IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received June 28, 2020, accepted July 4, 2020, date of publication July 8, 2020, date of current version July 21, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3008064

Fractional Fuzzy Inference System: The New Generation of Fuzzy Inference Systems

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This work was supported in part by the National Natural Science Foundation of China under Grant 41876098.

ABSTRACT This paper presents a new machinery of compositional rule of inference called fractional fuzzy inference system (FFIS). An FFIS is a fuzzy inference system (FIS) in which consequent parts of a rule base consist of a new type of membership functions called fractional membership functions. Fractional membership functions are characterized using fractional indices. There are two types of fractional indices. Each type can be either constant or dynamic. An FFIS intelligently considers not only the truth degrees of information included in membership functions, but also the volume of the information in the process of making a conclusion. In other words, the volume of information extracted from a membership function depends on the truth degree of information. Concretely, the higher the truth degree, the larger the volume of information that is involved in the process of making a conclusion. It is shown that typical FISs, e.g. Mamdani's or Larsen's FISs, are special cases of FFISs. Specifically, as the fractional indices approach one, the FFIS approaches a typical FIS. In addition, using two theorems proved in this paper, it is demonstrated that, independent of the problem in question, a typical FIS never leads to results which are more satisfactory than those obtained by the FFIS corresponding to the typical FIS provided that a particular set of fractional indices is taken into account. Put another way, it seems sound to expect that applying FFIS always leads to more satisfactory results than applying its corresponding FIS. It is also shown that FFIS grants a special dynamic to FIS which can be also customized according to a new concept called reaction trajectories map (RTM). Particularly, the RTM enables decision makers to select an FFIS more suitable for their purpose. Some more concepts such as the left and right orders of an FFIS and the fracture index are also introduced in this paper.

INDEX TERMS Fractional membership functions, fractional horizontal membership functions, fractional compositional rule of inference, fractional indices, reaction trajectories map, fractional translation rule.

I. INTRODUCTION

Motivated by Zadeh's paper [1] presenting a methodology to express a fuzzy algorithm, the basis of fuzzy logic control techniques was established by Mamdani's fuzzy inference system (FIS) applied on the fuzzy control of a steam engine in 1975. Successful application of Mamdani's FIS in control of Sendai subway in 1978 captured global attentions to fuzzy sets theory and fuzzy control. So far, much effort has been made to introduce different FISs by taking into account various translation rules of fuzzy if-then rules. Specifically, Mamdani's FIS uses minimum operator for the translation, whereas the algebraic product operator is applied in Larsen's FIS. There are also other types of FISs whose differences with other FISs correspond to consequent parts of a rule base. Concretely, in Takagi-Sugeno's FIS the consequent parts are considered as the crisp functions of inputs. One of the main reasons for introducing different FISs is to achieve more satisfactory results. In other words, the variety of FISs expand the space of fuzzy system outputs which, in turn, enable decision makers to choose an FIS that is more desirable for their purpose. Based on the compositional rule of inference proposed by Zadeh [1], FISs can be categorized in: 1. Compositional FISs, 2. Non-Compositional FISs. The compositional FISs, e.g. Mamdani's and Larsen's FISs, are those which are in accordance with the suggestion of Zadeh's. This paper concentrates on compositional FISs. Then, hereafter, by FISs we mean compositional FISs.

The motivation of this work comes from introducing a new machinery of translation rule or compositional rule of

The associate editor coordinating the review of this manuscript and approving it for publication was Fanbiao Li.

inference such that the space of fuzzy system outputs is expanded more than before without changing the structure of the system in question, e.g. gains, inputs, outputs, or the structure of fuzzy system, e.g. rule base and general structure of membership functions. Such a machinery of compositional rule of inference makes it possible to access parts of fuzzy systems output space that used to be inaccessible. That, in turn, leads to the discovery of new desirable outputs that were not observable or discoverable before. Fuzzy systems will eventually perform more satisfactory and efficiently.

So far, the consequent parts in FISs have been considered as typical membership functions whose domain is a universe of discourse and whose range is the membership degree, i.e. the interval [0, 1]. Simply put, the mechanism of reasoning in FISs has been applied on typical membership functions. As a result, considering typical membership functions, the FISs have been presented. In 2015 [2], Piegat and Landowski introduced a new representation of a typical membership function that they called horizontal membership function. The domain of a horizontal membership function relies on the square $[0, 1] \times [0, 1]$ coming from the membership degrees and a new variable called relative-distance-measure (RDM) variable. The range of horizontal membership function is the domain of its corresponding membership function, i.e. the universe of discourse. Piegat *et al.* primarily elaborated on their proposed approach called RDM fuzzy interval arithmetic enjoying horizontal membership functions in solving some problems of fuzzy mathematics, see [3]–[5]. The successful applications of RDM fuzzy interval arithmetic in comparison with other approaches such as standard interval arithmetic have been proved by Piegat and Landowski in [6]–[9]. Then, Mazandarani *et al.* by introducing a new framework of fuzzy calculus proved the effective applicability of horizontal membership functions and RDM fuzzy interval arithmetic in fuzzy dynamical systems, see [10]–[12] for more details.

In this paper, a new type of membership functions called fractional membership functions are introduced. A fractional membership function is originated from a new type of horizontal membership functions termed as fractional horizontal membership functions. A fractional horizontal membership function is characterized by an index called fractional index. Hence, the fractional indices impact on fractional membership functions. Corresponding to any typical membership function, a fractional membership function exists such that the fractional membership function approaches the typical membership function as the fractional index approaches one. Based on fractional membership functions considered in the consequent parts of a rule base, a new generation of fuzzy inference systems is introduced which is called fractional fuzzy inference system. It is demonstrated that any typical FIS, e.g. Mamdani's or Larsen's FISs, is a special case of fractional FISs. By two theorems, it is proved that a typical FIS never yields to results which are more satisfactory than those obtained by the FFIS corresponding to the typical FIS provided that a particular set of fractional indices is taken into account. Then, it seems sound to expect always more satisfactory results by applying fractional FISs for some sets of fractional indices. The point to be underscored is that no changes in the general structure of the system equipped with fuzzy system is made. As a matter of fact, if a fuzzy system is at disposal, then its corresponding fractional FIS can be applied without any changes in the rule base, general structure of membership functions, gains, inputs, outputs, and the structure of system in question.

The fractional indices play a major role in fractional FISs. The fractional indices can be constant or dynamic. How fractional indices should be determined is explained here. Moreover, some more detailed explanations are provided to clarify the process of designing the structure of a fractional FIS. It is shown that fractional FISs with dynamical indices are in fact FISs in which membership functions are dynamic internally. In addition, some new concepts such as horizontal translation rule, horizontal FIS, fractional compositional rule of inference, reaction trajectories map, the left and right orders of an FFIS, and the fracture index are introduced.

II. PRELIMINARIES

This section presents some necessary definitions and relations which will be used in this paper. Throughout this paper, the set of all real numbers is denoted by $\mathbb{R}$, the set of all type-1 fuzzy numbers (T1FNs) on $\mathbb{R}$ by E_1. By $[\tilde{A}]^\mu$ we show the well-known μ-level sets of a fuzzy set $\tilde{A}$ whose left and right end-points are indicated by $\underline{A}^\mu$ and $\overline{A}^\mu$, respectively. In addition, the triangular T1FN $\tilde{A}$ is characterized using the triple (a, b, c), $a \leq b \leq c$, as $\tilde{A} = (a, b, c)$; and $G(\sigma, m)$ denotes the Gaussian membership function $e^{\frac{-(x-m)^2}{2\sigma^2}}$ where σ and m are the standard variation and average of x.

Definition 1 [13]: A type-1 fuzzy set $\tilde{A} \in E_1$, $\tilde{A} : \mathbb{R} \to [0, 1]$ is called a type-1 fuzzy number if it is normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy subsets of the real numbers.

The T1FN $\tilde{A}$ can be represented in a parametric form by the ordered pair of functions $(\underline{A}^\mu, \overline{A}^\mu)$, $0 \leq \mu \leq 1$, satisfying the following properties:

1) $\underline{A}^\mu$ is a bounded, non-decreasing, left continuous function in (0, 1], and it is right continuous at $\mu = 0$,
2) $\overline{A}^\mu$ is a bounded, non-increasing, left continuous function in (0, 1], and it is right continuous at $\mu = 0$,
3) $\underline{A}^\mu \leq \overline{A}^\mu$.

Definition 2 [2], [10]–[12]: Let $\tilde{A} : [a, b] \subseteq \mathbb{R} \to E_1$. The horizontal membership function $A_{\mathcal{H}} : [0, 1] \times [0, 1] \to [a, b]$ is a representation of $\tilde{A}(x)$ as $A_{\mathcal{H}}(\mu, \alpha_A) = x$ in which $x \in [a, b]$, $\mu \in [0, 1]$ is the membership degree of x in $\tilde{A}(x)$, $\alpha_A \in [0, 1]$ is called relative-distance-measure (RDM) variable (or the horizontal index), and $A_{\mathcal{H}}(\mu, \alpha_A) = \underline{A}^\mu + (\overline{A}^\mu - \underline{A}^\mu)\alpha_A$.

Note 1: The horizontal membership function of $\tilde{A} \in E_1$ is also denoted by $\mathcal{H}(\tilde{A}) \triangleq A_{\mathcal{H}}(\mu, \alpha_A)$.

Note 2: The horizontal membership function of $\tilde{A} \in E_1$ can be also represented by $\mathcal{H}(\tilde{A}) \triangleq A_{\mathcal{H}}(\mu, \beta_A) = \overline{A}^{\mu} - (\overline{A}^{\mu} - \underline{A}^{\mu})\beta_A$, $\beta_A \in [0, 1]$.

Hereafter, without losing genrality, the α horizontal index is used in relations. The relations can be also written by the β horizontal index.

Note 3: The μ-level sets of $\tilde{A} \in E_1$ can be obtained using

$$\begin{aligned}
&\mathcal{H}^{-1}(A_{\mathcal{H}}(\mu, \alpha_A)) \\
&\quad = [\tilde{A}]^{\mu} \\
&\quad = \left[\inf_{\gamma \geq \mu} \min_{\alpha_A \in [0,1]} A_{\mathcal{H}}(\gamma, \alpha_A), \sup_{\gamma \geq \mu} \max_{\alpha_A \in [0,1]} A_{\mathcal{H}}(\gamma, \alpha_A) \right]. \quad (1)
\end{aligned}$$

Definition 3: Let $\tilde{A} : [a, b] \subseteq \mathbb{R} \to E_1$. The fractional horizontal membership function $A_{\mathcal{H}} : [0, 1] \times [0, \alpha_A^*] \to [c, d] \subseteq [a, b]$ is a fractional representation of $\tilde{A}(x)$ as $A_{\mathcal{H}}(\mu, \alpha_A) = x$ in which $x \in [c, d]$, $\mu \in [0, 1]$ is the membership degree of x in $\tilde{A}(x)$, $\alpha_A \in [0, \alpha_A^*]$ is the horizontal index, and $\alpha_A^* \in [0, 1]$ is called the fractional index. Moreover, $A_{\mathcal{H}}(\mu, \alpha_A) = \underline{A}^{\mu} + (\overline{A}^{\mu} - \underline{A}^{\mu})\alpha_A$.

Definition 4: The membership function $\tilde{A} \in E_1$ corresponding to a fractional horizontal membership function whose fractional index is α_A^*, is called the fractional membership function and denoted by $\tilde{A}_{\alpha^*}$.

The μ-level sets of the fractional membership function $\tilde{A}_{\alpha^*}$ is obtained using

$$\begin{aligned}
&\mathcal{H}^{-1}(A_{\mathcal{H}}(\mu, \alpha_A)) \\
&\quad = [\tilde{A}_{\alpha^*}]^{\mu} \\
&\quad = \left[\inf_{\gamma \geq \mu} \min_{\alpha_A \in [0,\alpha_A^*]} A_{\mathcal{H}}(\gamma, \alpha_A), \sup_{\gamma \geq \mu} \max_{\alpha_A \in [0,\alpha_A^*]} A_{\mathcal{H}}(\gamma, \alpha_A) \right]. \quad (2)
\end{aligned}$$

Note 4: The arrow "$\rightarrow$" in the upper of a linguistic term means that the fractional index of fractional membership function of the linguistic term is in the sense of α^*. For example, the linguistic term "positive big" (or PB) whose fractional index of fractional membership function is α^* is denoted by $\overrightarrow{PB}$. Analogously, the arrow "$\leftarrow$" in the upper of a linguistic term means that the fractional index of fractional membership function associated with the linguistic term is in the sense of β^*, e.g. $\overleftarrow{PB}$. Moreover, in the case that fractional index α^* or β^* is equal to one, the arrow in the upper of a linguistic term is omitted. Then, PB means that the fractional index of membership function of positive big is one.

Fig. 1 shows the horizontal membership function, fractional horizontal membership functions and fractional membership functions of a triangular fuzzy number associated with the linguistic term "positive big" (or PB).

Definition 5: The coordinate plane in which all fuzzy membership functions of linguistic terms of a linguistic variable exist is called membership functions plane. Moreover, the right half-plane (RHP) is the set of all points in the plane whose values are positive. Similarly, the left half-plane (LHP) is the set of all points in in the plane whose values are negative.

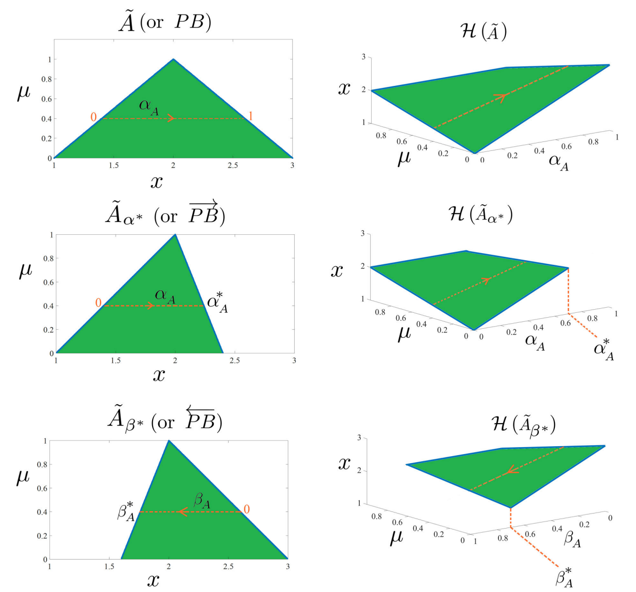

FIGURE 1. The fractional membership functions and fractional horizontal membership functions of the triangular fuzzy number $\tilde{A} = (1, 2, 3)$.

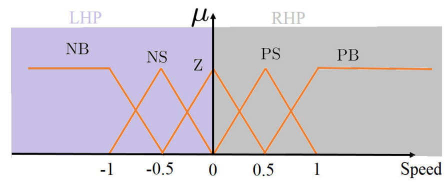

FIGURE 2. The membership functions plane.

Fig. 2 shows a membership functions plane consisting of 5 membership functions labeled as negative big (NB), negative small (NS), near zero (Z), positive small (PS), and positive big (PB). According to Definition 5, membership functions PB and PS are in the RHP, however, NB and NS are in the LHP.

III. FRACTIONAL FUZZY INFERENCE SYSTEMS

A fuzzy inference system is the process of obtaining a conclusion for a given input that may not have been encountered before. Thus, it plays a major role in the structure of a fuzzy system.

Note 5: In this paper we consider fuzzy systems with crisp inputs. In addition, for the sake of simplicity, fuzzy systems which are considered are those in which the consequent parts of the rule base consist of triangular or trapezoidal fuzzy numbers.

Let $R \triangleq (\tilde{A}, \tilde{B}, \mathcal{I})$ denote a fuzzy rule where $\tilde{A}, \tilde{B} \in E_1$ are membership functions describing linguistic terms (or values) of linguistic variables in the antecedent and consequent parts of the fuzzy rule, respectively. By combining the antecedent and consequent parts membership functions, the operator $\mathcal{I}$, as a translator or an interpreter, translates the fuzzy rule R into a fuzzy relation. Specifically, the following fuzzy rule

R: If X is $\tilde{A}$ Then Y is $\tilde{B}$,

might be interpreted as $\mathcal{I}(\tilde{A}(x), \tilde{B}(y)) = \tilde{A}(x) \wedge \tilde{B}(y)$ where x and y are the generic numerical values of X and Y,

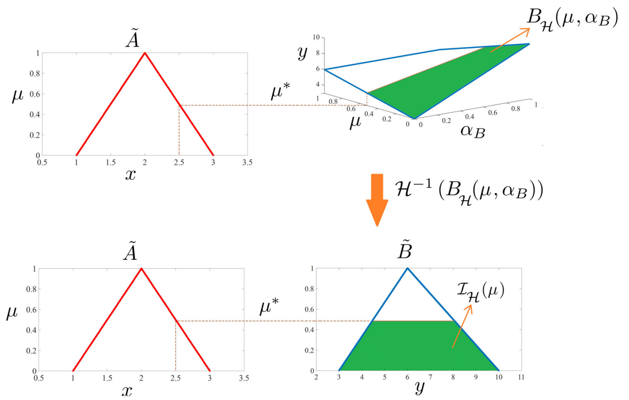

FIGURE 3. **The horizontal translation rule.**

respectively; and the base operator "$\wedge$" denotes the minimum operator (in Mamdani's FIS) or algebraic product operator (in Larsen's FIS). In the presence of an additional premise such as "X is x_0", which is known as a crisp input in the context of this paper, the compositional rule of inference asserts that we can infer the consequent part of the fuzzy rule from combining the truth degree of the antecedent part, i.e. μ^*, and the membership function of consequent part as $\mathcal{I}(\mu^*, \mu) = \mu^* \wedge \mu$ where $\mu^* = \tilde{A}(x_0)$, $\mu = \tilde{B}(y)$. Thus, the fuzzy rule R in the presence of a crisp input translates into $\mathcal{I}(\mu^*, \mu) = \mu^* \wedge \mu$. Hereafter, we denote $\mu^* = \tilde{A}(x)$ where the subscript of x has been suppressed for the sake of simplicity. The translation of a fuzzy rule can be also represented using horizontal membership functions which is called horizontal translation rule.

Definition 6: Consider the fuzzy rule R as

R: If X is $\tilde{A}$ Then Y is $\tilde{B}$,

and let $\mu^* = \tilde{A}(x)$ and $\mu = \tilde{B}(y)$. Then, the operator $\mathcal{I}_{\mathcal{H}}$ translates the fuzzy rule R horizontally as $\mathcal{I}_{\mathcal{H}}(\mu') \triangleq \mathcal{H}^{-1}\left(B_{\mathcal{H}}(\mu', \alpha_B)\right)$ where $\mu' = \mu$ or $\mu' = \frac{\mu}{\mu^*}$ provided that the operator acts as the minimum or algebraic product operator, respectively, and $0 \leq \mu \leq \mu^*$.

Fig. 3 illustrates the horizontal translation rule for the fuzzy rule R with a crisp input and by taking the minimum operator as the base operator into account, i.e. $\mu' = \mu, 0 \leq \mu \leq \mu^*$.

Proposition 1: The horizontal translation of a fuzzy rule is equivalent to the typical translation of the fuzzy rule, i.e. the operator $\mathcal{I}_{\mathcal{H}}(\mu')$ is equivalent to $\mathcal{I}(\mu^*, \mu) = \mu^* \wedge \mu$.

Proof: Suppose that $\tilde{B} = (b_1, b_2, b_3)$ is a triangular fuzzy number in the consequent part of a rule; and $\mu^* = \tilde{A}(x_0)$ is the membership degree of crisp input x_0 in the antecedent part. Let the base operator be minimum operator. Then, $\tilde{B}$ is clipped by μ^* as shown in Fig. 4. Thus, $\mathcal{I}(\mu^*, \mu) = \mu$, $0 \leqslant \mu \leqslant \mu^*$. According to Fig. 4, $\mathcal{I}(\mu^*, \mu) = \tilde{B}^*$ and $\tilde{B}^*$ is a trapezoidal membership function as $\tilde{B}^* = (b_1, b_2^*, b_3^*, b_3)$ in which

$$b_2^* = b_1 + (b_2 - b_1)\mu^*$$
$$b_3^* = b_3 - (b_3 - b_2)\mu^*$$

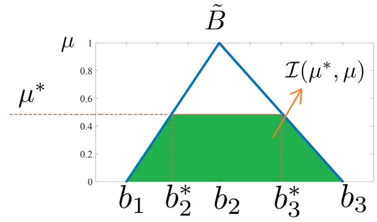

FIGURE 4. **A triangular fuzzy number clipped by a truth degree.**

Then, the horizontal membership function of $\tilde{B}^*$ is obtained as

$$\mathcal{H}(\tilde{B}^*) = b_1 + (b_2 - b_1)\mu + (b_3 - b_1)(1 - \mu)\alpha_B$$

where $0 \leqslant \mu \leqslant \mu^*$. Since $\mathcal{H}(\tilde{B}^*) = \mathcal{H}\left(\mathcal{I}(\mu^*, \mu)\right) = B_{\mathcal{H}}(\mu', \alpha_B)$, then the proof is complete. For the cases with the algebraic product operator as the base operator or the consequent part with a trapezoidal membership function, the proof is similar to the previous one and hence omitted. ■

Assume x_{01} and x_{02} are crisp inputs of a fuzzy system whose ith rule, R_i, $i = 1, \ldots, n$; has the following form:

R_i: If X_1 is $\tilde{A}_i^1$ and X_2 is $\tilde{A}_i^2$ Then Y is $\tilde{B}_i$.

Hence, the fuzzy inference system can be written as $\tilde{B} = \bigvee_i^n \mu_i^* \wedge \mu_i$ in which $\mu_i^* = \tilde{A}_i^1(x_{01}) \wedge \tilde{A}_i^2(x_{02})$, $\mu_i = \tilde{B}_i(y)$ and "$\bigvee$" denotes a union operator.

Definition 7: Let $\tilde{A}_i^j \in E_1, i = 1, \ldots, n; j = 1 \ldots, m$, be the membership function corresponding to a linguistic term of jth input in the ith rule of a rule base; and $\tilde{B}_i \in E_1$ be the membership function corresponding to a linguistic term in the consequent part. Then, the horizontal fuzzy inference system corresponding to the rule base with crisp inputs x_{0j} is defined as

$$\tilde{B} \triangleq \bigvee_i^n \mathcal{I}_{\mathcal{H}}(\mu_i') = \bigvee_i^n \mathcal{H}^{-1}\left(B_{i_{\mathcal{H}}}(\mu_i', \alpha_{B_i})\right) \tag{3}$$

where $\mu_i' = \mu_i$ (or $\mu_i' = \frac{\mu_i}{\mu_i^*}$), $\mu_i^* = \bigwedge_j^m \tilde{A}_i^j(x_{0j})$, and $0 \leq \mu_i \leq \mu_i^*$.

Note 6: In relation (3), $\alpha_{B_i} = \alpha_{B_j}$ if and only if the consequent part linguistic terms in the ith and jth rule are the same.

Proposition 2: The horizontal fuzzy inference system introduced in Definition 7 is equivalent to Mamdani's fuzzy inference system as long as $\mu_i' = \mu_i$; and is equivalent to Larsen's fuzzy inference system provided that $\mu_i' = \frac{\mu_i}{\mu_i^*}$.

Proof: Based on Proposition 1 the proof is straightforward. ■

So far, the processing of a rule base, in an FIS, has been based on extracting information from each rule using the membership function in the consequent part combined with a degree of truth of the antecedent part. This is also the case in horizontal FISs. But the horizontal FIS has been established based on horizontal membership functions which are a special case of fractional horizontal membership functions. Specifically, fractional horizontal membership functions are

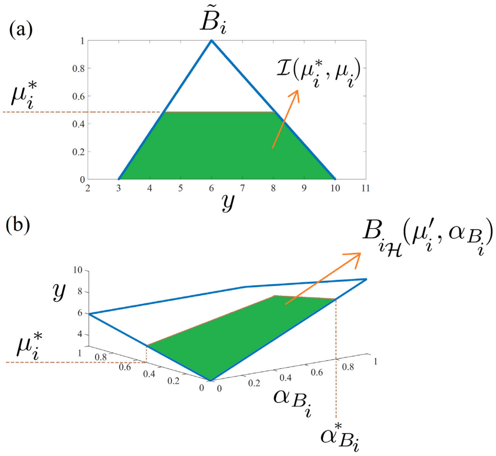

FIGURE 5. a) The typical inference result of a fuzzy rule. b) The fractional inference result of a fuzzy rule represented by a fractional horizontal membership function.

reduced to horizontal membership functions if fractional indices assume the integer number one. Inspired by this fact, a new generation of FISs can be established in which the processing of a rule base is not only based on the combination (or translation) involving the truth degree of the antecedent part, but also based on engaging fractional indices. As a way of illustration, let us consider the ith rule of a rule base as

R_i: If X is $\tilde{A}_i$ Then Y is $\tilde{B}_i$; $i = 1, \ldots, n$.

According to Definition 7, the horizontal FIS is as $\tilde{B} \triangleq \bigvee_i^n \mathcal{H}^{-1}\left(B_{i_{\mathcal{H}}}(\mu_i, \alpha_{B_i})\right)$ which is equivalent to the Mamdani's FIS $\tilde{B} \triangleq \bigvee_i^n \left(\tilde{A}_i(x_0) \wedge \tilde{B}_i(y)\right)$ where "$\wedge$" means the minimum operator and x_0 is a crisp input. Similar to Mamdani's FIS, in the horizontal FIS, each $B_{i_{\mathcal{H}}}$ is clipped only by the truth degree of the antecedent part, i.e. μ_i^*, while the horizontal indices α_{B_i} are free, i.e. $\alpha_{B_i} \in [0, 1]$. Simply put, the equivalency holds as long as the horizontal FIS is not induced by fractional indices $\alpha^*_{B_i}$. Nevertheless, by the aid of fractional horizontal membership functions, a new fuzzy inference system can be established enabling us to process the rule base using fractional indices of the consequent parts as well as truth degrees of the antecedent parts. Specifically, horizontal membership functions in the consequent parts can also be clipped by fractional indices which may be viewed as the induction of horizontal FIS by fractional indices. Thus, a two-degree freedom to process the information in the rule base as the knowledge base is provided. Indeed, in the sequel, it will be shown that the degree of freedom can increase based on the placement of fractional indices in a rule base. Fig. 5 shows the combination of the antecedent and consequent parts of the ith rule in two situations. The former, Fig. 5 (a), shows the ith rule inference result in which the consequent part membership function is clipped by μ_i^*. The latter depicted in Fig. 5 (b), presents the consequent part horizontal membership function that has been clipped not only by μ_i^*, but also by the fractional index. As a result, the new generation of fuzzy inference systems can be defined as follows:

Definition 8: Let μ_i^*; $i = 1, \ldots, n$, be the truth degree of the antecedent part of the ith rule of a rule base, and $\tilde{B}_i \in E_1$ be the membership function corresponding to a linguistic term in the consequent part of the ith rule. Then, the fuzzy inference system induced by fractional indices $\alpha^*_{B_i} \in [0, 1]$ is called fractional fuzzy inference system (or Mazandarani's FIS) and defined as $\tilde{B} \triangleq \bigvee_i^n \mathcal{H}^{-1}\left(B_{i_{\mathcal{H}}}(\mu_i', \alpha_{B_i})\right)$ where $\mu_i' = \mu_i$ when the base operator is minimum (or $\mu_i' = \frac{\mu_i}{\mu_i^*}$ when the base operator is algebraic product operator), $0 \leq \mu_i \leq \mu_i^*$, and $0 \leq \alpha_{B_i} \leq \alpha^*_{B_i}$.

The fractional FIS (FFIS) can be also defined abstractly as follows.

Definition 9: A fuzzy inference system in which the consequent parts of the rule base consist of fractional membership functions is called fractional fuzzy inference system.

The point that should be underscored is that the fractional fuzzy inference system might be regarded as a type of fuzzy if-then rule translation which may be called fractional translation rule. The FFIS might be also viewed as a new machinery of compositional rule of inference called fractional compositional rule of inference.

Proposition 3: The fractional FIS introduced in Definition 8 with fractional indices $\alpha^*_{B_i} = 1$ is equivalent to Mamdani's FIS as long as $\mu_i' = \mu_i$; and is equivalent to Larsen's FIS provided that $\mu_i' = \frac{\mu_i}{\mu_i^*}$.

Fig. 6 illustrates a simple FFIS in comparison with Mamdani's FIS. As is seen, the output membership function of the FFIS is a fraction of that of Mamdani's FIS. It is also easy to see that, the FFIS output coincides the output of Mamdani's FIS as long as the fractional indices assume the integer number one, i.e. $\alpha^*_{B_i} = 1$. Therefore, typical FISs, i.e. Mamdani's and Larsen's FISs are special cases of Mazandarani's FIS. Although in Definition 8 the FFIS has been presented based on $\alpha^*_{B_i}$ fractional indices, according to Note 2, it can also be presented based on either $\beta^*_{B_i}$ fractional indices or both form of fractional indices. Simply put, consequent parts of a rule base can be a set of fractional membership functions $\tilde{B}_{i\alpha^*_{B_i}}$ and $\tilde{B}_{j\beta^*_{B_j}}$ where i, j are rule numbers and $i \neq j$. In spite of this fact, for the sake of simplicity, the relations are presented only based on $\alpha^*_{B_i}$ fractional indices.

One of the important concepts which is originated from FFISs is what might be called the fracture index. The fracture index gives us a relative impression of the marginal behavior of an FFIS with respect to its corresponding FIS. In other words, as the fracture index approaches zero, the behavior of an FFIS approaches its corresponding FIS. The fracture index is defined based on the concepts of the left and right orders of an FFIS. In the sequel the mentioned concepts are presented.

Definition 10: The left and right orders of a fractional fuzzy inference system are defined respectively as $r = \frac{\sum_{i=1}^p \alpha_i^*}{N}$ and $s = \frac{\sum_{j=1}^q \beta_j^*}{N}$ where p and q are the number of consequent part fractional membership functions that are in the form of α^* and β^*; α_i^*, β_j^* are the values of fractional indices; and $N = p + q$ is the number of consequent part membership functions.

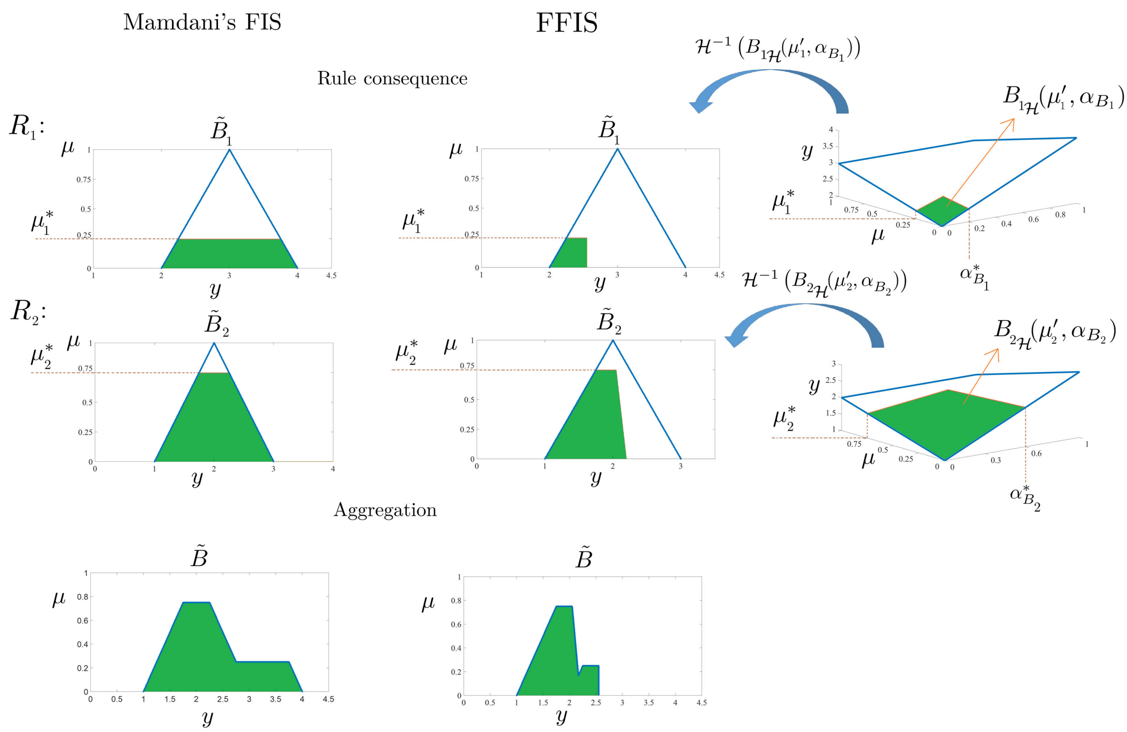

FIGURE 6. **Mamdani's and fractional fuzzy inference systems.**

Definition 11: The order of a fractional fuzzy inference system is defined by the pair (r, s) where r and s are the left and right orders of the fractional FIS.

Note 7: The order of an FFIS in which there are only fractional membership functions of the form α^* is defined solely based on the left order as $r = \frac{\sum_{i=1}^{N} \alpha_i^*}{N}$. Analogously, the order of an FFIS in which there are not any fractional membership functions of the form α^* is defined only based on right order, i.e. $s = \frac{\sum_{j=1}^{N} \beta_j^*}{N}$.

Definition 12: The fracture index of a fractional fuzzy inference system of order (r, s) denoted by γ is defined as $\gamma = 1 - (r + s)$.

It should be noted that according to Definition 10; $r, s \in [0, 1]$ and also, $0 \leq \gamma \leq 1$. When value one is assigned to each fractional index of an FFIS, the fractional FIS is reduced to a typical FIS. Hence, an FIS may be viewed as an FFIS whose fractional indices value is each equal to one, i.e. $\alpha_i^* = 1, \beta_j^* = 1, i = 1, \ldots, p; j = 1, \ldots, q$. Since a fractional membership function with the fractional index $\alpha^* = 1$ is the same as the one with the fractional index $\beta^* = 1$, then a typical FIS is equivalent with an FFIS in which there are only fractional membership functions of the form α^* with $\alpha_i^* = 1$. Thus, according to Definition 10 and Note 7, the order of a typical FIS is equal to one, i.e. $r = 1$. Therefore, typical FISs may be called first order fuzzy inference systems. Furthermore, based on Definition 12, the fracture index of a first order FIS (or a typical FIS, e.g. Mamdani's FIS) is equal to zero, i.e. $\gamma = 0$. It should be underscored that, as a whole, the relation $r + s = 1$ holds for the first order FISs. As it was mentioned before, the fracture index makes a sense of the similarity of the behavior of an FFIS with its corresponding FIS for small values of γ. Specifically, for small values of γ, as the fracture index approaches zero, the behavior of an FFIS gradually approaches its corresponding first order FIS. As a way of illustration, see Fig. 23 in Example 1.

Note 8: An FFIS is said to be corresponding to an FIS if and only if 1. The output of the FFIS partly approaches the FIS provided that γ approaches zero. 2. The output of the FFIS coincides the output of FIS for $\gamma = 0$.

Now, some questions arise: How fractional indices should be determined? What is the difference between the effect of the two forms of fractional indices, i.e. $\alpha_{B_i}^*$ and $\beta_{B_i}^*$, on an FFIS output? How an FFIS should be designed by the placement of fractional indices forms? In other words, which combination of fractional membership functions in the plane of consequent part membership functions should be considered? In the sequel, the questions are answered.

How fractional indices should be determined? An immediate answer is the use of an optimization algorithm - or a learning method - for characterizing the fractional indices. More concretely, assume $\hat{y}(t; \alpha^*), \alpha^* \triangleq (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$, $t \in [t_0, t_f] \subseteq \mathbb{R}$, is the output of a system which is controlled (or modelled) by an FFIS whose fractional indices are $\alpha_i^*; i = 1, \ldots, n$, and $y(t)$ is a desired (or taget) output. Then, with a cost function defined, e.g. $J(\alpha^*) = \int_{t_0}^{t_f} \left| y(t) - \hat{y}(t; \alpha^*) \right| dt$, the fractional indices are the solution of the optimization problem $\underset{\alpha^*}{minimize}\, J(\alpha^*)$ such that $0 \leq \alpha^* \leq 1$.

Theorem 1: Let $J \triangleq D\left(y(t), \hat{y}(t)\right)$ be a cost function where D denotes a metric defined on the space of real valued functions; $y(t)$ and $\hat{y}(t)$ are the target and approximated outputs of a system. Suppose that $\hat{y}(t)$ has been obtained by applying a typical FIS. In addition, let the approximated output obtained by applying an FFIS - corresponding to the typical FIS - be $\hat{y}(t; \alpha^*)$, $\alpha^* \triangleq (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$, such that the fractional indices $\alpha_i^*; i = 1, \ldots, n$, are the solution of the optimization problem $\underset{0 \le \alpha^* \le 1}{\text{minimize}}\ J(\alpha^*) = D\left(y(t), \hat{y}(t; \alpha^*)\right)$. Then, $J_{ffis} \le J_{fis}$ where $J_{ffis} = D\left(y(t), \hat{y}(t; \alpha^*)\right)$ and $J_{fis} = D\left(y(t), \hat{y}(t)\right)$.

Proof: Suppose the contrary, i.e. $J_{fis} < J_{ffis}$. Let $\alpha^* \triangleq (1, 1, \ldots, 1, \alpha_i^*, 1, \ldots, 1)$ and $0 \le \alpha_i^* < 1$ be the fractional indices for which $J_{ffis} = D\left(y(t), \hat{y}(t; \alpha^*)\right)$. Since α^* satisfies the optimization problem, then $\forall \hat{\alpha}^* \in [0, 1]$, $D\left(y(t), \hat{y}(t; \alpha^*)\right) \le D\left(y(t), \hat{y}(t; \hat{\alpha}^*)\right)$. Let $\hat{\alpha}^* = (1, 1, \ldots, 1)$, then according to Proposition 3 we have $J_{fis} = D(y(t), \hat{y}(t; \hat{\alpha}^*)$ meaning $J_{ffis} \le J_{fis}$. If $J_{ffis} = J_{fis}$, then the theorem is not contradicted. But, if $J_{ffis} < J_{fis}$, then the primary assumption is contradicted. Thus, the proof is complete. ■

In plain words, Theorem 1 states that applying a typical FIS (e.g. Mamadani's FIS) never leads to results which are more satisfactory than those obtained by applying a fractional FIS (or Mazandarani's FIS) corresponding to the typical FIS with a particular set of fractional indices.

Another answer to the question comes from the effect of fractional indices on the shape of membership functions and defuzzified output. Let us consider a triangular fuzzy number $\tilde{B}$ clipped by a truth degree in the level of μ^*. The output membership function - or the inference result - expands as the fractional index α_B^* approaches one; and contracts as the fractional index approaches zero. Accordingly, there is a large number of candidates for the crisp output, as α_B^* approaches one. The diversity of the candidates also decreases as α_B^* approaches zero. In simple terms, the exploration of a candidate for the crisp output becomes local. As a result, the smaller fractional index becomes, the more exploitation of a membership function becomes, and the more consistent crisp (or defuzzified) output with a membership function becomes as well. However, the more fractional index increases, the more exploration in a membership function increases. Furthermore, in a specific truth degree, μ^*, a small fractional index leads to the loss of a high fraction of information whose truth degree is up to μ^*. Clearly, in the case that the truth degree is high, a small fractional index causes a portion of high value information to be ignored in the exploitation. Hence, determining constant fractional indices by which a balance between exploration and exploitation is provided may not be an easy task, if an optimization algorithm is not applied. Nevertheless, it might be possible to characterize fractional indices dynamically. Concretely, a fractional index can be determined as a function of truth degree, i.e. $\alpha_B^* \triangleq f(\mu^*), f : [0, 1] \to [0, 1]$. Based on the aforementioned facts, the function $f(\mu^*)$ should be considered as a monotonically increasing function of μ^*. Specifically, by considering such a monotonically increasing function, an FFIS intelligently considers not only the truth degrees of information included in membership functions, but also the volume of the information in the process of making a conclusion. In other words, the volume of information extracted from a membership function depends on the truth degree of information. Concretely, a small truth degree results in a small fractional index which means the membership function contracts. Therefore, a small volume of information is engaged in the process of making a conclusion. Reciprocally, a high value of truth degree leads to the expansion of the membership function which accordingly results in engaging a high volume of information in the process of making a conclusion. In fact, the membership function has been given a dynamic whose behavior is expansive for high-value information and contractive for low-value information. Concretely, the higher the truth degree, the more valuable the information, and the larger the volume of information that is involved in the process of making a conclusion. Fig. 7 shows some of fractional indices as functions of μ^* and their effects on the shape of inference results for some amounts of μ^*. What should be underscored is that considering the function $f(\mu^*)$ as a monotonically increasing function would not be a general rule for every applications. One of the important criteria by which we can determine the function $f(\mu^*)$ is the reaction trajectories map (RTM) which will be explained later.

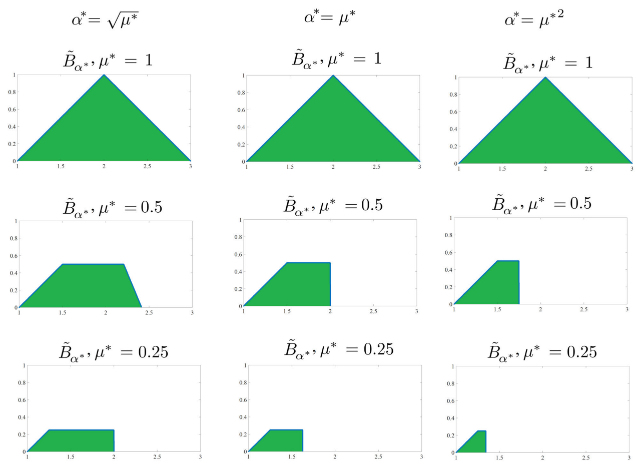

FIGURE 7. Dynamical fractional indices.

The next question then arises: What is the difference between the effect of two forms of fractional indices, i.e. α^* and β^*, on an FFIS output? Since α^* fractional index clips the membership function from the lower bound, defuzzification of fractional membership function leads to an output that decreases as the level of μ^* does. However, since fractional membership function is clipped by β^* from the upper bound, the crisp output increases as the level of μ^* decreases. As a way of illustration, let us consider the triangular fuzzy number $\tilde{B} = (1, 2, 3)$ whose corresponding fractional membership functions $\tilde{B}_{\alpha^*}$ and $\tilde{B}_{\beta^*}$ have been clipped by a truth

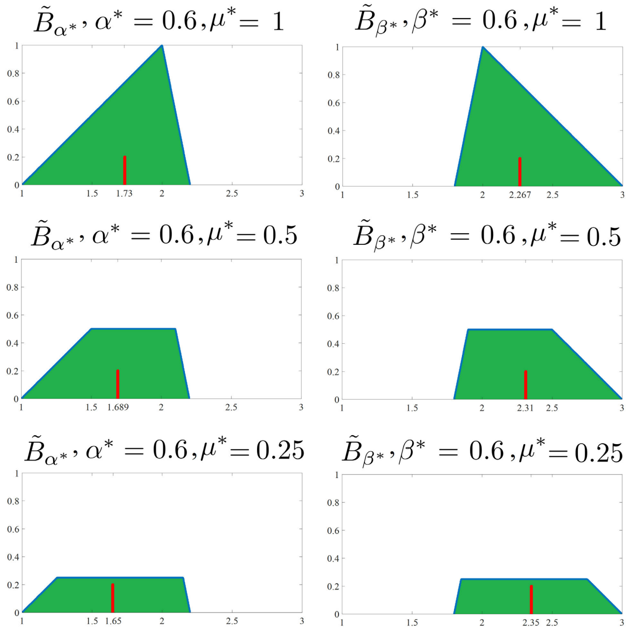

FIGURE 8. The comparison of defuzzified output obtained from the two forms of fractional indices. The red line indicates the defuzzified output.

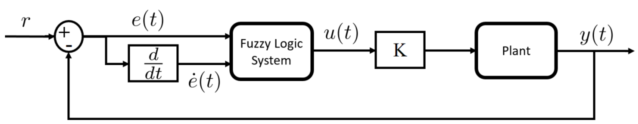

FIGURE 9. The fuzzy closed loop control system.

degree in the level of μ^*. Suppose that the fractional indices are constant, e.g. $\alpha^* = \beta^* = 0.6$ and the crisp output is obtained using the centroid of the area method in each μ^* level. As is seen in Fig. 8, the defuzzified output of $\tilde{B}_{\alpha^*}$ decreases as μ^* does, however, that of $\tilde{B}_{\beta^*}$ increases. As a result, the defuzzified output coming from a fractional membership function with β^* fractional index is greater than or equal to that coming from a fractional membership function with α^* fractional index.

We are now in the position to address the last question: Which combination of fractional membership functions in the plane of consequent part membership functions should be considered? In an FFIS the consequent parts membership functions are taken into account as fractional membership functions. In addition, there are two forms of fractional membership functions, i.e. the form that deals with α^* index and the form that deals with β^* index. Hence, in a fuzzy system whose membership functions plane corresponding to the consequent part consists of "m" membership functions, there are "2^m" combinations of fractional membership functions forms. For each combination, an FFIS might be considered. As a way of illustration, let us consider a fuzzy closed loop control system shown in Fig. 9.

The inputs of the fuzzy system are error and derivative of error denoted by $e(t)$ and $\dot{e}(t)$, respectively; and the

e \ $\dot{e}$	NB	NS	Z	PS	PB
NB	NB	NB	NB	NS	Z
NS	NB	NB	NS	Z	PS
Z	NB	NS	Z	PS	PB
PS	NS	Z	PS	PB	PB
PB	Z	PS	PB	PB	PB

FIGURE 10. The rule base of fuzzy system.

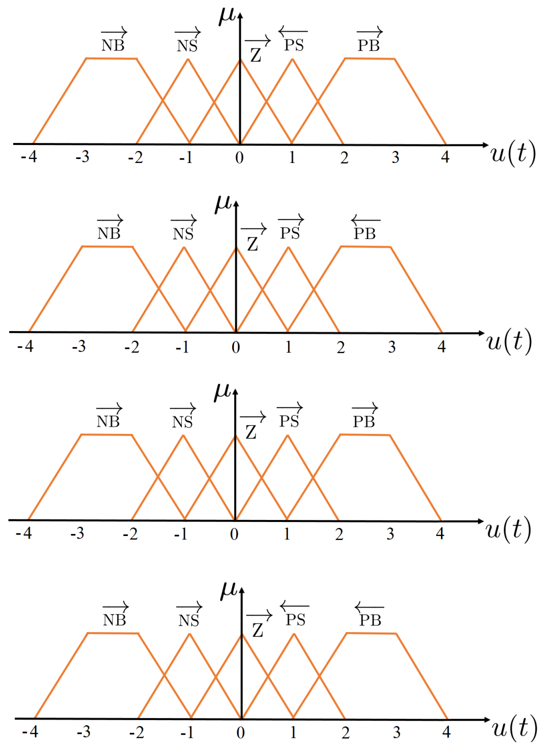

FIGURE 11. Four different combinations of fractional membership functions in the consequent part.

defuzzified output - or control signal - is $u(t)$. The output of the plant and the set-point have been denoted by $y(t)$ and r, respectively, and K is a gain. Suppose that each inputs and output are described by 5 linguistic terms, negative big (NB), negative small (NS), near zero (Z), positive small (PS), and positive big (PB). The rule base of fuzzy system has been given in Fig. 10. Since in the consequent part, $u(t)$ consists of 5 linguistic terms, or equivalently 5 membership functions, there are 32 different combinations of consequent part fractional membership functions forms. Fig. 11 shows 4 out of 32 combinations in the membership functions plane of $u(t)$ where the forms of fractional membership functions of NB, NS, and Z are the same but those of PS and PB are different. Let us now compare the outputs of these FFISs with this assumption that only the rules corresponding to the fractional membership functions $\overrightarrow{PB}$, $\overleftarrow{PB}$, $\overrightarrow{PS}$, and $\overleftarrow{PS}$ are involved. These rules have been also highlighted in the rule base shown in Fig. 10.

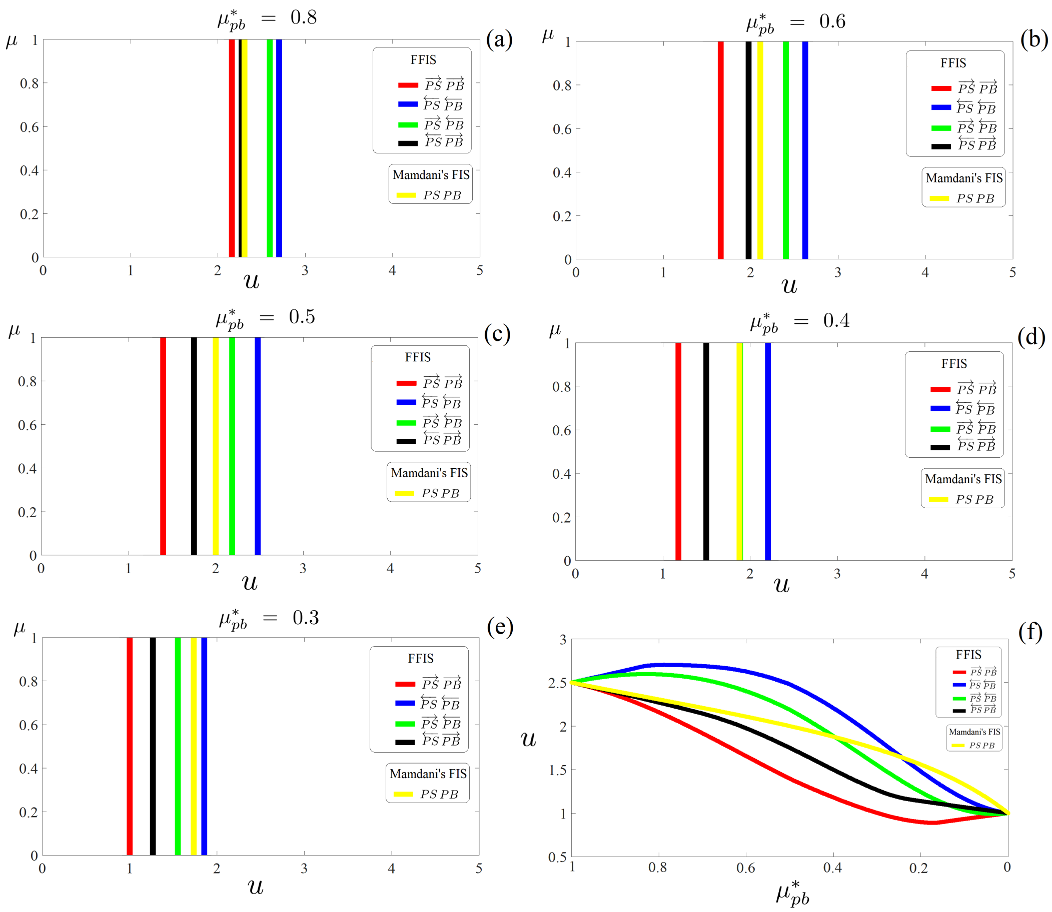

FIGURE 12. **Comparison of the output of Mamdani's FIS and that of four different combinations of fractional membership functions in FFIS - (a) to (e). The RTM in which fractional indices are linear functions of truth degree - (f).**

Suppose that μ^*_{pb} is the truth degree of antecedent part corresponding to the consequent part whose membership function has been interpreted as positive big. Additionally, let us assume that μ^*_{pb} decreases from one to zero. Correspondingly, $\mu^*_{ps} \triangleq 1 - \mu^*_{pb}$ is supposed to be the truth degree of antecedent part corresponding to the consequent part whose membership function has been interpreted as positive small; and thus it increases from zero to one. In other words, the motion direction of linguistic variable $u(t)$ can be viewed as if it is from the positive big to positive small. Simply put, the positive control signal is moving towards near zero which might happen when $e(t)$ and/or $\dot{e}(t)$ is moving towards near zero, i.e. from RHP to LHP. The defuzzified output, $u(t)$, of the FFISs and Mamdani's FIS have been illustrated in Fig. 12 - (a) to (e). The outputs have been obtained using the centroid of the area method. Moreover, the fractional indices have been considered as linear functions of the truth degree, i.e. $\alpha^*_{PB} = \mu^*_{pb}, \alpha^*_{PS} = \mu^*_{ps}, \beta^*_{PB} = \mu^*_{pb}$, and $\beta^*_{PS} = \mu^*_{ps}$. The RTM which is the trajectories of output with respect to the truth degree have been presented in Fig. 12- (f). As a matter of fact, an RTM includes information corresponding to the reaction of different forms of fractional membership functions when they interact with each other by varying the truth degree. In other words, the RTM carries a concept of fuzzy system reaction intensity in the face of $e(t)$ and $\dot{e}(t)$ variations. For more illustration let us consider the different reactions of fuzzy systems for the four combinations mentioned before. As is seen, the FFIS output decreases with maximum gradient when the consequent part linguistic terms are $\overrightarrow{PS}$ and $\overrightarrow{PB}$, i.e. both of them are in the form of α^* fractional membership functions. This is while the FFIS output decreases with minimum gradient when the consequent part linguistic terms are $\overleftarrow{PS}$ and $\overleftarrow{PB}$, i.e. both of them are in the form of β^* fractional membership functions. FFIS outputs corresponding to the consequent parts consisting of the membership functions $\overleftarrow{PS}, \overrightarrow{PB}$ or $\overrightarrow{PS}, \overleftarrow{PB}$ and Mamdani's FIS output take place between the outputs that decrease with the maximum and minimum gradients. Additionally, with $\overleftarrow{PS}, \overrightarrow{PB}$ or $\overrightarrow{PS}, \overleftarrow{PB}$ considered in the consequent parts, the outputs decrease with a gradient which may be either more or less than the gradient of output obtained by Mamdani's FIS.

Specifically, the use of α^* fractional membership functions leads to an increase in the gradient of the output that is decreasing. In fact, they push the output downward. However, the use of β^* fractional membership functions leads to a decrease in the gradient of the output that is decreasing, i.e. they pull the output upward. Analogously, the use of β^* fractional membership functions leads to an increase in the gradient of the output that is increasing, i.e. they push the output up. However, the use of α^* fractional membership functions results in a decrease in the gradient of the output that is increasing, i.e. they pull the output down. In the case that RHP membership functions of control signal are active and the control signal is increasing, then $e(t)$ and/or $\dot{e}(t)$ is in the RHP and is getting far from the desired point. Thus, by taking β^* fractional membership functions into account in the RHP, the control signal is pushed to increase with a high gradient. Simply put, the FFIS is more sensitive to the inputs linguistic terms being activated in the RHP towards PB, and shows a fast reaction. However, in the case that RHP membership functions of control signal are active and the control signal is decreasing, then $e(t)$ and/or $\dot{e}(t)$ is in the RHP and is approaching the desired point. Thus, the presence of β^* fractional membership functions in the RHP leads to an output which decreases with a low gradient. Specifically, the control signal decreases gradually and *conservatively*. Similarly, considering α^* fractional membership functions in the LHP results in a control signal that conservatively increases in the case that $e(t)$ and/or $\dot{e}(t)$ is in the LHP and is approaching the desired point. Moreover, the control signal decreases with a high gradient when $e(t)$ and/or $\dot{e}(t)$ is in the LHP and is getting far from the desired point. Eventually, for the closed loop control system shown in Fig. 9, the combination of fractional membership functions forms should be such that most of RHP membership functions are of the form that deals with β^* fractional index, and most of LHP membership functions are of the form that deals with α^* fractional index. In the case that all fractional membership functions in the LHP and RHP are of the forms of α^* and β^* fractional indices, respectively, the combination is called the aggressive combination. It should be noted that for control structures that are different from what has been shown in Fig. 9, other combinations of fractional membership function forms in the RHP and LHP might be taken into account based on the features of the two forms of fractional membership functions mentioned above. One remaining point is about fractional membership functions of linguistic terms that are neither in the RHP nor LHP, e.g. near zero. In this case, there is no significant difference between two forms of fractional membership functions on condition that membership function of the linguistic term is symmetric with respect to the origin. Otherwise, α^* fractional membership function might be assigned to the linguistic term providing that the membership function of the term leans towards the LHP. Analogously, β^* fractional membership function might be assigned to the term providing that the membership function of the term leans towards the RHP.

The point that should be underscored is that any combination of fractional membership functions forms is a design

$e \backslash \dot{e}$	NB	NS	Z	PS	PB
NB	$\overrightarrow{\text{NB}}$	$\overrightarrow{\text{NB}}$	$\overrightarrow{\text{NB}}$	$\overrightarrow{\text{NS}}$	Z
NS	$\overrightarrow{\text{NB}}$	$\overrightarrow{\text{NB}}$	$\overrightarrow{\text{NS}}$	Z	$\overleftarrow{\text{PS}}$
Z	$\overrightarrow{\text{NB}}$	$\overrightarrow{\text{NS}}$	Z	$\overleftarrow{\text{PS}}$	$\overleftarrow{\text{PB}}$
PS	$\overrightarrow{\text{NS}}$	Z	$\overleftarrow{\text{PS}}$	$\overleftarrow{\text{PB}}$	$\overleftarrow{\text{PB}}$
PB	Z	$\overleftarrow{\text{PS}}$	$\overleftarrow{\text{PB}}$	$\overleftarrow{\text{PB}}$	$\overleftarrow{\text{PB}}$

FIGURE 13. The rule base of the closed loop control system equipped with FFIS, the aggressive combination.

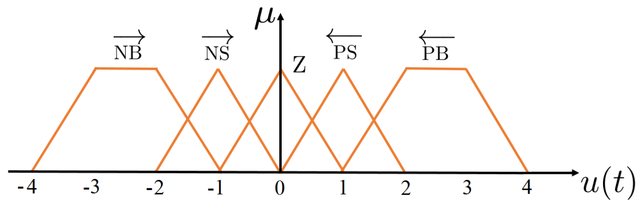

FIGURE 14. The plane of consequent part fractional membership functions of the closed loop control system that is an aggressive combination.

of fuzzy system. In this regard, the RTM plays a major role. In fact, the RTM sheds light on the intensity of reactions of any combination of fractional membership functions. Thus, the RTM enables us to make a decision on the design of fuzzy system based on our desirable intensity of reaction of any system counterpart (or consequent part).

Note 9: It is noteworthy to pinpoint that the fractional index for the fractional membership function of near zero should be determined such that the defuzzified output coincides the core of membership function if the truth degree combined with the fractional membership function is unity.

As mentioned above, the rule base and consequent part membership functions plane corresponding to FFIS applied in the closed loop system can be considered as shown in figures. 13 and 14. It should be noted that Fig. 14 shows an aggressive combination.

Consequently, in the sequel a theorem which is more general than Theorem 1 is given.

Theorem 2: Let $J \triangleq D\left(y(t), \hat{y}(t)\right)$ be a cost function where D denotes a metric defined on the space of real valued functions; $y(t)$ and $\hat{y}(t)$ are the target and approximated outputs of a system. Suppose that $\hat{y}(t)$ has been obtained by applying a typical FIS. In addition, let the approximated output obtained by applying a fractional FIS - corresponding to the typical FIS - be $\hat{y}(t; \alpha^*)$ where $\alpha^* \triangleq (f_1, f_2, \ldots, f_n)$, such that $f_i : [0, 1] \to [0, 1]$, $\alpha^* = \arg\left(\underset{\alpha^*}{minimize}\, J(\alpha^*)\right)$ and $J(\alpha^*) = D\left(y(t), \hat{y}(t; \alpha^*)\right)$. Then, $J_{ffis} \leq J_{fis}$ where $J_{ffis} = D\left(y(t), \hat{y}(t; \alpha^*)\right)$ and $J_{fis} = D\left(y(t), \hat{y}(t)\right)$.

Proof: The proof is similar to that of Theorem 1 and hence omitted. ∎

The importance of Theorem 2 comes from the fact that, independent of the problem in question, one can always be sure that, by applying FFIS, they can obtain a more satisfactory result than the result that would be obtained by applying a typical FIS on condition that the exceptional situation $J_{ffis} = J_{fis}$ does not happen. As a matter of fact, it is very unlikely that the optimal solution is a vector of functions which are all constant and equal to one. Simply put, it is almost impossible for the solution of the optimization problem to correspond to the typical FIS. As a result, by ignoring the exceptional case, there is always at least an FFIS which leads to results that are more satisfactory than those a typical FIS leads to. In other words, a typical FIS never yields to results which are more satisfactory than those obtained by the FFIS corresponding to the typical FIS provided that a particular set of fractional indices is taken into account.

In the sequel, we are going to show the superiority of the proposed FIS in comparison with other typical FISs by some examples. Although complex examples could have been presented, our aim, in this preliminary step, is to clarify the performance of FFIS without involving readers in the complexities of examples. Thus, the main goal in the following examples is to compare the performance of FFIS with the typical FISs under same settings, rather than to achieve a perfect control or fuzzy model. In the following, a particle swarm optimization (PSO) algorithm is employed where the following conditions are considered: 1. The maximum number of iterations is 100, 2. The population size is 50, 3. The constriction coefficients [14] are applied. 4. The cost function is integral of the absolute value of the error, i.e. $\int_0^{t_f} |e(t)|dt$, where the error is as $e(t) = y - \hat{y}$ in which $y(t)$ and $\hat{y}(t)$ are the target (or desired) and approximated (or real) outputs of the system under study in the time interval $[0, t_f]$.

Example 1: Consider the closed loop control system shown in Fig. 9. The plant is an inverted pendulum whose mathematical model is presented by the following nonlinear differential equations

$$\ddot{\theta}(t) = \frac{g\sin(\theta(t)) - 0.5aml\dot{\theta}^2(t)\sin(2\theta(t)) - a\cos(\theta(t))\bar{u}(t)}{\frac{4l}{3} - aml\cos^2(\theta(t))} \tag{4}$$

where θ is the pendulum angle (in radian) from the vertical axis, $g = 9.8(\frac{m}{s^2})$ is the gravity constant, $m = 2(Kg)$ is the mass of pendulum, $a = \frac{1}{m+M}$, $M = 8(Kg)$ is the mass of the cart, $2l = 4$ is the length of pendulum, and $\bar{u}$ is the force applied to the cart which is equal to $Ku(t)$ based on Fig. 9. The objective of the control system is to keep the pendulum upright. The rule base is the same as that has been shown in Fig. 13. The membership functions of inputs and output of the fuzzy system have been presented in Fig. 15.

Using the PSO algorithm and by considering Mamdani's FIS in the fuzzy system, the optimal value of the gain $K \in [0, 220]$ is obtained which is $K = 219.7$. The centroid of the area has been used as the defuzzification method. The FFIS is applied in two different cases: 1. With constant fractional indices, 2. With dynamical fractional indices. The constant fractional indices - which have been considered as arbitrary values - have been presented in Fig. 16. According to Definitions 10 and 12, the first, second and third set of fractional indices determine, respectively, the FFIS of orders $(\frac{1.8}{5}, \frac{0.75}{5})$, $(\frac{1}{5}, 0)$ and $(\frac{2}{5}, \frac{1}{5})$ with respected fracture index $\gamma = 0.49$, $\gamma = 0.8$ and $\gamma = 0.4$. The dynamical fractional indices of the LHP and RHP membership functions have been considered as quadratic functions of antecedent part truth degree. Concretely, they are in the form of $\alpha^* = f(\mu^*) = \mu^{*2}$ and $\beta^* = f(\mu^*) = \mu^{*2}$, see Fig. 16. Moreover, the base operator in the FFIS is the same as Mamdani's FIS. The result of applying Mamdani's, Larsen's and FFISs have been shown in figures 17, 18 and 19.

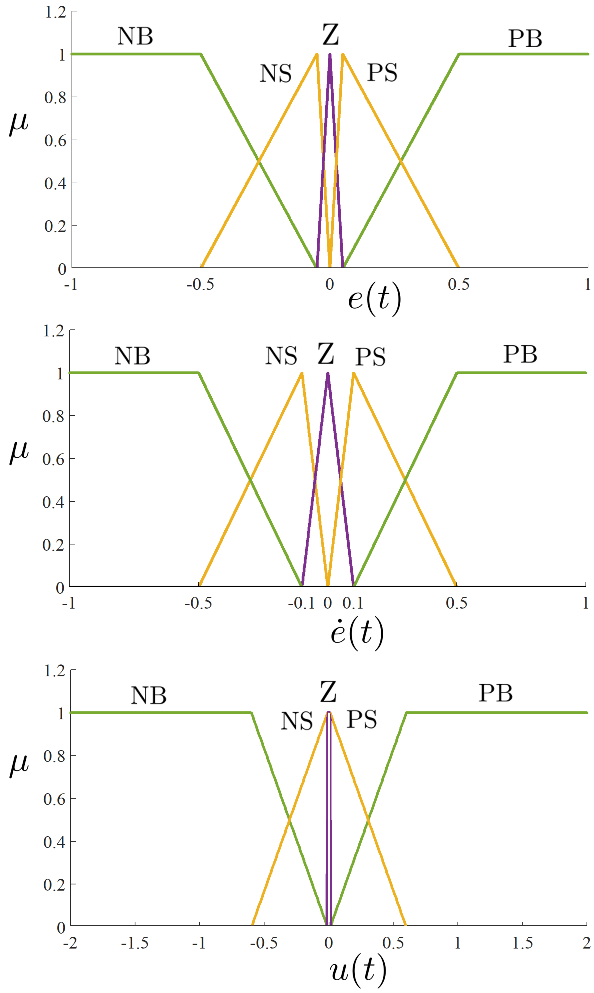

FIGURE 15. Membership functions of inputs and the output associated with Example 1.

As is seen, applying the FFIS is obviously more effective than Mamdani's and Larsen's FISs. In addition, Fig. 18 shows that the settling time of the system equipped with the FFIS is less than that of the system equipped with Mamdani's or Larsen's FIS. The other point that should be noted is that

	Constant	Constant	Constant	Dynamical
Fractional Indices	Set # 1	Set # 2	Set # 3	Set # 4
β^*_{PB}	0.5	0	0.5	μ^{*2}
β^*_{PS}	0.25	0	0.5	μ^{*2}
α^*_{Z}	1	1	1	1
α^*_{NS}	0	0	0.5	μ^{*2}
α^*_{NB}	0.8	0	0.5	μ^{*2}

FIGURE 16. Fractional indices associated with Example 1.

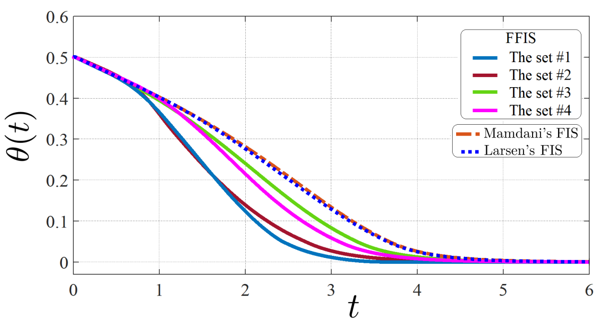

FIGURE 17. The angle trajectory of inverted pendulum controlled by different FISs.

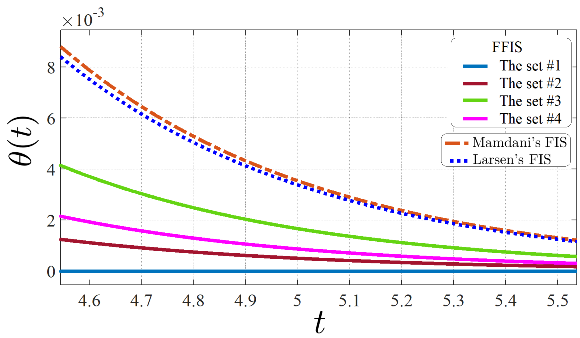

FIGURE 18. The angle trajectories of inverted pendulum in the time interval [0.46, 0.55].

the fractional fuzzy inference system whose fracture index is $\gamma = 0.4$ results in an output which is more similar to that of its corresponding typical FIS, i.e. Mamdani's FIS. The previous results have been obtained in the time interval [0, 6] with the sampling time "$\frac{6}{6000} = 0.001$". Now, let us consider the result of applying three different FISs in the time interval [0, 10] with the sampling time "$\frac{10}{1500} = 0.0067$" which is more than that considered on time interval [0, 6]. Figures 20 and 21 illustrate the results. Therefore, all the obtained results are in accordance with Theorems 1 and 2.

It should be noted that the structure of rule base and the system in question have been considered the same for the typical FISs and the FFIS. Additionally, it is noteworthy that in the real world applications achieving a small sampling time not only may not be always possible, but also may increase the costs involved. As is seen, by increasing the sampling

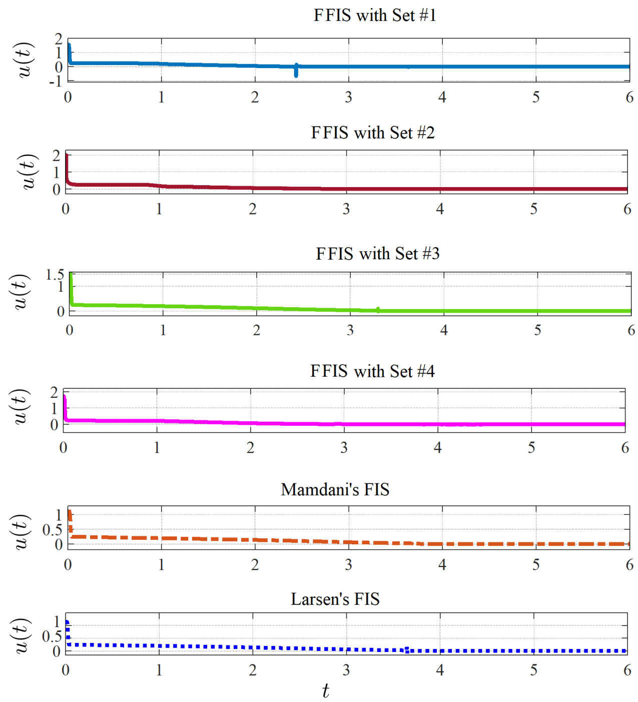

FIGURE 19. The control signal generated by different FISs.

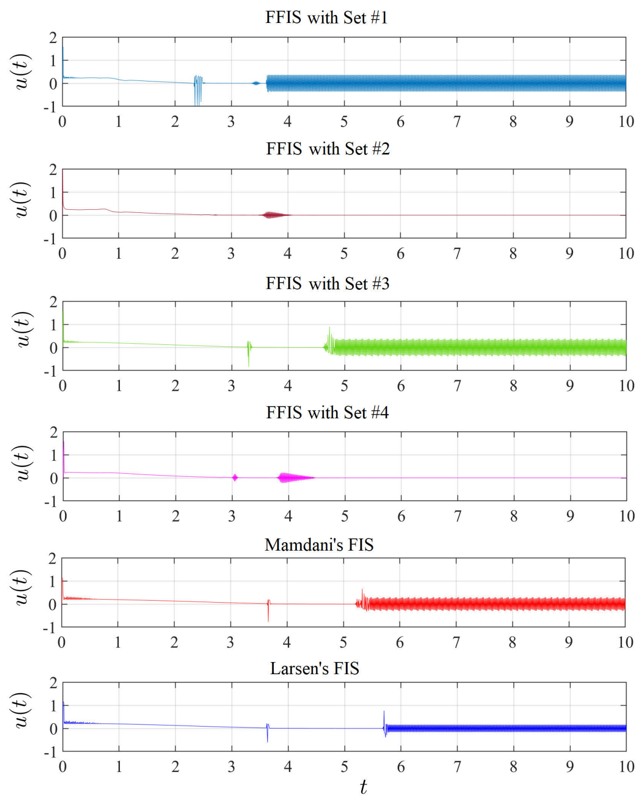

FIGURE 20. The control signal generated by different FISs with the increased sampling time.

time from "0.001" to "0.0067", Mamdani's and Larsen's FISs output control signals with high chattering. However, the FFIS is capable of sustain this situation. Simply put,

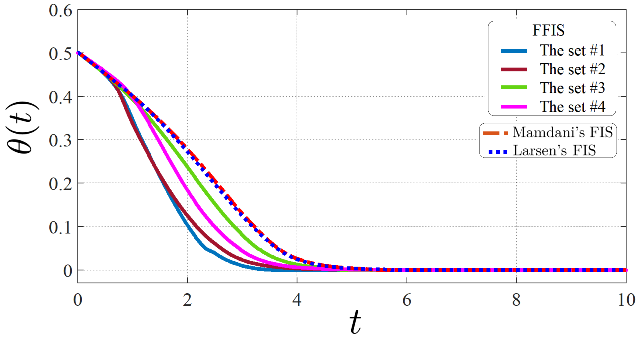

FIGURE 21. The angle trajectory of inverted pendulum controlled by different FISs with the increased sampling time.

Fractional Indices	Set # 1	Set # 2	Set # 3	Set # 4	Set # 5
β^*_{PB}	0.5	0.5	0.6	0.8	0.9
β^*_{PS}	0.5	0.6	0.7	0.7	0.9
α^*_{Z}	1	1	1	1	1
α^*_{NS}	0.5	0.6	0.7	0.9	0.9
α^*_{NB}	0.5	0.6	0.8	0.9	1
Fracture Index (γ)	0.4	0.34	0.24	0.14	0.06

FIGURE 22. The fractional FIS corresponding to Example 1 with different fracture indices.

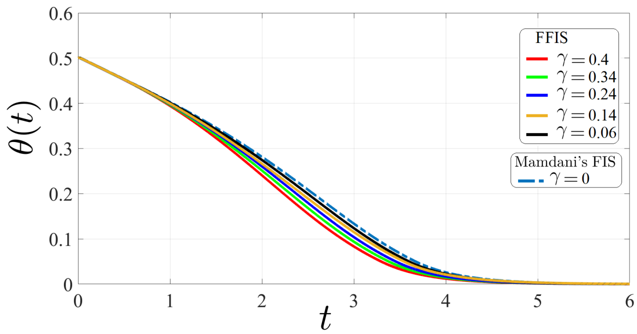

FIGURE 23. The effect of fracture index on the behavior of a fractional FIS.

there are sets of fractional indices by which the FFIS outputs the control signal without chattering or with a chattering alleviated to a great extent, in comparison with Mamdani's and Larsen's FISs outputs. It should be pinpointed that the results have been obtained based on the arbitrary fractional indices and that they are not optimal. Additionally, in order to show the effect of fracture index, an FFIS with different fracture indices shown in Fig. 22 have been considered and applied in the closed loop system. As was stated previously and illustrated in Fig. 23, the output of the system equipped with the FFIS comes close to that equipped with Mamdani's FIS as the fracture index approaches zero, for small values of γ. As a matter of fact, one of the advantages of fracture index is that it helps us to fine tune the output.

Example 2: The purpose of this example is to make a comparison between FFIS, Mamdani's and Larsen's FISs using the well-known Mackey-Glass chaotic time-series prediction

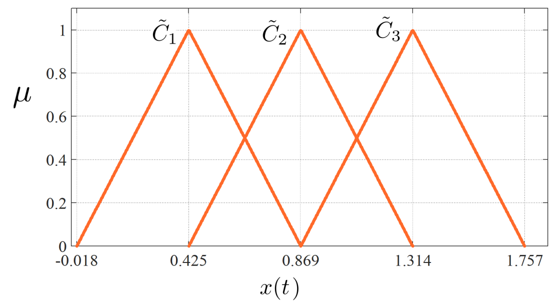

FIGURE 24. The membership functions of the output $x(t)$ corresponding to Example 2.

(1,1,1,1,2)	(2,1,1,1,2)	(3,1,1,1,2)	(1,2,1,1,2)	(2,2,1,1,2)	(3,2,1,1,2)	(1,3,1,1,2)	(2,3,1,1,2)	(3,3,1,1,2)
(1,1,2,1,2)	(2,1,2,1,2)	(3,1,2,1,2)	(1,2,2,1,2)	(2,2,2,1,2)	(3,2,2,1,2)	(1,3,2,1,2)	(2,3,2,1,2)	(3,3,2,1,1)
(1,1,3,1,2)	(2,1,3,1,2)	(3,1,3,1,2)	(1,2,3,1,2)	(2,2,3,1,2)	(3,2,3,1,1)	(1,3,3,1,2)	(2,3,3,1,1)	(3,3,3,1,1)
(1,1,1,2,2)	(2,1,1,2,2)	(3,1,1,2,2)	(1,2,1,2,2)	(2,2,1,2,2)	(3,2,1,2,2)	(1,3,1,2,2)	(2,3,1,2,2)	(3,3,1,2,2)
(1,1,2,2,2)	(2,1,2,2,2)	(3,1,2,2,2)	(1,2,2,2,2)	(2,2,2,2,2)	(3,2,2,2,2)	(1,3,2,2,3)	(2,3,2,2,2)	(3,3,2,2,1)
(1,1,3,2,2)	(2,1,3,2,2)	(3,1,3,2,2)	(1,2,3,2,3)	(2,2,3,2,2)	(3,2,3,2,2)	(1,3,3,2,3)	(2,3,3,2,2)	(3,3,3,2,2)
(1,1,1,3,2)	(2,1,1,3,2)	(3,1,1,3,2)	(1,2,1,3,2)	(2,2,1,3,2)	(3,2,1,3,2)	(1,3,1,3,3)	(2,3,1,3,2)	(3,3,1,3,2)
(1,1,2,3,2)	(2,1,2,3,2)	(3,1,2,3,3)	(1,2,2,3,3)	(2,2,2,3,3)	(3,2,2,3,2)	(1,3,2,3,3)	(2,3,2,3,2)	(3,3,2,3,2)
(1,1,3,3,3)	(2,1,3,3,3)	(3,1,3,3,3)	(1,2,3,3,3)	(2,2,3,3,3)	(3,2,3,3,2)	(1,3,3,3,3)	(2,3,3,3,2)	(3,3,3,3,2)

FIGURE 25. The rule base corresponding to Example 2.

benchmark dataset. The chaotic time-series considered in this example comes from the following delay differential equation:

$$\dot{x}(t) = \frac{0.2x(t-\tau)}{1 + x^{10}(t-\tau)} - 0.1x(t)$$

where $x(0) = 1.2$, $\tau = 17$ and $x(t) = 0$ for $t < 0$. Using the fourth-order Runge-Kutta method, 1000 input-output data pairs of the format: $[x(t-24), x(t-18), x(t-12), x(t-6), x(t)]$ have been extracted where $24 \leq t \leq 1023$, $x(t)$ is the output and the former four variables are inputs. Specifically, the FISs make an approximation of the function $f : \mathbb{R} \to \mathbb{R}$ such that $x(t) = f(x(t-24), x(t-18), x(t-12), x(t-6))$. In the 1000 input-output data pairs, the first 500 pairs are considered as the training dataset to build the fuzzy model of the time series, while the remaining 500 pairs have been used to test the validity of the fuzzy model. Here, 3 membership functions have been considered for each input and the output. The membership functions corresponding to the inputs $x(t-24)$, $x(t-18)$, and $x(t-12)$ are the same and have been taken into account as Gaussian membership functions $\tilde{A}_1 = G(0.274, 0.219)$, $\tilde{A}_2 = G(0.274, 0.766)$, and $\tilde{A}_3 = G(0.274, 1.314)$. The membership functions of $x(t-6)$ are as $\tilde{B}_1 = G(0.267, 0.246)$, $\tilde{B}_2 = G(0.267, 0.780)$, and $\tilde{B}_3 = G(0.267, 1.314)$. Fig. 24 shows the membership function of the output $x(t)$. By the use of the simplest fuzzy look-up table method, the fuzzy rules have been obtained. The number of fuzzy rules used for this case is 81 that have been presented in Fig. 25. In the figure, a rule like: if $x(t-24)$ is $\tilde{A}_i$ and $x(t-18)$ is $\tilde{A}_j$ and $x(t-12)$ is $\tilde{A}_k$ and $x(t-6)$ is $\tilde{B}_m$ then $x(t)$ is $\tilde{C}_p$ has been expressed in the format (i, j, k, m, p).

The performance criterion by which the comparison between FISs has been made is the 2-norm of the error

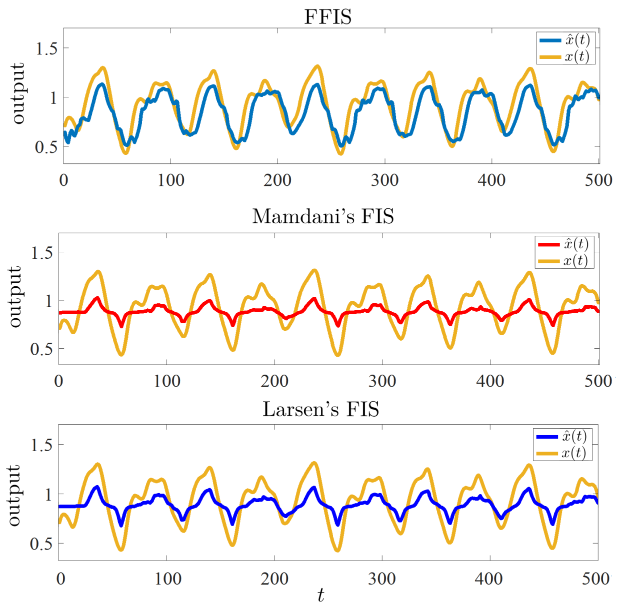

FIGURE 26. The results obtained from applying FISs on the training data.

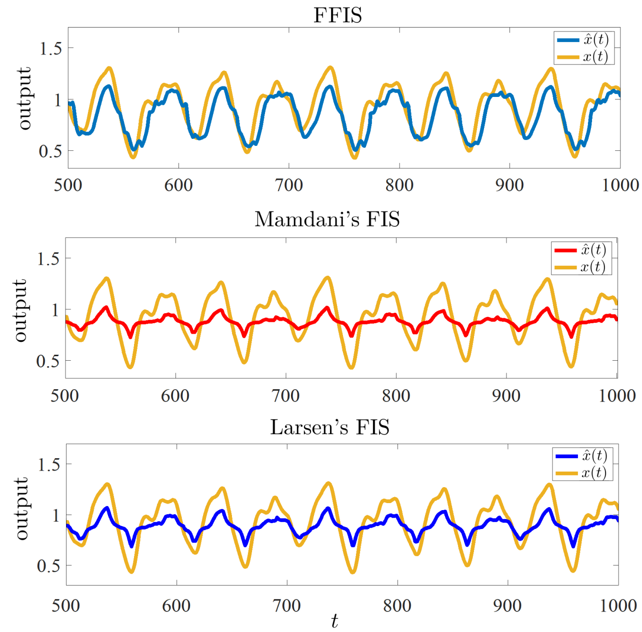

FIGURE 27. The results obtained from applying FISs on the test data.

$e(t) = x(t) - \hat{x}(t)$, $t_0 \le t \le t_n$, i.e. $J \triangleq ||e(t)||_2 = \sqrt{\sum_{t=t_0}^{t_n} e^2(t)}$, where $\hat{x}(t)$ is the estimated value of $x(t)$ coming from the fuzzy model. The "*AND*" operator in the antecedent parts has been taken into account as the algebraic product operator. Moreover, the base operator in FFIS is the same as Mamdani's FIS. Additionally, the fractional indices of the membership functions corresponding to the consequent parts have been considered, arbitrary, as $\alpha^*_{c_1} = 0.5$, $\alpha^*_{c_2} = 0$, and $\alpha^*_{c_3} = 0.6$ which means the fractional fuzzy inference system is of the order $\frac{1.1}{3}$ with the fracture index $\gamma = 0.63$. Figures 26, 27, and 28 show the results obtained by applying FISs on the training data, test data and all the data, respectively. As is seen, in the prediction of Mackey-Glass chaotic time-series, FFIS with the performance $J = 4.68$ outperforms Mamdani's and Larsen's FISs with the respected performances $J = 5.96$ and $J = 5.28$.

Example 3: To draw another comparison between the performance of typical FISs (Mamdani's and Larsen's FISs) and the fractional FIS, they are employed to predict the number of people who have been infected with Coronavirus Disease 2019 (COVID-19) in Europe. The dataset used in this example has been drawn from European Centre for Disease Prevention and Control (ECDC) [15], [16] and contains the data of the confirmed cases between 01-03-2020 and 22-06-2020. It is assumed that the data could be considered as time series data with 102 input-output data pairs of the format: $[x(t-12), x(t-8), x(t-4), x(t-1), x(t)]$ where $x(t)$ is the output and the former four variables are inputs.

Based on health experts recommendation and reports, about 14 days provides enough time to know whether a person has been infected by COVID-19 or not. Here,

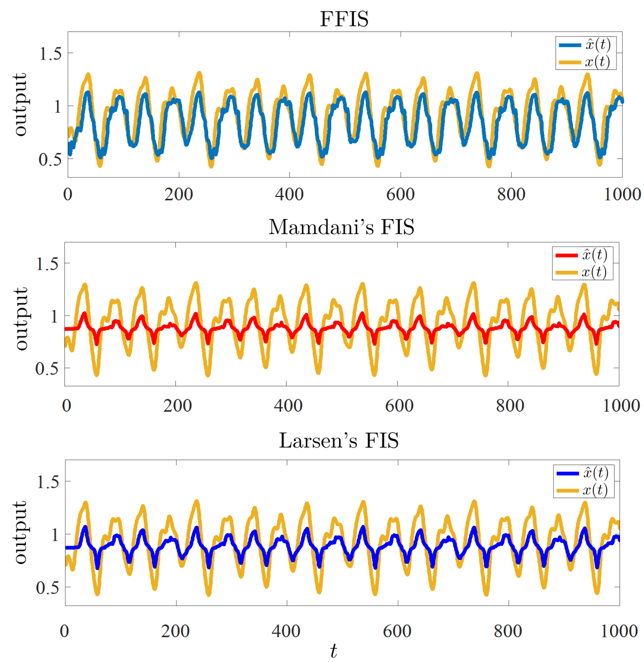

FIGURE 28. The results obtained from applying FISs on all the data.

Inputs	Membership Functions $\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$
$x(t-1)$	$G(7934, 5522)$	$G(7934, 21389)$	$G(7934, 37256)$
$x(t-4)$	$G(8620, 2777)$	$G(8620, 20020)$	$G(8620, 37260)$
$x(t-8)$	$G(9076, 954)$	$G(9076, 19110)$	$G(9076, 37260)$
$x(t-12)$	$G(9214, 402)$	$G(9214, 18830)$	$G(9214, 37260)$

FIGURE 29. The membership functions of the inputs corresponding to Example 3.

the maximum delay has been assumed to be 12 days. Analogous to what has been carried out in the previous

(1,1,1,1,1)	(2,1,1,1,2)	(3,1,1,1,2)	(1,2,1,1,1)	(2,2,1,1,2)	(3,2,1,1,2)	(1,3,1,1,2)	(2,3,1,1,2)	(3,3,1,1,3)
(1,1,2,1,1)	(2,1,2,1,2)	(3,1,2,1,2)	(1,2,2,1,2)	(2,2,2,1,2)	(3,2,2,1,3)	(1,3,2,1,2)	(2,3,2,1,3)	(3,3,2,1,3)
(1,1,3,1,2)	(2,1,3,1,2)	(3,1,3,1,3)	(1,2,3,1,2)	(2,2,3,1,3)	(3,2,3,1,3)	(1,3,3,1,2)	(2,3,3,1,3)	(3,3,3,1,3)
(1,1,1,2,1)	(2,1,1,2,2)	(3,1,1,2,2)	(1,2,1,2,1)	(2,2,1,2,2)	(3,2,1,2,3)	(1,3,1,2,2)	(2,3,1,2,2)	(3,3,1,2,3)
(1,1,2,2,1)	(2,1,2,2,2)	(3,1,2,2,2)	(1,2,2,2,2)	(2,2,2,2,2)	(3,2,2,2,3)	(1,3,2,2,2)	(2,3,2,2,2)	(3,3,2,2,3)
(1,1,3,2,2)	(2,1,3,2,2)	(3,1,3,2,2)	(1,2,3,2,2)	(2,2,3,2,2)	(3,2,3,2,3)	(1,3,3,2,2)	(2,3,3,2,2)	(3,3,3,2,3)
(1,1,1,3,1)	(2,1,1,3,2)	(3,1,1,3,2)	(1,2,1,3,2)	(2,2,1,3,2)	(3,2,1,3,2)	(1,3,1,3,2)	(2,3,1,3,2)	(3,3,1,3,2)
(1,1,2,3,2)	(2,1,2,3,2)	(3,1,2,3,2)	(1,2,2,3,2)	(2,2,2,3,2)	(3,2,2,3,2)	(1,3,2,3,2)	(2,3,2,3,2)	(3,3,2,3,2)
(1,1,3,3,2)	(2,1,3,3,2)	(3,1,3,3,2)	(1,2,3,3,2)	(2,2,3,3,2)	(3,2,3,3,3)	(1,3,3,3,2)	(2,3,3,3,2)	(3,3,3,3,3)

FIGURE 30. The rule base corresponding to Example 3.

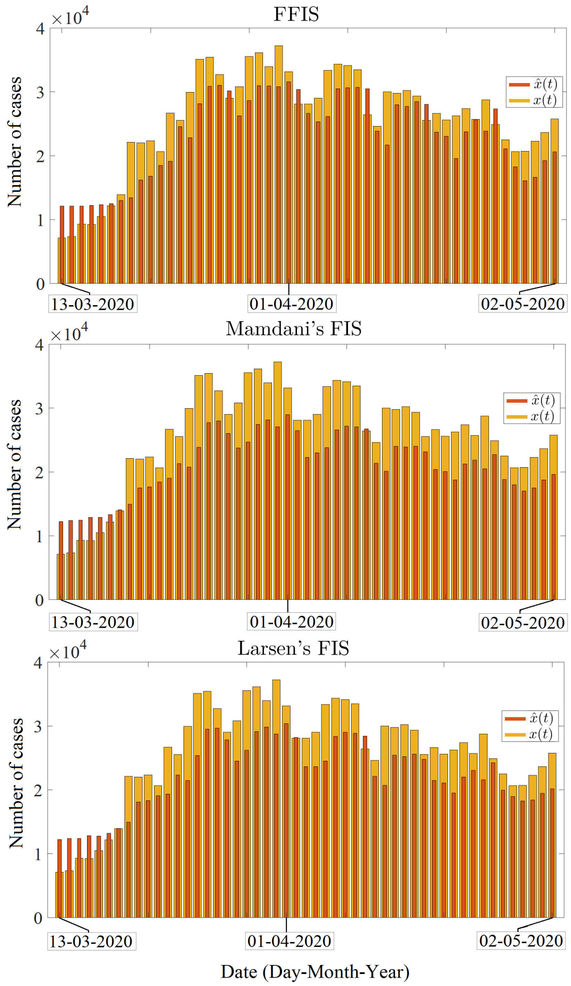

FIGURE 31. The results obtained from applying FISs on the training data.

example, in the 102 input-output data pairs, the first 51 pairs are considered as the training dataset to build the fuzzy model, while the remaining 51 pairs have been used to test the validity of the fuzzy model. Three membership functions have been considered for each input and the output. Fig. 29 shows the membership functions corresponding to the inputs. The membership functions that correspond to the output are triangular membership functions as $\tilde{B}_1 = (7111, 7111, 22180)$, $\tilde{B}_2 = (7111, 22180, 22180)$ and $\tilde{B}_3 = (22180, 37260, 37260)$. The number of fuzzy rules for this case is 81 illustrated in Fig. 30. The presentation of the rules follows the format that was explained in the previous example.

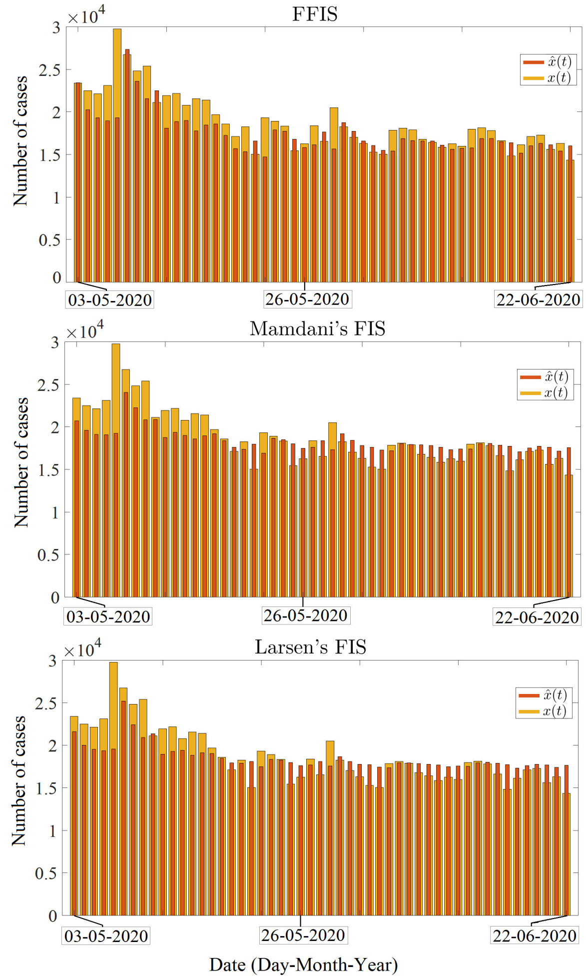

FIGURE 32. The results obtained from applying FISs on the test data.

The performance criteria by which the comparisons between FISs have been made are as $J_1 \triangleq \frac{||e(t)||_2}{10^4}$ and $J_2 \triangleq ||\frac{e(t)}{x(t)}||_2$ where $e(t) = x(t) - \hat{x}(t)$, and $\hat{x}(t)$ is the estimated value of $x(t)$ coming from the fuzzy model. The "*AND*" operator in the antecedent parts has been taken into account as the algebraic product operator. In addition, the base operator in FFIS is the same as Mamdani's FIS. Furthermore, the fractional indices of the membership functions corresponding to the consequent parts have been considered arbitrary; as $\alpha^*_{b_1} = 1, \alpha^*_{b_2} = 0$, and $\alpha^*_{b_3} = 1$. It should be noted that since $\alpha^*_{b_1} = \alpha^*_{b_3} = 1$, then, according to Proposition 3 and Note 4, the function of fuzzy rules of fractional FIS and Mamdani's FIS in which the consequent part membership functions are $\tilde{B}_1$ or $\tilde{B}_3$ is the same. Figures 31, 32, and 33 show the results obtained by applying FISs on the training data, test data and all the data, respectively. As is seen, in the prediction of the number of people who have been infected with COVID-19, FFIS with the performance criteria $J_1 = 3.57, J_2 = 1.87$ outperforms Mamdani's and Larsen's FISs whose respected performance criteria are $J_1 = 4.66, J_2 = 2.25$ and $J_1 = 4.02, J_2 = 1.94$.

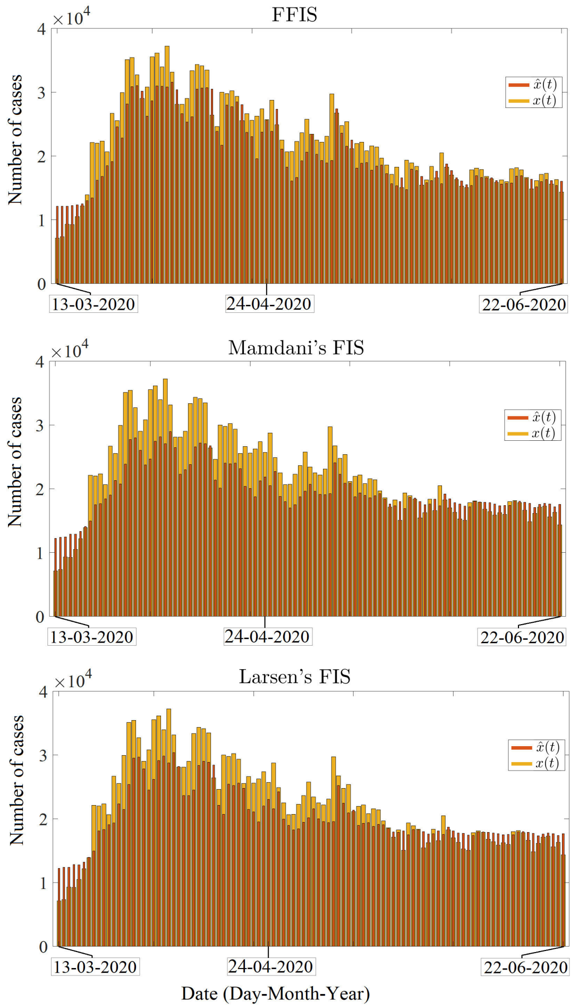

FIGURE 33. The results obtained from applying FISs on all the data.

Consequently, the obtained results are in accordance with Theorems 1 and 2.

IV. CONCLUSIONS

Fractional fuzzy inference system as the new generation of FISs was introduced in this paper. It was shown that an FFIS extracts the information included in a rule base in a way different from what a typical FIS does. With dynamical fractional indices considered, the information is extracted intelligently. In other words, the higher the truth degree of information, i.e. the more valuable the information, the more significant the volume of the information that is involved in the process of making a conclusion (or a decision). Moreover, by determining optimal fractional indices, the intelligent process tends towards the optimal use of information. Theorems 1 and 2 showed that there is always at least an FFIS whose application leads to results that are by far more desirable than those obtained by the typical FIS corresponding to the FFIS.

This is while, by replacing a typical FIS with an FFIS in a system, no changes are made in the general structure of the system. In this paper, the concept of the RTM was explained and illustrated with some figures as well. In fact, an RTM includes information about the different outputs of a fuzzy system where fractional membership functions assume different dynamical fractional indices. The information shows how fast or slow the output of the fuzzy system reacts to the error and its derivative while they are approaching or becoming far from the desired point. Thus, the RTM helps decision makers to design and choose their desired FFIS. It was also demonstrated that by the use of the concepts of the left and right orders of an FFIS, typical FISs may be viewed as the FISs of first order, i.e. integer order FISs. The concept of the fracture index defined based on the left and right orders enabled us to have an impression of the behavior of an FFIS with respect to its corresponding first order FIS. As a matter of fact, if the output of a first order FIS is at disposal, the fracture index of an FFIS, for small γ values, might serve as a means to place in evidence a similarity indicator of the FFIS output with the integer order FIS output. The other advantage of fracture index is that it might be applicable for making a fine tune of outputs as was illustrated in Fig. 23.

Eventually, FFISs open an uncharted territory to design fuzzy systems and revisit almost all the applications of fuzzy systems. Simply put, in the light of FFISs, so many applications can be considered in which applying FFIS yields to more satisfactory results than ever before. The fields in which FFIS can prove applicable include fault detection [17], control systems [18], [19], handoff decision algorithms [20], internet of thing [21], risk assessment [22], [23], evaluating the quality of experience [24], decision support systems [25], fuzzy clustering and classification [26], [27], fuzzy image processing [28], fuel cell stack problem [29], educational systems [30], fuzzy modelling [31], psychology [32], emotion categories [33], packet scheduling algorithms [34], multiobjective optimization problem [35], decision-making [36], heuristic algorithms [37], etc.

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