as

$$\delta\mathbf{x}_k = \Gamma(\mathbf{x}_k, T)(\mathbf{f}(\mathbf{x}_k) + \mathbf{v}_k), \tag{11}$$

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{w}_k, \tag{12}$$

where

$$\Gamma(\mathbf{x}_k, T) = \frac{1}{T} \int_0^T e^{[D\mathbf{f}(\mathbf{x}_k)]\tau} d\tau \tag{13}$$

with $D\mathbf{f}$ is the Jacobian matrix of $\mathbf{f}$, $\mathbf{v}_k$ and $\mathbf{w}_k$ are the sampled-data of $\bar{\mathbf{v}}(t)$ and $\bar{\mathbf{w}}(t)$, respectively. It should be noted that the state $\mathbf{y}_k$ in eq. (12) is the sampled-data of the measurable output $\bar{\mathbf{y}}$.

By substituting the delta operation δ defined as eq. (4) into eq. (11), we can derive that

$$\mathbf{x}_{k+1} = T\Gamma(\mathbf{x}_k, T)\mathbf{f}(\mathbf{x}_k) + \mathbf{x}_k + T\Gamma(\mathbf{x}_k, T)\mathbf{v}_k. \tag{14}$$

Let the estimation of the system state $\bar{\mathbf{x}}$ in the system (9) from the sampled-data of the measurable output $\mathbf{y}_k$ is $\hat{\mathbf{x}}_k$. In this study, the state $\hat{\mathbf{x}}_k$ is estimated by a nonlinear state estimator using extended Kalman filter [27], which has the algorithm as follows:

1. *State estimation*:

$$\hat{\mathbf{x}}_k^- = \Phi(\hat{\mathbf{x}}_{k-1}), \tag{15}$$

where

$$\Phi(\hat{\mathbf{x}}_{k-1}) = T\Gamma(\hat{\mathbf{x}}_{k-1}, T)\mathbf{f}(\hat{\mathbf{x}}_{k-1}) + \hat{\mathbf{x}}_{k-1}. \tag{16}$$

2. *Calculation for error covariance*:

$$\mathbf{P}_k^- = [D\Phi(\hat{\mathbf{x}}_{k-1})]\mathbf{P}_{k-1}[D\Phi(\hat{\mathbf{x}}_{k-1})]^T + [T\Gamma(\mathbf{x}_k, T)]\mathbf{Q}[T\Gamma(\mathbf{x}_k, T)]^T, \tag{17}$$

where $D\Phi(\hat{\mathbf{x}}_{k-1})$ is the Jacobian matrix of $\Phi(\hat{\mathbf{x}}_k)$ at $\hat{\mathbf{x}}_{k-1}$.

3. *Observational update for state estimate*:

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k[\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)], \tag{18}$$

where $\mathbf{K}_k$ is the Kalman gain, which is given by

$$\mathbf{K}_k$$
$$= \mathbf{P}_k^- D\mathbf{h}(\hat{\mathbf{x}}_k^-)^T \left[[D\mathbf{h}(\hat{\mathbf{x}}_k^-)]\mathbf{P}_k^-[D\mathbf{h}(\hat{\mathbf{x}}_k^-)]^T + \mathbf{R}\right]^{-1}, \tag{19}$$

where $D\mathbf{h}(\hat{\mathbf{x}}_k^-)$ is the Jacobian matrix of $\mathbf{h}(\hat{\mathbf{x}}_k)$ at $\hat{\mathbf{x}}_k^-$.

4. *Update for error covariance*:

$$\mathbf{P}_k = [\mathbf{I} - \mathbf{K}_k D\mathbf{h}(\hat{\mathbf{x}}_k^-)]\mathbf{P}_k^-. \tag{20}$$

Remark 4: The estimation error $\varepsilon = \bar{\mathbf{x}}(kT) - \hat{\mathbf{x}}_k$ of the proposed discrete-time state estimation is exponentially bounded when the following conditions are satisfied for every $k \geq 0$ [28]:

1. The Jacobian matrices $D\Phi(\hat{\mathbf{x}}_k)$ and $D\mathbf{h}(\hat{\mathbf{x}}_k)$ are bounded.
2. The covariance matric $\mathbf{P}_k(\hat{\mathbf{x}}_k)$ are positive and bounded.
3. The function $D\Phi(\hat{\mathbf{x}}_k)$ is nonsingular for $k \geq 0$.
4. Covariance matrices $\mathbf{Q}$, and $\mathbf{R}$ are positive.
5. There are positive real numbers χ_1, χ_2, γ_1, and γ_2 such that if $\|\mathbf{x}_k - \hat{\mathbf{x}}_k\| \leq \gamma_1$ and $\|\mathbf{x}_k - \hat{\mathbf{x}}_k\| \leq \gamma_2$, then

$$\|\Phi(\mathbf{x}_k) - \Phi(\hat{\mathbf{x}}_k) - D\Phi(\hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)\|$$
$$\leq \chi_1 \|\mathbf{x}_k - \hat{\mathbf{x}}_k\|, \tag{21}$$

and

$$\|\mathbf{h}(\mathbf{x}_k) - \mathbf{h}(\hat{\mathbf{x}}_k) - D\mathbf{h}(\hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)\|$$
$$\leq \chi_2 \|\mathbf{x}_k - \hat{\mathbf{x}}_k\|. \tag{22}$$

IV. SIMULATION RESULTS

In this session, the proposed discrete-time state estimation method is applied for the Van der Pol and Lorenz systems with the presence of noises. The simulations have been carried out to assess performances of the proposed method, which is based on the continualized discrete-time model, and the conventional method using the forward difference model. The simulations were calculated using Matlab/Simulink in a computer, whose processor is Intel Core i7-2600 CPU @3.4GHz.

A. VAN DER POL OSCILLATOR

The nonlinear van der Pol oscillator is one of an important model for the phenomena of limit-cycle oscillation [29], [30]. The van der Pol oscillator with the presence of noises is modeled by the following differential equation

$$\frac{d\bar{\mathbf{x}}(t)}{dt} = \begin{bmatrix} \bar{x}_2(t) \\ -\bar{x}_1(t) + \varepsilon(1 - \bar{x}_1(t)^2)\bar{x}_2(t) \end{bmatrix} + \bar{\mathbf{v}}(t), \tag{23}$$

where $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$, $\varepsilon > 0$ is a friction coefficient that characterizes the nonlinearity and $\bar{\mathbf{v}} \in \mathbb{R}^2$ is a system noise.

The measurable system output is given by

$$y = \bar{x}_1(t) + \bar{w}(t), \tag{24}$$

where $\bar{w}$ a measurement noise.

The state estimator of the above Van der Pol oscillator is given by eqs. (15)-(20), where the functions $\mathbf{f}$, $\mathbf{h}$, and their Jacobian matrices $D\mathbf{f}$ and $D\mathbf{h}$ are given by
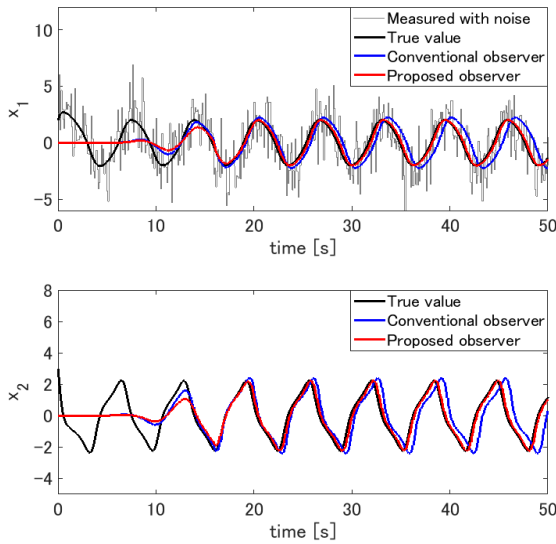
$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_2 \\ -x_1 + \varepsilon(1 - x_1^2)x_2 \end{bmatrix}, \tag{25}$$
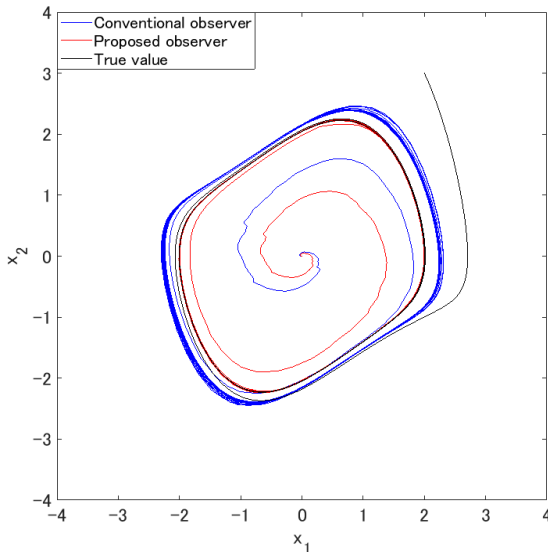
$$\mathbf{h}(\mathbf{x}) = x_1, \tag{26}$$

$$D\mathbf{f} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & \varepsilon(1 - x_1^2) \end{bmatrix}, \tag{27}$$

$$D\mathbf{f} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & \varepsilon(1 - x_1^2) \end{bmatrix}, \tag{28}$$
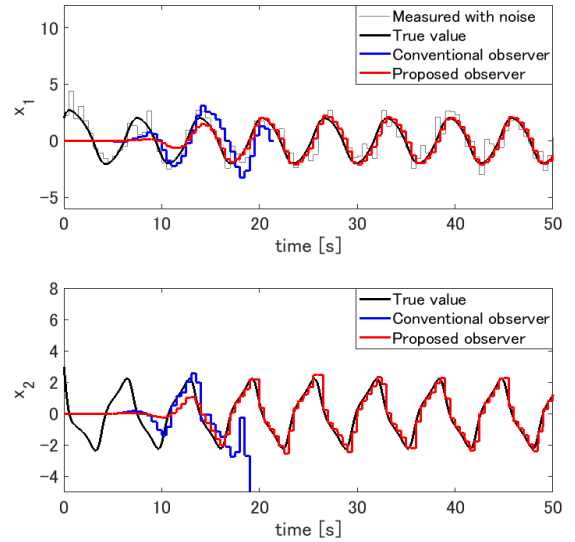
respectively.

FIGURE 1. Time responses of the proposed observer and the conventional observer for the van der Pol oscillator with $\varepsilon = 0.5$ and $T = 0.1$ s.
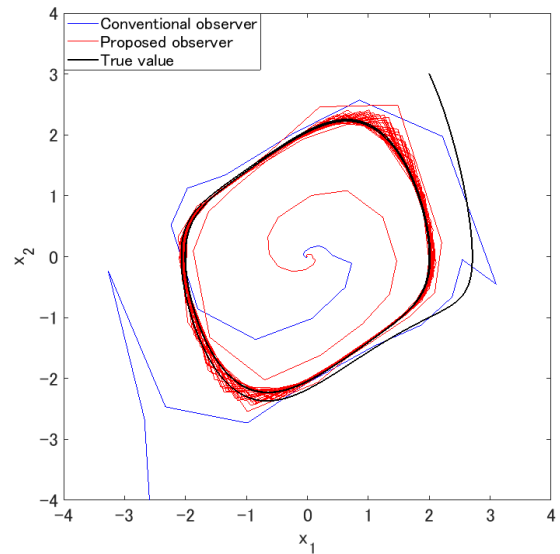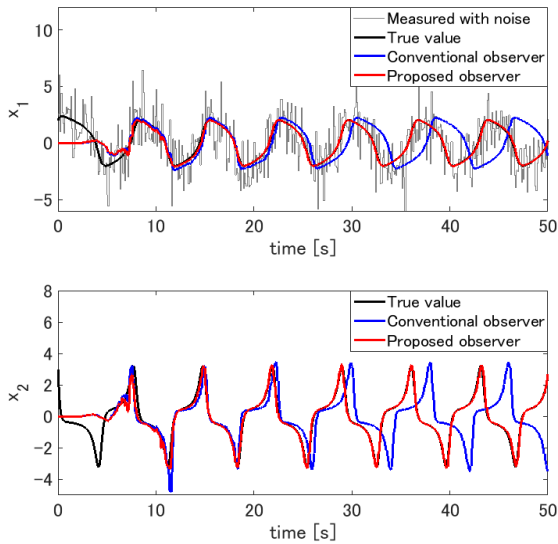

FIGURE 3. Time responses of the proposed observer and the conventional observer for the van der Pol oscillator with $\varepsilon = 0.5$ and $T = 0.5$ s.

FIGURE 2. Phase planes of the proposed observer and the conventional observer for the van der Pol oscillator with $\varepsilon = 0.5$ and $T = 0.1$ s.
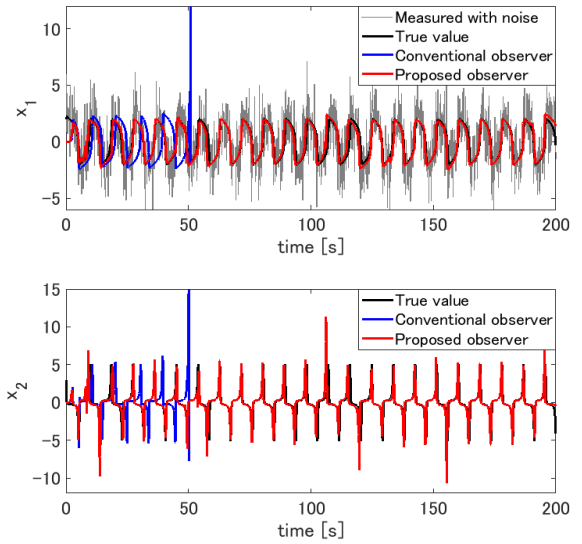

FIGURE 4. Phase planes of the proposed observer and the conventional observer for the van der Pol oscillator with $\varepsilon = 0.5$ and $T = 0.5$ s.

The simulations have been carried out for 100 seconds with the arbitrary noise covariance matrices of $\mathbf{Q} = [0.01; 0.01]$ and $\mathbf{R} = 0.5$. The initial state of the continuous-time system is $\bar{\mathbf{x}}_0 = (2, 3)$, while the initial conditions for the discrete-time state estimator are $\hat{\mathbf{x}}_0 = (0, 0)$ and $\mathbf{P}_0 = [0\ 0; 0\ 0]$. Figures 1 and 2 show the time responses and the phase planes of the proposed estimator using continualized discretization method, the conventional estimator using forward difference method, and the measurement with noises comparing with the true values, which are the responses of the continuous-time model without noises, for the coefficient of $\varepsilon = 0.5$ and sampling period $T = 0.1$ seconds. In this case,
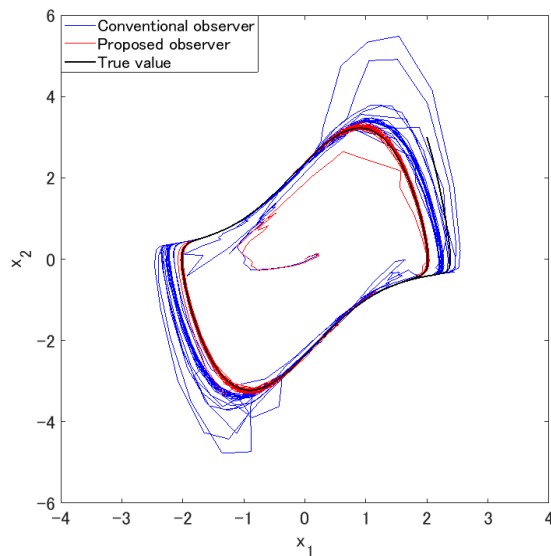
both the proposed and the conventional estimators yield the limit cycle responses that are close to the true values regardless of the difference in the initial condition. However, the responses of the proposed estimator are more exact both in-phase and amplitude. When the sampling interval T is increased to 0.5 seconds, as can be seen in Figs 3 and 4, the responses of the conventional estimator diverge after a short time, while the responses of the proposed estimator still tracking the true values of the Van der Pol system almost exactly. Figures 1 and 3 show that the high sampling frequency can improve the performances of the sampled-date system, by contrast, it also may increase the effect of the noise. Figures 5 and 6 show the responses of estimators for
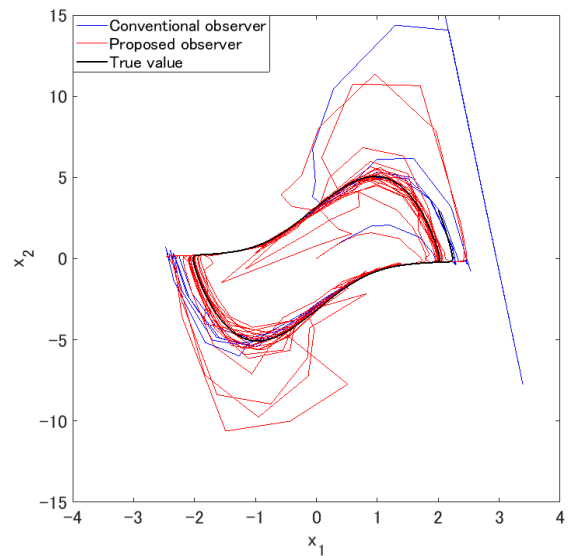
FIGURE 5. Time responses of the proposed observer and the conventional observer for the van der Pol oscillator with $\varepsilon = 1.5$ and $T = 0.1$ s.

FIGURE 7. Time responses of the proposed observer and the conventional observer for the van der Pol oscillator with $\varepsilon = 3$ and $T = 0.1$ s.

FIGURE 6. Phase planes of the proposed observer and the conventional observer for the van der Pol oscillator with $\varepsilon = 1.5$ and $T = 0.1$ s.

FIGURE 8. Phase planes of the proposed observer and the conventional observer for the van der Pol oscillator with $\varepsilon = 3$ and $T = 0.1$ s.

the sampling T of 0.1 seconds but with the friction coefficient ε of 1.5, which increases the nonlinearity of the system. The proposed estimator yields accurate responses and gives a better performance than the conventional one. When the coefficient ε is increased further to 3, the responses of the conventional estimator diverge, while the responses of the proposed estimator still track to the true values and preserve the limit cycle dynamics of the continuous-time system without noise, as showed in Figs. 7 and 8.

Table 1 summaries errors and computation times of the conventional and the proposed methods for the van der Pol oscillator with the conditions used in the simulations. The error is defined by an average of squared differences between

continuous-time and discrete-time responses as

$$error = \frac{T}{tl}\left[\sum_{k=0}^{kT=tl}\sum_{i=1}^{n}\left(\bar{x}_i(t)\big|_{t=kT} - x_{ik}\right)^2\right]^{1/2} \quad (29)$$

where tl is the length of the simulation time and n is the order of the system. In this study, tl has a value of 100 second. It should be noted that the differences between the initial values of the continuous-time states and the discrete-time states may affect to value of this error. The computation times of the conventional method and proposed method are measured by using *cputime* command in the Matlab. While the proposed method uses a longer CPU time, this time still enable us to estimate the true value of the system

TABLE 1. Conditions used for simulations of van del Pol oscillator.

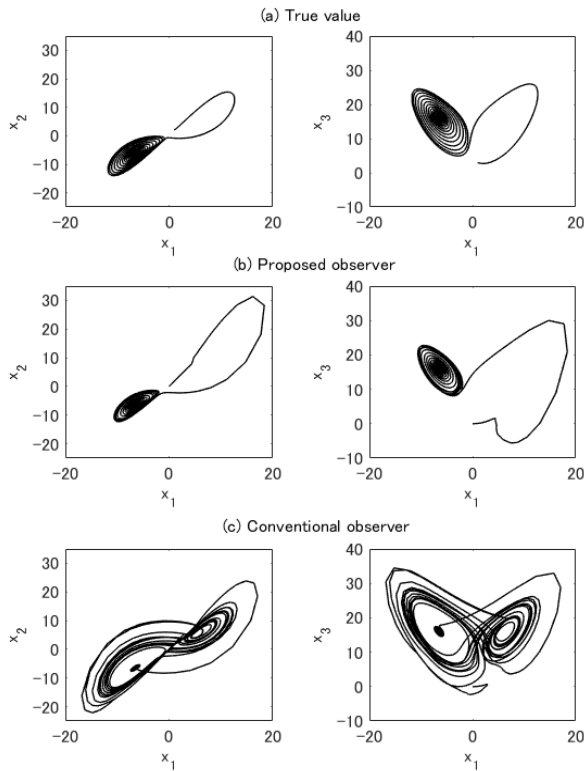| Case	ε	T (s)	Conventional method		Proposed method	
			Error	Compt. Time (s)	Error	Compt. Time (s)
1	0.5	0.1	0.073	0.02	0.024	42.45
2	0.5	0.5	infinity	0.2	0.054	8.62
3	1.5	0.1	0.093	0.02	0.018	41.78
4	3	0.1	infinity	1	0.042	39.67

FIGURE 9. Phase planes of the proposed observer and the conventional observer for the Lorenz system with $r = 17$ and $T = 0.02$ s.

online (by the real time) with a high accuracy comparing with the conventional method, especially for the cases of large sampling interval.

B. LORENZ SYSTEM

Consider the following Lorenz differential equation, which is usually used to model the atmospheric dynamics [31], [32], given by

$$
\begin{bmatrix}
\dfrac{d\bar{x}_1(t)}{dt} \\
\dfrac{d\bar{x}_2(t)}{dt} \\
\dfrac{d\bar{x}_3(t)}{dt}
\end{bmatrix}
=
\begin{bmatrix}
\sigma(\bar{x}_2 - \bar{x}_1) \\
r\bar{x}_1 - \bar{x}_2 - \bar{x}_1\bar{x}_3 \\
-b\bar{x}_3 + \bar{x}_1\bar{x}_2
\end{bmatrix}
+ \bar{v}(t),
\tag{30}
$$

where $\bar{v} \in R^3$ is a system noise. The state $\bar{x}_1$ is proportional to the convective velocity; the state $\bar{x}_2$ is proportional to the temperature difference between ascending and descending flows; and the state $\bar{x}_3$ is proportional to the mean convective heat flow. The traditional Lorenz system usually takes the
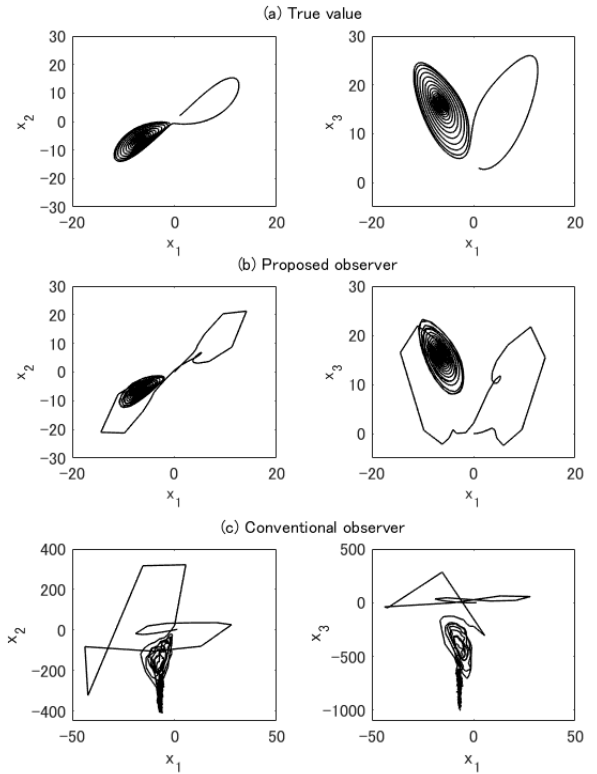

FIGURE 10. Phase planes of the proposed observer and the conventional observer for the Lorenz system with $r = 17$ and $T = 0.05$ s.

Prandtl number σ at 10 and the coefficient b, which is related to the wave number, at $8/3$. The reduced Rayleigh number r is variable.

In the present study, we assume that only the state $\bar{x}_1$ is measurable. Thus, the system output is given by

$$
y = \bar{x}_1(t) + \bar{w}(t).
\tag{31}
$$

The state estimator of the above Lorenz system is given by eqs. (15)-(20), where the functions $\mathbf{f}$, $\mathbf{h}$, and their Jacobian matrices $D\mathbf{f}$ and $D\mathbf{h}$ are given by

$$
\mathbf{f}(\mathbf{x}) =
\begin{bmatrix}
\sigma(\bar{x}_2 - \bar{x}_1) \\
r\bar{x}_1 - \bar{x}_2 - \bar{x}_1\bar{x}_3 \\
-b\bar{x}_3 + \bar{x}_1\bar{x}_2
\end{bmatrix},
\tag{32}
$$

$$
\mathbf{h}(\mathbf{x}) = x_1,
\tag{33}
$$

$$
D\mathbf{f} =
\begin{bmatrix}
-\sigma & \sigma & 0 \\
r - \bar{x}_3 & -1 & -\bar{x}_1 \\
\bar{x}_2 & \bar{x}_1 & -b
\end{bmatrix},
\tag{34}
$$

$$
D\mathbf{h} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},
\tag{35}
$$

respectively.

The simulations have been carried out with the noise covariance matrices of $\mathbf{Q} = [0.001; 0.001; 0.001]$ and $\mathbf{R} = 0.01$. The initial condition of the continuous-time state is $\bar{x}_0 = (1, 2, 3)$, while the initial conditions for the discrete-time state estimator are $\hat{\mathbf{x}}_0 = (0, 0, 0)$ and $\mathbf{P}_0 = [0\ 0\ 0;\ 0\ 0\ 0;\ 0\ 0\ 0]$. Figure 9 shows the phase planes (x_1, x_2) and (x_1, x_3) of the proposed estimator using continualized discretization method and the conventional estimator

FIGURE 11. Phase planes of the proposed observer and the conventional observer for the Lorenz system with $r = 28$ and $T = 0.02$s.
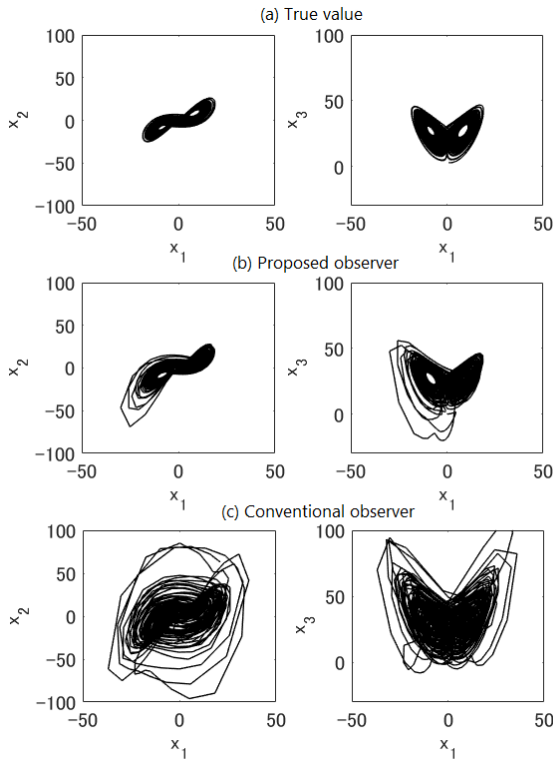

FIGURE 12. Phase planes of the proposed observer and the conventional observer for the Lorenz system with $r = 28$ and $T = 0.05$s.

using forward difference method comparing with the true values, which are the responses of the continuous-time model without noises, for the Rayleigh number $r = 17$ and the sampling period $T = 0.02$ seconds. In this case, the proposed estimator yields the responses that converge to the equilibrium point $\left(-8\sqrt{2/3}, \; -8\sqrt{2/3}, \; 16\right)$ of the original system immediately closing to the true values; while the responses of conventional estimator have a chaotic behavior before converging to the same equilibrium point. When the sampling interval T increases to 0.05 seconds, as can be seen in figure 10, the responses of the conventional estimator oscillate around to the point of $(-7, \; -300, \; -750)$, which is far different from the true equilibrium point, while the responses of the proposed estimator still track the true responses of the Lorenz system almost exactly. Figures 11 show the responses of the estimators for the sampling T of 0.02 seconds but with the Rayleigh number r of 28. In this case, both methods give responses that are relatively similar to the chaotic behavior of the true responses. However, the responses of the proposed estimator are more accurate than those of the conventional one. When the sampling interval T is increased to 0.05 seconds, as can be seen in Fig. 12, the responses of the conventional estimator diverge, while the responses of the proposed estimator still track to the true responses and preserve the chaotic behavior of the continuous-time system without noise. The errors and computation times of the conventional and the proposed methods for the Lorenz system
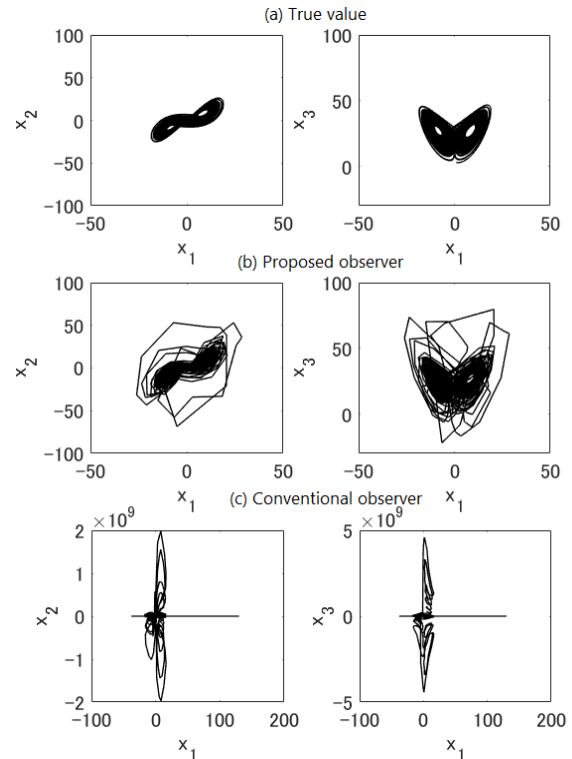
TABLE 2. Conditions used for simulations of lorenz system.

Case	r	T (s)	Conventional method		Proposed method	
			Error	Compt. time (s)	Error	Compt. time (s)
1	17	0.02	0.1087	0.13	0.0374	86.83
2	17	0.05	18.2	0.08	0.0658	38.77
3	28	0.02	0.352	0.17	0.292	86.58
4	28	0.05	12.9×10^6	0.17	0.4632	38.09

with the conditions used in the simulations are summarized by Table 2.

V. CONCLUSION

A discrete-time observer was proposed to estimate the states of the continuous-time nonlinear systems with the presence of noises. The proposed observer applies the extended Kalman filter for the discrete-time model using the continualized discretization method. The discrete-time model is described by a product of the discrete-time integration gain and the system function, which are the same as that of the continuous-time model. The discrete-time integration is a solution of the equation derived by the continualization process. The proposed observer was applied to estimate the states of van der Pol and Lorenz oscillators with the presence of noises. The simulation results showed that the proposed observer gives better performances in retaining the complex system dynamics such as

limit cycle and chaos than the conventional one based on the forward-difference method even for relatively low sampling frequencies. Although the present study considers the noises to be Gaussian and applies the EKF for nonlinear systems, the idea in this study is extendable to systems, where the noises are non-Gaussian, using the set-valued state estimation methods. It is expected that the proposed observer will contribute to a wide range of applications in digital control and system analysis.

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