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Dynamic Maintenance Strategy for Word-of-Mouth Marketing

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ABSTRACT After-sale service is an integrable part of marketing activities. The after-sale service experience of a consumer can be measured by the dynamic maintenance (DM) strategy used by the merchant, i.e., the instantaneous fractions of the maintenance cost paid by the merchant in the total maintenance cost at all time. This paper aims to develop an optimal DM strategy for a word-of-mouth (WOM) marketing campaign. First, we propose an individual-based WOM propagation model in which the effect of the DM strategy is accounted for. Second, we convert the original problem into an optimal control problem, where the objective functional stands for the expected marketing profit, each optimal control stands for an optimal DM strategy. Third, we derive the optimality system for the optimal control problem. By solving the optimality system, we get a potential optimal control. Next, through comparative experiments we conclude that the DM strategy associated with the potential optimal control outperforms most DM strategies in terms of the expected marketing profit. Therefore, we recommend this potential DM strategy. Finally, we examine the effect of some factors on the expected marketing profit for the potential DM strategy. Our findings help to enhance the marketing profit of a WOM marketing campaign.

INDEX TERMS WOM marketing, marketing profit, dynamic maintenance strategy, WOM propagation model, optimal control model.

I. INTRODUCTION

Consumers tend to share their feelings about a product or service with their friends, forming a word-of-mouth (WOM) about the product or service [1]. With the popularity of online social networks (OSNs), nowadays WOMs can propagate much more rapidly than ever before [2], [3]. This phenomenon has been utilized by merchants to promote the sale of their products or services, forming what is called word-of-mouth (WOM) marketing or viral marketing [4].

After-sale service is an integrable part of marketing activities [5]–[9]. The after-sale service experience of each buyer about a product forms a part of the WOM of the product. In order to maximize the marketing profit by means of WOM propagation, the merchant has to improve the after-sale service experiences of buyers of his products [10]–[12]. This experience can be measured by the dynamic maintenance (DM) strategy used by the merchant, i.e., the instantaneous fractions of the maintenance cost paid by the merchant in the total maintenance cost at all time. Therefore, the merchant faces the following problem:

Dynamic maintenance (DM) problem: Seek a DM strategy so that the resulting marketing profit is maximized.

To our knowledge, to date this problem has not been addressed in a systematic manner. This paper is devoted to the study of this problem. Our main contributions are sketched below.

- We propose an individual-based WOM propagation model in which the effect of the DM strategy is accounted for. Thereby, we estimate the expected marketing profit for a DM strategy. On this basis, we reduce the DM problem to an optimal control problem (i.e., the DM model), where the objective functional stands for the expected marketing profit, each optimal control stands for an optimal DM strategy.

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- We derive the optimality system for the DM model. By solving the optimality system, we get a potential optimal control for the DM model. Through comparative experiments we find that the potential optimal DM strategy associated with the potential optimal control outperforms most DM strategies in terms of the expected marketing profit. Therefore, we recommend this DM strategy. Finally, we examine the effect of some factors on the expected marketing profit for the potential optimal DM strategy.

Our findings help to enhance the marketing profit of a WOM marketing campaign. The subsequent materials are organized in this way: Section 2 reviews the related work. Section 3 models the DM problem as an optimal control problem. Section 4 derives the optimality system for the optimal control problem. In Section 5, the potential optimal DM strategy obtained by solving the optimality system is justified through comparative experiments. Section 6 examines the influence of some factors on the expected marketing profit for the potential optimal DM strategy. Finally, Section 7 closes this work.

II. RELATED WORK

In this section, we review the related work, with the goal of highlighting the novelty of this work.

A. WOM PROPAGATION MODEL

The key to addressing the DM problem is to estimate the expected marketing profit in the presence of WOM propagation. For this purpose, we have to introduce a WOM propagation model in which the effect of the DM strategy is accounted for. The existing WOM propagation models can be classified into three categories: population-based, network-based, and individual-based.

A population-based WOM propagation model classifies OSN individuals based on their states [13]–[16]. For instance, [16] introduced a population-based positive/negative WOM mixed propagation model. Unfortunately, such WOM propagation models only apply to homogeneous OSNs. A network-based WOM propagation model classifies OSN individuals based on their states and their influences in OSNs [17], [18]. Unfortunately, such WOM propagation models are only applicable to some special OSNs such as scale-free networks.

An individual-based WOM propagation model classifies each OSN individual into a few classes based on his state [19]–[22]. Inspired by [16], [20] suggested an individual-based positive/negative WOM mixed propagation model. Reference [21] proposed an individual-based WOM propagation model with static influence-based discount pricing mechanism. Later, [19] generalized this model by considering dynamic influence-based discount pricing mechanism. Recently, [22] advised an individual-based WOM propagation model with dynamic competitive mechanism. Such WOM propagation models enjoy the striking advantage that they apply to all OSNs. Unfortunately, neither of these WOM propagation models takes into account the effect of the quality of after-sale service on WOM.

In the present paper, we introduce a novel individual-based WOM propagation model in which the effect of the DM strategy is accounted for. On this basis, we estimate the expected marketing profit for a DM strategy.

B. OPTIMAL CONTROL APPROACH TO WOM MARKETING

Optimal control theory, which is the theory of finding a control scheme of a dynamic system so as to achieve a specific optimality criterion [23], has been widely applied to marketing researches such as advertising [24], [25] and influential maximization [26]–[28].

Recently, optimal control theory has been applied to the maximization of marketing profit by means of WOM propagation. For instance, [19] studied the marketing profit maximization problem in the framework of influence-based discount pricing, [20] dealt with the problem in the presence of both positive and negative WOMs, and [22] addressed this problem in the context of competitive marketing. To our knowledge, the marketing profit maximization problem in the presence of after-sale service has not been addressed through optimal control approach.

The present paper is devoted to the marketing profit maximization by taking into account the effect of after-sale service. Based on our proposed WOM propagation model, we model and study the DM problem through optimal control approach. Our results contribute to enhancing the marketing profit for a WOM marketing campaign.

III. THE MODELING OF THE DM PROBLEM

This section is devoted to the modeling of the DM problem. First, we introduce a set of terms and notations that will be used later. Then, we describe a WOM propagation model. Finally, we reduce the DM problem to an optimal control problem.

A. BASIC TERMS AND NOTATIONS

Suppose a merchant intends to promote the sale of a given product in the predetermined time horizon $[0, T]$ by means of WOM propagation. Let $V = \{v_1, v_2, \ldots, v_N\}$ denote the associated target market, i.e., the set of buyers and potential buyers of the product. Let $G = (V, E)$ denote the WOM propagation network for the target market V, i.e., $\{v_i, v_j\} \in E$ stands for that the individuals v_i and v_j are friends in a certain online social network. Let $\mathbf{A} = \left(a_{ij}\right)_{N\times N}$ denote the adjacency matrix for G, i.e., $a_{ij} = 1$ or 0 according as $(v_i, v_j) \in E$ or not.

The merchant needs to provide all buyers of the product with after-sale service. The quality of the after-sale service enjoyed by a buyer at a given time can be measured by the instantaneous fraction of the maintenance cost paid by the merchant in the total maintenance cost at that time; the higher the fraction is, the higher the quality of service will be. Let $\theta(t)$ denote the instantaneous fraction of the maintenance cost paid by the merchant in the total maintenance cost at time t. We refer to the function θ defined by $\theta(t)$ $(t \in [0, T])$ as a *dynamic maintenance (DM) strategy*. For ease in implementation, let

$$\Theta = \{\theta \in L[0, T] : \theta(t) \in [0, 1], t \in [0, T]\}, \tag{1}$$

be the set of admissible DM strategies, where $L[0, T]$ denote the set of Lebesgue integrable functions defined on the interval $[0, T]$ (see [29]).

B. A WOM PROPAGATION MODEL

At any time in the time horizon $[0, T]$, an individual in the target market may have bought the product or not (waiting). Further, a buyer of the product may have given a score for the quality of the after-sale service (active) or not (silent). Obviously, the average score given by an active buyer at time t is increasing with $\theta(t)$. Let $X_i(t) = 0, 1$, and 2 denote that the individual v_i is waiting, silent, and active at time t, respectively. Then the vector $\mathbf{X}(t) = (X_1(t), \cdots, X_N(t))$ stands for the state of the target market at time t.

Let us introduce a pair of assumptions as follows.

(A_1) A silent buyer becomes active at an average rate of $\alpha > 0$.

(A_2) Affected by an active neighboring buyer, a waiting individual becomes a (silent) buyer at time t at an average rate of $\beta\theta(t)$, where β is a positive constant.

It follows from these assumptions that the state of the individual v_i varies according to Fig. 1.

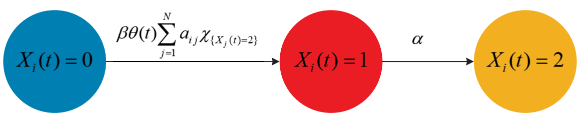

FIGURE 1. State transition diagram of the individual v_i at time t.

Let $W_i(t)$, $S_i(t)$, and $A_i(t)$ denote the probability of the individual v_i being waiting, silent, and active at time t, respectively. Since $W_i(t) = 1 - S_i(t) - A_i(t)$, the vector $\mathbf{E}(t) = (\mathbf{S}(t), \mathbf{A}(t)) = (S_1(t), \cdots, S_N(t), A_1(t), \cdots, A_N(t))$ stands for the expected state of the target market at time t. We have the following result.

Theorem 1: The evolution of the expected state of the target market over time obeys the ordinary differential system (2), as shown at the bottom of the page.

Proof: Let χ_S denote the characteristic function for the set S. The waiting individual v_i becomes a silent buyer at time t at an average rate of $\beta\theta(t)\sum_{j=1}^{N} a_{ij}\chi_{\{X_j(t)=2\}}$, and the silent buyer v_i becomes active at time t at an average rate of α. Let $\mathbb{E}[\cdot]$ denote the expectancy of a random variable. Then

$$\begin{aligned}\frac{dS_i(t)}{dt} &= \mathbb{E}[\beta\theta(t)\sum_{j=1}^{N} a_{ij}\chi_{\{X_j(t)=2\}}] \times [1 - S_i(t) - A_i(t)] \\ &\quad - \alpha S_i(t) \\ &= \beta\theta(t)[1 - S_i(t) - A_i(t)]\sum_{j=1}^{N} a_{ij}A_j(t) - \alpha S_i(t).\end{aligned} \tag{3}$$

Similarly, we can prove $\frac{dA_i(t)}{dt} = \alpha S_i(t)$. $\square$

This system as an individual-based WOM propagation model characterizes the effect of the DM strategy on the evolution of the expected state of the target market over time. This model can be recast in matrix notation as

$$\begin{cases}\dfrac{d\mathbf{E}(t)}{dt} = f(\mathbf{E}(t), \theta(t)), \quad 0 \le t \le T, \\ \mathbf{E}(0) = E_0.\end{cases} \tag{4}$$

C. THE OPTIMAL CONTROL MODELING OF THE DM PROBLEM

Let c_1 denote the unit price of the product, c_2 the average cost per unit time for maintaining each item of the product.

Theorem 2: The expected net profit gained by the merchant in the time horizon $[0, T]$ is

$$\begin{aligned}\mathcal{P}(\theta) = c_1\beta\int_0^T \theta(t)\sum_{i=1}^{N}[1 - S_i(t) - A_i(t)]\sum_{j=1}^{N} a_{ij}A_j(t)dt \\ - c_2\int_0^T \theta(t)\sum_{i=1}^{N}[S_i(t) + A_i(t)]\,dt.\end{aligned} \tag{5}$$

Proof: It follows from Theorem 1 that the individual v_i buys the product in the time horizon $[0, T]$ with a probability of $\beta\int_0^T \theta(t)[1 - S_i(t) - A_i(t)]\sum_{j=1}^{N} a_{ij}A_j(t)dt$. So, the expected gross profit gained by the merchant from selling an item of the product to the individual v_i is $c_1\beta\int_0^T \theta(t)[1 - S_i(t) - A_i(t)]\sum_{j=1}^{N} a_{ij}A_j(t)dt$. Hence, the expected gross profit gained by the merchant in the time horizon $[0, T]$ is $c_1\beta\int_0^T \theta(t)\sum_{i=1}^{N}[1 - S_i(t) - A_i(t)]\sum_{j=1}^{N} a_{ij}A_j(t)dt$. On the other hand, the expected cost paid by the merchant for maintaining the item bought by the individual v_i is $c_2\int_0^T \theta(t)[S_i(t) + A_i(t)]dt$. Hence, the expected maintenance cost paid by the merchant in the time horizon $[0, T]$ is $c_2\int_0^T \theta(t)\sum_{i=1}^{N}[S_i(t) + A_i(t)]dt$. The claim follows. $\square$

Therefore, we model the DM problem as the optimal control problem (6), as shown at the bottom of the next page.

We refer to the optimal control problem (6) as the *DM model*. This model is characterized by the 7-tuple

$$\mathcal{M} = (G, T, \alpha, \beta, c_1, c_2, \mathbf{E}_0). \tag{7}$$

$$\begin{cases}\dfrac{dS_i(t)}{dt} = \beta\theta(t)[1 - S_i(t) - A_i(t)]\sum_{j=1}^{N} a_{ij}A_j(t) - \alpha S_i(t), & 0 \le t \le T,\ 1 \le i \le N, \\ \dfrac{dA_i(t)}{dt} = \alpha S_i(t), & 0 \le t \le T,\ 1 \le i \le N, \\ \mathbf{E}(0) = \mathbf{E}_0 \end{cases} \tag{2}$$

IV. A METHOD FOR SOLVING THE DM MODEL

In the previous section, we reduced the DM problem to the DM model. In this section, we derive a method for solving this model.

According to optimal control theory, the Hamiltonian for the DM model is as shown in Eq. (8), as shown at the bottom of the page, where $(\boldsymbol{\lambda}, \boldsymbol{\mu}) = (\lambda_1, \cdots, \lambda_N, \mu_1, \cdots, \mu_N)$ is the associated adjoint. We have the following result.

Theorem 3: Suppose θ is an optimal control for the DM model (7), $\mathbf{E}$ is the solution to the corresponding model (2). Then there exists an adjoint $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ such that the system (9), as shown at the bottom of the page, holds. Moreover, let

$$\begin{aligned} g(\mathbf{E}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) &= c_1\beta \sum_{i=1}^{N}[1 - S_i(t) - A_i(t)] \sum_{j=1}^{N} a_{ij}A_j(t) \\ &\quad - c_2 \sum_{i=1}^{N}[S_i(t) + A_i(t)] \\ &\quad + \beta \sum_{i=1}^{N} \lambda_i(t)[1 - S_i(t) - A_i(t)] \sum_{j=1}^{N} a_{ij}A_j(t). \end{aligned} \tag{10}$$

Then

$$\theta(t) \in \arg \max_{\widetilde{\theta}\in[0,1]} g(\mathbf{E}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))\widetilde{\theta}, \quad 0 \le t \le T. \tag{11}$$

As a result, $\theta(t) = 0$ or 1 according as $g(\mathbf{E}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) < 0$ or > 0.

Proof: According to Pontryagin Maximum Principle, there exits $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ such that

$$\begin{cases} \dfrac{d\lambda_i(t)}{dt} = -\dfrac{\partial H(E(t), \theta(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))}{\partial S_i}, \\ \dfrac{d\mu_i(t)}{dt} = -\dfrac{\partial H(E(t), \theta(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))}{\partial A_i}, \\ 0 \le t \le T, \quad 1 \le i \le N. \end{cases} \tag{12}$$

The first $2N$ equations in the system (9) follow by direct calculations. Since the terminal cost is unspecified and the final state is free, we have $\boldsymbol{\lambda}(T) = \boldsymbol{\mu}(T) = \mathbf{0}$. Based on Pontryagin Maximum Principle, we have

$$\theta(t) \in \arg \max_{\widetilde{\theta}\in\Theta} H(E(t), \widetilde{\theta}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)), \quad 0 \le t \le T. \tag{13}$$

Eqs. (11) follow by direct calculations. □

By optimal control theory, Eqs. (2), Eqs. (9), and Eqs. (11) constitute the optimality system for the DM model. By solving the optimality system, we get a control for the DM model. Since this control is not necessarily optimal, we refer to it as the *potential optimal control* for the DM model.

V. EXAMPLES OF THE POTENTIAL OPTIMAL CONTROL

In the previous section, we introduced the notion of potential optimal control for the DM model. In this section, we give a few examples of the potential optimal control.

A. EXPERIMENT DESIGN

First, below we describe an algorithm (i.e., the POC algorithm) for computing the potential optimal control for the DM model (see [30]). Here, POC is the acronym of the phrase "potential optimal control", $||\cdot||_1$ stands for the 1-norm of a function. In all the following experiments, we set $\epsilon = 10^{-6}$, $K = 10^4$.

Second, we describe an algorithm (i.e., the RC algorithm) for randomly and uniformly generating a control for the DM model as follows. Here, RC is the acronym of the phrase "random control". In all the following experiments, we set $n = 1000$.

Next, let us describe three WOM propagation networks. First, since many real-world networks are scale-free (i.e., with a power-law degree distribution) [31], we use Pajek (a well-known social network analysis software) [32]

$$\max_{\theta\in\Theta} \mathcal{P}(\theta) = c_1\beta \int_0^T \theta(t) \sum_{i=1}^{N}[1 - S_i(t) - A_i(t)] \sum_{j=1}^{N} a_{ij}A_j(t)dt - c_2 \int_0^T \theta(t) \sum_{i=1}^{N} [S_i(t) + A_i(t)]\, dt$$

$$\text{subject to} \quad \begin{cases} \dfrac{d\mathbf{E}(t)}{dt} = f(\mathbf{E}(t), \theta(t)), \quad 0 \le t \le T, \\ \mathbf{E}(0) = \mathbf{E}_0 \end{cases} \tag{6}$$

$$\begin{aligned} H(\mathbf{E}, \theta, \lambda, \mu) = c_1\beta\theta \sum_{i=1}^{N}(1 - S_i - A_i) \sum_{j=1}^{N} a_{ij}A_j - c_2\theta \sum_{i=1}^{N}(S_i + A_i) \\ + \sum_{i=1}^{N} \lambda_i \left[\beta\theta(1 - S_i - A_i) \sum_{j=1}^{N} a_{ij}A_j - \alpha S_i\right] + \alpha \sum_{i=1}^{N} \mu_i S_i \end{aligned} \tag{8}$$

$$\begin{cases} \dfrac{d\lambda_i(t)}{dt} = c_1\beta\theta(t) \sum_{j=1}^{N} a_{ij}A_j(t) + c_2\theta(t) + \left[\beta\theta(t) \sum_{j=1}^{N} a_{ij}A_j(t) + \alpha\right] \lambda_i(t) - \alpha\mu_i(t), \\ \dfrac{d\mu_i(t)}{dt} = c_1\beta\theta(t) \sum_{j=1}^{N} a_{ij} \left[2A_j(t) + S_j(t) - 1\right] + c_2\theta(t) + \left[\beta\theta(t) \sum_{j=1}^{N} a_{ij}A_j(t)\right] \lambda_i(t) \\ \qquad -\beta\theta(t) \sum_{j=1}^{N} a_{ij} \left[1 - S_j(t) - A_j(t)\right] \lambda_j(t), \\ 0 \le t \le T, 1 \le i \le N, \quad \boldsymbol{\lambda}(T) = \boldsymbol{\mu}(T) = \mathbf{0} \end{cases} \tag{9}$$

Algorithm 1 POC

Input: a DM model $\mathcal{M} = (G, T, \alpha, \beta, c_1, c_2, \mathbf{E}_0)$, convergence error ϵ, maximum number K of iterations.

Output: the potential optimal control θ.

```
k ← 0; θ^(0)(t) ← 0, t ∈ [0, T];
repeat
    k ← k + 1;
    forward compute E using Eqs. (2) with θ = θ^(k−1); E^(k) ← E;
    backward compute λ and μ using Eqs. (9) with θ = θ^(k−1) and E = E^(k); λ^(k) ← λ; μ^(k) ← μ;
    compute θ using Eqs. (11) with E = E^(k), λ = λ^(k), and μ = μ^(k); θ^(k) ← θ;
until ‖θ^(k) − θ^(k−1)‖_1 < ϵ or k ≥ K;
return θ^(k).
```

Algorithm 2 RC

Input: a DM model $\mathcal{M} = (G, T, \alpha, \beta, c_1, c_2, \mathbf{E}_0)$,, positive integer n.

Output: a control θ.

```
t_0 ← 0;
for k = 1 to n do
    t_k := t_{k−1} + T/n;
end for
for k = 0 to n − 1 do
    randomly and uniformly generate a number η ∈ [0, 1];

    θ(t) ← η, t_k ≤ t < t_{k+1};
end for
θ(T) ← θ(t^(n−1));
return θ.
```

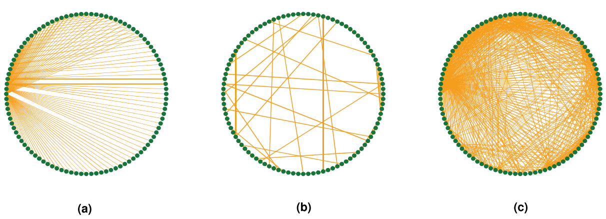

FIGURE 2. Three WOM propagation networks: (a) a scale-free network G_{SF}, (b) a small-world network G_{SW}, (c) an email network G_{EM}.

to generate a scale-free network with 100 nodes, denoted G_{SF} and plotted in Fig. 2(a). Second, since many real-world networks are small-world (i.e., with a relatively small diameter) [31], we use Pajek to generate a small-world network with 100 nodes, denoted G_{SW} and exhibited in Fig. 2(b). Finally, consider a realistic email network [33], denoted G_{EM} and displayed in Fig. 2(c). Finally, in all the following experiments, let $\mathbf{S}_0 = (0.1, \cdots, 0.1)$, $\mathbf{A}_0 = (0, \cdots, 0)$, $\mathbf{E}_0 = (\mathbf{S}_0, \mathbf{A}_0)$.

B. EXPERIMENTS

Experiment 1: Consider the DM model $\mathcal{M}_1 = (G_{SF}, 10, 0.2, 0.1, 800, 1, \mathbf{E}_0)$. *By executing the POC algorithm on* $\mathcal{M}_1$*, we get the potential optimal control for* $\mathcal{M}_1$*, denoted* θ^* *and plotted in Fig. 3(a). By repeatedly executing the RC algorithm on* $\mathcal{M}_1$*, we get 100 controls, denoted* θ_1 *through* θ_{100}*. Fig. 3(b) exhibits* $\mathcal{P}(\theta)$, $\theta \in \{\theta^*\} \bigcup \{\theta_1, \cdots, \theta_{100}\}$. *It is seen that* $\mathcal{P}(\theta^*) > \mathcal{P}(\theta_k)$ *for all* k.

Experiment 2: Consider the DM model $\mathcal{M}_2 = (G_{SW}, 10, 0.1, 0.2, 700, 1, \mathbf{E}_0)$. *By running the POC algorithm on* $\mathcal{M}_2$*, we get the potential optimal control for* $\mathcal{M}_2$*, denoted* θ^* *and plotted in Fig. 4(a). By repeatedly executing the RC algorithm on* $\mathcal{M}_1$*, we get 100 controls, denoted* θ_1 *through* θ_{100}*. Fig. 4(b) displays* $\mathcal{P}(\theta)$, $\theta \in \{\theta^*\} \bigcup \{\theta_1, \cdots, \theta_{100}\}$. *It is seen that* $\mathcal{P}(\theta^*) > \mathcal{P}(\theta_k)$ *for all* k.

Experiment 3: Consider the DM model $\mathcal{M}_3 = (G_{EM}, 10, 0.1, 0.1, 750, 1, \mathbf{E}_0)$. *By performing the POC algorithm on* $\mathcal{M}_3$*, we get the potential optimal control for* $\mathcal{M}_3$*, denoted* θ^* *and plotted in Fig. 5(a). By repeatedly executing the RC algorithm on* $\mathcal{M}_3$*, we get 100 controls, denoted* θ_1 *through* θ_{100}*. Fig. 5(b) demonstrates* $\mathcal{P}(\theta)$, $\theta \in \{\theta^*\} \bigcup \{\theta_1, \cdots, \theta_{100}\}$. *It is seen that* $\mathcal{P}(\theta^*) > \mathcal{P}(\theta_k)$ *for all* k.

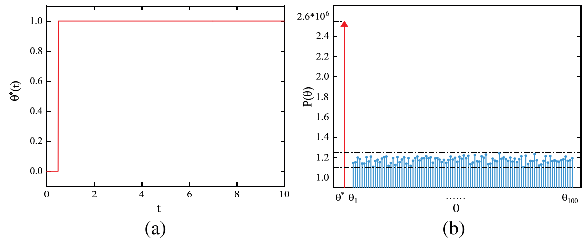

FIGURE 3. The results in Experiment 1: (a) the optential optimal control θ^*, (b) $\mathcal{P}(\theta)$, $\theta \in \{\theta^*\} \bigcup \{\theta_1, \cdots, \theta_{100}\}$.

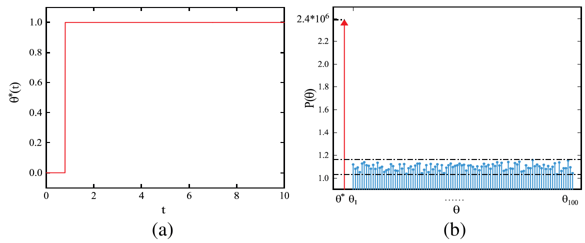

FIGURE 4. The results in Experiment 2: (a) the optential optimal control θ^*, (b) $\mathcal{P}(\theta)$, $\theta \in \{\theta^*\} \bigcup \{\theta_1, \cdots, \theta_{100}\}$.

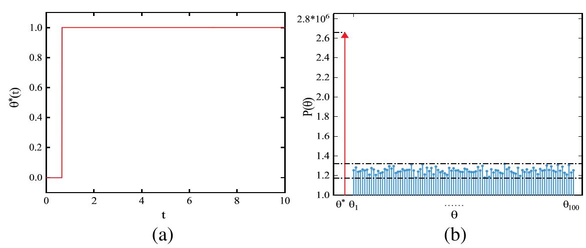

FIGURE 5. The results in Experiment 3: (a) the optential optimal control θ^*, (b) $\mathcal{P}(\theta)$, $\theta \in \{\theta^*\} \bigcup \{\theta_1, \cdots, \theta_{100}\}$.

We conclude from the above three experiments and 1,000 similar experiments that the DM strategy associated with the potential optimal control outperforms most DM strategies in terms of the expected marketing profit.

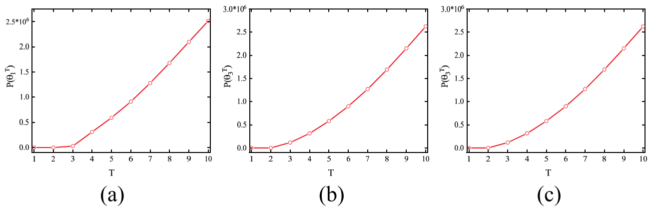

FIGURE 6. (a) $\mathcal{P}(\theta_1^T)$ versus T, (b) $\mathcal{P}(\theta_2^T)$ versus T, (c) $\mathcal{P}(\theta_3^T)$ versus T in Experiment 4.

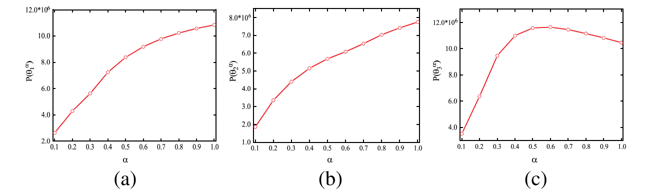

FIGURE 7. (a) $\mathcal{P}(\theta_1^\alpha)$ versus α, (b) $\mathcal{P}(\theta_2^\alpha)$ versus α, (c) $\mathcal{P}(\theta_3^\alpha)$ versus α in Experiment 5.

Therefore, we recommend this DM strategy. For convenience, we refer to the DM strategy associated with the potential optimal control as the *potential DM strategy*.

VI. FURTHER DISCUSSIONS

In this section we examine the influence of some factors on the expected marketing profit for the potential optimal DM strategy through computer experiments. First, we examine the influence of the maintenance period T.

Experiment 4: Consider three sets of DM models as follows.

(a) *Let* $\mathcal{M}_1^T = (G_{SF}, T, 0.2, 0.1, 800, 1, \mathbf{E}_0)$, $T \in \mathcal{T} = \{1, 2, \cdots, 10\}$. *By running the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_1^T, $T \in \mathcal{T}$. *Fig. 6(a) shows* $\mathcal{P}(\theta_1^T)$ *versus* T. *It is seen that* $\mathcal{P}(\theta_1^T)$ *is increasing with* T.

(b) *Let* $\mathcal{M}_2^T = (G_{SW}, T, 0.1, 0.2, 700, 1, \mathbf{E}_0)$, $T \in \mathcal{T} = \{1, 2, \cdots, 10\}$. *By executing the POC algorithm on these models, we get their respective potential optimal controls, denoted* θ_2^T, $T \in \mathcal{T}$. *Fig. 6(b) shows* $\mathcal{P}(\theta_2^T)$ *versus* T. *It is seen that* $\mathcal{P}(\theta_2^T)$ *is increasing with* T.

(c) *Let* $\mathcal{M}_3^T = (G_{SF}, T, 0.1, 0.1, 750, 1, \mathbf{E}_0)$, $T \in \mathcal{T} = \{1, 2, \cdots, 10\}$. *By performing the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_3^T, $T \in \mathcal{T}$. *Fig. 6(c) shows* $\mathcal{P}(\theta_3^T)$ *versus* T. *It is seen that* $\mathcal{P}(\theta_3^T)$ *is increasing with* T.

Based on these experiments and 1,000 similar experiments, we conclude that the expected marketing profit for the potential optimal DM strategy is increasing with the maintenance period. This conclusion demonstrates that the merchant should properly extend the maintenance period.

Second, we look into the influence of the comment rate α.

Experiment 5: Consider three sets of DM models as follows.

(a) *Let* $\mathcal{M}_1^\alpha = (G_{SF}, 10, \alpha, 0.14, 1000, 1, \mathbf{E}_0)$, $\alpha \in A = \{0.1, 0.2, \cdots, 1.0\}$. *By executing the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_1^α, $\alpha \in A$. *Fig. 7(a) shows* $\mathcal{P}(\theta_1^\alpha)$ *versus* α. *It is seen that* $\mathcal{P}(\theta_1^\alpha)$ *is increasing with* α.

(b) *Let* $\mathcal{M}_2^\alpha = (G_{SW}, 10, \alpha, 0.12, 1000, 1, \mathbf{E}_0)$, $\alpha \in A = \{0.1, 0.2, \cdots, 1.0\}$. *By running the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_2^α, $\alpha \in A$. *Fig. 7(b) shows* $\mathcal{P}(\theta_2^\alpha)$ *versus* α. *It is seen that* $\mathcal{P}(\theta_2^\alpha)$ *is increasing with* α.

(c) *Let* $\mathcal{M}_3^\alpha = (G_{EM}, 10, \alpha, 0.1, 1000, 1, \mathbf{E}_0)$, $\alpha \in A = \{0.1, 0.2, \cdots, 1.0\}$. *By performing the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_3^α, $\alpha \in A$. *Fig. 7(c) shows* $\mathcal{P}(\theta_3^\alpha)$ *versus* α. *It is seen that* $\mathcal{P}(\theta_3^\alpha)$ *is first increasing then decreasing with* α.

Based on these experiments and 1,000 similar experiments, we conclude that the expected marketing profit for the potential optimal DM strategy is first increasing then decreasing with the comment rate.

Next, let us investigate the influence of the WOM strength β.

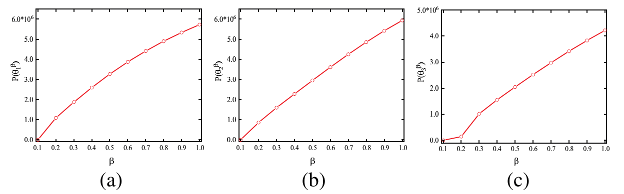

FIGURE 8. (a) $\mathcal{P}(\theta_1^\beta)$ versus β, (b) $\mathcal{P}(\theta_2^\beta)$ versus β, (c) $\mathcal{P}(\theta_3^\beta)$ versus β in Experiment 6.

Experiment 6: Consider three sets of DM models as follows.

(a) *Let* $\mathcal{M}_1^\beta = (G_{SF}, 10, 0.03, \beta, 1000, 1, \mathbf{E}_0)$, $\beta \in B = \{0.1, 0.2, \cdots, 1.0\}$. *By running the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_1^β, $\beta \in B$. *Fig. 8(a) shows* $\mathcal{P}(\theta_1^\beta)$ *versus* β. *It is seen that* $\mathcal{P}(\theta_1^\beta)$ *is increasing with* β.

(b) *Let* $\mathcal{M}_2^\beta = (G_{SW}, 10, 0.03, \beta, 1000, 1, \mathbf{E}_0)$, $\beta \in B = \{0.1, 0.2, \cdots, 1.0\}$. *By performing the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_2^β, $\beta \in B$. *Fig. 8(b) shows* $\mathcal{P}(\theta_2^\beta)$ *versus* β. *It is seen that* $\mathcal{P}(\theta_2^\beta)$ *is increasing with* β.

(c) *Let* $\mathcal{M}_3^\beta = (G_{SW}, 10, 0.01, \beta, 1000, 1, \mathbf{E}_0)$, $\beta \in B = \{0.1, 0.2, \cdots, 1.0\}$. *By executing the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_3^β, $\beta \in B$. *Fig. 8(b) shows* $\mathcal{P}(\theta_3^\beta)$ *versus* β. *It is seen that* $\mathcal{P}(\theta_3^\beta)$ *is increasing with* β.

Based on these experiments and 1,000 similar experiments, we conclude that the expected marketing profit for the potential optimal DM strategy is increasing with the WOM strength.

Finally, let us inspect the influence of the ratio of c_1 to c_2.

Experiment 7: Consider three sets of DM models as follows.

(a) *Let* $\mathcal{M}_1^c = (G_{SF}, 10, 0.1, 0.1, c, 1, \mathbf{E}_0)$, $c \in C = \{1000, 1100, \cdots, 2000\}$. *By running the POC*

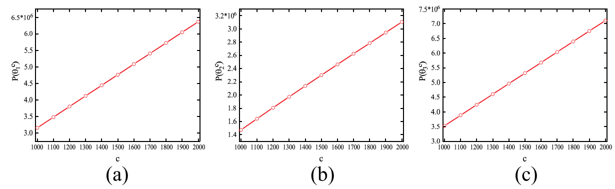

FIGURE 9. (a) $\mathcal{P}(\theta_1^c)$ versus c, (b) $\mathcal{P}(\theta_2^c)$ versus c, (c) $\mathcal{P}(\theta_3^c)$ versus c in Experiment 7.

algorithm on these models, we get a set of potential optimal controls, denoted θ_1^c, $c \in C$. *Fig. 9(a) shows* $\mathcal{P}(\theta_1^c)$ *versus c. It is seen that* $\mathcal{P}(\theta_1^c)$ *is increasing with c.*

(b) *Let* $\mathcal{M}_2^c = (G_{SW}, 10, 0.2, 0.1, c, 1, \mathbf{E}_0)$, $c \in C = \{1000, 1100, \cdots, 2000\}$. *By performing the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_2^c, $c \in C$. *Fig. 9(b) shows* $\mathcal{P}(\theta_2^c)$ *versus c. It is seen that* $\mathcal{P}(\theta_2^c)$ *is increasing with c.*

(c) *Let* $\mathcal{M}_3^c = (G_{EM}, 10, 0.1, 0.1, c, 1, \mathbf{E}_0)$, $c \in C = \{1000, 1100, \cdots, 2000\}$. *By executing the POC algorithm on these models, we get a set of potential optimal controls, denoted* θ_3^c, $c \in C$. *Fig. 9(c) shows* $\mathcal{P}(\theta_3^c)$ *versus c. It is seen that* $\mathcal{P}(\theta_3^c)$ *is increasing with c.*

Based on these experiments and 1,000 similar experiments, we conclude that the expected marketing profit for the potential optimal DM strategy is increasing with the ratio of the unit price of the product and the maintenance cost per unit time of the product. This conclusion shows that the merchant should properly enhance the unit price of the product.

VII. CONCLUSIONS AND REMARKS

This paper has addressed the problem of maximizing the marketing profit for a WOM marketing campaign by dynamically adjusting the after-sale maintenance strategy. First, we have reduced the problem to an optimal control problem. Second, we have derived a potential optimal maintenance strategy by solving the optimality system for the optimal control problem. Finally, we have justified our maintenance strategy through comparative experiments. Therefore, we have recommended the derived maintenance strategy.

There are many related issues that are yet to be resolved. First, the combined effect of the maintenance strategy and the discount pricing strategy on WOM marketing should be examined [19], [21]. Second, this work may be adapted to competitive WOM marketing campaigns [22], [34]–[37]. Next, in the presence of competing rivals, it is appropriate to deal with the marketing profit maximization problem in the framework of game theory [22], [37], [38]. Finally, the used methodology may be applied to some other areas such as rumor control [39]–[42] and defense of advanced persistent threat [43]–[47].

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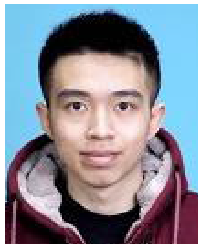

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