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Approximate Nonlinear Discrete-Time Models Based on B-Spline Functions

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ABSTRACT We consider the discretization of continuous-time nonlinear systems described by normal forms. In particular, we consider the case when the input to the system is generated by a B-spline hold device to obtain an approximate discrete-time model. It is shown that the corresponding sampled-data model and its accuracy (measured in terms of the local truncation error) depend on the smoothness of the input and on the applied integration strategy, namely, the truncated Taylor series expansion. Moreover, the sampling zero dynamics of the discrete-time model are asymptotically characterized as the sampling period goes to zero, and it is shown that these zero dynamics converge to the asymptotic sampling zeros of the linear case.

INDEX TERMS B-spline functions, nonlinear systems, normal forms, truncated Taylor series, zero dynamics.

I. INTRODUCTION

The theory of normal forms has been studied for continuous and discrete-time nonlinear systems [1]–[4]. Normal forms allow us to rewrite the system as a chain of integrators followed by nonlinear dynamics that link the input to the state. This particular structure is called prime form [2] and can be used to characterize the resulting sampled-data model under different assumptions on the system input (see, for example, [5]–[9]).

For nonlinear systems the exact sampled-data model is not usually available. Thus, our motivation has been to study the effect of numerical integration methods on the obtained approximate sampled-data model. In order to solve the differential equations, these models have been developed based on truncated series expansion, e.g., Taylor or Lie series methods [6]–[8], [10], [11].

When discretizing a continuous-time system, the corresponding sampled-data model includes extra zeros (linear case) or zero dynamics (nonlinear case), that depend on the hold device used to generate the system input [12]. The inter-sample behavior is typically assumed to be piecewise constant, i.e., using a zero-order hold (ZOH) device [6]–[8], [11]. In [6], the nonlinear sampling zero dynamics were described for the first time. In [8], a truncated Taylor series expansion was proposed to discretize nonlinear systems, explicitly characterizing the sampling zero dynamics. Based on the latter result, [7], [11] proposed a more accurate sampled-data model for relative degree two, in order to guarantee the stability of the resulting model.

Higher-order hold devices have been used to model the input when discretizing a continuous-time system. In particular, Generalized Hold Functions (GHF) can be used to arbitrary place the zeros of the sampled-data model, without restricting the sampling period to be small [13]. The properties of GHF have been used, for example, in [14] to shift the sampling zeros asymptotically to the origin for fast sampling rates. However, the use of this kind of holds can give misleading results [15].

More recently, it has been shown that the smoothness of the input has a direct impact in the resulting sampled-data model. In [5] an approximate model has been developed considering a B-spline generalized hold for systems with no continuous-time zero dynamics. In fact, the Euler-Frobenius polynomials that characterize asymptotic sampling zeros also appear in B-spline interpolation [16], [17].

In this paper we extend the results presented in [5] to n-th order nonlinear systems with relative degree r. We show that the numerical integration method, namely a truncated Taylor series expansion can exploit the B-spline assumption on the

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system input to obtain a more accurate model than the usual one based on ZOH. In fact, the order of the B-spline hold has a direct impact on the corresponding sampled-data model for which the asymptotic sampling zero dynamics can be characterized, as the sampling period goes to zero. Moreover, we study the accuracy of the applied integration strategy by analyzing the local truncation error associated with the state vector.

The remainder of the paper is organized as follows: in Section II we present the B-spline generalized hold device used to generate the system input. Then, in Section III we introduce the theory of normal forms for continuous-time systems. The normal forms theory for discrete-time system is presented in Section IV and we recall important results for the linear case. The proposed approximate (nonlinear) sampled-data model is developed in Section V, analyzing its accuracy and the asymptotic zero dynamics. Finally, in Section VI we present conclusions.

II. B-SPLINE FUNCTIONS

Linear and nonlinear sampled-data models depend not only on the applied integration strategy but also on the knowledge of (or assumption about) the intersample behavior of the signals. In this paper, we consider the case when only the discrete-time input sequence is known. There are different options to interpolate the continuous-time input from the discrete-time sequence. When interpolating with high-order degree polynomials one can have the Runge's phenomenon, that can be avoided using piecewise polynomials functions such as B-splines [17]. These are smooth, positive and well-behaved functions and have minimal support [18]. Moreover, B-splines have been shown to be related to the Euler-Frobenius polynomials [17] that characterize asymptotic sampling zeros. Hence we use B-splines in the hold device to generate the input or, equivalently, to describe an assumption about its smoothness.

An ℓ-th order B-spline function is defined in the time-domain and in the $\mathcal{Z}$-domain, respectively, as follows:

$$\tilde{\beta}_\ell(t) = \frac{1}{h^\ell}\sum_{p=0}^{\ell+1}\frac{(-1)^p}{\ell!}\binom{\ell+1}{p}(t-ph)^\ell\mu(t-ph) \tag{1}$$

$$\mathcal{Z}\left\{\tilde{\beta}_\ell(kh)\right\} = \frac{1}{\ell!}\frac{B_\ell(z)}{z^\ell}, \tag{2}$$

where $B_\ell(z)$ is the Euler-Frobenius polynomial [19], [20] of order ℓ.

Lemma 2.1: When the input to a linear time-invariant continuous-time system is generated by an ℓ-th order B-spline hold, then the hold can be expressed as the hybrid system shown in Fig. 1, where $F_\ell(z)$ is a digital pre-filter given by

$$F_\ell(z) = \tfrac{1}{h^\ell}\left(1-z^{-1}\right)^\ell. \tag{3}$$

Moreover, $u^{(\ell)}(t)$ corresponds to the ℓ-th derivative of $u(t)$ and all the higher-order derivatives are equal to zero, i.e.,

$$u^{(\ell+1)}(t) = u^{(\ell+2)}(t) = \cdots = 0. \tag{4}$$

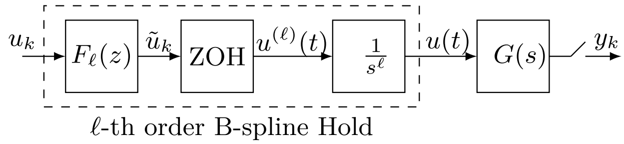

FIGURE 1. Hybrid system equivalent to an ℓ-th order B-spline Hold [21].

Additionally, the first ℓ-th derivatives of $u(t)$ at the sampling instants can be computed as follows:

$$\mathcal{Z}\left\{\frac{d^i}{dt^i}\,\tilde{\beta}_\ell\Big|_{t=kh}\right\} = \frac{1}{h^i}\frac{(z-1)^i}{z^\ell}\frac{B_{\ell-i}(z)}{(\ell-i)!}. \tag{5}$$

Proof: See [5] and [21]. □

III. NORMAL FORMS

In this paper we consider the class of nonlinear systems affine in the input, i.e.,

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{6a}$$

$$y(t) = h(x(t)), \tag{6b}$$

where $x(t)$ is the state vector, the vector fields $f(x(t))$ and $g(x(t))$, and the output function $h(x(t))$ are analytic in an open set $\mathcal{M} \in \mathbb{R}$ containing the origin [22].

Assumption 3.1: The system (6) has an equilibrium point $x_0 = 0$. Then, note that $f(0) = 0$ and $g(0) \neq 0$, otherwise $\dot{x}(t) = 0$ for any $u(t)$ [2], [4].

By applying a local coordinate transformation, the system given by (6) can be expressed in normal form as follows [22]:

$$\dot{\zeta}(t) = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{r-1} & \\ 0 & & & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right]\zeta(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}(a + bu(t)) \tag{7a}$$

$$\dot{\eta}(t) = c(\zeta, \eta) \tag{7b}$$

$$y(t) = \zeta_1(t). \tag{7c}$$

The local coordinate transformation in the neighborhood of $x_0 = 0$ is given by

$$\Phi(x) = [\zeta(t);\ \eta(t)]^T, \tag{8}$$

where $\zeta(t) = [\phi_1(x), \ldots, \phi_r(x)]^T$ and $\eta(t) = [\phi_{r+1}(x), \ldots, \phi_n(x)]^T$. Note that model (7) can be expressed into an external part, given by $\zeta(t)$, and an internal part, given by $\eta(t)$ [23].

Remark 3.2: The local coordinate transformation (8) does not (locally) change the equilibrium point. Thus, the proposed model (7) evolves in the neighborhood of $\phi_0 = 0$. Thus, fundamental properties such as the controllability of the system remain invariant [3], [4].

Considering that $\Phi(x)$ is close to the origin, we have that

$$a(\phi) = a(0) + a_1\vec{\phi} + \tfrac{1}{2}\vec{\phi}^T a_2\vec{\phi} + \ldots \tag{9a}$$

$$b(\phi) = b(0) + b_1\vec{\phi} + \tfrac{1}{2}\vec{\phi}^T b_2\vec{\phi} + \ldots \tag{9b}$$

Remark 3.3: Considering Assumption 3.1 and Remark 3.2, we have that $a(0) = 0$ and $b(0) \neq 0$. Thus,

$$a(\phi) = \bar{a}(\phi) = a_1\vec{\phi} + \frac{1}{2}\vec{\phi}^T a_2\vec{\phi} + \dots \quad (10a)$$

$$b(\phi) = b_0 + \bar{b}(\phi) \quad (10b)$$

Definition 3.4: The zero dynamics of the nonlinear system (7) are given by $\eta(t)$. If we set $\zeta(t) = 0$, this implies $y(t) = 0$. Thus, the continuous-time zero dynamics are given by

$$\dot{\eta}(t) = c(0, \eta), \quad (11)$$

for any initial condition $\phi(0, \eta(0))$, and for an input

$$u_{zd}(t) = -\frac{a(0, \eta)}{b(0, \eta)}. \quad (12)$$

IV. DISCRETE-TIME SYSTEMS

In the first part of this section we present important concepts that will be used later in Section V to develop the main result of the paper, namely, an approximate sampled-data model for nonlinear systems.

A. DISCRETE-TIME SYSTEMS

We consider a discrete-time nonlinear system affine in the input u_k, similar to (6), using the shift operator:

$$x_{k+1} = F_q(x_k) + G_q(x_k)u_k \quad (13a)$$

$$y_k = H_q(x_k), \quad (13b)$$

where we have used the notation $x_k = x(kh)$ and h is the sampling period. Using the local coordinate transformation in (8), then (13) can be written as

$$\zeta_{k+1} = a(\zeta_k, \eta_k) + b(\zeta_k, \eta_k)u_k \quad (14a)$$

$$\eta_{k+1} = c(\zeta_k, \eta_k) \quad (14b)$$

$$y_k = \zeta_{1,k}. \quad (14c)$$

In addition, the necessary condition over the input u_k in order to characterize the internal zero dynamics of system (14) is given by [8]:

$$(u_{zd})_k = -\frac{a(0, \eta_k)}{b(0, \eta_k)}. \quad (15)$$

We are interested in characterizing the discrete-time zero dynamics due to the sampling process that have no continuous-time counterpart. Firstly, in the next subsection, we present the sampled-data model for an n-th order integrator.

B. N-TH ORDER INTEGRATOR

The results presented in this section are based on [5].

Theorem 4.1: Consider the continuous-time system $G(s) = s^{-n}$, where $n > 0$, and consider that the input to the system is generated by a B-spline ℓ-th order hold, as shown in Fig. 1. The corresponding sampled-data model is given by

$$\bar{x}_{k+1} = A_q\bar{x}_k + B_q u_k \quad (16a)$$

$$y_k = C_q\bar{x}_k \quad (16b)$$

where matrices A_q, B_q and C_q are given in (24).

Proof: An n-th order integrator can be written in normal form as follows

$$y(t) = x_1(t) \quad (17a)$$

$$\dot{x}_1(t) = x_2(t) \quad (17b)$$

$$\vdots$$

$$\dot{x}_n(t) = u(t) \quad (17c)$$

$$\vdots$$

$$\dot{x}_{n+\ell}(t) = u^{(\ell)}(t). \quad (17d)$$

Thus, our interest is to obtain the discrete-time model when the applied integration strategy is a truncated Taylor series expansion [5], [8] of order $(n + \ell)$, i.e.,

$$x_1(kh+\tau) = x_1(kh) + \tau\dot{x}_1(kh) + \dots + \frac{\tau^{(n+\ell)}}{(n+\ell)!}x_1^{(n+\ell)}(kh) \quad (18a)$$

$$x_2(kh+\tau) = x_2(kh) + \tau\dot{x}_2(kh) + \dots + \frac{\tau^{n+\ell-1}}{(n+\ell-1)}x_2^{(n+\ell-1)}(kh) \quad (18b)$$

$$\vdots$$

$$x_{n+\ell}(kh+\tau) = x_{n+\ell}(kh) + \tau\dot{x}_{n+\ell}(kh), \quad (18c)$$

where $0 \leq \tau < h$. Notice that, according to Lemma 2.1, $(\ell + 1)$-th and higher order derivatives are all equal to zero. Then, model (18) can be expressed as

$$x_{k+1}\& = A_q^1 x_k + B_q^1\bar{u}(kh) \quad (19a)$$

$$y_k = C_q x_k \quad (19b)$$

where the matrices are

$$A_q^1 = \begin{bmatrix} 1 & h & \cdots & \frac{h^{n-1}}{(n-1)!} \\ 0 & 1 & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad (20a)$$

$$B_q^1 = \begin{bmatrix} \frac{h^n}{n!} & \frac{h^{n+1}}{(n+1)!} & \cdots & \frac{h^{n+\ell}}{(n+\ell)!} \\ \frac{h^{n-1}}{(n-1)!} & \frac{h^n}{n!} & \cdots & \frac{h^{n+\ell-1}}{(n+\ell-1)!} \\ \vdots & \vdots & \vdots & \vdots \\ h & \frac{h^2}{2} & \cdots & \frac{h^{\ell+1}}{(\ell+1)!} \end{bmatrix} \quad (20b)$$

$$C_q = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \quad (20c)$$

Then, $u(kh)$ and its ℓ-th derivatives can be computed using (2) and (5). Then, $\bar{u}(kh)$ is given by

$$\bar{u}_k(kh) = \begin{bmatrix} u(kh) \\ \dot{u}(kh) \\ \vdots \\ u^{\ell}(kh) \end{bmatrix} = \underbrace{\left.\begin{bmatrix} \tilde{\beta}_\ell(t) \\ \frac{d}{dt}\tilde{\beta}_\ell(t) \\ \vdots \\ \frac{d^\ell}{dt^\ell}\tilde{\beta}_\ell(t) \end{bmatrix}\right|_{t=kh}}_{M} \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_{k-\ell} \end{bmatrix}. \tag{21}$$

Note that $u(kh)$ and $u^{(i)}(kh)$, for $i = 1, \dots, \ell$ depend on u_k and previous input samples, which can be included in the discrete-time model as additional states. Thus, we define $(\xi_1)_{k+1} = u_k$ and $(\xi_j)_k = u_{k-j}$. The additional ℓ states can be represented as follows:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_\ell \end{bmatrix}_{k+1} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}}_{E} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_\ell \end{bmatrix}_k + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_k \tag{22}$$

Then, using matrix E in (22), it is possible to expand (19) in order to include all the information available about $u(t)$. The augmented state vector is then defined as $\bar{x}_k = [x;\ \xi]_k^T$ and the state-space representation is given by

$$\bar{x}_{k+1} = A_q\bar{x}_k + B_q u_k \tag{23a}$$
$$y_k = C_q\bar{x}_k. \tag{23b}$$

where C_q is defined in (20c), the matrices

$$A_q = \left[\begin{array}{c|c} A_q^1 & B_q^1 M_1 \\ \hline 0_{\ell\times n} & E_{\ell\times\ell} \end{array}\right], \quad B_q = \begin{bmatrix} B_q^1 M_0 \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{24}$$

and $M_0 = M_{1:\ell+1,1}$ (i.e., the first column of M), and $M_1 = M_{1:\ell+1,2:\ell+1}$. Moreover, due to the matrix M structure, we have that

$$B_q^1 M_0 = \frac{h^n}{(n+\ell)!}. \tag{25}$$

□

It can be shown that the discrete-time transfer function associated with (19) is given by (see, for example, [21], [24])

$$G_q(z) = \frac{h^n}{(n+\ell)!}\frac{B_{n+\ell}(z)}{z^\ell(z-1)^n}. \tag{26}$$

Our interest is to characterize the zeros that appear in the sampled-data model due to the sampling process and to extend this result to the sampling zero dynamics that appear in the nonlinear case. As a first step, the following theorem characterizes the sampling zeros of an n-th order integrator.

Theorem 4.2: The discrete-time model for an n-th order integrator defined in (19) and (26) can be expressed in normal form as:

$$w_{1,k+1} = q_{11}w_{1,k} + Q_{12}\chi_k + \frac{h^n}{(n+\ell)!}u_k \tag{27a}$$
$$\chi_{k+1} = Q_{21}w_{1,k} + Q_{22}\chi_k, \tag{27b}$$

where χ_k represents the $(w_{2:n+\ell})_k$ states, and where the eigenvalues of matrix Q_{22} are the sampling zeros of (26), i.e.,

$$B_{n+\ell}(z) = \det(zI - Q_{22}). \tag{28}$$

Proof: Consider the similarity transformation $w_k = T\bar{x}_k$, where

$$T = \left[\begin{array}{c|c} 1 & 0 \\ \hline T_{21} & I_{n+\ell-1} \end{array}\right] \tag{29}$$

and

$$T_{21} = -\frac{(B_q)_{2:n+\ell,1}}{(B_q^1 M_0)_{1,1}} = -\frac{(n+\ell)!}{h^n}(B_q)_{2:n+\ell,1}. \tag{30}$$

Applying T to (23), the following state-space representation is obtained:

$$\tilde{A}_q = TA_qT^{-1} = Q = \left[\begin{array}{c|c} q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] \tag{31}$$
$$= \left[\begin{array}{c|c} -A_{12}T_{21} & A_{12} \\ \hline -(T_{21}A_{12}+A_{22})T_{21} & T_{21}A_{12}+A_{22} \end{array}\right]. \tag{32}$$

Using matrix A_q in (24), we have that

$$A_{12} = \begin{bmatrix} h & \frac{h^2}{2} & \cdots & \frac{h^{n-1}}{(n-1)!} & (B_q^1 M_1)_{1,1:\ell} \end{bmatrix} \tag{33}$$
$$A_{22} = \begin{bmatrix} (A_q^1)_{2:n,2:n} & (B_q^1 M_1)_{2:n,1:\ell} \\ 0_{1:\ell,2:n} & E_{\ell\times\ell} \end{bmatrix}. \tag{34}$$

Then, $\tilde{B}_q$ and $\tilde{C}_q$ are given by

$$\tilde{B}_q = TB_q = \begin{bmatrix} \frac{h^n}{(n+\ell)!} & 0 & \cdots & 0 \end{bmatrix}^T \tag{35}$$
$$\tilde{C}_q = C_qT^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{36}$$

Thus, the discrete-time model for an n-th order integrator is given by the following state-space representation:

$$w_{k+1} = \tilde{A}_q w_k + \tilde{B}_q u_k \tag{37a}$$
$$y_k = \tilde{C}_q w_k, \tag{37b}$$

To obtain the (sampling) zeros polynomial of (37), and therefore of (26), we compute the numerator polynomial as follows [25]:

$$N(z) = \det\begin{bmatrix} zI - \tilde{A}_q & -\tilde{B}_q \\ \tilde{C}_q & 0 \end{bmatrix} \tag{38}$$
$$= \left[\begin{array}{cc|c} z - q_{11} & -Q_{12} & -\frac{h^n}{(n+\ell)!} \\ & & 0 \\ -Q_{21} & zI - Q_{22} & \vdots \\ \hline 1 \quad 0 & \cdots & 0 \end{array}\right]. \tag{39}$$

Then, computing the determinant along the last column, (28) is readily obtained. □

V. AN APPROXIMATE SAMPLED-DATA MODEL FOR NONLINEAR SYSTEMS

In this section we present the main result of the paper, namely, we obtain an approximate sampled-data model when the samples of the input are interpolated using B-spline functions and when the integration strategy is a truncated Taylor series expansion. We also characterize the extra zero dynamics that appear due to the sampling process.

We first introduce an assumption in order to obtain a sampled-data model that preserves the input affine property.

Assumption 5.1: Let us assume that the nonlinear system in (7) satisfies:

$$\frac{\partial b(\zeta_k, \eta_k)}{\partial \zeta_{r,k}} = 0 \tag{40a}$$

$$\frac{\partial^i a(\zeta_k, \eta_k)}{(\partial \zeta_{r,k})^i} = 0; \quad \forall\, i > 1. \tag{40b}$$

Theorem 5.2: Consider the n-th order nonlinear system (6), where the input is generated by an ℓ-th order B-spline hold. Using the local coordinate transformation (8), the system can be represented in normal form as follows

$$\begin{aligned} w_{k+1} &= \left(\tilde{A}_q + \frac{h^{r+\ell}}{(r+\ell)!}\tilde{A}_2(w_k)\right) w_k \\ &\quad + \frac{h^{r+\ell}}{(r+\ell)!}\tilde{A}_3(w_k) + \tilde{B}_q(w_k) u_k \end{aligned} \tag{41a}$$

$$y_k = \tilde{C}_q\, w_k, \tag{41b}$$

where matrices $\tilde{A}_q$ and $\tilde{C}_q$ are defined in (32) and (36), respectively, and where $\tilde{A}_2(w_k)$, $\tilde{A}_3(w_k)$, $\tilde{B}_q(w_k)$ are given later in (54), (55), (56).

Proof: The continuous-time system (6) can be written in normal form as in (7). Applying the integration strategy shown in (18) up to order $r + \ell$ and considering $\tau = h$, we obtain the following sampled-data model

$$\begin{aligned} \hat{\zeta}_1(kh+h) &= \hat{\zeta}_1(kh) + h\hat{\zeta}_2(kh) + \frac{h^2}{2}\hat{\zeta}_3(kh) + \dots \\ &\quad + \frac{h^r}{r!}\frac{d}{dt}\hat{\zeta}_r(kh) + \dots + \frac{h^{r+\ell}}{(r+\ell)!}\frac{d^{\ell+1}}{dt^{\ell+1}}\hat{\zeta}_r(\alpha_1) \end{aligned} \tag{42a}$$

$$\begin{aligned} \hat{\zeta}_2(kh+h) &= \hat{\zeta}_2(kh) + \dots + \frac{h^{r-1}}{(r-1)}\frac{d}{dt}\hat{\zeta}_r(kh) + \\ &\quad + \dots + \frac{h^{r+\ell-1}}{(r+\ell-1)!}\frac{d^{\ell+1}}{dt^{\ell+1}}\hat{\zeta}_r(\alpha_2) \end{aligned} \tag{42b}$$

$$\vdots$$

$$\begin{aligned} \hat{\zeta}_r(kh+h) &= \hat{\zeta}_r(kh) + h\frac{d}{dt}\zeta_r(kh) + \frac{h^2}{2}\frac{d^2}{dt^2}\hat{\zeta}_r(kh) + \\ &\quad + \dots + \frac{h^{\ell+1}}{(\ell+1)!}\frac{d^{\ell+1}}{dt^{\ell+1}}\hat{\zeta}_r(\alpha_r) \end{aligned} \tag{42c}$$

$$\hat{\eta}(kh+h) = \hat{\eta}(kh) + h\, c(\hat{\zeta}, \hat{\eta})\Big|_{t=\alpha_{r+1}}. \tag{42d}$$

This model is exact for some unknown time instants $kh \leq \alpha_i < kh + h$; $i = 1, \dots, r+1$. Replacing the time instants α_i by kh, we obtain an approximate discrete-time model given by

$$\begin{aligned} \zeta_1(kh+h) &= \zeta_1(kh) + h\zeta_2(kh) + \frac{h^2}{2}\zeta_3(kh) + \dots \\ &\quad + \frac{h^r}{r!}\frac{d}{dt}\zeta_r(kh) + \dots + \frac{h^{r+\ell}}{(r+\ell)!}\frac{d^{\ell+1}}{dt^{\ell+1}}\zeta_r(kh) \end{aligned} \tag{43a}$$

$$\begin{aligned} \zeta_2(kh+h) &= \zeta_2(kh) + \dots + \frac{h^{r-1}}{(r-1)}\frac{d}{dt}\zeta_r(kh) + \\ &\quad + \dots + \frac{h^{r+\ell-1}}{(r+\ell-1)!}\frac{d^{\ell+1}}{dt^{\ell+1}}\zeta_r(kh) \end{aligned} \tag{43b}$$

$$\vdots$$

$$\begin{aligned} \zeta_r(kh+h) &= \zeta_r(kh) + h\frac{d}{dt}\zeta_r(kh) + \frac{h^2}{2}\frac{d^2}{dt^2}\hat{\zeta}_r(kh) + \\ &\quad + \dots + \frac{h^{\ell+1}}{(\ell+1)!}\frac{d^{\ell+1}}{dt^{\ell+1}}\zeta_r(kh) \end{aligned} \tag{43c}$$

$$\eta(kh+h) = \eta(kh) + h\, c(\zeta, \eta)|_{t=kh}. \tag{43d}$$

where, based on Assumption 5.1, we have neglected the derivatives of $b(\zeta_k, \eta_k)$ and higher-order derivatives of $a(\zeta_k, \eta_k)$. In addition, considering that $\zeta(t)$ is close to the origin, we assume $a(\zeta_k, \eta_k) \approx 0$ (see Remark 3.3). Thus, the approximate sampled-data model in (43) can be expressed in the following state-space form:

$$\zeta_{k+1} = A_q\zeta_k + A_3(\zeta_k, \eta_k) + B_2(\zeta_k, \eta_k)\bar{u}_k \tag{44a}$$

$$\eta_{k+1} = \eta_k(\zeta_k, \eta_k) + hc(\zeta_k, \eta_k), \tag{44b}$$

where $\bar{u}_k$ is given by (21) and

$$\begin{aligned} &A_3(\zeta_k, \eta_k) \\ &= \begin{bmatrix} \dfrac{h^r}{r!} + \dfrac{h^{r+1}}{(r+1)!}\dfrac{\partial a}{\partial \zeta_r} + \dots + \dfrac{h^{r+\ell}}{(r+\ell)!}\left(\dfrac{\partial a}{\partial \zeta_r}\right)^{\ell} \\ \vdots \\ h + \dfrac{h^2}{2}\dfrac{\partial a}{\partial \zeta_r} + \dots + \dfrac{h^{\ell+1}}{(\ell+1)!}\left(\dfrac{\partial a}{\partial \zeta_r}\right)^{\ell} \end{bmatrix} \\ &\quad \times a(\zeta_k, \eta_k). \end{aligned} \tag{45}$$

Remark 5.3: Notice that, according to the integration strategy applied to each state component $\zeta_i(kh+h)$, the neglected terms appear in the $(r+i)$-th and higher-order derivatives. Moreover, the derivatives of $a(\zeta, \eta)$ that are not considered in the expansion are of order ℓ.

Note that matrix (46), as shown at the bottom of the next page, can be split into a linear part, given by (20b), and a matrix with the nonlinearities factorized by $h^{(r+\ell)}/(r+\ell)!$, i.e.,

$$B_2(\zeta_k, \eta_k) = \underbrace{B_q^1 b_0}_{B_q^1} + \frac{h^{r+\ell}}{(r+\ell)!}B_q^2(\zeta_k, \eta_k). \tag{47}$$

Similarly to the linear case, the input and its derivatives at the sampling instants can be included in the model as additional states (see (22)). Thus, we define $\bar{\zeta}_k = [\zeta_k;\ \xi_k]^T$ as the augmented state vector and then (44) can be written as follows:

$$\bar{\zeta}_{k+1} = \bar{A}_q(\zeta_k, \eta_k)\bar{\zeta}_k + \frac{h^{r+\ell}}{(r+\ell)!}\bar{A}_3(\zeta_k, \eta_k) + \bar{B}_q u_k \quad \text{(48a)}$$

$$\eta_{k+1} = \eta_k + hc(\zeta_k, \eta_k), \quad \text{(48b)}$$

where

$$\bar{A}_q(\zeta_k, \eta_k) = A_q + \frac{h^{r+\ell}}{(r+\ell)!}A_2(\zeta_k, \eta_k), \quad (49)$$

where A_q is given by (24) and

$$A_2(\zeta_k, \eta_k) = \begin{bmatrix} 0_{r\times r} & B_q^2(\zeta_k, \eta_k)M_1 \\ 0_{\ell\times r} & 0_{\ell\times\ell} \end{bmatrix}, \quad (50)$$

$$\bar{A}_3(\zeta_k, \eta_k) = \begin{bmatrix} A_3(\zeta_k, \eta_k) \\ 0_{\ell\times 1} \end{bmatrix}, \quad (51)$$

$$\bar{B}_q(\zeta_k, \eta_k) = \begin{bmatrix} B_2(\zeta_k, \eta_k)M_0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (52)$$

We apply now the (linear) similarity transformation $w_k = T\xi_k$, where T is defined in (29), and T_{21} is given by

$$T_{21} = -\frac{(B_q)_{2:r+\ell}}{B_q^1 M_0} = -\frac{(r+\ell)!}{b_0 h^r}(B_q)_{2:r+\ell}. \quad (53)$$

We are interested in obtaining a sampled-data model that can be defined in the neighborhood of the origin. Thus, using the results presented in (10), the new approximate sampled-data model is given by the submatrices of $\tilde{A_q}$ shown in (33)-(34) and

$$\begin{aligned}\tilde{A}_2(w_k) &= TA_2(T^{-1}w_k) \\ &= \tilde{Q}(w_k) = \begin{bmatrix} \tilde{q}_{11}(w_k) & \tilde{Q}_{12}(w_k) \\ \tilde{Q}_{21}(w_k) & \tilde{Q}_{22}(w_k) \end{bmatrix} \quad (54)\end{aligned}$$

$$\tilde{A}_3(w_k) = T\bar{A}_3(T^{-1}w_k) \quad (55)$$

$$\begin{aligned}\tilde{B}_q(w_k) &= T\bar{B}_q(T^{-1}w_k) \\ &= \left[\frac{h^r}{(r+\ell)!}b(w_k)\ 0 \cdots 0 \right]^T \quad (56)\end{aligned}$$

□

A. LOCAL VECTOR TRUNCATION ERROR

In this section we analyze the local vector truncation error [26] between the exact and the proposed approximate sampled-data models defined in (42) and (43), respectively.

Theorem 5.4: Consider the continuous-time nonlinear system in (6) having an input generated by a B-spline generalized hold. Then, the approximate sampled-data model in (43) has a Local Vector Truncation Error of the order of $(h^{r+2}, h^{r+1}, \cdots, h^3, h^2)$.

Proof: From the definition of Local Vector Truncation Error in [26], we have that the difference between the exact and the approximate discrete-time model is given by

$$\hat{e}_1 = \hat{\zeta}_1(kh+h) - \zeta_1(kh+h) = 0 \quad \text{(57a)}$$
$$\vdots$$
$$\hat{e}_r = \hat{\zeta}_r(kh+h) - \zeta_r(kh+h) = 0 \quad \text{(57b)}$$
$$\hat{e}_{r+1} = \hat{\eta}(kh+h) - \eta(kh+h) = 0. \quad \text{(57c)}$$

Then, considering Remark 5.3, the differences are given by

$$\begin{aligned}\hat{e}_1 &= \frac{h^{r+1}}{(r+1)!}\frac{d^2}{dt^2}\Big(\hat{\zeta}_r(kh) - \zeta_r(kh)\Big) + \cdots \\ &\quad + \frac{h^{r+\ell}}{(r+\ell)!}\frac{d^{r+\ell}}{dt^{r+\ell}}\Big(\hat{\zeta}_r(\alpha_1) - \zeta_r(kh)\Big) \quad \text{(58a)} \\ \hat{e}_2 &= \frac{h^r}{(r)!}\frac{d^2}{dt^2}\Big(\hat{\zeta}_r(kh) - \zeta_r(kh)\Big) + \cdots \\ &\quad + \frac{h^{r+\ell-1}}{(r+\ell-1)!}\frac{d^{r+\ell}}{dt^{r+\ell}}\Big(\hat{\zeta}_r(\alpha_2) - \zeta_r(kh)\Big) \\ &\ \ \vdots \quad \text{(58b)} \\ \hat{e}_r &= \frac{h^2}{2}\frac{d^2}{dt^2}\Big(\hat{\zeta}_r(kh) - \zeta_r(kh)\Big) + \cdots \\ &\quad + \frac{h^{r+\ell}}{(r+\ell)!}\frac{d^{\ell+1}}{dt^{\ell+1}}\Big(\hat{\zeta}_r(\alpha_r) - \zeta(kh)\Big) \quad \text{(58c)} \\ \hat{e}_{r+1} &= h\left(c(\hat{\zeta}, \hat{\eta})\Big|_{t=\alpha_{r+1}} - c(\zeta, \eta)|_{t=kh} \right). \quad \text{(58d)}\end{aligned}$$

Since $\hat{e}_i$ includes terms of the order of h^{r+2-i}, $i = 1, \ldots r+1$, and higher, we can notice that the error is of the order of h^{r+2-i}. Thus, following the ideas presented in [26],

$$B_2(\zeta_k, \eta_k) = b(\zeta_k, \eta_k)$$
$$\times \begin{bmatrix} \frac{h^r}{r!} + \frac{h^{r+1}}{(r+1)!}\frac{\partial a}{\partial \zeta_r} + \cdots + \frac{h^{r+\ell}}{(r+\ell)!}\left(\frac{\partial a}{\partial \zeta_r}\right)^\ell & \frac{h^{r+1}}{(r+1)!} + \frac{h^{r+2}}{(r+2)!}\left(\frac{\partial a}{\partial \zeta_r}\right) + \cdots + \frac{h^{r+\ell}}{(r+\ell)!}\left(\frac{\partial a}{\partial \zeta_r}\right)^{\ell-1} & \cdots & \frac{h^{r+\ell}}{(r+\ell)!} \\ \frac{h^{r-1}}{(r-1)!} + \frac{h^r}{r!}\frac{\partial a}{\partial \zeta_r} + \cdots + \frac{h^{r+\ell-1}}{(r+\ell-1)!}\left(\frac{\partial a}{\partial \zeta_r}\right)^\ell & \frac{h^r}{r!} + \frac{h^{r+1}}{(r+1)!}\left(\frac{\partial a}{\partial \zeta_r}\right) + \cdots + \frac{h^{r+\ell-1}}{(r+\ell-1)!}\left(\frac{\partial a}{\partial \zeta_r}\right)^{\ell-1} & \cdots & \frac{h^{r+\ell-1}}{(r+\ell-1)!} \\ \vdots & \vdots & & \vdots \\ h + \frac{h^2}{2}\frac{\partial a}{\partial \zeta_r} + \cdots + \frac{h^{\ell+1}}{(\ell+1)!}\left(\frac{\partial a}{\partial \zeta_r}\right)^\ell & \frac{h^2}{2} + \frac{h^3}{3}\frac{\partial a}{\partial \zeta_r} + \cdots + \frac{h^{\ell+1}}{(\ell+1)!}\left(\frac{\partial a}{\partial \zeta_r}\right)^{\ell-1} & \cdots & \frac{h^{\ell+1}}{(\ell+1)!} \end{bmatrix}. \quad (46)$$

the approximate sampled-data model (43) has a local vector truncation error of order $(h^{r+2}, h^{r+1}, \cdots, h^3, h^2)$. □

From the above result, we can see that the (local) error between the output of model (43) and the output of the true system is of the order of h^{r+2} (see (14)). Moreover, it can be noticed that the approximate model based on B-spline functions is more accurate than the models considered in [26] and in [8], because the local truncation error in each state component $\zeta_i(kh+h)$ is of a higher order in the sampling period h. Thus, the assumption about the smoothness of the system input has been exploited to improve the accuracy of the obtained model.

Remark 5.5: Notice that other splines may be used as interpolating functions [18]. In this case, the truncated Taylor series expansion has to be modified, leading to a different approximate model.

B. ASYMPTOTIC ZERO DYNAMICS

In this section we characterize the zero dynamics of the corresponding nonlinear discrete-time model. Theorem 5.6 shows that these zero dynamics are asymptotically the same as the sampling zeros of an n-th order integrator.

Theorem 5.6: Consider the approximate discrete-time model (41). The associated sampling zero dynamics can be asymptotically characterized, as the sampling period goes to zero, in terms of the eigenvalues of matrix Q_{22}, i.e.,

$$\bar{\chi}_{k+1} = Q_{22}\bar{\chi}_k \tag{59a}$$

$$\eta_{k+1} = \eta_k, \tag{59b}$$

where $\bar{\chi}_k = (w_{2:r+\ell})_k$.

Proof: We impose the zero dynamics condition to model (41), i.e., $y_k = w_k = 0$. Thus,

$$\begin{bmatrix} 0 \\ \hline w_2 \\ \vdots \\ w_{r+\ell} \end{bmatrix}_{k+1} = \left(Q + \frac{h^{r+\ell}}{(r+\ell)!}\tilde{Q}(w_k)\right)\begin{bmatrix} 0 \\ \hline w_2 \\ \vdots \\ w_{r+\ell} \end{bmatrix}_k + \frac{h^{r+\ell}}{(r+\ell)!}\tilde{A}_3(w_k) + \tilde{B}_q(w_k)(u^{zd})_k. \tag{60}$$

where, Q and $\tilde{Q}(w_k)$ are given by (31)-(32) and (54), respectively. Then, solving for the first row in (60), we have that

$$(u^{zd})_k = -\frac{(r+\ell)!}{h^r b(w_k)}\Big[\Big(Q_{12} + \frac{h^{r+\ell}}{(r+\ell)!}\tilde{Q}_{12}(w_k)\Big)\bar{\chi}_k + \frac{h^{r+\ell}}{(r+\ell)!}(\tilde{A}_3(w_k))_{1,1}\Big]. \tag{61}$$

For the remaining equations and considering that $\bar{\zeta}_k(t)$ evolves close to the origin, we have that

$$\bar{\chi}_{k+1} = \left(Q_{22} + \frac{h^{r+\ell}}{(r+\ell)!}\tilde{Q}_{22}(w_k)\right)\bar{\chi}_k + \frac{h^{r+\ell}}{(r+\ell)!}\tilde{A}_3(w_k) \tag{62}$$

$$\eta_{k+1} = \eta_k + hc(\zeta_k, \eta_k). \tag{63}$$

Considering that $h \to 0$, then (59b) is readily obtained. □

Theorem 5.6 shows that the nonlinear zero dynamics of the approximate discrete-time model proposed in (41) can be split into two parts: a linear subsystem given by Q_{22}, which represents the asymptotic zero dynamics that appear due to the sampling process, and a nonlinear subsystem, which is the discretization of the continuous-time zero dynamics of (6). Moreover, notice that the eigenvalues of the linear part, namely of Q_{22}, are the sampling zeros of an n-th order integrator (see Theorem 4.2).

In the following example, we show how the proposed sampled-data model and the associated zero dynamics are obtained.

Example 5.7: Consider an n-th order nonlinear continuous-time system (6) with relative degree $r = 2$. The latter can be expressed in normal form as in (7). We assume that the input is generated by a B-spline first-order hold given by

$$u(t) = u_{k-1} + \frac{u_k - u_{k-1}}{h}(t - kh); \quad kh \le t < (k+1)h. \tag{64}$$

We obtain the proposed approximate dynamics sampled-data model applying the integration strategy (18) of order $r + \ell = 3$, i.e.,

$$\zeta_{1,k+1} = \zeta_{1,k} + h\zeta_{2,k} + \frac{h^2}{2}\dot{\zeta}_2(kh) + \frac{h^3}{3!}\ddot{\zeta}_2(kh) \tag{65a}$$

$$\zeta_{2,k+1} = \zeta_{2,k} + h\dot{\zeta}_2(kh) + \frac{h^2}{2}\ddot{\zeta}_2(kh) \tag{65b}$$

$$\eta_{k+1} = \eta_k + hc(\zeta, \eta), \tag{65c}$$

where

$$\ddot{\zeta}_2(kh) = b(\zeta,\eta)\dot{u}(kh) + \left(\frac{\partial a(\zeta,\eta)}{\partial \zeta_2} + \frac{\partial b(\zeta,\eta)}{\partial \zeta_2}u(kh)\right) \times (a(\zeta,\eta) + b(\zeta,\eta)u(kh)) \tag{66a}$$

Considering Assumption (5.1), the first two equations in (65) can be expressed as follows,

$$\zeta_{1,k+1} = \zeta_{1,k} + h\zeta_{2,k} + \left(\frac{h^2}{2} + \frac{h^3}{6}\frac{\partial a(\zeta,\eta)}{\partial \zeta_2}u(kh)\right)a(\zeta,\eta) + \left[\left(\frac{h^2}{2} + \frac{h^3}{6}\frac{\partial a(\zeta,\eta)}{\partial \zeta_2}\right)b(\zeta,\eta) \quad \frac{h^3}{6}b(\zeta,\eta)\right] \times \begin{bmatrix} u(kh) \\ \dot{u}(kh) \end{bmatrix} \tag{66b}$$

$$\zeta_{2,k+1} = \zeta_{2,k} + \left(h + \frac{h^2}{2}\frac{\partial a(\zeta,\eta)}{\partial \zeta_2}\right)a(\zeta,\eta) + \left[\left(h + \frac{h^2}{2}\frac{\partial a(\zeta,\eta)}{\partial \zeta_2}\right)b(\zeta,\eta) \quad \frac{h^2}{2}b(\zeta,\eta)\right] \times \begin{bmatrix} u(kh) \\ \dot{u}(kh) \end{bmatrix}, \tag{66c}$$

where $u(kh)$ and its first derivative are defined according to (2) and (5), respectively, i.e.,

$$\bar{u}(kh) = \begin{bmatrix} 0 & 1 \\ \frac{1}{h} & -\frac{1}{h} \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix}. \tag{67}$$

Then, the augmented state-space model is given by

$$\bar{\zeta}_{k+1} = \begin{bmatrix} 1 & h & \frac{h^2}{6} b(\zeta,\eta)\left(h\frac{\partial a(\zeta,\eta)}{\partial \zeta_2} + 2\right) \\ 0 & 1 & \frac{h}{2} b(\zeta,\eta)\left(h\frac{\partial a(\zeta,\eta)}{\partial \zeta_2} + 1\right) \\ 0 & 0 & 0 \end{bmatrix} \bar{\zeta}_k + \begin{bmatrix} \left(\frac{h^2}{2} + \frac{h^3}{6}\frac{\partial a(\zeta,\eta)}{\partial \zeta_2}\right) \\ \left(h + \frac{h^2}{2}\frac{\partial a(\zeta,\eta)}{\partial \zeta_2}\right) \\ 0 \end{bmatrix} a(\zeta,\eta) + \begin{bmatrix} \frac{h^2}{6} b(\zeta,\eta) \\ \frac{h}{2} b(\zeta,\eta) \\ 1 \end{bmatrix} u_k \tag{68a}$$

$$\eta_{k+1} = \eta_k + hc(\zeta,\eta). \tag{68b}$$

The result in (10) allows us to split model (68) into a linear and a nonlinear part. Then, applying the similarity transformation T in (29) and $T_{21} = -[3/h, \quad -6/(h^2 b_0)]^T$, based on Remark 3.3, and considering that $\bar{\zeta}_k(t)$ evolves close to the origin, the following state-space representation is obtained:

$$w_{k+1} = \tilde{A}_1 w_k + \tilde{A}_2(w) w_k + \tilde{A}_3(w) + \tilde{B}(w) u_k \tag{69a}$$

$$\eta_{k+1} = \eta_k + hc(\zeta,\eta), \tag{69b}$$

where,

$$\tilde{A}_1 = \begin{bmatrix} 6 & h & \frac{h^2}{3} b_0 \\ -\frac{12}{h} & -2 & -\frac{h}{2} b_0 \\ -\frac{36}{h^2 b_0} & -\frac{6}{h b_0} & -2 \end{bmatrix}, \tag{70}$$

$$\tilde{A}_2(w) = \begin{bmatrix} 0 & 0 & \frac{h^3}{6} b(w)\frac{\partial a(w)}{\partial w_2} + \frac{h^2}{3}\bar{b}(w) \\ 0 & 0 & -\frac{h}{2}\bar{b}(w) \\ 0 & 0 & -h\frac{\partial a(w)}{\partial w_2} \end{bmatrix}, \tag{71}$$

$$\tilde{A}_3(w) = \begin{bmatrix} \frac{h^2}{6}\left(h\frac{\partial a(w)}{\partial w_2} + 3\right) \\ -\frac{h}{2} \\ 0 \end{bmatrix} a(w), \tag{72}$$

$$\tilde{B}(w) = \begin{bmatrix} \frac{h^2}{6} b(w) \\ 0 \\ 0 \end{bmatrix}. \tag{73}$$

Applying the condition for the zero dynamics, we obtain

$$\bar{\chi}_{k+1} = \begin{bmatrix} -2 & -\frac{h}{2} b_0 \\ -\frac{6}{h b_0} & -2 \end{bmatrix} \bar{\chi}_k + \begin{bmatrix} 0 & -\frac{h}{2}\bar{b}(w) \\ 0 & -h\frac{\partial a(w)}{\partial w_2} \end{bmatrix} \bar{\chi}_k - \begin{bmatrix} \frac{h}{2} \\ 0 \end{bmatrix} a(w) \tag{74a}$$

$$\eta_{k+1} = \eta_k + hc\,(0, \chi_k, \eta_k) \tag{74b}$$

Note that the eigenvalues of the first matrix in (74a) correspond to the roots of the Euler-Frobenius polynomial $B_3(z) = z^2 + 4z + 1$. Thus, following (28), the system (74a) can be expressed as:

$$\bar{\chi}_{k+1} = Q_{22}\bar{\chi}_k + \begin{bmatrix} 0 & -\frac{h}{2}\bar{b}(w) \\ 0 & -h\frac{\partial a(w)}{\partial w_2} \end{bmatrix} \bar{\chi}_k - \begin{bmatrix} \frac{h}{2} \\ 0 \end{bmatrix} a(w). \tag{75}$$

Considering $h \to 0$, we notice that the zero dynamics converge to the asymptotic sampling zeros of the linear case.

VI. CONCLUSION

An approximate sampled-data model for a class of nonlinear systems affine in the input has been proposed based on normal forms, B-spline functions and the truncated Taylor series expansion. The resulting model includes extra zero dynamics that can be asymptotically characterized, as the sampling period goes to zero, in terms of the asymptotic sampling zeros found in the linear case. Also, it has been shown that the location of the zero dynamics depends not only in the integration strategy but also on the smoothness of the system input described using B-spline functions. The accuracy of the approximate model has been also studied. In particular, the local error truncation of each element of the state vector has been characterized.

These results give further insights in approximate solutions for nonlinear systems under sampling and the (asymptotic) connection with the linear case. Moreover, the smoothness of the input to the system plays an important role in the location of the zero dynamics, which can be exploited, for example, in the design of discrete-time control laws [27].

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