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A New Non-Full Rank Algorithm for the IMC-Derived *d*-Step MIMO Structures in the Pole-Free State Space

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ABSTRACT In this paper, the powerful issue of a non-full rank inverse model control investigation is provided. It is broadly known, that the inverse-based methodology is associated with the full-rank control systems only. However, following the recently obtained authors' results in this matter, it should be concluded, that for single-delayed non-full rank state-space plants, the inverse model control-related perfect control expression can also be established. It is shown here, that for all right- and left-oriented multivariable LTI non-full rank systems, including the square ones, governed by the discrete-time state-space domain arranging the zero-reference value, the maximum-speed pole-free instance of such inverse model control strategy can be achieved. Thus, the new non-full rank algorithm does not coincide with the z-transfer-function scenario, which additionally sounds the intriguing peculiarity. The innovative content of the manuscript, that has never been seen before, is strongly supported by the numerical examples. Henceforth, the presented methodology covers the entire set of LTI MIMO state-space-oriented plants in the discrete-time domain, which is also a consequence of conducted research investigation in the past.

INDEX TERMS Non-full rank, perfect control procedure, Moore–Penrose inverse, generalized inverses, skeleton factorization, pole-free delayed plants, discrete-time state-space MIMO structures.

I. INTRODUCTION

The inverse model control (IMC) has widely been investigated over the past decades due to its application in many scientific and engineering fields [1]–[4], [5]–[8], [9]–[12], [13], [14], [15]–[18], [19]–[22], [23]–[25]. The unique behaviors of the discussed technique such as robustness, maximum-speed or minimum-energy properties make it attractive in many practical implementations [1], [9], [12], [23], [26]–[28]. The widely known IMC-based perfect control law, which constitutes the deterministic instance of the minimum variance control (MVC), has not only been associated with the full-rank square MIMO systems, including SISO ones [1]. In fact, the multivariable nonsquare right-invertible plants, i.e. systems with greater number of input than output variables, can also be subjected by such noise-free control law [3]. In such a case, through the application of nonunique parameter/polynomial right inverses, we can redefine the special properties of the closed-loop state-feedback plant [1], [9], [26]. The IMC methodology, both in the input-output and the state-space domains, has necessitated the system to be under full-rank regulation, hence the non-full rank instance has not been considered, in general [29], [30].

Since in the non-full rank scenario the pseudoinverse formula supported by the so-called skeleton factorization mechanism has to be engaged in the inverse model control design, the perfect control law requirements cannot structurally be fulfilled [8]. So, this case has been thrown away from the further investigations by the world control society. However, following the recent authors' results it is certain, that for some single-delayed state-space plant being under zero-setpoint, the non-full rank perfect control structure can also be successfully obtained [12]. It is interesting to note, that this outcome is not valid in the input-output-oriented transfer-function model description, what additionally underlines the variability of the perfect control scheme [29]. Nevertheless, the new analytical results presented in the paper confirm, that for all right- and left-oriented non-full rank multivariable LTI discrete-time state-space systems with time delay $d \geqslant 1$ having zero-reference value, the IMC-based maximum-speed

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pole-free perfect control object can clearly be established. This phenomenon, which has never been seen before, sheds a new light on the generally accepted control and systems theory canons. An extension of the manuscript's results on the other control strategies seems to be expected. Henceforth, every control plant with arbitrary time delay d occupied by the non-full rank adverse behavior can now be treated as the accessible one.

A remaining of the manuscript is organized as follows. The essential symbols and abbreviations are introduced in Section II. Next, the paradigm of the state-space perfect control law is explained in Section III. The most important Section IV deals with the problem of the maximum-speed pole-free non-full rank inverse model control design. The newly obtained results are confirmed by the simulation examples of Section V. Finally, the conclusions and open problems of Section VI summarize the paper, successfully.

II. PRELIMINARIES

In order to eliminate any confusions and misconceptions, the main symbols and abbreviations enforced in the work are presented below.

TABLE 1. Table of symbols.

Notation	
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	– parameter system's matrices,
$\mathbf{I_n}$	– n-identity matrix,
d	– time delay of a plant,
q^{-1}	– backward shift operator,
z	– complex operator,
$\underline{\beta}(q^{-1})$	– polynomial matrix,
$\underline{\det}(.)$	– determinant symbol,
$(.)^{\xi}$	– appropriate inverse formula,
$(.)^{\mathrm{R}}$	– any nonunique right inverse,
$(.)^{\mathrm{R}}_{\sigma}$	– nonunique polynomial right σ-inverse,
$(.)^{\mathrm{R}}_{0}$	– unique minimum-norm right T-inverse,
$(.)^{\mathrm{L}}_{0}$	– unique least-squares left T-inverse,
$(.)^{\delta}$	– unique Moore–Penrose T-pseudoinverse,
$\Vert . \Vert^2$	– norm symbol,
DOFs	– degrees of freedom,
IMC	– inverse model control,
LTI	– linear time-invariant,
MIMO	– multi-input/multi-output.

Having the fundamentals covering the crucial motivation as well as mathematical background, let us switch now to the overall investigation of the IMC-related deterministic perfect control procedure. Next section effectively discovers the steps to be undertaken during the complex design process of the closed-loop state-feedback control scheme.

III. CONTROL SYSTEM DESCRIPTION

Following the introduction section, we consider the multi-input/multi-output LTI plants governed by the discrete-time state-space structure

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)q^{(-d+1)}, \quad \mathbf{x}(0) = \mathbf{x_0},$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k), \tag{1}$$

with the system's parameter matrices as $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times n_u}$, $\mathbf{C} \in \mathbb{R}^{n_y\times n}$ and vectors $\mathbf{x}(k) \in \mathbb{R}^{n}$, $\mathbf{u}(k) \in \mathbb{R}^{n_u}$, $\mathbf{y}(k) \in \mathbb{R}^{n_y}$. The n-state, n_u-input and n_y-output nomenclature denotes the numbers of state, input and output variables, respectively. The $\mathbf{x_0}$ indicates an initial condition of the state vector $\mathbf{x}(k)$, whilst k and q^{-1} stand for the discrete time and backward shift operator, respectively. The considered $d \geqslant 1$ deals with the time delay of a plant.

Remark 1: It should be assumed, that in further investigations the system is controllable.

Remark 2: The poles of the plant governed by the state-space framework (1) *can be calculated in accordance with the following canon*

$$\det(z\mathbf{I_n} - \mathbf{A}) = 0, \tag{2}$$

where $\mathbf{I_n}$ defines the n-dimensional identity matrix, whereas z means some complex operator.

The IMC-based perfect control algorithm is the deterministic instance of the well-known MVC strategy, which guarantees, that we receive the reference value on the system's output just after time delay of the plant. In order to obtain such formula, we have to fulfill the perfect control law requirement in the following manner

$$J = \min_{\mathbf{u}(k)} \left\{ \left\Vert \mathbf{y}(k+d) - \mathbf{y_{ref}}(k+d) \right\Vert^2 \right\}, \tag{3}$$

where $\mathbf{y}(k+d)$ is the d-step deterministic output predictor while $\mathbf{y_{ref}}(k+d) \in \mathbb{R}^{n_y}$ stands for an arbitrary reference value enabled after $(k+d)$-step.

Throughout minimizing the performance index (3) we receive the complete state-space-oriented perfect control expression [1]

$$\mathbf{u}(k) = (\mathbf{CB})^{\xi}\Big[\mathbf{y_{ref}}(k+d) - \mathbf{C}\Big(\sum_{p=1}^{d-1}\mathbf{A}^p\mathbf{B}\mathbf{u}(k-p) + \mathbf{A}^d\mathbf{x}(k)\Big)\Big], \tag{4}$$

where symbol $(.)^{\xi}$ denotes an appropriate inverse formula of the product of $\mathbf{CB} \in \mathbb{R}^{n_y\times n_u}$. Naturally, in case of $\mathbf{CB}$ being of full (normal) rank we have to apply any right inverse as $(\mathbf{CB})^{\mathrm{R}}$.

Remark 3: It should be emphasized, that in order to fulfill the condition (3)*, the full rank plant has to be square or right invertible ($n_u \geqslant n_y$), see Refs. [8], [9].*

Remark 4: It is clear, that the IMC-oriented perfect control structure can also be established for full rank plants governed by the right-invertible input-output models. For more details see Ref. [29].

Remark 5: Notice, that in the non-full rank scenario, the general perfect control law (3) *cannot structurally be achieved. This important exception will be discussed in detail in the next section.*

In the literature, we can find a plethora of unique/nonunique generalized inverse formulas precisely dedicated to the parameter/polynomial square/non-square full (normal) rank matrices [8], [31]. At this point, the most important ones should be recalled in forms of the S-, σ-, τ- and H-inverse, in order to involve their in the perfect control design process [8], [29], [32]. Naturally, each of them causes some different behavior of the closed-loop control system. The most intriguing one is the recently introduced nonunique polynomial right σ-inverse of $\mathbf{CB}$ as follows

$$(\mathbf{CB})_{\sigma}^{\mathrm{R}} = \underline{\beta}^{\mathrm{T}}(q^{-1})\big[\mathbf{CB}\,\underline{\beta}^{\mathrm{T}}(q^{-1})\big]^{-1}, \tag{5}$$

attaches the infinite number of the DOFs-derived $\underline{\beta}(q^{-1})$ forms. Thus, we can obtain the infinite number of inverses and consequently different peculiarities of the controlled plant [1], [9].

Remark 6: It is interesting to note, that the nonunique polynomial right σ-inverse (5) *covers all right-inverse formulas, in general. Moreover, the analytical confirmation of the direct relationship between the popular S-inverse and σ-inverse can be found in Ref. [32].*

However, while employment of the generalized right σ-inverse with polynomial DOFs seems to be rather obvious, such operation cannot be applied to the non-full IMC-oriented instances. This fact additionally stresses the complexity of the perfect control methodology. Therefore, in order to significantly extend the well-known canons of control and systems theory, the next section definitely solves the eternal non-full rank problem mainly engaged in the inverse-based multivariable control. This constitutes a main accomplishment of the paper.

IV. NON-FULL RANK IMC STRUCTURE

At the beginning of this section it should already be reminded, that the rank of the system being under perfect control force is strictly associated with the rank of the $\mathbf{CB}$ product. In other words, if the $\mathbf{CB}$ matrix is of the full/non-full rank, then the perfect control plant is also characterized as the full/non-full rank, respectively.

Pursuing this line, in the non-full rank scenario, the special class of the inverses has to be involved into the inverse model control design procedure [8]. Therefore, in the case of the non-full rank $\mathbf{CB}$ matrix meeting the relation

$$\det\Big(\mathbf{CB}(\mathbf{CB})^{\mathrm{T}}\Big) = 0, \tag{6}$$

we have to apply the well-known unique Moore–Penrose T-pseudoinverse marked as $(\mathbf{CB})^{\delta}$ [33]. Based on the so-called skeleton factorization we can explicitly obtain its general expression

$$(\mathbf{CB})^{\delta} = \mathbf{B}_0^{\mathrm{R}}\mathbf{C}_0^{\mathrm{L}}, \tag{7}$$

where symbols $(.)_0^{\mathrm{R}}$ and $(.)_0^{\mathrm{L}}$ denote the unique minimum-norm right and unique least-squares left T-inverse formulas, respectively [8].

Remark 7: Observe, that in the pseudoinverse (7), *the respective Moore–Penrose right and left T-inverses in the forms of $\mathbf{B}_0^{\mathrm{R}} = \mathbf{B}^{\mathrm{T}}\big[\mathbf{B}\mathbf{B}^{\mathrm{T}}\big]^{-1}$ and $\mathbf{C}_0^{\mathrm{L}} = \big[\mathbf{C}^{\mathrm{T}}\mathbf{C}\big]^{-1}\mathbf{C}^{\mathrm{T}}$ are commonly used.*

Remark 8: Notice, that the expression (7) *is valid for any size of the* $\mathbf{CB}$ *product. In such a case we have* $\dim[\mathbf{C}] = n_y \times n$ *and* $\dim[\mathbf{B}] = n \times n_u$ *with* $n < \min(n_u, n_y)$ *being a non-full rank. Of course, we should preserve here the conditions:* $\det(\mathbf{C}) \neq 0$ *and* $\det(\mathbf{B}) \neq 0$.

Naturally, despite the fact that

$$\mathbf{B}\mathbf{B}_0^{\mathrm{R}} = \mathbf{I_n} \quad \text{and} \quad \mathbf{B}_0^{\mathrm{R}}\mathbf{B} \neq \mathbf{I_{n_u}}, \tag{8}$$

as well as

$$\mathbf{C}\mathbf{C}_0^{\mathrm{L}} \neq \mathbf{I_{n_y}} \quad \text{and} \quad \mathbf{C}_0^{\mathrm{L}}\mathbf{C} = \mathbf{I_n}, \tag{9}$$

we sill receive

$$\mathbf{CB}(\mathbf{CB})^{\delta} \neq \mathbf{I_{n_y}} \quad \text{and} \quad (\mathbf{CB})^{\delta}\mathbf{CB} \neq \mathbf{I_{n_u}}, \tag{10}$$

for the non-full ranked $\mathbf{CB}$, in general.

Below the crucial theorem related to the IMC-originated control systems is demonstrated.

Theorem 1: For non-full rank systems described by the Eqs. (1) *with* $\det(\mathbf{C}) \neq 0$ *and* $\det(\mathbf{B}) \neq 0$, *the perfect control strategy* (4) *with non-zero setpoint* $\mathbf{y_{ref}}(k+d)$ *cannot be established, since the product of* $\mathbf{CB}$ *is of non-full rank.*

Proof: Immediately, after substitution the formula (4) involving the Moore–Penrose inverse as in Eq. (7) to the state-space representation (1). Thus, we obtain $\mathbf{y}(k+d) \neq \mathbf{y_{ref}}(k+d)$, in general case. □

Remark 9: It should also be mentioned, that the above theorem concerning the non-full rank state-space plants can easily be extended to cover the objects described by the input-output structures. Therefore, in both systems' domains, the perfect control law requirement formed as

$$\mathbf{y}(k+d) = \mathbf{y_{ref}}(k+d), \tag{11}$$

is also not fulfilled.

Indeed, in the transfer-function plant description, the non-full rank perfect control scheme cannot be given, in general [29]. However, for the non-full rank systems governed by the state-space framework, being under $\mathbf{y_{ref}}(k+d) = \mathbf{0}$, the unique peculiarity of the perfect control formula appears. For zero reference value, the IMC-related law (3) simplifies itself to the expression

$$J = \min_{\mathbf{u}(k)}\left\{\left\|\mathbf{y}(k+d)\right\|^2\right\}, \tag{12}$$

for which the perfect control algorithm, or rather perfect regulation, ends up with

$$\mathbf{u}(k) = -(\mathbf{CB})^{\delta}\Big[\mathbf{C}\Big(\sum_{p=1}^{d-1}\mathbf{A}^p\mathbf{B}\mathbf{u}(k-p) + \mathbf{A}^d\mathbf{x}(k)\Big)\Big]. \tag{13}$$

On the first sight, in spite of the application of the simplified perfect control expression (13), we are still predicting

$$\mathbf{y}(k+d) \neq \mathbf{0}, \tag{14}$$

since the output of the system equals

$$\mathbf{y}(k+d) = \mathbf{C}\big[\mathbf{A}\mathbf{x}(k+d-1) + \mathbf{B}\mathbf{u}(k+d-1)q^{(-d+1)}\big], \tag{15}$$

or rather

$$\mathbf{y}(k+d) = \mathbf{C}\big[\mathbf{Ax}(k+d-1) + \mathbf{Bu}(k)\big], \tag{16}$$

or finally

$$\mathbf{y}(k+d) = \mathbf{CAx}(k+d-1) - \mathbf{CB}(\mathbf{CB})^{\delta}\Big[\mathbf{C}\Big(\sum_{p=1}^{d-1}\mathbf{A}^{p}\mathbf{Bu}(k-p) + \mathbf{A}^{d}\mathbf{x}(k)\Big)\Big], \tag{17}$$

under decisive relationship $\mathbf{CB}(\mathbf{CB})^{\delta} \neq \mathbf{I}_{\mathbf{n_y}}$.

In addition, this statement is confirmed by the formula

$$\mathbf{Ax}(k+d-1) = \mathbf{A}\Big(\mathbf{Ax}(k+d-2) + \mathbf{Bu}(k-1)\Big), \tag{18}$$

which can further be extended to the structure

$$\mathbf{Ax}(k+d-1) = \mathbf{A}\Big(\mathbf{A}\Big(\mathbf{Ax}(k+d-3) + \mathbf{Bu}(k-2)\Big) + \mathbf{Bu}(k-1)\Big), \tag{19}$$

and finally to

$$\mathbf{Ax}(k+d-1) = \mathbf{A}\Big(\mathbf{A}\Big(\ldots(\mathbf{Ax}(k) + \mathbf{Bu}(k-d+1)) \ldots + \mathbf{Bu}(k-2)\Big) + \mathbf{Bu}(k-1)\Big), \tag{20}$$

giving rise to the introduction of the following expression

$$\mathbf{Ax}(k+d-1) = \Big(\sum_{p=1}^{d-1}\mathbf{A}^{p}\mathbf{Bu}(k-p) + \mathbf{A}^{d}\mathbf{x}(k)\Big). \tag{21}$$

Henceforth, according to the above investigation, the system's output (17) can be rewritten to the compact formula

$$\mathbf{y}(k+d) = \Big[\mathbf{I}_{\mathbf{n_y}} - \mathbf{CB}(\mathbf{CB})^{\delta}\Big] * \Big[\mathbf{C}\Big(\sum_{p=1}^{d-1}\mathbf{A}^{p}\mathbf{Bu}(k-p) + \mathbf{A}^{d}\mathbf{x}(k)\Big)\Big], \tag{22}$$

which after occupation by the peculiarity derived from the chosen relation of (10), holds the Eq. (14), perhaps.

However, after application of the new perfect control formula (13) to the state equation of the framework (1), our point of view brightens.

In fact, the proper mathematical consideration should sensibly solve the main manuscript's complicated problem. The subsequent issues totally coincide with expected final result. We start with the following

Theorem 2: For systems described by the Eqs. (1) *with* $\det(\mathbf{B}) \neq 0$ *and* $\det(\mathbf{C}) \neq 0$*, the perfect control structure* (13) *can be established, since the product of* $\mathbf{CB}$ *can now be both full or non-full rank.*

Proof: Observe, that the state equation of the framework (1) can be rewritten in the following manner

$$\mathbf{x}(k+d) = \mathbf{Ax}(k+d-1) + \mathbf{Bu}(k), \tag{23}$$

and subsequently, substituted by the formula (13), giving rise to the introduction of supporting relation

$$\mathbf{x}(k+d) = \mathbf{Ax}(k+d-1) - \mathbf{B}(\mathbf{CB})^{\delta}\Big[\mathbf{C}\Big(\sum_{p=1}^{d-1}\mathbf{A}^{p}\mathbf{Bu}(k-p) + \mathbf{A}^{d}\mathbf{x}(k)\Big)\Big]. \tag{24}$$

Thus, in accordance with the expression (21), we obtain

$$\mathbf{x}(k+d) = \Big[\mathbf{I}_{\mathbf{n}} - \mathbf{B}(\mathbf{CB})^{\delta}\mathbf{C}\Big] * \Big(\sum_{p=1}^{d-1}\mathbf{A}^{p}\mathbf{Bu}(k-p) + \mathbf{A}^{d}\mathbf{x}(k)\Big), \tag{25}$$

and finally, the skeleton factorization-based expression

$$\mathbf{x}(k+d) = \Big[\mathbf{I}_{\mathbf{n}} - \mathbf{BB}_{0}^{\mathrm{R}}\mathbf{C}_{0}^{\mathrm{L}}\mathbf{C}\Big] * \Big(\sum_{p=1}^{d-1}\mathbf{A}^{p}\mathbf{Bu}(k-p) + \mathbf{A}^{d}\mathbf{x}(k)\Big), \tag{26}$$

ends the proof with output equation of structure (1) equal to

$$\mathbf{Cx}(k+d) = \mathbf{0}. \tag{27}$$

□

For the completeness, let us analyze the stability behavior of the non-full rank IMC systems. This important property is complexly discussed by the following

Theorem 3: Consider the d-step delayed plants defined by the environment (1) *being under general perfect control algorithm* (4)*. The stability feature of the IMC-oriented scheme is determined by the expression*

$$\det(z\mathbf{I}_{\mathbf{n}} - \mathbf{A} + \mathbf{B}(\mathbf{CB})^{\delta}\mathbf{CA}) = 0. \tag{28}$$

Proof: The generalized multivariable perfect control law with non-zero reference value $\mathbf{y_{ref}}(k+d)$ and different time delay $d \geq 1$ can concurrently, without loss a generality, be analyzed in terms of one-step delay plant with $\mathbf{y_{ref}}(k+d) = \mathbf{0}$.

Since we have

$$\mathbf{u}(k) = -\mathbf{Kx}(k), \tag{29}$$

with $\mathbf{K} = (\mathbf{CB})^{\delta}\mathbf{CA}$, the proof ends successfully. □

Finally, in order to recapitulate the analytical accomplishment of the manuscript, the following closing theorem is suggested.

Theorem 4: Consider the linear time-invariant multivariable system with time delay $d \geqslant 1$ governed by the discrete-time state-space framework (1)*. Then, for* $\mathbf{y_{ref}}(k+d) = \mathbf{0}$ *we can clearly establish the maximum-speed/maximum-accuracy pole-free non-full rank IMC-based perfect control structure.*

Proof: Immediately, after considering the entire investigations, in particular the expressions (26) and (27). □

It is astonishing, that the perfect control-oriented inverse model control design can clearly be given in the non-full rank state-space domain. Furthermore, the new theory includes the state-feedback plants with any time delay. The innovation, which has never been seen before significantly extends the issues presented in Ref. [12].

Remark 10: Notice, that the Theorem 4 takes into account any size of the matrix $\mathbf{CB}$ *provided by* $n_u > n$ *and* $n_y > n$.

It should additionally be emphasized, that the new presented results are only justifiable in the state-space framework. Despite the fact, that there obviously is the direct relationship between the state-space and transfer-function domains, these outcomes are not valid in the input-output description. Accordingly, this important observation highlights the complex nature of the perfect control strategy.

The representative numerical examples of the next section exhibit the big potential of the new breakthrough theory.

V. SIMULATION EXAMPLES

We remark first, that the perfect control design for the full rank MIMO systems is well-defined and broadly explored. Therefore, we only take into account the non-full rank scenarios being under the **CB** matrix with non-full rank.

A. CASE $n_u < n_y$ WITH TIME DELAY $d = 1$

In the first instance, we examine the LTI single-delayed system governed by the discrete-time state-space framework (1) with:

$$\mathbf{A} = \begin{bmatrix} 0.4 & -0.3 \\ 0.2 & -0.2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.5 & -0.3 & 0.8 \\ 0.3 & -0.4 & 0.2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0.1 & -0.5 \\ 0.1 & -1 \\ 0.4 & 0.7 \\ -1.4 & 0.9 \end{bmatrix} \text{ and } \mathbf{x_0} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}.$$

In such scenario, the general perfect control law (4) specializes to the following form

$$\mathbf{u}(k) = (\mathbf{CB})^{\delta}\Big[\mathbf{y_{ref}}(k+1) - \mathbf{CAx}(k)\Big]. \tag{30}$$

The plots of the state $\mathbf{x}(k)$, control $\mathbf{u}(k)$ and output $\mathbf{y}(k)$ signals are shown in Fig. 1, respectively. It is clear, that for $\mathbf{y_{ref}}(k) \neq \mathbf{0}$, the outputs of the plant do not reach the reference values at any time. Nevertheless, the control formula (30) arranging $\mathbf{y_{ref}}(k+d) = \mathbf{0}$ simplifies to the subsequent expression

$$\mathbf{u}(k) = -(\mathbf{CB})^{\delta}\mathbf{CAx}(k). \tag{31}$$

Now, after engaging the above equation (31) to the control design process, together with the consideration of the new defined Theorem 4, we obtain the non-full rank pole-free maximum-speed/maximum-accuracy IMC-oriented closed-loop perfect control system, which variables are presented in Fig. 2. Obviously, the plant's output achieves the zero-setpoint after $d = 1$, which fulfills the perfect control requirement (12) in the best way.

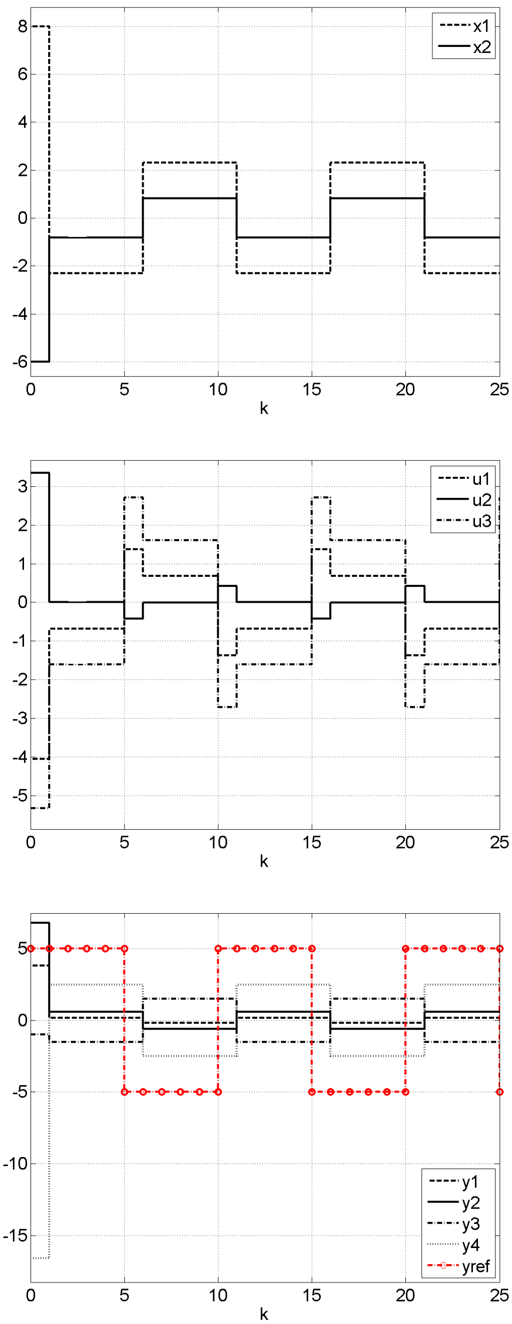

FIGURE 1. Runs of the non-full rank perfect control under $\mathbf{y_{ref}}(k) \neq \mathbf{0}$, case: **CB** with $n_u|_{=3} < n_y|_{=4}$.

B. CASE $n_u = n_y$ WITH TIME DELAY $d = 2$

In the second case, we consider the LTI discrete-time state-space system (1) with time delay $d = 2$ described by:

$$\mathbf{A} = \begin{bmatrix} 0.4 & -0.3 \\ 0.2 & -0.2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.5 & -0.3 & 0.8 \\ 0.3 & -0.4 & 0.2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0.1 & -0.5 \\ 0.1 & -1 \\ 0.4 & 0.7 \end{bmatrix} \text{ and } \mathbf{x_0} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}.$$

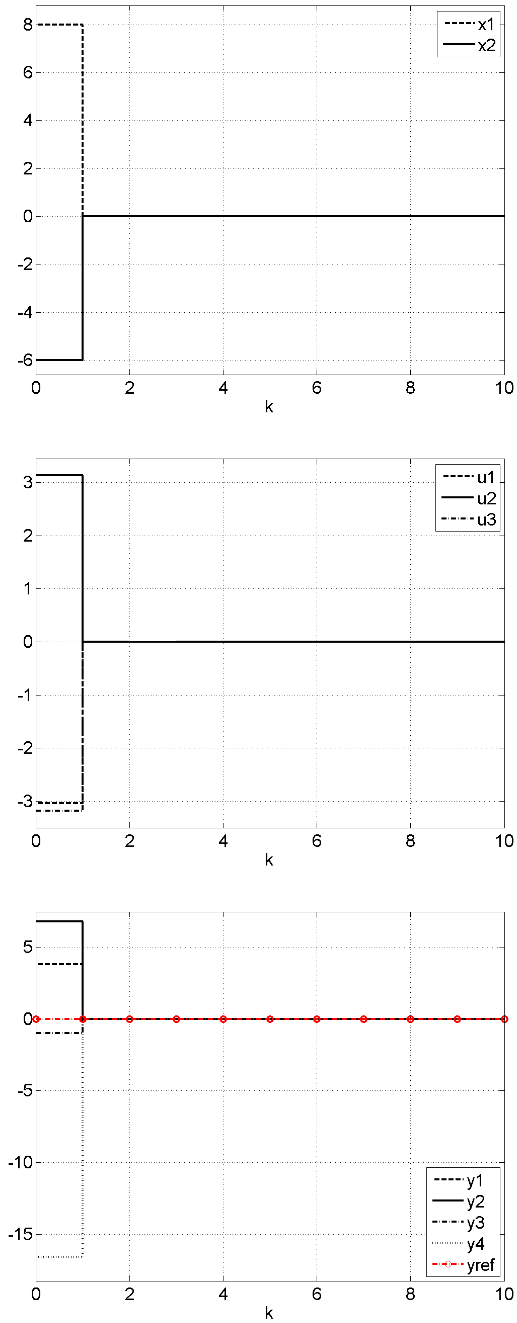

FIGURE 2. Runs of the non-full rank perfect control under $\mathbf{y}_{\mathbf{ref}}(k) = \mathbf{0}$, case: CB with $n_u|_{=3} < n_y|_{=4}$.

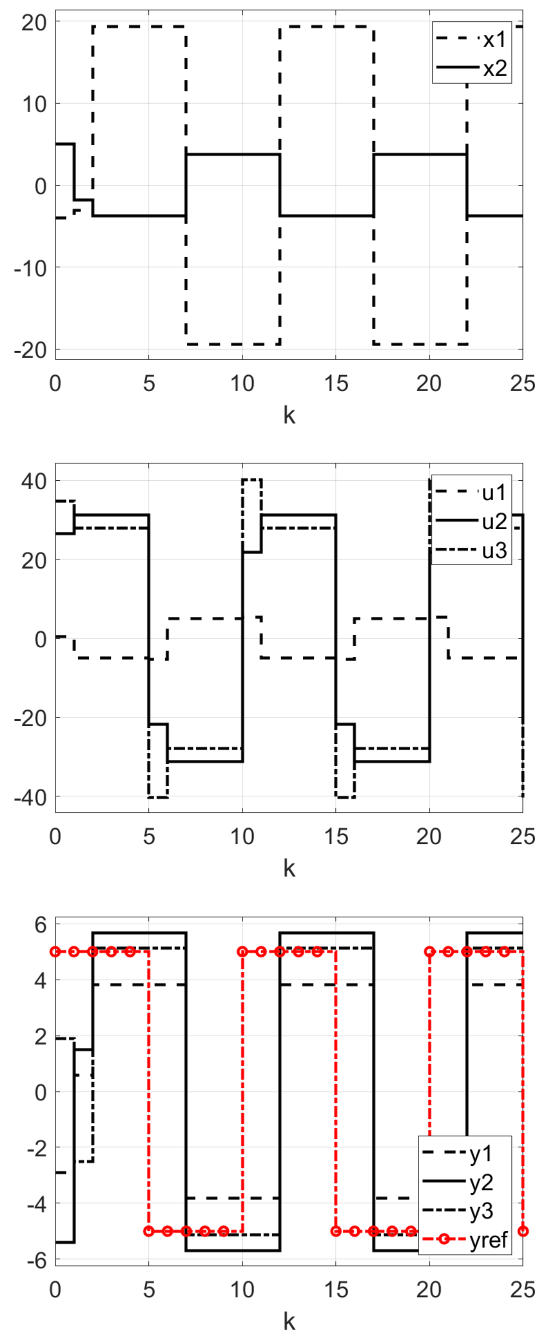

FIGURE 3. Runs of the non-full rank perfect control under $\mathbf{y}_{\mathbf{ref}}(k) \neq \mathbf{0}$, case: CB with $n_u|_{=3} = n_y|_{=3}$.

In this example, the complex perfect control strategy (4) is reduced to the following rule

$$\mathbf{u}(k) = (\mathbf{CB})^{\delta}\Big[\mathbf{y}_{\mathbf{ref}}(k+2) - \mathbf{CABu}(k-1) - \mathbf{CA}^2\mathbf{x}(k)\Big]. \tag{32}$$

The runs of the state, control and output variables are depicted in Fig. 3, respectively.

As previously, for $\mathbf{y}_{\mathbf{ref}}(k+d) \neq \mathbf{0}$, the control system cannot obtain the reference value on the output. However, the control approach (32) respecting $\mathbf{y}_{\mathbf{ref}}(k+d) = \mathbf{0}$ can be rewritten to the expression

$$\mathbf{u}(k) = -(\mathbf{CB})^{\delta}\Big[\mathbf{CABu}(k-1) + \mathbf{CA}^2\mathbf{x}(k)\Big], \tag{33}$$

for which we receive the stable non-full rank IMC-based perfect control scheme. The signals of the system are exhibited in Fig. 4. Naturally, the plant's output reach the zero-reference value after $d = 2$, which ideally realizes the said control law condition (12).

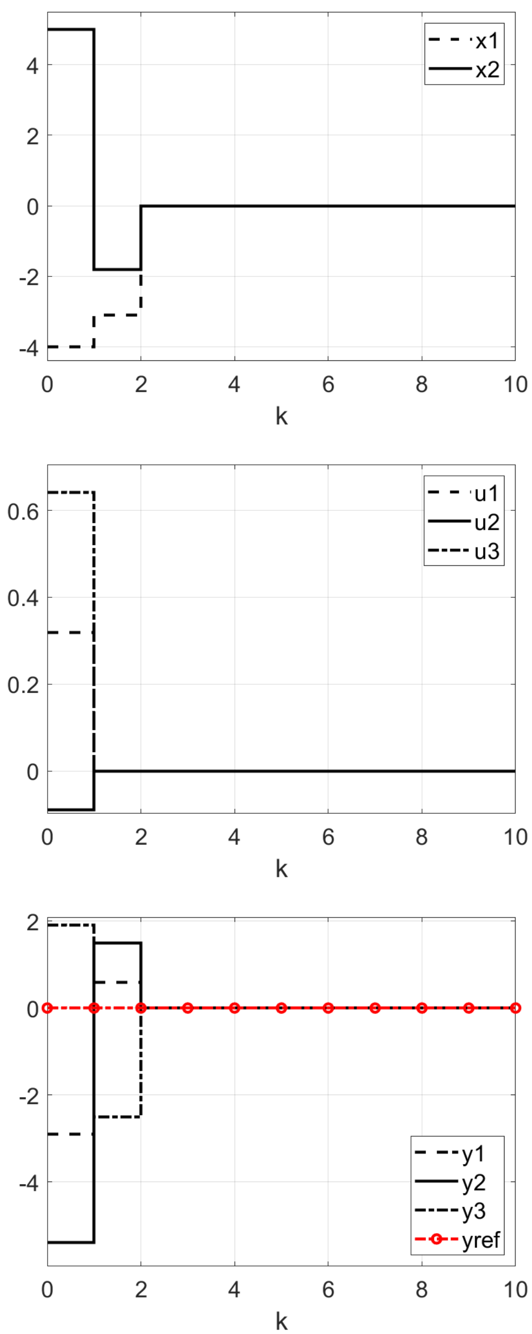

FIGURE 4. Runs of the non-full rank perfect control under $\mathbf{y}_{\mathbf{ref}}(k) = \mathbf{0}$, case: CB with $n_u|_{=3} = n_y|_{=3}$.

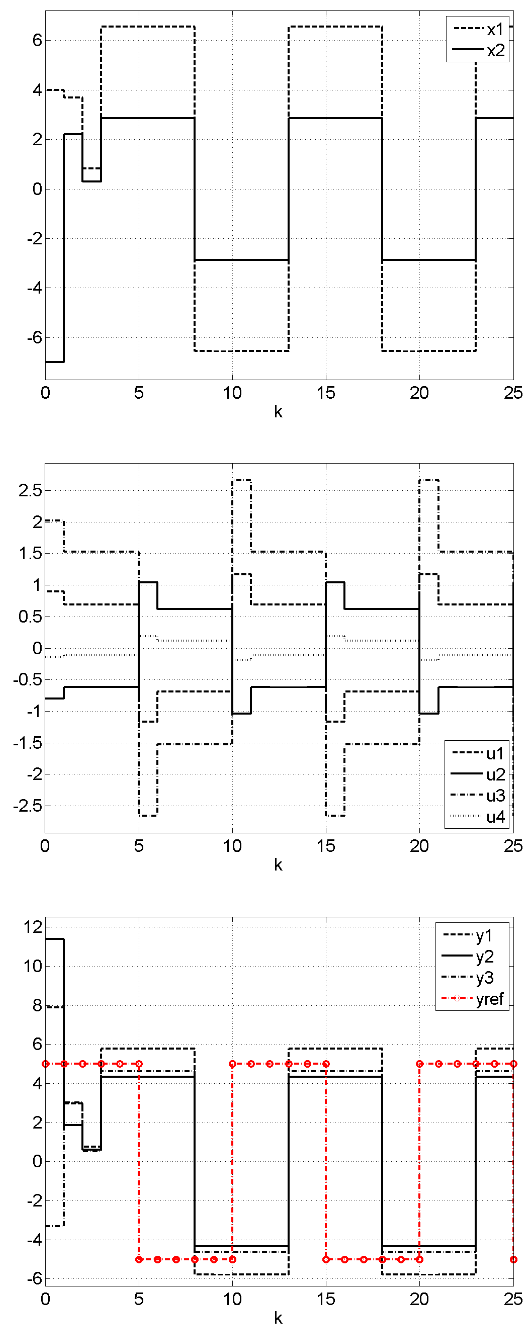

FIGURE 5. Runs of the non-full rank perfect control under $\mathbf{y}_{\mathbf{ref}}(k) \neq \mathbf{0}$, case: CB with $n_u|_{=4} > n_y|_{=3}$.

C. CASE $n_u > n_y$ WITH TIME DELAY $d = 3$

In the last example, we investigate the LTI system with time delay $d = 3$ represented by the matrices of the discrete-time state-space structure (1) as:

$$\mathbf{A} = \begin{bmatrix} 0.4 & -0.3 \\ 0.2 & -0.2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.5 & -0.3 & 2.8 & 0.1 \\ 1.3 & -1.4 & 0.2 & -0.5 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0.1 & -0.5 \\ 0.1 & -1 \\ 0.4 & 0.7 \end{bmatrix} \text{ and } \mathbf{x_0} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}.$$

In such scenario, the general perfect control algorithm (4) comes down to the following formula

$$\mathbf{u}(k) = (\mathbf{CB})^{\delta}\Big[\mathbf{y}_{\mathbf{ref}}(k+3) - \mathbf{CABu}(k-1) - \mathbf{CA}^2\mathbf{Bu}(k-2) - \mathbf{CA}^3\mathbf{x}(k)\Big]. \quad (34)$$

The charts of $\mathbf{x}(k)$, $\mathbf{u}(k)$ and $\mathbf{y}(k)$ are presented in Fig. 5, respectively. Of course, for $\mathbf{y}_{\mathbf{ref}}(k+d) \neq \mathbf{0}$, the plant's outputs cannot arrive at the reference value at any time. Nevertheless, the control expression (34), for $\mathbf{y}_{\mathbf{ref}}(k+d) = \mathbf{0}$,

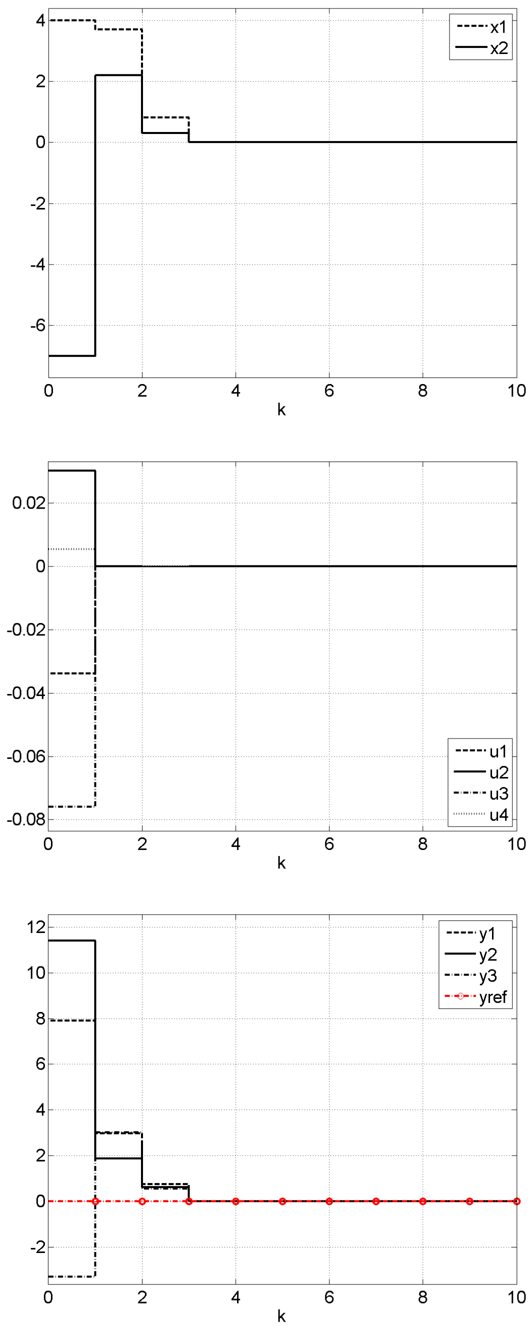

FIGURE 6. **Runs of the non-full rank perfect control under $\mathbf{y_{ref}}(k) = \mathbf{0}$, case: CB with $n_u|_{=4} > n_y|_{=3}$.**

simplifies itself to the succeeding relation

$$\mathbf{u}(k) = -(\mathbf{CB})^{\delta}\Big[\mathbf{CABu}(k-1) + \mathbf{CA}^2\mathbf{Bu}(k-2) + \mathbf{CA}^3\mathbf{x}(k)\Big]. \quad (35)$$

Therefore, after taking into consideration the new Theorem 4 supported by the control equation (35), we arrive at the non-full rank IMC-related perfect control design. The variable runs of the system are shown in Fig. 6. Again, the fundamental perfect control rule (12) has been held here in terms of $\mathbf{y}(k) = 0$, for $k \geq 3$.

VI. CONCLUSIONS AND OPEN PROBLEMS

The complex analytical study in the field of the inverse model control design for the multivariable LTI non-full rank state-space systems described by the maximum-speed/maximum-accuracy discrete-time framework has been presented in the manuscript. From now on it is clear, that the IMC-based perfect control strategy accompanied by the arbitrary time delay d can structurally be established for any non-full rank state-space plant being under zero-reference value. Nevertheless, the presented result is only valid in the state-space framework, so, the following open problems should immediately be defined. It could be interesting to extend the new postulated approach to the class of systems governed by the input-output domains. This issue, together with the studies covering $\mathbf{y_{ref}}(k+d) \neq \mathbf{0}$, constitutes a key research challenge, finally providing the unified IMC-related non-full rank perfect control theory in the nearest future. Last but not least, the complex practical implementation of the theoretical methods presented throughout the paper will be presented shortly.

REFERENCES

[1] W. P. Hunek and T. Feliks, "A new geometric-oriented minimum-energy perfect control design in the IMC-based state-space domain," *IEEE Access*, vol. 8, pp. 41733–41739, 2020, doi: 10.1109/ACCESS.2020.2977278.
[2] C. Fu and W. Tan, "Linear active disturbance rejection control for processes with time delays: IMC interpretation," *IEEE Access*, vol. 8, pp. 16606–16617, 2020, doi: 10.1109/ACCESS.2020.2967806.
[3] M. Krok, W. P. Hunek, and T. Feliks, "Switching perfect control algorithm," *Symmetry*, vol. 12, no. 5, p. 816, May 2020, doi: 10.3390/sym12050816.
[4] D. Romeres, M. Zorzi, R. Camoriano, S. Traversaro, and A. Chiuso, "Derivative-free online learning of inverse dynamics models," *IEEE Trans. Control Syst. Technol.*, vol. 28, no. 3, pp. 816–830, May 2020, doi: 10.1109/TCST.2019.2891222.
[5] X. Wang, Y. Lin, B. Wang, W. Liu, and K. Bai, "Output voltage control of BESS inverter in stand-alone micro-grid based on expanded inverse model," *IEEE Access*, vol. 8, pp. 3781–3791, 2020, doi: 10.1109/ACCESS.2019.2962530.
[6] W. P. Hunek, "Perfect control for right-invertible Grünwald-Letnikov plants—An innovative approach to practical implementation," *Fractional Calculus Appl. Anal.*, vol. 22, no. 2, pp. 424–443, Apr. 2019, doi: 10.1515/fca-2019-0026.
[7] D. Y. Dube and H. G. Patel, "Discrete time minimum variance control of satellite system," in *Mathematical Modelling and Scientific Computing With Applications*, S. Manna, B. N. Datta, and S. S. Ahmad, Eds. Singapore: Springer, 2020, pp. 337–346.
[8] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*. New York, NY, USA: Springer, 2006.
[9] W. P. Hunek and T. Feliks, "A geometric-based approach to the maximum-speed state and output variables for some class of IMC structures," in *Proc. 6th Int. Conf. Control, Decis. Inf. Technol. (CoDIT)*, Paris, France, Apr. 2019, pp. 1385–1389, doi: 10.1109/CoDIT.2019.8820536.
[10] A. Inoue, M. Deng, A. Yanou, and T. Henmi, "Multi-variable generalized minimum variance control with time-delay using interactor matrix," in *Proc. Int. Conf. Adv. Mech. Syst. (ICAMechS)*, Aug. 2019, pp. 81–86, doi: 10.1109/ICAMechS.2019.8861635.
[11] F. Stinga and M. Marian, "Estimation and nonlinear predictive control for an induction machine," in *Proc. 6th Int. Conf. Control, Decis. Inf. Technol. (CoDIT)*, Paris, France, Apr. 2019, pp. 494–499, doi: 10.1109/CoDIT.2019.8820507.

[12] W. P. Hunek and T. Feliks, "A new extension of inverse model control design to non-full rank state-space plants," in *Proc. Eur. Control Conf. (ECC)*, to be published.
[13] L. Qida, T. Shubin, and Y. Yongkuan, "Performance evaluation of generalized minimum variance multi-disturbance control system," in *Proc. Chin. Control Decis. Conf. (CCDC)*, Jun. 2019, pp. 3286–3290, doi: 10.1109/CCDC.2019.8833345.
[14] S.-H. Lee and C. C. Chung, "Reference redesigned perfect tracking control, with application to servo control system," in *Proc. 53rd IEEE Conf. Decis. Control*, Los Angeles, CA, USA, Dec. 2014, pp. 4542–4547, doi: 10.1109/CDC.2014.7040098.
[15] H. Toshani, M. Farrokhi, and Y. Alipouri, "Constrained generalised minimum variance controller design using projection-based recurrent neural network," *IET Control Theory Appl.*, vol. 11, no. 2, pp. 143–154, Jan. 2017, doi: 10.1049/iet-cta.2016.0141.
[16] D. Horla, "Minimum variance adaptive control of a servo drive with unknown structure and parameters," *Asian J. Control*, vol. 15, no. 1, pp. 120–131, Jan. 2013, doi: 10.1002/asjc.479.
[17] J.-R. Li and M.-G. Gan, "A novel robust perfect tracking control method for nonlinear servo systems," in *Proc. 37th IEEE Chin. Control Conf. (CCC)*, Wuhan, China, 2018, pp. 3790–3795, doi: 10.23919/ChiCC.2018.8482835.
[18] T. Williams and P. J. Antsaklis, "Decoupling," in *The Control Handbook* (Electrical Engineering Handbook), W. S. Levine, Ed. Boca Raton, FL, USA: CRC Press, ch. 50, 1996, pp. 745–804.
[19] Z. Kowalczuk and P. Suchomski, "Continuous-time predictive control of delay systems," in *Proc. 4th IEEE Int. Symp. Methods Models Automat. Robot. (MMAR)*, Międzyzdroje, Poland, Aug. 1997, pp. 461–470.
[20] T. Moir and M. Grimble, "Weighted minimum variance controller for nonsquare multivariable systems," *IEEE Trans. Autom. Control*, vol. AC-31, no. 10, pp. 976–977, Oct. 1986.
[21] E. F. Camacho and C. Bordons, *Model Predictive Control*. London, U.K.: Springer-Verlag, 2004.
[22] M. Cychowski, *Robust Model Predictive Control*. Berlin, Germany: VDM Verlag Dr. Müller AG und Co. KG, 2009.
[23] T. Zhang, H. G. Li, Z. Y. Zhong, and G. P. Cai, "Hysteresis model and adaptive vibration suppression for a smart beam with time delay," *J. Sound Vib.*, vol. 358, pp. 35–47, Dec. 2015, doi: 10.1016/j.jsv.2015.08.017.
[24] A. J. Telmoudi, M. Soltani, L. Chaouech, and A. Chaari, "Parameter estimation of nonlinear systems using a robust possibilistic C-regression model algorithm," *Proc. Inst. Mech. Eng., I, J. Syst. Control Eng.*, vol. 234, no. 1, pp. 134–143, Jan. 2020, doi: 10.1177/0959651818756246.
[25] S. Dadhich and W. Birk, "Analysis and control of an extended quadruple tank process," in *Proc. 13th IEEE Eur. Control Conf. (ECC)*, Strasbourg, France, 2014, pp. 838–843, doi: 10.1109/ECC.2014.6862290.
[26] W. P. Hunek, D. Paczko, T. Feliks, and M. Krok, "A norm-based approach to the minimum-energy multivariable perfect control design," in *Proc. 22nd Int. Conf. Syst. Theory, Control Comput. (ICSTCC)*, Sinaia, Romania, Oct. 2018, pp. 7–11, doi: 10.1109/ICSTCC.2018.8540653.
[27] L. Noueili, W. Chagra, and M. Ksouri, "New iterative learning control algorithm using learning gain based on σ inversion for nonsquare multi-input multi-output systems," *Model. Simul. Eng.*, vol. 2018, pp. 1–9, Jun. 2018, doi: 10.1155/2018/4195938.
[28] T. Zhang, B. T. Yang, H. G. Li, and G. Meng, "Dynamic modeling and adaptive vibration control study for giant magnetostrictive actuators," *Sens. Actuators A, Phys.*, vol. 190, pp. 96–105, Feb. 2013, doi: 10.1016/j.sna.2012.11.001.
[29] W. P. Hunek, *Towards a General Theory of Control Zeros for LTI MIMO Systems*. Opole, Poland: Opole Univ. Technology Press, 2011.
[30] A. A. Ahmed, B. K. Koh, and Y. I. Lee, "A comparison of finite control set and continuous control set model predictive control schemes for speed control of induction motors," *IEEE Trans. Ind. Informat.*, vol. 14, no. 4, pp. 1334–1346, Apr. 2018, doi: 10.1109/TII.2017.2758393.
[31] E. Boasso, "Further results on the (b, c)-inverse, the outer inverse $A^{(2)}_{\tau,s}$ and the Moore-Penrose inverse in the Banach context," *Linear Multilinear Algebra*, vol. 67, no. 5, pp. 1006–1030, 2018, doi: 10.1080/03081087.2018.1441798.
[32] W. P. Hunek and P. Majewski, "Relationship between S- and σ-inverse for some class of nonsquare matrices—An initial study," in *Proc. IEEE 15th Int. Conf. Control Automat. (ICCA)*, Edinburgh, U.K., Jul. 2019, pp. 1156–1160, doi: 10.1109/ICCA.2019.8899957.
[33] X. Chen, H. Zhao, H. Sun, and S. Zhen, "Adaptive robust control based on Moore-penrose generalized inverse for underactuated mechanical systems," *IEEE Access*, vol. 7, pp. 157136–157144, 2019, doi: 10.1109/ACCESS.2019.2950211.

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