IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received June 28, 2020, accepted July 25, 2020, date of publication July 29, 2020, date of current version August 10, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3012871

An Improved Interior Point Algorithm for Quantile Regression

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This work was supported in part by the Natural Science Foundation of Hunan Province of China under Grant 2018JJ2071, and in part by the Social Science Foundation of Hunan Province of China under Grant 18YBA080.

ABSTRACT Quantile regression is a powerful statistical technique for estimating the quantiles of a conditional distribution on the values of covariates. It has been widely used in many fields. In this paper, an improved interior point algorithm for quantile regression is proposed. The algorithm introduces multiple centrality corrections technique into the interior point algorithm for quantile regression. The purpose of introducing the multiple centrality corrections technique is to reduce the overall solution time required to solve a quantile regression problem. The computational experiments results constitute evidence of the improvement obtained with the use of multiple centrality correction technique combined with the interior point algorithm.

INDEX TERMS Quantile regression, multiple centrality corrections, interior point algorithm.

I. INTRODUCTION

Most applied statistics can be regarded as an exposition of linear models and their associated least squares methods. The reason why the least squares method is popular is that the computational simplicity and its many excellent properties when the observed errors are normally distributed. However, in reality, the basic assumptions of the least squares method are often not satisfied, and the estimation obtained by the least squares method in this case will no longer have excellent properties.

Quantile regression is a powerful statistical technique used to estimate and draw inferences about conditional quantile functions. Median regression [1] as introduced in the 19th century is a special case. In contrast to conventional mean regression that minimizes sums of squares residuals, median regression minimizes sums of absolute residuals; quantile regression simply replaces symmetric absolute loss by asymmetric linear loss.

Quantile regression was first proposed by Koenker and Bassett [2], since their groundbreaking work, quantile regression has provoked great concern in theoretical and empirical respects. Quantile regression is insensitive to outliers, and it can give a more complete picture of the conditional distribution than a single estimate of the center. Due to the significant advantages of quantile regression, it has become an attractive statistical tool in regression analysis. It has been widely used in economy, finance, environment, eletrical engineering and other fields (for example [3]–[7]). Applications of quantile regression to problems with large sample sizes are almost routine. In statistics, as the data increases, the model becomes more complex and the pressure on calculations increase. A lot of innovations are possible in this case, as is the calculation of quantile regression.

Linear programming and the related simplex method, like many other important statistical ideas, emerged during the World War II. Danzig mentioned in his memoir [8] that his simplex method thought appeared in 1947 when he tried to solve a class of military planning problems. Charnes *et al* [9] seemed to be the first to explicitly use simplex method to solve the median regression problem. Barrodale and Roberts [10] proposed a modification of the simplex method for linear programming, which is applicable to the primal formulation of the median regression problem. The algorithm of Barrodale and Roberts [10] was the first to exploit the bounded variables dual form of the median regression problem. Some slight modifications of the Barrodale and Roberts algorithm [10] are described in Koenker and d'Orey [11]. The modified algorithm can be used to compute quantile regression models of

The associate editor coordinating the review of this manuscript and approving it for publication was Geng-Ming Jiang.

Koenker and Bassett [2] and the associated empirical quantile functions. In practical applications, simplex method performs very well for medium-scale quantile regression problems. However, for problems with more than a few thousand observations, simplex method begin to live up to their slothful theoretical reputation [12]. This may be partially due to the theoretical results of the worst-case performance of the simplex method, which show that for certain pathological problems, the number of simplex iterations required for the solution can increase exponentially as the size of the problem increases. In summary, these have strongly promoted the exploration of improved methods for large-scale quantile regression problems.

Simplex method is a non-polynomial algorithm. It might have to make a very large number of steps which depends exponentially on the problem dimension [13]. The first linear programming polynomial algorithm was proposed by Khachiyan [14]. Following Khachiyan's work [14], the ellipsoid method was the only algorithm for solving linear programming whose runtime had been proved to be polynomial until Karmarkar's algorithm [15]. Mathematician Karmarkar [15] proposed a new polynomial-time algorithm for linear programming, which has polynomial computational complexity. Although a single iteration of the Karmarkar's algorithm is expensive, optimality is achieved after a relatively small number of iterations, which makes the algorithm computationally attractive. The publication of Karmarkar's article [15] opened up a new field of research, now known as interior point methods. The basic idea of interior point algorithm is to find a new interior point which makes the objective function descend along the feasible direction from an initial point in the feasible region, and then from the new interior point, find the subsequent interior point which makes the objective function descend along the feasible direction, and iterate repeatedly, so that the objective function gradually tends to the optimal value. Following Karmarkar [15], many interior point algorithms have been proposed. These approaches, as shown by Karmarkar [15] and subsequent authors, provide significantly better worst-case performance than simplex algorithms, and show impressive practical performance in the large-scale linear programming that appears in commercial and extensive numerical experiments. Among these numerous interior point algorithms, the primal-dual interior point algorithm has been proved theoretically to have polynomial computational complexity, fast convergence and good robustness, so it has become one of the most widely used and efficient algorithms. Kojima *et al.* [16] developed the theoretical background of this method and gave the first complexity results. Kojima *et al.* [17] made further progress. They provided good theoretical results and extra safeguards for the primal-dual algorithm, which could be translated into computational practice.

Among the various primal-dual interior point methods, the predictor-corrector interior point algorithms are the most applicable and efficient methods both theoretically and computationally. The first predictor corrector interior point algorithm was proposed by Mehrotra [18]. After that, Yu *et al.* [19] presented a polynomial predictor-corrector interior-point algorithm for convex quadratic programming based on a modified predictor-corrector interior-point algorithm. Zhao [20] provided a unified way to enlarge the neighborhoods of predictor–corrector interior-point algorithms for linear programming. Xiaoni and Sanyang [21] presented a globally convergent infeasible-interior-point predictor-corrector algorithm the second-order cone programming by using the Alizadeh– Haeberly–Overton search direction. Yang *et al.* [22] presented a Mehrotra-type predictor–corrector infeasible-interior-point method for symmetric optimization based on a new one-norm neighborhood. A corrector-predictor interior-point algorithm is proposed for symmetric optimization by Pirhaji *et al.* [23]. Interior point method is a very appealing approach to the optimal power flow (OPF) problem mainly due to its speed of convergence and ease of handling inequality constraints. Therefore, Capitanescu *et al.* [24] analyzed the ability of the predictor–corrector algorithm to solve various classical OPF problems. Pinheiro *et al.* [25] proposed a predictor–corrector primal–dual modified log-barrier interior–exterior point method with global convergence and cubic fitting strategies for solving the Reactive Optimal Power Flow (ROPF) problem.

Mehrotra-type predictor–corrector algorithm was especially noteworthy for its efficiency and fast convergence, such that new variations started to be developed from it. In addition, Mehrotra-type predictor–corrector algorithms are the base of the interior point methods software packages due to its practical efficiency [26]. Therefore, Portnoy and Koenker [12] applied the Mehrotra's predictor corrector interior point algorithm to the quantile regression problems.

In practice, what really reduces the efficiency of the primal-dual algorithm is the huge difference between complementary products. If the order of complementary products is different, then the complementary products are either too small or too large relative to their average value, which is undesirable. There is a theoretical basis for the view that large difference between complementary products will lead to an iteration away from the region where Newton's method for centering converges fast. This, in practice, is reflected in the fact that only small steps are allowed in the primal and dual spaces. Gondzio [27] proposed a remedy to this problem. He proposed the multiple centrality corrections (mcc) technique in a primal-dual method for linear programming, aiming to improve the centrality of the current iteration and achieve larger stepsizes in primal and dual spaces. According to a large number of numerical experimental results in Gondzio [27], the use of multiple centrality corrections technique can reduce computation time compared to the widely used predictor-corrector interior point algorithm. The variation of this version is considered to be one of the most effective variations of interior point algorithm. Following Gondzio [27], there has been intense interest in multiple centrality corrections technique. See, for example,

Torres and Quintana [28] extended the multiple centrality corrections technique developed by Gondzio [27] from linear programming to nonlinear OPF. And a large number of numerical results show that mcc technology is fast and robust, and is superior to the successful prediction correction technology. Wu and Chang [29] proposed a fuzzy multiple centrality correction algorithm based on the multiple centerality correction technology to perform multiple corrections on complementarity conditions in the nonlinear predictor-corrector primal-dual interior point algorithm so as to realize the optimal power flow. Capitanescu *et al.* [30] presented and texted the mcc interior point algorithm on various OPF variants. The results obtained suggest that the mcc algorithm is a highly viable alternative to the successful predictor corrector algorithm. Min and Shengsong [31] presented a new algorithm to solve nonlinear optimal power flow problems. The OPF problem is solved by a primal–dual interior point method with multiple centrality corrections as a sequence of linearized trust region sub-problems. The computational results show that the proposed algorithm is very effective to OPF applications. And comparison with the predictor–corrector primal–dual interior point method is also made to demonstrate the superiority of the multiple centrality corrections technique. Colombo and Gondzio [32] revisited the technique of multiple centrality correctors and presented further development of multiple centrality correctors for interior point methods. Huang and Jiang [33] presented the nonlinear weight multiple centrality corrections interior point method for optimal power flow. Berti *et al.* [34] presented a proposal for a variation of the predictor-corrector interior point method with multiple centrality corrections.

The multiple centrality corrections techniques are intended for the current point to progress in the neighbor of the central path. Accordingly, it makes it possible to obtain larger steps in the next iteration, moving in the direction of convergence. This version is considered by many researchers to be one of the most effective changes to the predictor-corrector interior point method [34]. Therefore, in this paper, we introduce the multiple centrality corrections techniques into the interior point algorithm for quantile regression to improve the current interior point algorithm for quantile regression. The computational experiment results constitute evidence of the improvement obtained with the use of multiple centrality corrections technique combined with the interior point algorithm.

This remainder of this article is organized as follows: Section II is devoted to describing quantile regression and converting it to the linear programming form. In Section III, multiple centrality corrections technique is described and incorporated into the predictor-corrector interior point algorithm for quantile regression. The computational experiments results are presented in Section IV. Lastly, Conclusions are given in Section V.

II. QUANTILE REGRESSION

Quantile regression is one of the research frontiers of econometrics. It constitutes a family of statistical techniques intended to estimate and draw inferences about conditional quantile functions. It is an extension of mean regression. In recent years, it has become more and more useful not only in econometrics, but also in biomedicine, finance, eletrical engineering and environmental science.

A. QUANTILE REGRESSION

If X is a real random variable, it can be described by its distribution function

$$F(x) = P(X \leq x). \tag{1}$$

Meanwhile, for any $0 < \tau < 1$, the τth quantile of X can be expressed as

$$F^{-}(\tau) = inf\{x : F(x) \geq \tau\}, \tag{2}$$

where median $F^{-1}(1/2)$ plays the ''middle'' role.

The quantile arises from a simple optimization problem: a point estimation of a random variable with an F distribution function. If the loss function is piecewise linear

$$\rho_\tau(u) = u(\tau - I(u < 0)), \tag{3}$$

for any $0 < \tau < 1$, we can find $\hat{x}$ to get the minimum expected loss value. The earliest reasearch in this issue was Fox and Rubin [15]. Then, let's find the minimum of the following objective function

$$E_{\rho_\tau}(X - \hat{x}) = (\tau - 1)\int_{-\infty}^{\hat{x}} (x - \hat{x})dF(x) + \tau \int_{\hat{x}}^{+\infty} (x - \hat{x})dF(x). \tag{4}$$

Taking the derivative with respect to $\hat{x}$ and setting the derivate to 0, we get

$$F(\hat{x}) - \tau = 0. \tag{5}$$

Because F is monotonic, any element from set $\{x : F(x) = \tau\}$ will minimize the expected loss value. When equation (5) has a unique solution, let $\hat{x} = F^{-1}(\tau)$, otherwise, take the minimum value from the solution set as the solution. For the asymmetrical linear loss function, the optimal point estimation is the quantile. In the case of symmetry, the quantile is the median.

Considering the quantile as a solution to a simple optimization problem, we seek a more general method to eatimate the quantile function model. The ordinary least squares method provides a temple for this idea. The sample mean is the solution to the problem

$$\min_{\mu \in R} \sum_{i=1}^{n} (y_i - \mu)^2. \tag{6}$$

If x is given, the conditional mean of y can be expressed as $\mu(x) = x^T\beta$, and we can eatimate β by solving

$$\min_{\beta \in R} \sum_{i=1}^{n} (y_i - x_i^T\beta)^2. \tag{7}$$

Similarly, since the τth quantile of the sample is the solution to the problem

$$\min_{\alpha \in R} \sum_{i=1}^{n} \rho_\tau(y_i - \alpha), \tag{8}$$

we can then express the τth conditional quantile function as

$$Q_y(\tau|x) = x^T \beta(\tau), \tag{9}$$

and we can get $\hat{\beta}(\tau)$ by solving

$$\min_{\beta \in R^n} \sum_{i=1}^{n} \rho_\tau(y_i - x_i^T \beta). \tag{10}$$

The above content is the central idea in Koenker and Bassett [2].

B. LINEAR PROGRAMMING FORM FOR QUANTILE REGRESSION

As shown in the previous subsection, quantile regression places asymmetric weight on positive and negative residuals, and then solves the modified L_1 problem (10). Our goal is to find an estimation of the coefficient β. Since the objective function is not differentiable, the traditional method of differentiating the objective function is no longer applicable. A feasible method for estimating the parameters β of quantile regresssion is the linear programming method.

Let both u and v denote n dimensional vectors, and $u = [y - X\beta]_+$, $v = [X\beta - y]_+$, where $y = (y_1, y_2, \ldots, y_n)^T$ and $[z]_+$ represents the non-negative part of z, i.e. $(u, v) \in R_+^{2n}$. Then the modified L_1 problem (10) can be rewritten in linear programming form

$$\min_{(u,v,\beta)} O_p^T \beta + \tau e^T u + (1 - \tau e^T v), \tag{11}$$

under the constraints

$$\begin{cases} X^T \beta + u - v = y \\ (u, v) \in R_+^{2n}, \end{cases} \tag{12}$$

where e represents an n dimensional unit vector, and O_p represents p dimensional zero vector.

Let matrix A be expressed as

$$A = [X \vdots I \vdots -I], \tag{13}$$

then linear programming problem (11) and (12) become

$$\min_{(u,v,\beta)} \tau e^T u + (1 - \tau) e^T v$$
$$s.t. \begin{cases} Ab = y \\ (u, v) \in R_+^{2n}, \end{cases} \tag{14}$$

where $b = (\beta^T, u^T, v^T)^T$. According to duality theory, the dual programming of (14) can be written as

$$\max y^T d, \tag{15}$$

subject to

$$A^T d = c, \tag{16}$$

where $d = (d_1, d_2, \ldots, d_n)$, c is a $p + 2n$ dimensional vector

$$c = (0, 0, \ldots, 0, \tau, \tau, \ldots, \tau, (1 - \tau), (1 - \tau), \ldots, (1 - \tau)). \tag{17}$$

After simplication, the dual programming becomes

$$\max y^T d$$
$$s.t. \begin{cases} X^T d = 0 \\ d \in [\tau - 1, \tau]^n. \end{cases} \tag{18}$$

With $a = d + 1 - \tau$, the dual programming can be further simplified to

$$\max y^T a$$
$$s.t. \begin{cases} X^T a = (1 - \tau) X^T e \\ d \in [0, 1]^n, \end{cases} \tag{19}$$

where $[0, 1]^n$ represents the n field Cartesian product of the unit interval. The dual programming formulation accords well with the standard formula of interior point algorithm for linear programming with bounded variables.

III. MULTIPLE CENTRALITY CORRECTIONS TECHNIQUE

The real breakthrough in the study of interior point algorithm was in 1984. Karmarkar [15] proposed a new polynomial-time algorithm for linear programming, which has polynomial computational complexity. Following Karmarkar, there are many variations of interior point algorithms. Among these numerous interior point algorithms, the primal-dual interior point algorithm has been proved theoretically to have polynomial computational complexity, fast convergence and good robustness, so it has become one of the most widely used and efficient algorithms. Among the various primal-dual interior point methods, the predictor-corrector interior point algorithm proposed by Mehrotra [18] is the most applicable and efficient methods both theoretically and computationally.

In the multiple centrality corrections techniques, special attention is paid to restoring the centrality of the next iteration, while increasing the stepsizes in the primal and dual spaces. The multiple centrality corrections techniques achieve two complementary goals: (1) larger step size is achieved in the primal and dual spaces; (2) the centrality of the current iteration has been improved. This multiple centrality corrections technique is considered by many researchers to be one of the most effective changes to the predictor-corrector interior point method [34].

In this paper, we introduce the multiple centrality corrections technique into the interior point algorithm for quantile regression to improve the current interior point algorithm for quantile regression. The computational experiment results constitute evidence of the improvement obtained with the use of multiple centrality corrections technique combined with the interior point algorithm. Next, We briefly describe the predictor-corrector interior point method.

A. PREDICTOR-CORRECTOR INTERIOT POINT ALGORITHM

First, we consider the following linear programming with bounded variables

$$\max_x c^T x \quad s.t. \begin{cases} Ax = b \\ x + s = u \\ x, s \geq 0, \end{cases} \tag{20}$$

where $A \in R^{m\times n}$, $rank(A) = m$, $c, x, s, u \in R^n$ and $b \in R^m$. According to duality theory, the corresponding dual programming of (20) can be written as

$$\max_{(y,z,w)} c^T y - u^T w \quad s.t. \begin{cases} A^T y + z - w = c' \\ z, w \geq 0, \end{cases} \tag{21}$$

where $z, w \in R^n$ and $y \in R^m$. The KKT (Karush-Kuhn-Tucker) conditions associated with (20) and (21) can be expressed as

$$\begin{aligned} Ax &= b \\ x + s &= u \\ A^T y + z - w &= c \\ XZe &= \mu e \\ SWe &= \mu e \end{aligned} \tag{22}$$

where $X = diag(x)$, (that is, if $i = j, X_{i,j} = x_i$, otherwise, $X_{i,j} = 0$), $Z = diag(z)$, $S = diag(s)$ and $W = diag(w)$.

Almost all current codes based on the primal- dual method are performed in a more or less similar way: in each iteration they factor the KKT equations using some direct method, and then solve the predictor term and the corrector term twice in the Newton step [18]. Predictor-corrector interior point algorithms seek the solution of linear programming problem by applying Newton's method to (22). The predictor direction is obtained by solving the following linear equation system with $\mu = 0$

$$\begin{bmatrix} A & 0 & 0 & 0 & 0 \\ I & 0 & I & 0 & 0 \\ 0 & A^T & 0 & I & -I \\ Z & 0 & 0 & X & 0 \\ 0 & 0 & W & 0 & S \end{bmatrix} \begin{bmatrix} \Delta_a x \\ \Delta_a y \\ \Delta_a s \\ \Delta_a z \\ \Delta_a w \end{bmatrix} = \begin{bmatrix} b - Ax \\ u - x - s \\ c - A^T y - z + w \\ \mu\text{e-XZe} \\ \mu e - SWe \end{bmatrix}, \tag{23}$$

where direction Δ_a is also known as the affine-scaling direction, which is responsible for optimization. The maximum feasible affine-scaling primal stepsize (α_P) and dual stepsize (α_D) must ensure that the primal and dual variables stay feasible. The corrector direction Δ_c is the solution of the following linear equation system

$$\begin{bmatrix} A & 0 & 0 & 0 & 0 \\ I & 0 & I & 0 & 0 \\ 0 & A^T & 0 & I & -I \\ Z & 0 & 0 & X & 0 \\ 0 & 0 & W & 0 & S \end{bmatrix} \begin{bmatrix} \Delta_c x \\ \Delta_c y \\ \Delta_c s \\ \Delta_c z \\ \Delta_c w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mu e - \Delta X_a \Delta Z_a e \\ \mu e - \Delta S_a \Delta W_a e \end{bmatrix}, \tag{24}$$

where $\Delta X_a = diag(\Delta_a x)$, $\Delta Z_a = diag(\Delta_a z)$, $\Delta S_a = diag(\Delta_a s)$, $\Delta W_a = diag(\Delta_a w)$ and $\mu > 0$ is the centering parameter, which prevents the complementary products $x_i z_i$ and $s_i w_i$ from converging to zero. The choice of μ can be refered to Mehrotra [18].

The predictor-corrector direction is given by $\Delta = \Delta_a + \Delta_c$, where Δ_a is responsible for reducing primal and dual infeasibilities as well as the duality gap, while Δ_c is responsible for keeping the current iteration away from the boundary of the feasible domain. The iteration will stop when the duality gap is less than a specified tolerance.

B. MULTIPLE CENTRALITY CORRECTIONS TECHNIQUE

In practice, what really reduces the efficiency of the primal-dual interior point algorithm is the huge difference between complementary products. If the order of the complementary products is different, then the complementary products are either too small or too large relative to their average value, which is undesirable. There is a theoretical basis for the view that large difference between complementary products will lead to an iteration away from the region where Newton's method for centering converges fast. This, in practice, is reflected in the fact that only small steps are allowed in the primal and dual spaces.

Jansen *et al.* [35] remedied this problem by defining a series of traceable targets. The sequence goes from an arbitrary internal point to a point near the center path. The algorithm generates iterates that simultaneously get closer to optimality and closer to centrality. Based on Jansen *et al.* [35], Gondzio [27] proposed the multiple centrality corrections technique in a primal-dual method for linear programming, aiming to improve the centrality of the next iteration and achieve larger stepsizes in primal and dual spaces.

Suppose that a predictor direction Δ_p has been determined for a primal solution (x, s) and dual solution (y, z, w) at a given iteration, and that maximum feasible primal stepsize (α_P) and dual stepsize (α_D) to maintain the nonnegativity of primal and dual variables have been computed. We want to find a corrector direction Δ_m, which allows for larger stepsizes $\widetilde{\alpha}_P$ and $\widetilde{\alpha}_D$ in primal and dual spaces for direction $\Delta_p + \Delta_m$.

Assume that the required increase of stepsizes is δ_a, then the stepsizes α_P and α_D expand to

$$\begin{aligned}\widetilde{\alpha}_P &= min(\alpha_P + \delta_a, 1)\\ \widetilde{\alpha}_D &= min(\alpha_D + \delta_a, 1)\end{aligned} \tag{25}$$

The trial point is often defined as

$$\begin{aligned}(\widetilde{x}, \widetilde{s}) &= (x, s) + \widetilde{\alpha}_P(\Delta_P x, \Delta_P s)\\ (\widetilde{y}, \widetilde{z}, \widetilde{w}) &= (y, z, w) + \widetilde{\alpha}_D(\Delta_P y, \Delta_P z, \Delta_P w).\end{aligned} \tag{26}$$

In order to improve the centrality of the next iteration $(\hat{x}, \hat{s}, \hat{y}, \hat{z}, \hat{w})$, a requirement is attached to the Δ_m term, which is to drive the trial point (26) back near the central path. There's a lot of freedom in choosing a target that is close to the central path. One natural guess is to drive the trial point to the analytic center by defining the target as

$$v = (\mu e, \mu e) \in R^{2n}. \tag{27}$$

However, the point (27) is usually an unreachable target. In practice, we usually compute the complementarity products for the trial point (26)

$$\widetilde{v} = (\widetilde{X}\widetilde{z}, \widetilde{S}\widetilde{w}) \in R^{2n}, \tag{28}$$

and then project them on a hypercube $H = [\beta_{min}\mu, \beta_{max}\mu]^{2n}$ to define the target

$$v_t = \pi(\widetilde{v}|H) \in R^{2n}, \tag{29}$$

where $\beta_{min}\mu$ and $\beta_{max}\mu$ are the relative threshold values for outlier complementarity products. Then the corrector direction Δ_m can be computed from

$$\begin{bmatrix} A & 0 & 0 & 0 & 0\\ I & 0 & I & 0 & 0\\ 0 & A^T & 0 & I & -I\\ Z & 0 & 0 & X & 0\\ 0 & 0 & W & 0 & S \end{bmatrix} \begin{bmatrix} \Delta_m x\\ \Delta_m y\\ \Delta_m s\\ \Delta_m z\\ \Delta_m w \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ v_t - \widetilde{v} \end{bmatrix}. \tag{30}$$

The correcting process can be easily repeated the desired number of times. It continues until

$$\hat{\alpha}_P < \alpha_P + r\delta_a \quad or \quad \hat{\alpha}_D < \alpha_D + r\delta_a, \tag{31}$$

where r is the minimum acceptable increase of stepsizes.

The steps of the multiple centrality correction technique can be described as follows:

Step (1): Initialization parameters. Δ_P is the predictor direction (Mehrotra's predictor corrector direction $\Delta = \Delta_a + \Delta_c$); α_P and α_D are the stepsizes along α_P in the primal and dual spaces, respectively; K is the maximum number of corrections allowed; Set corrections counter $k = 0$.

Step (2): If $k < K$, go to Step (3).

Step (3): Computer the trial point (26).

Step (4): Define the target (29).

Step (5): Computer the corrector Δ_m from (30).

Step (6): Perform the ratio test for the composite direction $\Delta = \Delta_p + \Delta_m$. If (31) is not satisfied, then k = k + 1, $\Delta_P = \Delta$. Otherwise, $\Delta = \Delta_P$ and terminate corrections.

The algorithm proposed in this paper introduces multiple centrality corrections technique into the interior point algorithm for quantile regression. It has the following advantages. First, it uses real objects and trial points whose centrality quality can be measured at low cost: if the latter is found unsatisfactory, the point is reliably corrected. What we want to emphasize here is that the main purpose of the revision is to improve centrality. Another advantage of this method is that the calculation of each corrector term requires the same effort.This effort is determined by the solution of the equation system (30). The remaining operations, that is, calculating the trial point in the complementary product space, projecting them onto H, calculating the new direction Δ, performing ratio tests on it, doesn't contribute much.

IV. COMPUTATIONAL EXPERIMENTS

At present, computational methods for quantile regression mainly include simplex algorithm, interior point method, smoothing method and proximal algorithm. Each of them has its own advantages. None of them can fully dominate the others. In this paper, we introduce the multiple centrality corrections techniques to the interior point method for quantile regression to improve the current interior point method for quantile regression. Therefore, in this paper, only the comparison between the interior point method combing the multiple centrality corrections techniques for quantile regression and the current interior point method for quantile regression is given.

In this section, we present the computational experiments performed when incorporating the multiple centrality corrections technique in the predictor-corrector interior point algorithm for quantile regression. Three experiments are constructed to verify the performance of the modified interior point algorithm for quantile regression. The computational experiments were run on a Lenovo G470 with 2.50 GHz and 10240MB RAM, with Windows 7 operating system. All solution times in this article are in seconds.

A. TEST DATA

PCx [36] is a linear programming solver that implements a variant of predictor-corrector interior point algorithm with multiple centrality corrections technique. The results reported in this article were obtained by PCx. MPS (Mathematical Programming System) is a file format for presenting and achieving linear programming and mixed integer programming problems. Almost all commercial LP solvers accept this format, as does PCx. Therefore, in this article, we first convert quantile regression problem into a linear programming form, then use MPS format file to represent the linear programming, and finally use PCx to solve it.

Portnoy and Koenker's approach [12] was designed for "long, thin problems", that is for problems with large sample size, but only a relatively small number of parameters. When the parametric dimension of the model is large, the

original implementation of interior point methods can be quite slow [37]. Therefore, without loss of generality, in this article, we take the parameter dimension $p = 5$ as an example.

In this section, we illustrate the performance of the proposed modified interior point algorithm for quantile regression by comparing it with Portnoy and Koenker's interior point method [12] for quantile regression through different simulations. In three computational experiments in this article, data were drawn from Gaussian distribution, t distribution and Chi-square distribution, respectively.

B. OBTAINED RESULTS

To obtain a better sense of the performance of the predictor-corrector interior point algorithm with multiple centrality corrections technique for quantile regression. we present three computational experiments performed when incorporating multiple centrality corrections technique in predictor-corrector interior point algorithm for quantile regression.

1) EXPERIMENT 1

In this experiment, we selected the observed values of samples subject to Gaussian distribution. In the following Figures 1 to 5, we give the solution time using the predictor-corrector interior point algorithm with multiple centrality corrections technique for quantile regression at quantiles 0.1, 0.25, 0.5, 0.75 and 0.9, respectively. For comparison purpose, we also provide the solution time using the predictor-corrector interior point algorithm [12] for quantile regression. These two algorithms are represented by "mcc" and "pc", respectively. The difference between the two algorithms lies in the multiple centrality corrections technique of Gondzio [27], and the other settings in PCx are used by default.

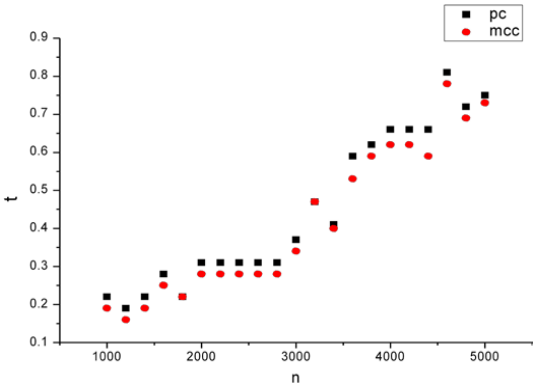

FIGURE 1. Timing comparison of two algorithms for quantile regression: times are in seconds for Gaussian data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.1$.

In Figures 1 to 5, timing comparison of two algorithms at quantiles 0.1, 0.25, 0.5, 0.75 and 0.9 is presented, respectively. When $\tau = 0.1$, according to the solution time of the two algorithms, mcc shows reductions in solution time in 19 test problems, of which, 14 problems display reduction

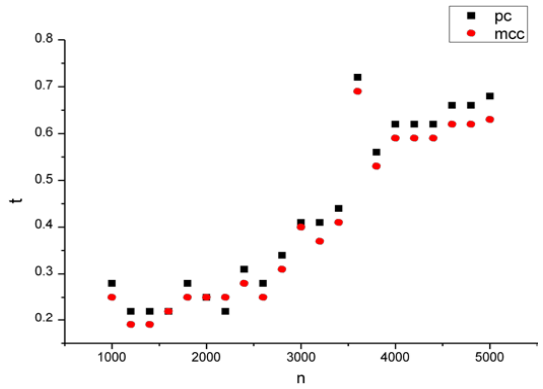

FIGURE 2. Timing comparison of two algorithms for quantile regression: times are in seconds for Gaussian data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.25$.

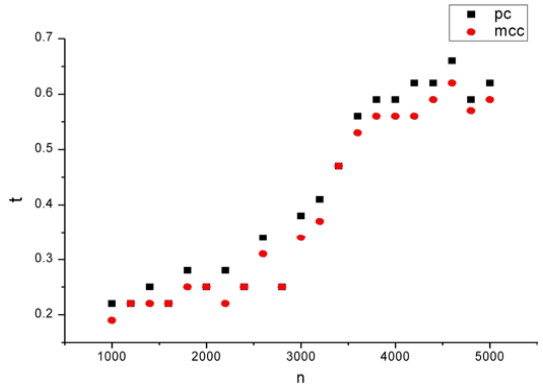

FIGURE 3. Timing comparison of two algorithms for quantile regression: times are in seconds for Gaussian data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.5$.

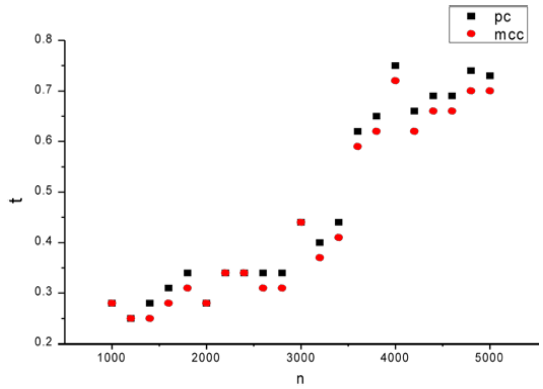

FIGURE 4. Timing comparison of two algorithms for quantile regression: times are in seconds for Gaussian data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.75$.

rates greater than 5%, the largest reduction is observed at design point $n = 1200$, corresponding to a reduction rate of approximately 15.8%. We can see very intuitively from Figure 1, there are 19 black squares above the corresponding red circles. When $\tau = 0.25$, mcc shows reductions in solution time in 18 test problems, of which, 13 problems display reduction rates greater than 5%. The largest reduction in

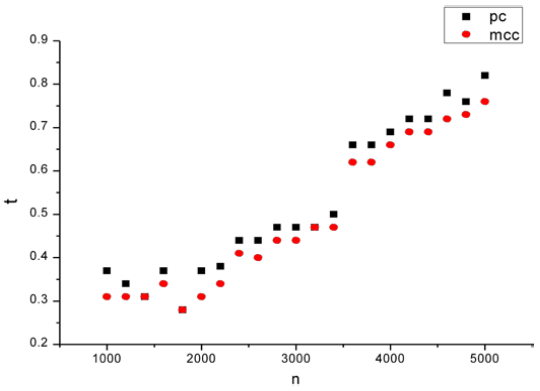

FIGURE 5. Timing comparison of two algorithms for quantile regression: times are in seconds for Gaussian data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.9$.

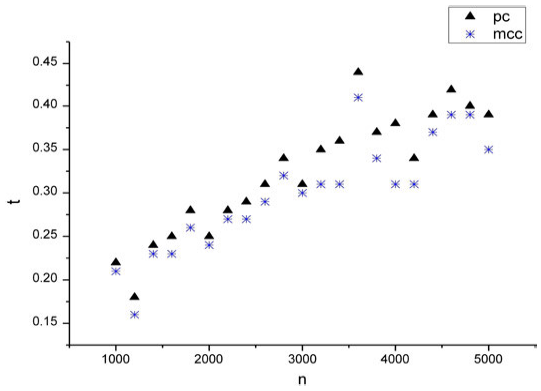

FIGURE 6. Timing comparison of two algorithms for quantile regression: times are in seconds for t data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.1$.

solution time is observed at design points in n : 1200, 1400, corresponding to a reduction rate of approximately 13.6%. Only when $n = 2200$, mcc shows increment in solution time. As shown in Figure 2, the red circle is above the corresponding black square when $n = 2200$. When $\tau = 0.5$ mcc shows reductions in solution time in 15 test problems, of which, 11 problems display reduction rates greater than 5%. The largest reduction in solution time is observed at $n = 2200$, corresponding to a reduction rate of approximately 21.4%. As can be seen from Figure 4, there are 15 black squares above the corresponding red circles. This indicates that mcc gets reductions in solution time in 15 test problems. When $\tau = 0.9$, mcc shows reductions in solution time in 18 test problems. The largest reduction in solution time is observed at design point $n = 1000$, corresponding to a reduction rate of approximately 16.2%. As shown in Figure 5, there are 18 black squares above the corresponding red circles. At the remaining three design points, the black squares and the corresponding red circles coincide, indicating that the solution time required by the two algorithms is the same at these three design points.

2) EXPERIMENT 2

In this experiment, we selected the observed values of samples subject to Student t distribution. Similar to the previous Experiment 1, in the following Figures 6 to 10, the comparison of solution time of the two algorithms at quantiles 0.1, 0.25, 0.5, 0.75 and 0.9 is presented, respectively.

As shown in Figure 6, there are 21 black UpTriangles above the corresponding blue asterisks, indicating that mcc shows reductions in solution time in 21 test problems. When $\tau = 0.1$, according to the solution time of the two algorithms, 14 problems display reduction rates greater than 5%, the largest reduction is observed at $n = 4000$, corresponding to a reduction rate of approximately 18.4%. When $\tau = 0.25$, mcc shows reductions in solution time in 21 test problems, of which, 16 problems display reduction rates greater than 5%. The largest solution time reduction rate is approximately 19.4%. From Figure 7, we can see very intuitively

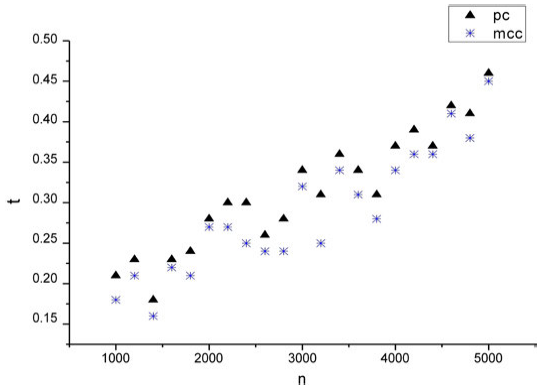

FIGURE 7. Timing comparison of two algorithms for quantile regression: times are in seconds for t data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.25$.

that all black UpTriangles are above the corresponding blue asterisks, which indicates that mcc takes less solution time. As can be seen from Figure 8, all black UpTriangles are above the corresponding blue asterisks, this indicates that mcc gets reduction in solution time in 21 test problems. When $\tau = 0.5$, according to the solution time of the

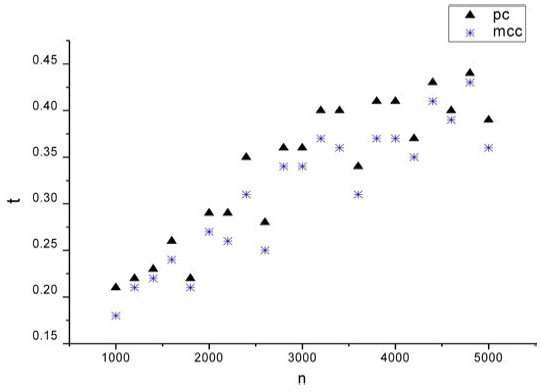

FIGURE 8. Timing comparison of two algorithms for quantile regression: times are in seconds for t data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.5$.

two algorithms, the largest reduction in solution time is observed at design point $n = 1000$, corresponding to a reduction rate of approximately 14.3%. Similarly, in Figure 9 and 10, all black UpTriangles are above the corresponding blue asterisks. The maximum solution time reduction rate is 9.5% and 12.8% respectively.

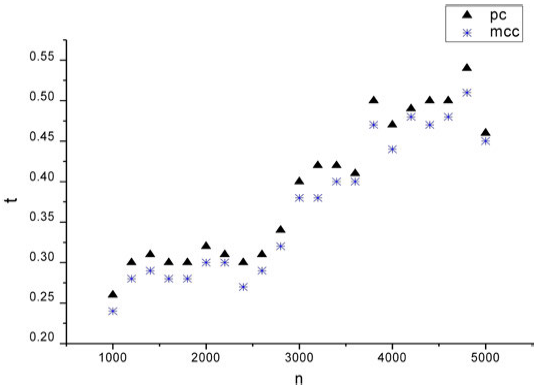

FIGURE 9. Timing comparison of two algorithms for quantile regression: times are in seconds for t data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.75$.

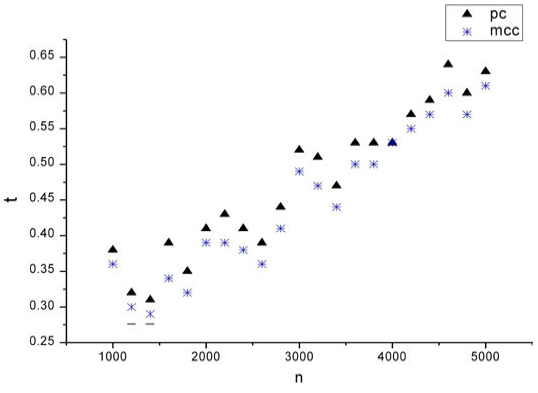

FIGURE 10. Timing comparison of two algorithms for quantile regression: times are in seconds for t data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.9$.

3) EXPERIMENT 3

In this experiment, we selected the observed values of samples subject to Chi-square distribution. Similar to the previous experiments 1 and 2, in the following Figures 11 to 15, we give the comparison of the results obtained by the two algorithms at quantiles 0.1, 0.25, 0.5, 0.75 and 0.9, respectively.

In Figures 11 to 15, we present a comparison of the solution time of the two algorithms for Chi-square data at quantiles 0.1, 0.25, 0.5, 0.75 and 0.9, respectively. When $\tau = 0.1$, according to the solution time of the two algorithms, mcc shows reductions in solution time in 19 test problems, of which, 15 problems display reduction rates greater than 5%, the largest reduction is observed at design point

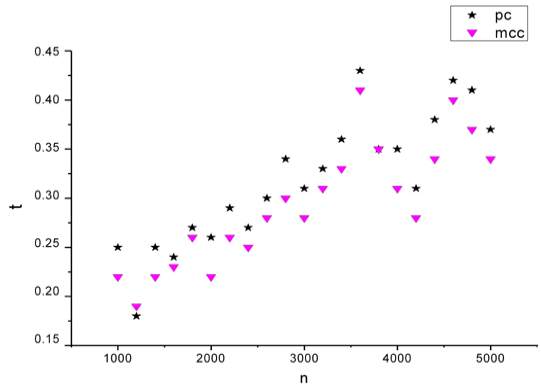

FIGURE 11. Timing comparison of two algorithms for quantile regression: times are in seconds for Chi-square data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.1$.

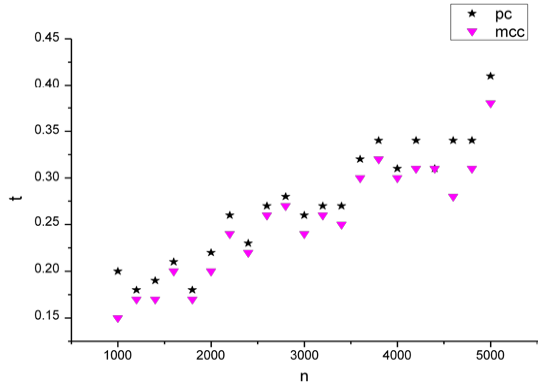

FIGURE 12. Timing comparison of two algorithms for quantile regression: times are in seconds for Chi-square data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.25$.

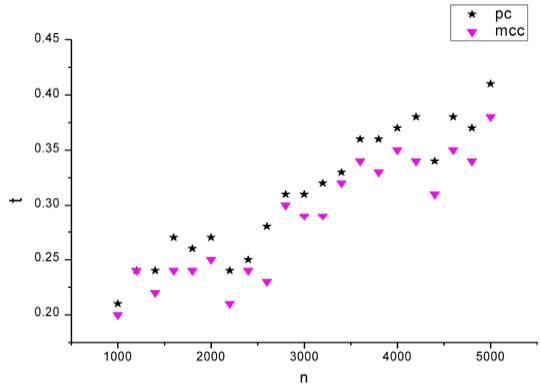

FIGURE 13. Timing comparison of two algorithms for quantile regression: times are in seconds for Chi-square data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.5$.

$n = 2000$, corresponding to a reduction rate of approximately 15.4%. From Figure 11, we can see very intuitively that there are 19 black stars are above the corresponding magenta DownTriangles. In Figure 11, when $n = 3800$, the black star coincides with the corresponding magenta DownTriangle, and the solution time of the two algorithms is the same; when $n = 1200$, the magenta DownTriangle

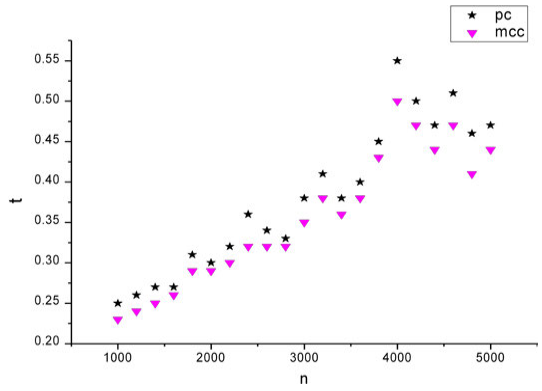

FIGURE 14. Timing comparison of two algorithms for quantile regression: times are in seconds for Chi-square data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.75$.

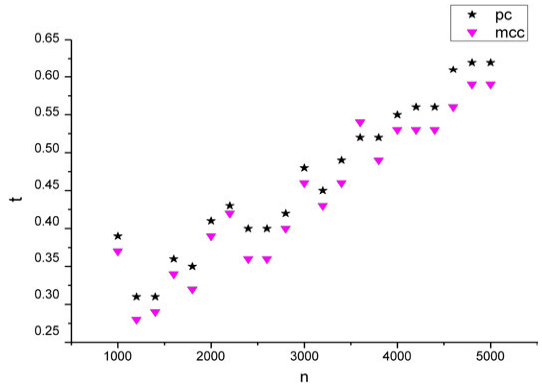

FIGURE 15. Timing comparison of two algorithms for quantile regression: times are in seconds for Chi-square data. Timings were made at 21 design points in n : 1000, 1200, . . . , 4800, 5000, $\tau = 0.9$.

is above the corresponding black star, pc takes less solution time. When $\tau = 0.25$, mcc shows reductions in solution time in 20 test problems, of which, 14 problems display reduction rates greater than 5%. The largest reduction in solution time is observed at design point in $n = 1000$, corresponding to a reduction rate of approximately 25%. As shown in Figure 12, there are 20 black stars above the corresponding magenta DownTriangles. When $\tau = 0.5$, mcc shows reductions in solution time in 20 test problems, of which, 16 problems display reduction rates greater than 5%. The largest reduction in solution time is observed at $n = 2600$, which is approximately 17.9%. As shown in Figure 14, all black stars are above the corresponding magenta DownTriangles. When $\tau = 0.75$, the maximum solution time reduction rate is 11.1%. When $\tau = 0.9$, mcc shows reductions in solution time in 20 test problems, of which, 12 problems display reduction rates greater than 5%, the largest solution time reduction rate is 10.0%. As shown in Figure 15, there are 20 black stars above the corresponding magenta DownTriangles.

V. CONCLUSION

In this paper, we introduce the multiple centrality corrections technique to predictor–corrector interior point algorithm for quantile regression. Multiple centrality corrections technique can increase the step sizes in primal and dual spaces and improve the centrality of the current iteration. The computational experiments results show that compared with the previous interior point method for quantile regression, the calculation time required to calculate the quantile regression problem is reduced after the introduction of the multiple centrality corrections technique. This proves the superiority of using multiple centrality corrections technique combined with interior point algorithm for quantile regression.

Due to the numerous calculation methods of quantile regression, the improvement techniques of the quantile regression calculation method proposed in this paper are still very limited. Therefore, using more abundant methods to improve the efficiency of the algorithms for quantile regression is the focus of future research. In addition, applying quantile regression to different practical fields is also the focus of future research.

ACKNOWLEDGMENT

The authors would like to thank the editor and reviews for their valuable suggestions to improve the quality of this manuscript.

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