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Dynamics and Chaos Control for a Discrete-Time Lotka-Volterra Model

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ABSTRACT Bifurcation theory (center manifold and Ljapunov–Schmidt reduction, normal form theory, universal unfolding, calculation of bifurcation diagrams) has become an important and very useful means in the solution of nonlinear stability problems in many branches of engineering. The present study deals with qualitative behavior of a two-dimensional discrete-time system for interaction between prey and predator. The discrete-time model has more chaotic and rich dynamical behavior as compare to its continuous counterpart. We investigate the qualitative behavior of a discrete-time Lotka-Volterra model with linear functional response for prey. The local asymptotic behavior of equilibria is discussed for discrete-time Lotka-Volterra model. Furthermore, with the help of bifurcation theory and center manifold theorem, explicit parametric conditions for directions and existence of flip and Hopf bifurcations are investigated. Moreover, two chaos control methods, that is, OGY feedback control and hybrid control strategy, are implemented. Numerical simulations are provided to illustrate theoretical discussion and their effectiveness.

INDEX TERMS Lotka-Volterra model, stability, flip bifurcation, Hopf bifurcation, chaos control.

I. INTRODUCTION AND PRELIMINARIES

For the many different deterministic non-linear dynamic systems (physical, mechanical, technical, chemical, ecological, economic, and civil and structural engineering), the discovery of irregular vibrations in addition to periodic and almost periodic vibrations is one of the most significant achievements of modern science. An in-depth study of the theory and application of non-linear science will certainly change one's perception of numerous non-linear phenomena and laws considerably, together with its great effects on many areas of application. As the important subject matter of non-linear science, bifurcation theory, singularity theory and chaos theory have developed rapidly in the past two or three decades. They are now advancing vigorously in their applications to mathematics, physics, mechanics and many technical areas worldwide, and they will be the main subjects of our concern.

Various type of interactions such as predation, mutualism, competition and parasitism are found in biological species. Predator-prey interaction is one of these relationships. Due to universal existence and involvement in daily life, prey-predator systems play key role in ecology and mathematical biology. The classical Lotka-Volterra mathematical model was first proposed by Lotka [1] and Volterra [2]. Later on the classical Lotka-Volterra model was modified and extended by including density-dependent prey growth functions and implementing various type of functional responses [3].

The intake rate for a predator as a function of food density is called a functional response in ecology [4]. Moreover, the dynamical behavior of predator-prey model is affected due to implementation of a certain functional response [5]. Arguing as in [6], in general there are three main types of functional responses, which are called Holling's type I, II, and III. For further information related to these three Holling type functional responses we refer to [3]. In case of overlapping generations, predator-prey interaction is described by differential equations.

On the other hand, difference equations are used to discuss the predator-prey models with non-overlapping generations. Furthermore, there is a greater chance of complex, chaotic

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and bifurcating behavior of discrete-time models as compare to their continuous counterparts. Many researchers have studied stability of fixed points, Neimark-Sacker bifurcation, period-doubling bifurcation, and chaos control of predator-prey systems which are governed by difference equations. For example, [4] Ghaziani *et al*. investigated Hopf and flip bifurcations for a discrete-time predator-prey model with Holling type-II functional response. Flip and Hopf bifurcations are discussed in [7] for a discrete version of a predator-prey model. A non-standard finite difference scheme was used by Mickens [8] for a class of prey-predator system. Agiza *et al*. [9] discussed complex chaotic dynamics for a predator-prey system with Holling type-II functional response. Murakami [10] studied stability and Neimark-Sacker bifurcation for a discrete prey-predator model. Moreover, Din [11] performed chaos control and bifurcation analysis for a prey-predator system of Leslie-Gower type. Similarly, Hopf bifurcation, flip bifurcation, and chaos control are investigated in [12] for a model of a discrete predator–prey system with Allee effects.

Zhao *et al*. [13] reported flip bifurcation and Neimark-Sacker bifurcation for a class of discrete prey-predator interaction. Singh *et al*. [14] explored local dynamics and bifurcation analysis for a class of Leslie-Gower type predator-prey model with predator partially dependent on prey. Asheghi [15] discussed plant-herbivore type predator-prey interaction for non-overlapping generations. Ghaziani *et al*. [4] reported bifurcating behavior for a discrete predator–prey model with Holling functional response. Moreover, piecewise constant argument is implemented in [16] to investigate bifurcation and chaos control for a predator-prey system. Keeping in view the herd behavior for prey population, Salman *et al*. [17] implemented a functional response of square root type in order to discuss bifurcation and chaos control for a discrete prey-predator model. Chen *et al*. [18] studied bifurcation and chaos control for a prey-predator system with implementing Holling functional response of type-IV. Crowley–Martin functional response was implemented by Ren *et al*. [19] in order to investigate Hopf bifurcation, flip bifurcation, and chaos for a discrete prey-predator model. Li and Wang [20] discussed global behavior and existence of periodic solutions for a discrete predator–prey system. Matouk and Elsadany [21] explored bifurcating behavior for a generalized predator-prey system. Gu and Huang [22] investigated global bifurcations of a discrete predator–prey system.

For similar type of investigations, we refer to [23]–[29] and references therein. Recently, Ma *et al*. [30] studied stability, bifurcation and chaos control for a host-parasitoid system with a Beverton-Holt growth function for a host population and Hassell-Varley framework. Shabbir *et al*. [31] proposed a new class of prey-predator interaction for non-overlapping generations with implementation of cannibalism in prey population.

The classical Lotka-Volterra prey-predator model with linear functional response of prey is governed by the following nonlinear differential system [32]:

$$\begin{aligned} \frac{dN(\tau)}{d\tau} &= N(\tau)\left(\alpha - \beta N(\tau) - \gamma P(\tau)\right), \\ \frac{dP(\tau)}{d\tau} &= P(\tau)\left(\gamma\delta N(\tau) - D\right), \end{aligned} \tag{1}$$

where $N(\tau)$ and $P(\tau)$ are population densities of prey and predator, respectively, at time τ, α denotes maximum per capita growth rate for prey species, β represents the strength for intra-specific competition of prey population, γ represents the strength for intra-specific competition between prey and its predator, δ denotes the conversion rate of prey into predator, and per capita death rate of predator species is represented by D. Moreover, all these parameters are positive constants.

The non-dimensional form of (1) can be obtained by using the following transformations:

$$x(t) = \frac{\alpha\delta N(\tau)}{D}, \quad y(t) = \frac{\gamma P(\tau)}{D}, \ t = D\tau.$$

Furthermore, introducing the new parameters $a = \frac{\alpha}{D}$, $b = \frac{\beta}{\alpha\delta}$ and $c = \frac{\gamma}{\alpha}$, we obtain the following non-dimensional form of (1):

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left(a - bx(t) - y(t)\right), \\ \frac{dy(t)}{dt} &= y(t)\left(cx(t) - 1\right), \end{aligned} \tag{2}$$

where a, b and c are positive constants.

In genetics of population overlapping generations alludes to breeding systems in which more than one mating generation is present at any one time, and differential equations are appropriate for modeling of such interactions. On the other hand, in the system in which this is not the case there must be non–overlapping generations (or discrete-time generations) where every mating generation ends just one breeding season, and difference equations or discrete counterparts of differential equations are suitable for such interactions. Overlapping generations are considered normal rather than exceptional. Generations of overlapping type are found in species which survive for several years, and procreate several times. Generations of non–overlapping type are detected in many species where the adult generation lasts after one breeding season. The examples of non-overlapping generations are univoltine insects, and some annual plants. On the other hand, synchronized life cycles with non–overlapping of generations in both prey and predator suggest that the discrete–time model is suitable for describing prey–predator interaction. It is familiar that there are two methods for construction of discrete–time systems related to population dynamics starting from a continuous model governed by differential equations. The first method consists of procedure to follow the mechanism rudimentary in the continuous system and formulating a discrete–time model which directly demonstrates the phenomenon, and the second instead is inspired by the discretization of the continuous system. In most of the cases second method is frequently used to study rich dynamics, chaotic behavior, bifurcation analysis and chaos control for discrete-time models. Piecewise constant arguments and Euler forward approximations are most frequently used numerical schemes to obtain discrete–time counterparts of predator–prey models. In this paper, we implement piecewise constant arguments method for the discretization of system (2). The coming investigation reveals that piecewise constant arguments method is more appropriate to discuss rich dynamics, chaotic behavior, bifurcation analysis and chaos control for discrete counterpart of system (2). According to [33], [34] piecewise constant arguments method is a better choice for discretization of continuous classes of predator-prey interaction.

In [32], Elsadany and Matouk studied fractional-order discretized counterpart of (2). They discussed local dynamics of equilibria and Neimark-Sacker bifurcation is demonstrated through implementation of numerical simulations only. On the other hand, theoretical discussions related to Neimark-Sacker bifurcation, period-doubling bifurcation and chaos control are clearly missing in [32]. In order to compensate this deficiency, we apply piecewise constant arguments method to obtain a simple looking discrete-time model for non-overlapping generations but with comprehensive theoretical discussion.

Arguing as in [11], [12], [16], [35], one can apply the piecewise constant arguments method for differential equations to obtain the following discrete-time predator-prey system:

$$\begin{aligned} x_{n+1} &= x_n e^{a-bx_n-y_n}, \\ y_{n+1} &= y_n e^{cx_n-1}, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{3}$$

Moreover, related to parametric selection in system (3) for numerical simulation the following Remark is stated:

Remark 1: The parameter values for system (3) are not from some field studies. We choose the parameter values to illustrate the dynamics, bifurcations and chaos control of system (3) in numerical simulation section.

We investigate qualitative behavior of model (3) including existence of equilibria and their local stability, Hopf bifurcation, flip bifurcation and chaos control. The main contribution and novelty of this article are summarized as follows:

- A novel discrete-time model for a class of predator-prey interaction is obtained via implementation of piecewise constant arguments method.
- The obtained model is a good representative of predator-prey interaction for non-overlapping generations.
- Local stability analysis for steady-states of discrete-time model is carried out.
- It is demonstrated that system (3) undergoes two types of bifurcations (period-doubling bifurcation and Neimark-Sacker bifurcation) at its interior fixed point.
- Hybrid control method and OGY control strategy are implemented for controlling chaotic and bifurcating behavior of system (3).

Furthermore, in Section II, we investigate local asymptotic behavior of steady-states for the model (3). With the implementation of center manifold theorem and bifurcation theory of normal forms, flip bifurcation is discussed at positive fixed point of system (3) in Section III. Neimark-Sacker bifurcation is performed is Section IV for positive steady-state of system (3). OGY feedback and hybrid control methods are introduced in Section V. At the end, numerical simulations are provided in Section VI in order to illustrate our theoretical discussion.

II. LOCAL STABILITY OF STEADY-STATES

In order to obtain the steady-states of (3), we consider the following two-dimensional algebraic system:

$$x = x\exp(a - bx - y), \quad y = y\exp(cx - 1). \tag{4}$$

Then, there are three solutions of (4), namely, $O = (0, 0)$, $E = \left(\frac{a}{b}, 0\right)$ and $P = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$, which are three equilibria of system (3). Next, we assume that $ac > b$, then P is the unique positive equilibrium point of system (3). Furthermore, denote $J(O)$ as variational matrix of (3) evaluated at trivial equilibrium point O, then $J(O)$ is computed as follows:

$$J(O) = \begin{bmatrix} e^a & 0 \\ 0 & \frac{1}{e} \end{bmatrix},$$

which reveals that $O = (0, 0)$ is a saddle equilibrium point. Next, assume that $J(E)$ denotes the variational matrix of (3) evaluated at boundary equilibrium point E, then we have

$$J(E) = \begin{bmatrix} 1 - a & -\frac{a}{b} \\ 0 & e^{\frac{ac}{b}-1} \end{bmatrix}.$$

Then, the following Lemma gives the dynamics for the boundary equilibrium point E of system (3).

Lemma 1: For steady-state $E = \left(\frac{a}{b}, 0\right)$ the following statements holds true:

(a) $E = \left(\frac{a}{b}, 0\right)$ is a repeller (source) if and only if $a > 2$ and $ac > b$.

(b) $E = \left(\frac{a}{b}, 0\right)$ is a saddle point if and only if $0 < a < 2$ and $ac > b$, or $a > 2$ and $ac < b$.

(c) $E = \left(\frac{a}{b}, 0\right)$ is non-hyperbolic point if and only if $a = 2$, or $ac = b$.

(d) $E = \left(\frac{a}{b}, 0\right)$ is a sink if and only if $0 < a < 2$ and $ac < b$.

For $c = 1.9$, $a \in [0, 5]$ and $b \in [0, 8]$ topological classification of boundary fixed point $E = \left(\frac{a}{b}, 0\right)$ is depicted in Fig. 1. Next, we suppose that $ac > b$, then variational matrix $J(P)$ of (3) at steady-state $P = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ is computed as follows:

$$J(P) = \begin{bmatrix} 1 - d\frac{b}{c} & -\frac{1}{c} \\ ac - b & 1 \end{bmatrix}.$$

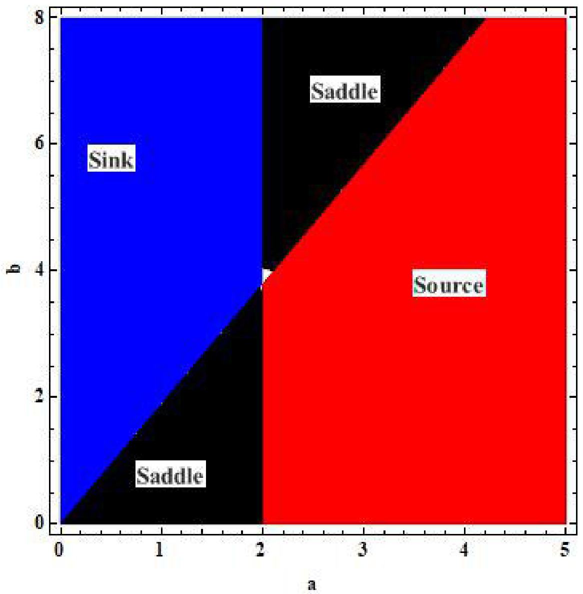

FIGURE 1. Topological classification of $E = \left(\frac{a}{b}, 0\right)$ at $c = 1.9$, $a \in [0, 5]$ and $b \in [0, 8]$.

On the other hand, one can compute characteristic polynomial for $J(P)$ as follows:

$$\mathbb{P}(\lambda) = \lambda^2 - \left(2 - \frac{b}{c}\right)\lambda + 1 + a - \frac{2b}{c}. \quad (5)$$

Keeping in view the relations between coefficients and roots of a quadratic equation, we have the following Lemma [36]–[41]:

Lemma 2: Consider $\mathbb{P}(\tau) = \tau^2 - \alpha\tau + \beta$ a quadratic polynomial with $\mathbb{P}(1) > 0$, and τ_1, τ_2 represent roots for $\mathbb{P}(\tau) = 0$, then the following conditions are satisfied:

(*a*) $|\tau_1| < 1$ and $|\tau_2| < 1$ if and only if $\mathbb{P}(-1) > 0$ and $\mathbb{P}(0) < 1$.

(*b*) $|\tau_1| < 1$ and $|\tau_2| > 1$, or $|\tau_1| > 1$ and $|\tau_2| < 1$ if and only if $\mathbb{P}(-1) < 0$.

(*c*) $|\tau_1| > 1$ and $|\tau_2| > 1$ if and only if $\mathbb{P}(-1) > 0$ and $\mathbb{P}(0) > 1$.

(*d*) $\tau_1 = -1$ and $|\tau_2| \neq 1$ if and only if $\mathbb{P}(-1) = 0$ and $\mathbb{P}(0) \neq \pm 1$.

(*e*) τ_1 and τ_2 are conjugate complex numbers with $|\tau_1| = |\tau_2| = 1$ if and only if $\alpha^2 - 4\beta < 0$ and $\mathbb{P}(0) = 1$.

On the other hand, from (5) it follows that $\mathbb{P}(1) = \frac{ac-b}{c} > 0$ because $ac > b$. Thus, one can apply Lemma 2 to prove the following results for positive fixed point of system (3).

Lemma 3: Assume that $ac > b$, then the following statements hold true:

(i) The steady-state $P = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ of (3) is a sink if and only if

$$0 < a \leq 2, \quad \frac{b}{a} < c < \frac{2\,b}{a},$$

or

$$2 < a < 8, \quad \frac{3\,b}{4+a} < c < \frac{2\,b}{a}.$$

(ii) $P = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ is a saddle point if and only if

$$a > 2 \quad \text{and} \quad \frac{b}{a} < c < \frac{3\,b}{4+a}.$$

(iii) $P = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ is a source or repeller if and only if

$$0 < a \leq 8, \quad c > \frac{2\,b}{a},$$

or

$$a > 8, \quad c > \frac{3\,b}{4+a}.$$

(iv) Assume that λ_1 and λ_2 be distinct real roots of (5), then $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if the following conditions are satisfied:

$$a \in \left(2, \infty\right) \setminus \{8\}, \quad \text{and } c = \frac{3\,b}{4+a}.$$

(v) Suppose that λ_1 and λ_2 be conjugate complex roots of (5), then $|\lambda_1| = |\lambda_2| = 1$ if the following conditions are satisfied:

$$0 < a < 8, \; c = \frac{2b}{a}.$$

For $c = 1.5$, $a \in [0, 8]$ and $b \in [0, 8]$ topological classification of positive fixed point $P = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ is depicted in Fig. 2, in which blue, red and green regions represent sink, source and saddle, respectively. Furthermore, yellow region represents where positive fixed point does not exist, that is, $ac < b$.

III. PERIOD-DOUBLING BIFURCATION

This section deals with period-doubling bifurcation about positive fixed point of (3). For this, bifurcation theory of normal forms and center manifold theorem [42] are implemented

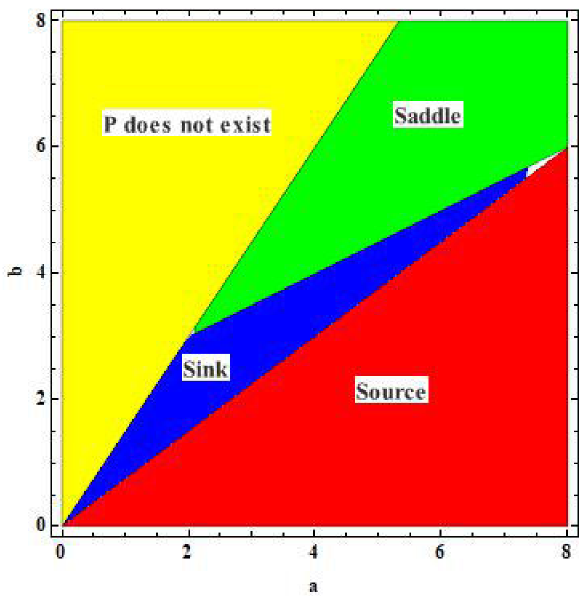

FIGURE 2. Topological classification of $P = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ at $c = 1.5$, $a \in [0, 8]$ and $b \in [0, 8]$.

in order to discuss parametric conditions for existence and direction of flip bifurcation at fixed point $P = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ of system (3). We assume that

$$\left(2 - \frac{b}{c}\right)^2 - 4\left(1 + a - \frac{2b}{c}\right) > 0. \tag{6}$$

Then, it follows that characteristic polynomial (5) has distinct real roots, say λ_1 and λ_2. Furthermore, we suppose that

$$c \equiv c_0 = \frac{3\,b}{a+4}.$$

Then, roots of characteristic polynomial (5) are $\lambda_1 = -1$ and $\lambda_2 = \frac{5-a}{3}$. Moreover, $|\lambda_2| \neq 1$ under the restriction that $a \neq 2,\ 8$. Next, we consider the following set

$$\Omega_{FB} = \left\{(a, b, c_0) : a \in \left(2, \infty\right) \setminus \{8\}, \quad \text{and} \quad c_0 = \frac{3\,b}{4+a},\ b > 0\right\}.$$

Suppose that $(a, b, c_0) \in \Omega_{FB}$, then the system (3) is expressed as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} X \exp\left(a - bX - Y\right) \\ Y \exp\left(c_0 X - 1\right) \end{pmatrix}. \tag{7}$$

Denote by $\bar{c}$ as bifurcation parameter with $|\bar{c}| \ll 1$. Then, for the aforementioned map (7), we have the following corresponding perturbed mapping:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} X \exp\left(a - bX - Y\right) \\ Y \exp\left((c_0 + \bar{c})X - 1\right) \end{pmatrix}. \tag{8}$$

Then, the unique positive fixed point for the map (8) is given by $\left(\frac{1}{c_0+\bar{c}}, \frac{a(c_0+\bar{c})-b}{c_0+\bar{c}}\right)$. Next, we consider the transformations $x = X - x^*$ and $y = Y - y^*$, where $x^* = \frac{1}{c_0+\bar{c}}$ and $y^* = \frac{a(c_0+\bar{c})-b}{c_0+\bar{c}}$, then we get the following map whose fixed point is shifted at origin (0, 0):

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{bmatrix} 1 - \frac{b}{c_0} & -\frac{1}{c_0} \\ ac_0 - b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{pmatrix} f_1(x, y, \bar{c}) \\ f_2(x, y, \bar{c}) \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} f_1(x, y, \bar{c}) &= \left(\frac{b(b-2c_0)}{2c_0}\right)x^2 + \left(\frac{b-c_0}{c_0}\right)xy \\ &\quad + \frac{1}{2c_0}y^2 + \frac{1}{c_0^3}\bar{c}^2 + \frac{b}{c_0^2}\bar{c}x + \frac{1}{c_0^2}\bar{c}y \\ &\quad - \left(\frac{b^2(b-3c_0)}{6c_0}\right)x^3 \\ &\quad - \left(\frac{b(b-2c_0)}{2c_0}\right)x^2y - \left(\frac{b-c_0}{2c_0}\right)xy^2 \\ &\quad - \frac{1}{6c_0}y^3 - \frac{b}{c_0^3}x\bar{c}^2 - \frac{1}{c_0^3}y\bar{c}^2 - \frac{b^2}{2c_0^2}x^2\bar{c} \\ &\quad - \frac{1}{2c_0^2}y^2\bar{c} - \frac{b}{c_0^2}xy\bar{c} \\ &\quad - \frac{1}{c_0^4}\bar{c}^3 + O\left((|x| + |y| + |\bar{c}|)^4\right), \end{aligned}$$

and

$$\begin{aligned} f_2(x, y, \bar{c}) &= \frac{1}{2}c_0(ac_0 - b)x^2 + c_0xy - \frac{b}{c_0^3}\bar{c}^2 \\ &\quad + a\bar{c}x + \frac{1}{6}c_0^2(ac_0 - b)x^3 + \frac{1}{2}c_0^2\,x^2y \\ &\quad + \left(ac_0 - \frac{1}{2}b\right)x^2\bar{c} + xy\bar{c} + \frac{b}{c_0^4}\bar{c}^3 \\ &\quad + O\left((|x| + |y| + |\bar{c}|)^4\right). \end{aligned}$$

Consider the transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} u \\ v \end{pmatrix}, \tag{10}$$

where

$$T = \begin{bmatrix} -\frac{1}{c_0} & -\frac{1}{c_0} \\ \frac{b}{c_0} - 2 & \frac{b}{c_0} - 1 + \frac{5-a}{3} \end{bmatrix}$$

be a non-singular matrix. Then, from (9) and (10), it follows that

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \frac{5-a}{3} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_3(u, v, \bar{c}) \\ f_4(u, v, \bar{c}) \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} &f_3(u, v, \bar{c}) \\ &= \left(\frac{b(2a^3\bar{c} - 6a^2b + 9a^2\bar{c} - 30ab + 48a\bar{c})}{2(8-a)(a+4)^2}\right)x^2 \\ &\quad + \left(\frac{b(84b + 176\bar{c})}{2(8-a)(a+4)^2}\right)x^2 \\ &\quad + \left(\frac{2a^2\bar{c} - 6ab + 13a\bar{c} - 15b + 20\bar{c}}{(8-a)(a+4)}\right)xy \\ &\quad + \left(\frac{a\bar{c} - 3b + 4\bar{c}}{b(8-a)}\right)y^2 \\ &\quad + \left(\frac{b^3(a^3 + 3a^2 - 33a - 62)}{(8-a)(a+4)^3}\right)x^3 \\ &\quad + \left(\frac{3b^2(2a^2 + 4a - 25)}{2(8-a)(a+4)^2}\right)x^2\,y \\ &\quad + \left(\frac{3b(1+a)}{(8-a)(a+4)}\right)xy^2 \\ &\quad + \left(\frac{2a^2\bar{c} - 15ab + 16a\bar{c} - 24b + 32\bar{c}}{3b(8-a)}\right)\bar{c}x \\ &\quad + \left(\frac{1}{8-a}\right)y^3 + \left(\frac{2(a\bar{c} - 3b + 4\bar{c})(a+4)}{3b^2(8-a)}\right)\bar{c}y \\ &\quad + \left(\frac{(a-2)(a\bar{c} - 3b + 4\bar{c})(a+4)^2}{27b^3(a-8)}\right)\bar{c}^2 \\ &\quad + O\left((|u| + |v| + |\bar{c}|)^4\right), \end{aligned}$$

$$
\begin{aligned}
&f_4(u, v, \bar{c}) \\
&\quad= \left(\frac{b(a^4\bar{c} - 3a^3b + 10a^3\bar{c} - 21a^2\bar{c} - 18ab)}{6(a-8)(a+4)^2}\right)x^2 \\
&\quad - \left(\frac{b(176a\bar{c} + 60b + 16\bar{c})}{6(a-8)(a+4)^2}\right)x^2 \\
&\quad + \left(\frac{a^3\bar{c} - 3a^2b + 6a^2\bar{c} + 3ab}{3(a-8)(a+4)}\right)xy \\
&\quad - \left(\frac{9a\bar{c} - 21b - 68\bar{c}}{3(a-8)(a+4)}\right)xy + \left(\frac{a-2}{6(a-8)}\right)y^3 \\
&\quad + \left(\frac{(a\bar{c} - 3b + 4\bar{c})(a-2)}{6b(a-8)}\right)y^2 \\
&\quad + \left(\frac{b^2(a^3 - 12a - 11)}{2(a-8)(a+4)^2}\right)x^2 y \\
&\quad + \left(\frac{b(1+a)(a-2)}{2(a-8)(a+4)}\right)xy^2 \\
&\quad + \left(\frac{a^3\bar{c} - 3a^2b + 6a^2\bar{c} - 33ab + 24b - 32\bar{c}}{9b(a-8)}\right)\bar{c}x \\
&\quad + \left(\frac{(a-2)b^3(a^3 + 3a^2 - 24a - 134)}{6(a-8)(a+4)^3}\right)x^3 \\
&\quad + \left(\frac{(a+4)(a-2)(a\bar{c} - 3b + 4\bar{c})}{9b^2(a-8)}\right)\bar{c}y \\
&\quad + \left(\frac{2(a+4)^2(a\bar{c} - 3b + 4\bar{c})}{9b^3(8-a)}\right)\bar{c}^2 \\
&\quad + O\left((|u| + |v| + |\bar{c}|)^4\right), \\
&x \\
&\quad = -\frac{1}{c_0}(u+v), \text{ and} \\
&y \\
&\quad = -\left(2 - \frac{b}{c_0}\right)u + \left(\frac{2-a}{3} + \frac{b}{c_0}\right)v.
\end{aligned}
$$

Furthermore, we implement the center manifold theorem [42]. For this, we assume that $W^C(0,0,0)$ denotes the center manifold for the map (11) which is evaluated at $(0,0)$ in a small neighborhood of $\bar{c} = 0$. Then, $W^C(0,0,0)$ is approximated as follows:

$$
\begin{aligned}
W^C(0,0,0) = \{(u, v, \bar{c}) \in \mathbb{R}^3 : v = m_1u^2 + m_2u\bar{c} \\
+ m_3\bar{c}^2 + O\left((|u| + |\bar{c}|)^3\right)\},
\end{aligned}
$$

where

$$
m_1 = 0, \quad m_2 = \frac{(a+4)(5a-4)}{b(a-8)(a-2)}, \quad \text{and}
$$
$$
m_3 = \frac{2(a+4)^2}{b^2(a-8)(a-2)}.
$$

Moreover, we define the following mapping which is restricted to the center manifold $W^C(0,0,0)$:

$$
\begin{aligned}
F : u \to -u + s_1u^2 + s_2u\bar{c} + s_3u^2\bar{c} + s_4u\bar{c}^2 + s_5u^3 \\
+ O\left((|u| + |\bar{c}|)^4\right),
\end{aligned}
$$

where

$$
s_1 = 0, \quad s_2 = -\frac{(a+4)^2}{b(a-8)}, \; s_3 = \frac{(a+4)(a^2 + 20a - 8)}{2b(a-2)(a-8)},
$$
$$
s_4 = -\frac{(7a^2 - 6a + 32)(a+4)^2}{b^2(a-2)(a-8)^2},
$$

and

$$
s_5 = \frac{(a+4)^3(a^3 - 12a^2 - 54a + 40)}{27\,b^3(a-8)^2(a-2)}.
$$
$$
l_1 = \left(\frac{\partial^2 f_1}{\partial u \partial \bar{c}} + \frac{1}{2}\frac{\partial F}{\partial \bar{c}}\frac{\partial^2 F}{\partial u^2}\right)_{(0,0)} = -\frac{(a+4)^2}{b(a-8)},
$$

and

$$
l_2 = \left(\frac{1}{6}\frac{\partial^3 F}{\partial u^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial u^2}\right)^2\right)_{(0,0)} = s_1^2 + s_5.
$$

Moreover, in closed form l_2 is calculated as follows:

$$
l_2 = \frac{(a+4)^3(a^3 - 12a^2 - 54a + 40)}{27\,b^3(a-8)^2(a-2)}.
$$

Then, the following Lemma gives the parametric conditions for existence and direction of flip bifurcation for system (3) at its positive fixed point.

Theorem 1: Suppose that $l_1 \neq 0$ and $l_2 \neq 0$, then system (3) undergoes flip bifurcation at its positive steady-state $\left(\frac{1}{c}, \frac{ac-b}{c}\right)$ when parameter c varies in small neighborhood of $c_0 = \frac{3\,b}{a+4}$. Furthermore, if $l_2 > 0$, then the period-two orbits that bifurcate from $\left(\frac{1}{c}, \frac{ac-b}{c}\right)$ are stable, and if $l_2 < 0$, then these orbits are unstable.

IV. NEIMARK-SACKER BIFURCATION

This section is related to investigation for Hopf bifurcation at positive fixed point $\left(\frac{1}{c}, \frac{ac-b}{c}\right)$ of system (3). For this, we assume that $P = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ is non-hyperbolic fixed point such that Jacobian matrix at P have complex conjugate eigenvalues, and absolute values of these eigenvalues are equal to one. These are necessary conditions for existence of Neimark-Sacker at fixed point P. For some other investigations of Hopf bifurcation for 2-dimensional discrete-time systems, we refer to [36]–[40], [43]–[50] and references therein.

Next, keeping in view the bifurcation theory of normal forms [51]–[55], the following Theorem is presented which provides explicit parametric conditions for direction and existence of Hopf bifurcation about positive fixed point $\left(\frac{1}{c}, \frac{ac-b}{c}\right)$ of model (3).

Theorem 2: Suppose that $b^2 + 4\,bc - 4\,ac^2 < 0$ and $0 < a < 8$ such that $a \neq 2, 4$, then system (3) undergoes Hopf bifurcation at $\left(\frac{1}{c}, \frac{ac-b}{c}\right)$ when c varies in a small neighborhood of $c_1 = \frac{2b}{a}$. Moreover, assume that $c > c_1$, then an attracting invariant closed curve bifurcates from $\left(\frac{1}{c}, \frac{ac-b}{c}\right)$.

Proof: For necessary conditions for existence of Neimark-Sacker bifurcation at positive fixed point $\left(\frac{1}{c}, \frac{ac-b}{c}\right)$ of model (3), we assume that:

$$c \equiv c_1 = \frac{2b}{a}, \quad \text{and } 0 < a < 8. \tag{12}$$

Assume that

$$\Omega_{NS} = \left\{(a, b, c_1) : c_1 = \frac{2b}{a},\ 0 < a < 8,\ b > 0,\ ac_1 > b\right\}.$$

Suppose that $(a, b, c_1) \in \Omega_{NS}$ with $c_1 = \frac{2b}{a}$. Then, system (3) is described by the following 2-dimensional map:

$$\begin{pmatrix} H \\ P \end{pmatrix} \to \begin{pmatrix} H \exp(a - bH - P) \\ P \exp(c_1 H - 1) \end{pmatrix}. \tag{13}$$

Moreover, $\left(\frac{1}{c_1}, \frac{ac_1-b}{c_1}\right)$ is positive fixed point for map (13) provided that $ac_1 > b$. Next, denote by $\tilde{c}$ as bifurcation parameter and taking into account the following perturbation corresponding to the map (13):

$$\begin{pmatrix} H \\ P \end{pmatrix} \to \begin{pmatrix} H \exp(a - bH - P) \\ P \exp((c_1 + \tilde{c})H - 1) \end{pmatrix}, \tag{14}$$

where $|\tilde{c}| \ll 1$ represents a small perturbation in c_1. Assume that $(x^*, y^*) = \left(\frac{1}{c_1+\tilde{c}}, \frac{a(c_1+\tilde{c})-b}{c_1+\tilde{c}}\right)$ denotes fixed point for (14).

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 - \dfrac{b}{c_1+\tilde{c}} & -\dfrac{1}{c_1+\tilde{c}} \\ a(c_1+\tilde{c}) - b & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned} g_1(x, y) &= \left(\frac{b(b - 2(c_1+\tilde{c}))}{2(c_1+\tilde{c})}\right)x^2 \\ &\quad + \left(\frac{b - (c_1+\tilde{c})}{(c_1+\tilde{c})}\right)xy + \frac{1}{2(c_1+\tilde{c})}y^2 \\ &\quad - \left(\frac{b^2(b - 3(c_1+\tilde{c}))}{6(c_1+\tilde{c})}\right)x^3 \\ &\quad - \left(\frac{b(b - 2(c_1+\tilde{c}))}{2(c_1+\tilde{c})}\right)x^2 y \\ &\quad - \left(\frac{b - (c_1+\tilde{c})}{2(c_1+\tilde{c})}\right)xy^2 \\ &\quad - \frac{1}{6(c_1+\tilde{c})}y^3 + O\left((|x| + |y|)^4\right), \end{aligned}$$

and

$$\begin{aligned} g_2(x, y) &= \frac{1}{2}(c_1+\tilde{c})(a(c_1+\tilde{c}) - b)x^2 \\ &\quad + (c_1+\tilde{c})xy + \frac{1}{6}(c_1+\tilde{c})^2(a(c_1+\tilde{c}) - b)x^3 \\ &\quad + \frac{1}{2}(c_1+\tilde{c})^2 x^2 y + O\left((|x| + |y|)^4\right). \end{aligned}$$

Moreover, the characteristic equation for the variational matrix of (15) evaluated at its fixed point (0, 0) is computed as follows:

$$\nu^2 - \left(2 - \frac{b}{c_1+\tilde{c}}\right)\nu + 1 + a - \frac{2b}{c_1+\tilde{c}} = 0. \tag{16}$$

Suppose that $(a, b, c_1) \in \Omega_{NS}$, then the complex conjugate roots of (16) are given as follows:

$$\nu_1 = \frac{2(c_1+\tilde{c}) - b - i\sqrt{4(c_1+\tilde{c})\left(a(c_1+\tilde{c}) - b\right) - b^2}}{2(c_1+\tilde{c})},$$

and

$$\nu_2 = \frac{2(c_1+\tilde{c}) - b + i\sqrt{4(c_1+\tilde{c})\left(a(c_1+\tilde{c}) - b\right) - b^2}}{2(c_1+\tilde{c})}.$$

Furthermore, we have that

$$|\nu_1| = |\nu_2| = \sqrt{1 + a - \frac{2\,b}{c_1+\tilde{c}}},$$

and

$$\left(\frac{d|\nu_2|}{d\tilde{c}}\right)_{\tilde{c}=0} = \left(\frac{d|\nu_1|}{d\tilde{c}}\right)_{\tilde{c}=0} = \frac{a^2}{4b} > 0.$$

Next, taking $F(\tilde{c}) = 2 - \frac{b}{c_1+\tilde{c}}$, then we have $F(0) = 2 - \frac{b}{c_1}$. Assume that $(a, b, c_1) \in \Omega_{NS}$, then it follows that

$$-2 < F(0) = 2 - \frac{b}{c_1} < 2.$$

Moreover, we suppose that $F(0) \neq 0, 1$, that is, $2 - \frac{b}{c_1} \neq 0$ and $2 - \frac{b}{c_1} \neq 1$. Therefore, one has $F(0) \neq \pm 2, 0, 1$ and this implies that $\nu_1^k, \nu_2^k \neq 1$ for all $k = 1, 2, 3, 4$ at $\tilde{c} = 0$. Thus, we conclude that the roots of (16) do not lie in the intersection of the unit circle with the coordinate axes at $\tilde{c} = 0$ if the following conditions are satisfied:

$$b \neq 2c_1, \quad b \neq c_1. \tag{17}$$

Furthermore, we consider the following transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -\dfrac{1}{c_1+\tilde{c}} & 0 \\ \kappa - 1 + \dfrac{b}{c_1+\tilde{c}} & -\omega \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{18}$$

where

$$\kappa := \frac{a-4}{4}, \quad \text{and } \omega := \frac{\sqrt{a(8-a)}}{4}.$$

Using transformation (18), the normal form for (15) is computed as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \kappa & -\omega \\ \omega & \kappa \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \tilde{p}(u, v) \\ \tilde{q}(u, v) \end{pmatrix}, \tag{19}$$

where

$$\begin{aligned} &\tilde{p}(u, v) \\ &= -\left(\frac{b(ab - 2a\tilde{c} - 4b)}{2a}\right)x^2 + \left(\frac{a\tilde{c} + 2b - ab}{a}\right)xy \\ &\quad - \frac{1}{2}y^2 + \left(\frac{b^2(ab - 3a\tilde{c} - 6b)}{6a}\right)x^3 \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{b(ab-2a\tilde{c}-4b)}{2a}\right)x^2\,y+\left(\frac{ab-a\tilde{c}-2b}{2a}\right)xy^2\\
&+\frac{1}{6}y^3+O((|u|+|v|)^4),\\
&\tilde{q}(u,v)\\
&=-\left(\frac{b(a\tilde{c}\kappa+ab-a\tilde{c}+2b\kappa-2b)(ab-2a\tilde{c}-4b)}{2a\omega(a\tilde{c}+2b)}\right)x^2\\
&\quad-\left(\frac{(a\tilde{c}+b)(a\tilde{c}+2b)^2}{6a^2\omega}\right)x^2\\
&\quad-\left(\frac{(ab-a\tilde{c}-2b)(a\tilde{c}\kappa+ab-a\tilde{c}+2b\kappa-2b)}{a\omega(a\tilde{c}+2b)}\right)xy\\
&\quad-\left(\frac{a\tilde{c}+2b}{a\omega}\right)xy\\
&\quad-\left(\frac{a\tilde{c}\kappa+ab-a\tilde{c}+2b\kappa-2b}{2\omega(a\tilde{c}+2b)}\right)y^2\\
&\quad-\left(\frac{b^2(3a\tilde{c}+6b-ab)(a\tilde{c}\kappa+ab-a\tilde{c}+2b\kappa-2b)}{6a\omega(a\tilde{c}+2b)}\right)x^3\\
&\quad+\left(\frac{(a\tilde{c}+b)(a\tilde{c}+2b)^2}{6a^2\omega}\right)x^3\\
&\quad-\left(\frac{b(2a\tilde{c}+4b-ab)(a\tilde{c}\kappa+ab-a\tilde{c}+2b\kappa-2b)}{2a\omega(a\tilde{c}+2b)}\right)x^2y\\
&\quad+\left(\frac{(a\tilde{c}+2b)^2}{2a^2\omega}\right)x^2\,y\\
&\quad-\left(\frac{(a\tilde{c}+2b-ab)(a\tilde{c}\kappa+ab-a\tilde{c}+2b\kappa-2b)}{2a\omega(a\tilde{c}+2b)}\right)xy^2\\
&\quad+\left(\frac{a\tilde{c}\kappa+ab-a\tilde{c}+2b\kappa-2b}{6\omega(a\tilde{c}+2b)}\right)y^3\\
&\quad+O((|u|+|v|)^4),
\end{aligned}
$$

Furthermore, it follows from Fig. 3 that $L(a) < 0$ for all $0 < a < 8$. Thus an attracting invariant closed curve bifurcates from the equilibrium point $\left(\frac{1}{c}, \frac{ac-b}{c}\right)$ whenever $c > c_1$. □

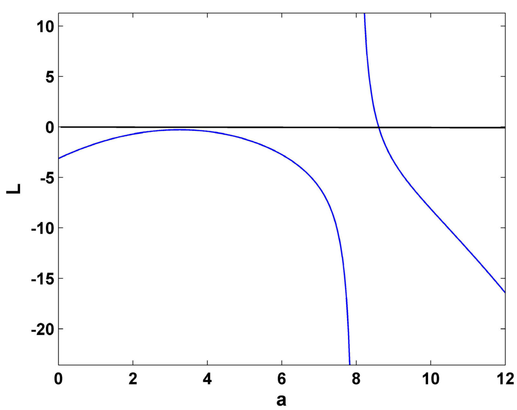

FIGURE 3. Plot of $L(a)$ for $0 \le a \le 12$.

V. CHAOS CONTROL

In this section, our aim is to apply two chaos control methods to system (3). These chaos control methods have been most frequently used strategies for controlling bifurcating and chaotic behaviors of discrete-time models [11], [12], [16], [29]–[31], [41], [45], [47], [48], [56]. The first chaos control method which is known as Ott-Grebogi-Yorke (OGY) [57] method is considered as pioneer control method for discrete models with some drawbacks [58]. Particularly, OGY method may be ineffective for discrete-time models which are discrete counterparts of continuous systems with an application of Euler approximation [47], [48]. But nevertheless, this method is effective and applicable in our case. Secondly, hybrid control method [59] is also implemented to system (3). Recently, in [16] a modified hybrid control method is proposed. On the other hand, in [60] an exponential type chaos control method is presented for controlling chaos and bifurcations in discrete-time systems. The main contribution of this section is to obtain mathematical criterion for controllability of system (3) via OGY and hybrid control methods.

For the application of OGY method, model (3) is written in the following form:

$$
\begin{aligned}
x_{n+1} &= x_n e^{a-bx_n-y_n} = f(x_n, y_n, c),\\
y_{n+1} &= y_n e^{cx_n-1} = g(x_n, y_n, c),
\end{aligned} \tag{20}
$$

where c is used for chaos control parameter. Moreover, it is assumed that $c \in \left(\hat{c}-\eta, \hat{c}+\eta\right)$, where $\eta > 0$ and $\hat{c}$ denotes nominal value of c. Furthermore, we consider $(x^*, y^*) = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ as positive fixed point of system (3). Then, one can approximate system (20) in neighborhood of the fixed point $(x^*, y^*) = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ as follows:

$$
\begin{bmatrix} x_{n+1}-x^* \\ y_{n+1}-y^* \end{bmatrix} \approx J(x^*, y^*, \hat{c}) \begin{bmatrix} x_n-x^* \\ y_n-y^* \end{bmatrix} + B[c-\hat{c}], \tag{21}
$$

where

$$
J(x^*, y^*, \hat{c}) = \begin{bmatrix} \frac{\partial f\left(x^*, y^*, \hat{c}\right)}{\partial x} & \frac{\partial f\left(x^*, y^*, \hat{c}\right)}{\partial y} \\ \frac{\partial g\left(x^*, y^*, \hat{c}\right)}{\partial x} & \frac{\partial g\left(x^*, y^*, \hat{c}\right)}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-\frac{b}{\hat{c}} & -\frac{1}{\hat{c}} \\ a\hat{c}-b & 1 \end{bmatrix}
$$

and

$$
B = \begin{bmatrix} \frac{\partial f\left(x^*, y^*, \hat{c}\right)}{\partial c} \\ \frac{\partial g\left(x^*, y^*, \hat{c}\right)}{\partial c} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{a\hat{c}-b}{\hat{c}^2} \end{bmatrix}.
$$

Next, we consider the following matrix for controllability:

$$
C = [B : JB] = \begin{bmatrix} 0 & \frac{b-a\hat{c}}{\hat{c}^3} \\ \frac{a\hat{c}-b}{\hat{c}^2} & \frac{a\hat{c}-b}{\hat{c}^2} \end{bmatrix}. \tag{22}
$$

Then, it must be noted that system (20) is controllable if C has rank 2. According to assumption for existence of positive equilibrium we have $a\hat{c}-b \neq 0$ and this implies that rank of

C is 2. Furthermore, we set $[c-\hat{c}] = -K\begin{bmatrix} x_n - x^* \\ y_n - y^* \end{bmatrix}$, where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then system (21) is written as follows

$$\begin{bmatrix} x_{n+1} - x^* \\ y_{n+1} - y^* \end{bmatrix} \approx [J - BK]\begin{bmatrix} x_n - x^* \\ y_n - y^* \end{bmatrix}. \tag{23}$$

In this case, the corresponding control system of (3) is given as follows

$$\begin{aligned} x_{n+1} &= x_n \exp\left(a - bx_n - y_n\right), \\ y_{n+1} &= y_n \exp\left(\left(\hat{c} - k_1(x_n - x^*) - k_2(y_n - y^*)\right)x_n - 1\right). \end{aligned} \tag{24}$$

Moreover, the positive equilibrium point (x^*, y^*) of (24) is locally stable if and only if absolute values of both eigenvalues of $J - BK$ are less than one. Moreover, the matrix $J - BK$ is given as follows:

$$J - BK = \begin{bmatrix} 1 - \frac{b}{\hat{c}} & -\frac{1}{\hat{c}} \\ a\hat{c} - b - \frac{(a\hat{c} - b)k_1}{\hat{c}^2} & 1 - \frac{(a\hat{c} - b)k_2}{\hat{c}^2} \end{bmatrix}.$$

The characteristic equation for the matrix $J - BK$ is given as follows

$$\begin{aligned} \mathbb{P}(\lambda) = \lambda^2 - \left(2 - \frac{b}{\hat{c}} + \frac{bk_2}{\hat{c}^2} - \frac{ak_2}{\hat{c}}\right)\lambda + 1 + a - \frac{2b}{\hat{c}} \\ - \frac{ak_1}{\hat{c}^2} + \frac{bk_2}{\hat{c}^2} + \frac{abk_2}{\hat{c}^2} - \frac{ak_2}{\hat{c}} + \frac{b(k_1 - bk_2)}{\hat{c}^3} = 0. \end{aligned} \tag{25}$$

Assume that λ_1 and λ_2 represent the roots of (25), then it follows that

$$\lambda_1 + \lambda_2 = 2 - \frac{b}{\hat{c}} + \frac{bk_2}{\hat{c}^2} - \frac{ak_2}{\hat{c}}, \tag{26}$$

and

$$\begin{aligned} \lambda_1\lambda_2 = 1 + a - \frac{2b}{\hat{c}} - \frac{ak_1}{\hat{c}^2} + \frac{bk_2}{\hat{c}^2} + \frac{abk_2}{\hat{c}^2} - \frac{ak_2}{\hat{c}} \\ + \frac{b(k_1 - bk_2)}{\hat{c}^3}. \end{aligned} \tag{27}$$

Moreover, we take $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. Then, the lines of marginal stability for (24) are computed as follows:

$$\begin{aligned} L_1 : a - \frac{2b}{\hat{c}} - \frac{ak_1}{\hat{c}^2} + \frac{bk_2}{\hat{c}^2} \\ + \frac{abk_2}{\hat{c}^2} - \frac{ak_2}{\hat{c}} + \frac{b(k_1 - bk_2)}{\hat{c}^3} = 0, \end{aligned} \tag{28}$$

$$L_2 : \hat{c}^2 - k_1 + bk_2 = 0, \tag{29}$$

and

$$\begin{aligned} L_3 : (4 + a)\hat{c}^3 - \hat{c}^2(3b + 2ak_2) + b(k_1 - bk_2) \\ + \hat{c}(-ak_1 + (2 + a)bk_2) = 0. \end{aligned} \tag{30}$$

Then, stability region for (24) is triangular region bounded by L_1, L_2 and L_3 in k_1k_2-plane.

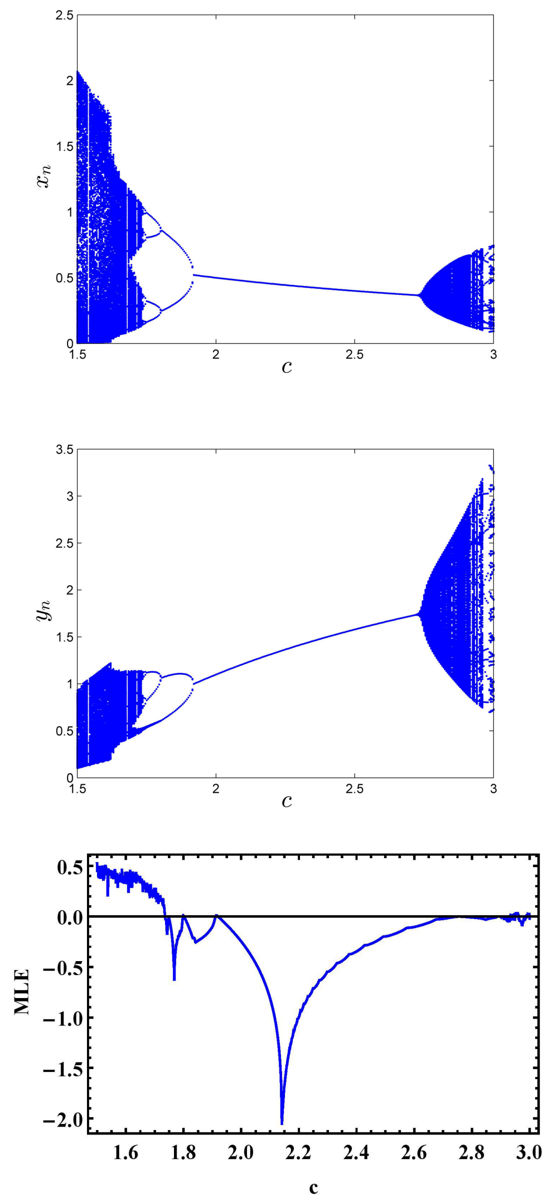

FIGURE 4. Bifurcation diagrams and maximum Lyapunov exponents for system (3) with $(a, b) = (3.5, 4.8)$, $(x_0, y_0) = (0.518, 1.0129)$ and $c \in [1.5, 3]$.

Secondly, we apply hybrid control method to system (3). This hybrid control method consists of parameter perturbation and feedback control [40]. Then, an application of hybrid control method yields the following control system:

$$\begin{aligned} x_{n+1} &= \alpha x_n e^{a - bx_n - y_n} + (1 - \alpha)x_n, \\ y_{n+1} &= \alpha y_n e^{cx_n - 1} + (1 - \alpha)y_n, \end{aligned} \tag{31}$$

where $\alpha \in]0, 1[$ denotes external control parameter for system (31). On the other hand, the Jacobian matrix for (31) at $(x^*, y^*)\left(\frac{1}{c}, \frac{ac-b}{c}\right)$ is computed as follows:

$$\begin{bmatrix} \alpha\left(1 - \frac{b}{c}\right) + 1 - \alpha & -\frac{\alpha}{c} \\ \alpha(ac - b) & 1 \end{bmatrix}. \tag{32}$$

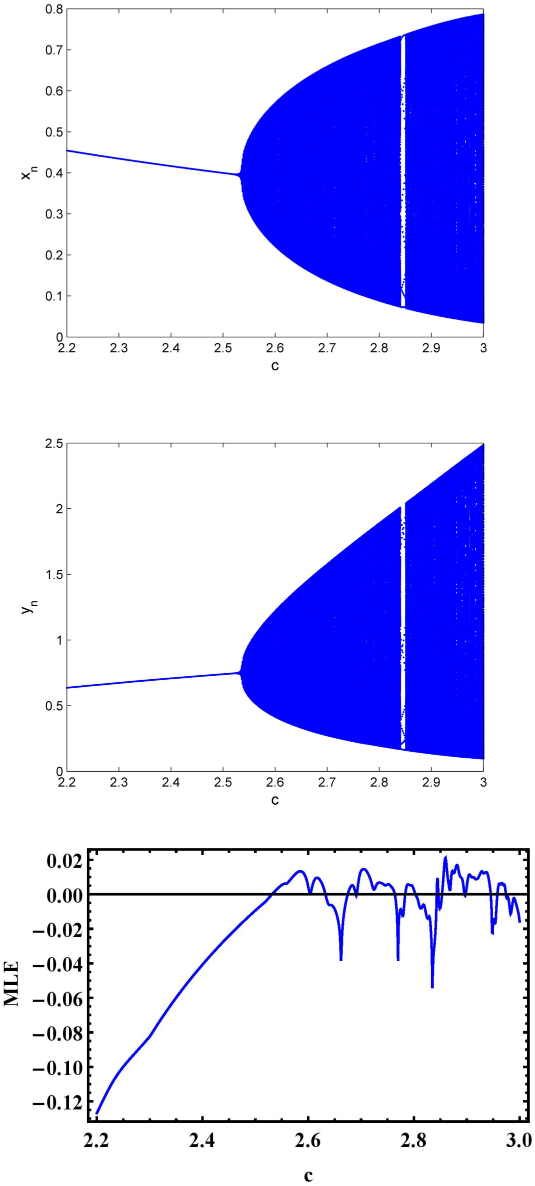

FIGURE 5. **Bifurcation diagrams and maximum Lyapunov exponents for system (3) with $(a, b) = (1.5, 1.9)$, $(x_0, y_0) = (0.4, 0.74)$ and $c \in [2.2, 3]$.**

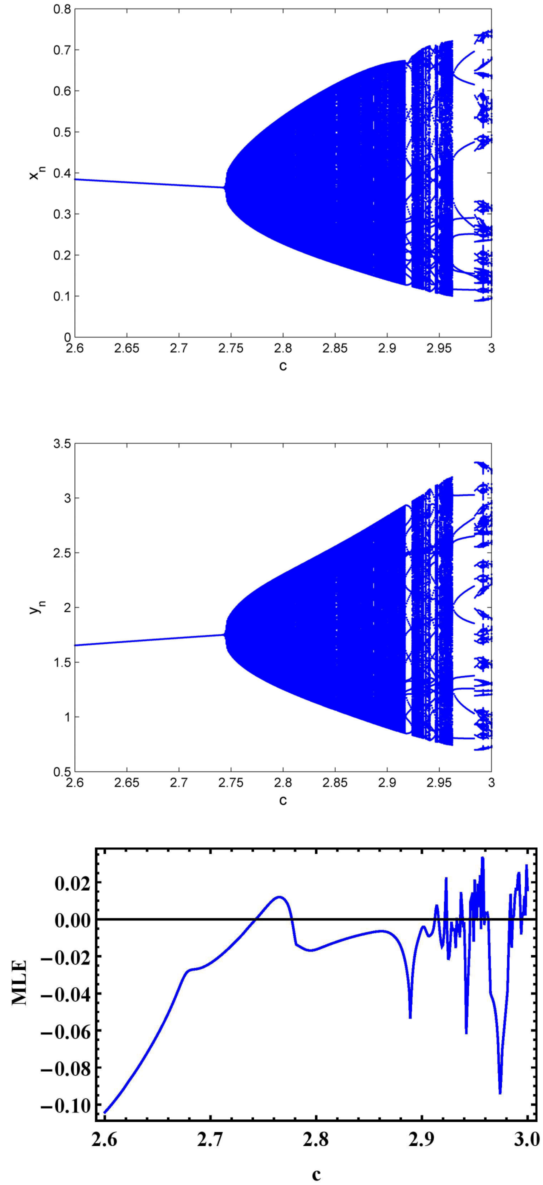

FIGURE 6. **Bifurcation diagrams and maximum Lyapunov exponents for system (3) with $(a, b) = (3.5, 4.8)$, $(x_0, y_0) = (0.365, 1.748)$ and $c \in [2.6, 3]$.**

Furthermore, the following Lemma gives parametric conditions for local stability of fixed point $(x^*, y^*) = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ for the controlled system (31).

Lemma 4: The positive equilibrium $(x^*, y^*) = \left(\frac{1}{c}, \frac{ac-b}{c}\right)$ of the controlled system (31) is locally asymptotically stable if and only if the following condition holds true:

$$\left|2 - \frac{b\alpha}{c}\right| < 2 - \frac{b\alpha}{c} + a\alpha^2 - \frac{b\alpha^2}{c} < 2.$$

VI. NUMERICAL SIMULATION AND DISCUSSION

In this section, some interesting numerical simulations related to system (3) are presented for particular choice of parametric values. These numerical results give evidence of flip bifurcation, Hopf bifurcation for system (3) at its positive steady-state, and show effectiveness of chaos control strategies.

Example 1: First, we choose $a = 3.5$, $b = 4.8$, $c \in [1.5, 3]$ with initial conditions $x_0 = 0.518$ and $y_0 = 1.0129$. Then, system (3) undergoes both flip bifurcation and Hopf bifurcation as c varies in small neighborhoods of $c_0 \approx 1.92$ and $c_1 \approx 2.74286$, respectively. At $(a, b, c) = (3.5, 4.8, 1.92)$ the unique positive steady-state of the system (3) is given by $(0.520833, 1)$, and characteristic equation of (3) at this steady-state is computed as follows

$$\lambda^2 + 0.5\lambda - 0.5 = 0.$$

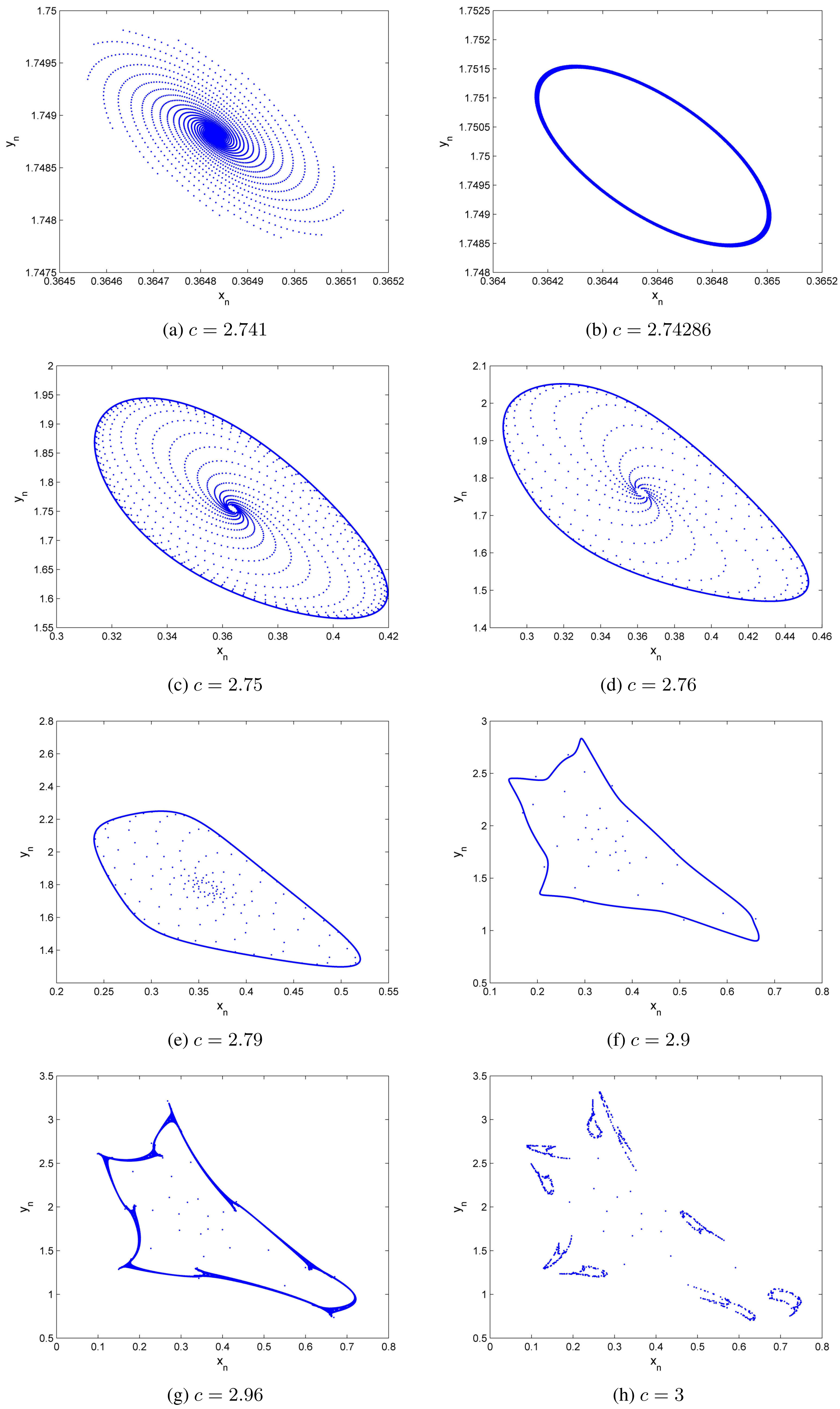

FIGURE 7. Phase portraits of system (3) with $a = 3.5$, $b = 4.8$, $c = 2.741, 2.74286, 2.75, 2.76, 2.79, 2.9, 2.96, 3$ $x_0 = 0.365$, $y_0 = 1.748$.

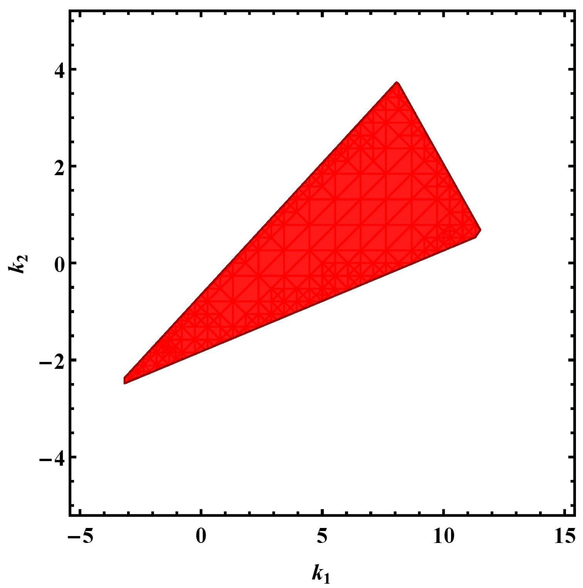

FIGURE 8. Stability region of the controlled system (34).

Moreover, roots of aforementioned equation are $\lambda_1 = -1$ and $\lambda_2 = 0.5$. Thus $(a, b, c) = (3.5, 4.8, 1.92) \in \Omega_{FB}$. Furthermore, $l_1 = 2.604166665 > 0$ and $l_2 = -1.1773757 < 0$, which proves the correctness of Theorem 1. Similarly, at $(a, b, c) = (3.5, 4.8, 2.74286)$ the unique positive steady-state of the system (3) is given by (0.364583, 1.75), and characteristic equation of (3) at this steady-state is computed as follows

$$\lambda^2 - 0.250002\lambda + 1 = 0.$$

Moreover, roots of aforementioned equation are $\lambda_1 = 0.125001 + 0.992158\,i$ and $\lambda_2 = 0.125001 - 0.992158\,i$ both with modulus one. Thus $(a, b, c) = (3.5, 4.8, 2.74286) \in \Omega_{NS}$. Furthermore, the first Lyapunov exponent for these parametric values is given by $L = -0.293403 < 0$, which proves the correctness of Theorem 2. Bifurcation diagrams and maximum Lyapunov exponents are depicted in Fig. 4.

Example 2: Taking the parameters $a = 1.5$, $b = 1.9$, $c \in [2.2, 3]$ and initial conditions $(x_0, y_0) = (0.4, 0.74)$ for system (3), then it undergoes Hopf bifurcation at $c \approx 2.533333333$. At $(a, b, c) = (1.5, 1.9, 2.5333333333)$ the unique positive steady-state of (3) is given by $(x^*, y^*) = (0.394737, 0.75)$. The characteristic equation for the variational matrix of system (3) evaluated at $(x^*, y^*) = (0.394737, 0.75)$ is computed as follows:

$$\lambda^2 - 1.2499999999901314\lambda + 1 = 0. \quad (33)$$

Moreover, the complex conjugate roots of (33) are given by $\lambda_1 = 0.625 - 0.780625\,i$ and $\lambda_2 = 0.625 + 0.780625\,i$ with modulus $|\lambda_{1,2}| = 1$. In this case the first Lyapunov exponent L is given by $L = \frac{800-542\,a+121\,a^2-8\,a^3}{32(a-8)} = -1.11659 < 0$. The bifurcation diagrams and maximum Lyapunov exponents (MLE) are depicted in Fig. 5.

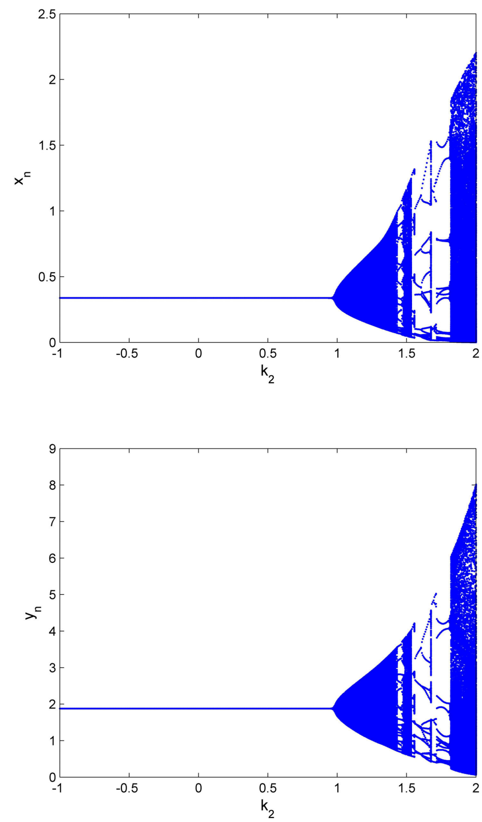

FIGURE 9. Bifurcation diagrams for the controlled system (34) with $k_1 = 3$, $(x_0, y_0) = (0.3, 1.87)$ and $k_2 \in [-1, 2]$.

Example 3: Next we take $a = 3.5$, $b = 4.8$, $c \in [2.6, 3]$ and initial values $(x_0, y_0) = (0.365, 1.748)$. Then bifurcation diagrams and MLE are depicted in Fig. 6. Moreover, phase portraits of system (3) for $a = 3.5$, $b = 4.8$, $(x_0, y_0) = (0.365, 1.748)$ and with different values of c are depicted in Fig. 7.

Next, we want to apply OGY feedback control method for $(a, b, c) = (3.5, 4.8, 2.96)$. Under these parametric conditions system has unique positive unstable equilibrium point $(x^*, y^*) = (0.337838, 1.87838)$ and corresponding phase portrait of (3) is shown in Fig. 7g. We want to move unstable equilibrium point towards stable trajectory. For this, taking $\hat{c} = 2.96$ and the corresponding controlled system is written as:

$$\begin{aligned} x_{n+1} &= x_n \exp\left(3.5 - 4.8x_n - y_n\right), \\ y_{n+1} &= y_n \exp\left(\left(2.96 - k_1(x_n - x^*) - k_2(y_n - y^*)\right)x_n - 1\right), \end{aligned} \quad (34)$$

where $(x^*, y^*) = (0.337838, 1.87838)$, and k_1 and k_2 are feedback gains. Furthermore, in this case we have

$$J = \begin{pmatrix} -0.621622 & -0.337838 \\ 5.56 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ 0.634587 \end{pmatrix},$$

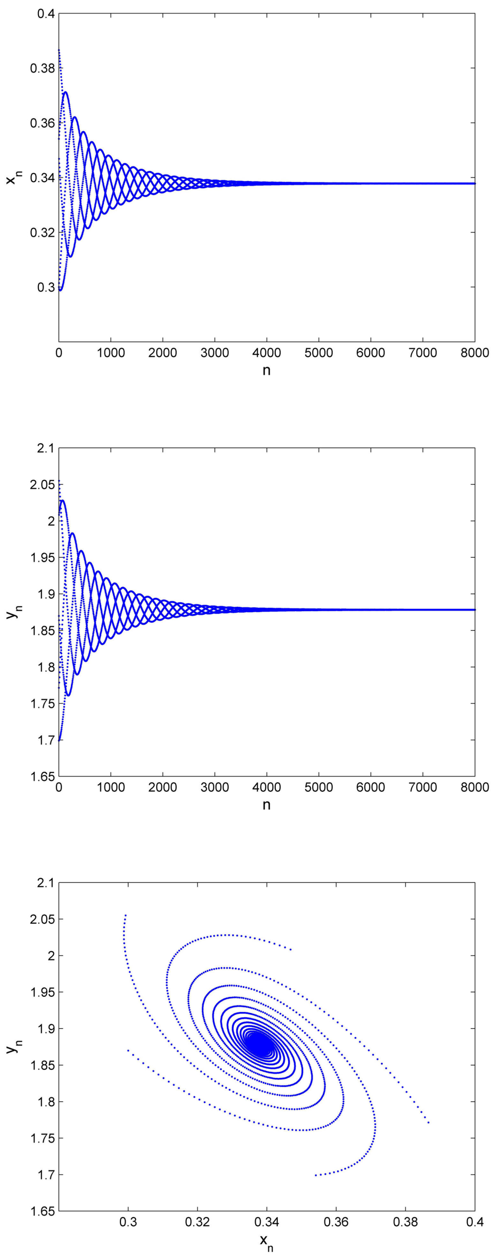

FIGURE 10. Plots for the controlled system (36) with $\alpha = 0.862$ and $(x_0, y_0) = (0.3, 1.87)$.

$$C = \begin{pmatrix} 0 & -0.214388 \\ 0.634587 & 0.634587 \end{pmatrix},$$

and variational matrix of the controlled system (34) is computed as follows

$$J - BK = \begin{pmatrix} -0.621622 & -0.337838 \\ 5.56 - 0.634587k_1 & 1 - 0.634587k_2 \end{pmatrix}.$$

The characteristic equation for variational matrix $J - BK$ is calculated as follows:

$$\lambda^2 - (0.378378 - 0.634587k_2)\lambda + 1.25676 - 0.214388k_1 + 0.394473k_2 = 0. \quad (35)$$

The roots of (35) lie inside the unit open disk if $0.208333\,k_1 < 1.82533 + k_2$, $-3.5043 < k_1 \leq 8.09379$, $0.650885 + k_2 < 0.543478\,k_1$, or $0.208333\,k_1 < 1.82533 + k_2$, $8.09379 < k_1 < 11.6236$, $0.892857\,k_1 + k_2 < 10.9745$. Moreover, the lines of marginal stability for the controlled system (34) are computed as follows

$$L_1 : 0.394473\,k_2 = 0.214388\,k_1 - 0.256757,$$
$$L_2 : 0.214388\,k_1 = 1.87838 + 1.02906\,k_2,$$

and

$$L_3 : 0.214388\,k_1 + 0.240114\,k_2 = 2.63514.$$

It is easy to see that stable eigenvalues lie within the triangular region bounded by the straight lines L_1, L_2, L_3 for the controlled system (34) (see Fig. 8). In particular, if we take $k_1 = 3$, then positive equilibrium point $(x^*, y^*) = (0.337838, 1.87838)$ of the controlled system (34) is locally stable if and only if $-1.20033 < k_2 < 0.97955$. Taking $k_1 = 3$ and $k_2 \in [-1, 2]$, then bifurcation diagrams for the controlled system (34) are depicted in Fig. 9.

Secondly, for the same parametric values, that is, $(a, b, c) = (3.5, 4.8, 2.96)$, we discuss the validity of hybrid control strategy. In this case the controlled system (31) is rewritten as follows:

$$x_{n+1} = \alpha x_n e^{3.5-4.8x_n-y_n} + (1-\alpha)x_n,$$
$$y_{n+1} = \alpha y_n e^{-1+2.96x_n} + (1-\alpha)y_n. \quad (36)$$

Moreover, the characteristic equation for the variational matrix of the system (36) at $(x^*, y^*) = (0.337838, 1.87838)$ is given by:

$$\lambda^2 - (2 - 1.62162\alpha)\lambda + 1 - 1.62162\alpha + 1.87838\alpha^2 = 0. \quad (37)$$

Moreover, the roots of (36) lie inside the open unit disk if and only if $0 < \alpha < 0.863309$. For $\alpha = 0.862$ and $(x_0, y_0) = (0.3, 1.87)$ the plots for the controlled system (36) are depicted in Fig. 10.

VII. CONCLUDING REMARKS

In this paper, piecewise constant argument is implemented in order to obtain a discrete-time predator-prey model with linear functional response. The proposed model is governed by two-dimensional planar system of difference equations in exponential form, which has more rich dynamics and chaotic behavior as compare to its continuous counterpart. Stability analysis of equilibria is investigated. Moreover, system (3) undergoes flip bifurcation and Hopf bifurcation as c is taken as bifurcation parameter. Both critical coefficients in case of flip (period-doubling) bifurcation and Lyapunov first coefficient in case of Neimark-Sacker (Hopf) bifurcation are calculated in closed forms. The numerical simulation results are provided including the interesting dynamical behaviors such as invariant cycles and chaotic sets. Two chaos control strategies are implemented in order to control chaotic behavior of the system (3). The effectiveness of these chaos control strategies is illustrated through numerical simulations. Moreover, one other possible discritization of system (2) was studied in [38] via Euler approximation. But Euler method is simple numerical method perhaps not enough appropriate for population models of predator-prey type [61].

FUTURE WORK

Keeping in view the dynamical consistency for discrete counterpart of continuous predator-prey model, one may apply a nonstandard finite difference scheme to system (2) as follows:

$$x_{n+1} = \frac{(1+ha)\,x_n}{1+hbx_n+hy_n},$$
$$y_{n+1} = \frac{y_n\,(1+hcx_n)}{1+h}, \tag{38}$$

where $h > 0$ is step size for nonstandard finite difference scheme. Dynamical study of system (38) will be our future work for investigation. Moreover, it is interesting to add delay effect in system (2) to see the bifurcating behavior of resulting system [62]. On the other hand, it is also interesting to extend these results for higher dimension and to see possible hidden attractors and complex behavior for resulting system [63]–[67].

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