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Interval Type-2 Trapezoidal Fuzzy Decision-Making Method With Consistency-Improving Algorithm and DEA Model

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ABSTRACT Interval type-2 fuzzy sets (IT2FSs) are useful and valuable tool to describe the decision makers' qualitative evaluation information. This paper designs a novel interval type-2 trapezoidal fuzzy decision-making (IT2TFDM) method, in which the local consistency adjustment strategy (LCAS) and interval type-2 fuzzy data envelopment analysis (DEA) are presented. First, in order to sufficiently describe the uncertain evaluation information, the definition of IT2TrFPRs is introduced, which is followed by the presentation of order consistency and multiplicative consistency for IT2TrFPRs. Then, an approach is proposed to check whether an IT2TrFPR is multiplicative consistent, and we construct a convergent consistency-improving algorithm, wherein the LCAS is utilized to retain the preference evaluation information of decision makers as much as possible. Furthermore, in order to determine the interval type-2 trapezoidal fuzzy priority weight vector, we investigate an interval type-2 trapezoidal fuzzy DEA model. An IT2TFDM method is proposed to obtain the reliable ranking of the alternatives. Finally, a fog-haze influence factor selection problem is given to show the practicality of the proposed IT2TFDM method, and the comparative analysis is carried out to clarify its validity and merits.

INDEX TERMS Decision-making method, interval type-2 trapezoidal fuzzy preference relations, multiplicative consistency, local consistency adjustment strategy, fuzzy data envelopment analysis.

I. INTRODUCTION

In practice group decision-making (GDM) problems [1], [2], it is convenient for decision makers (DMs) to describe their evaluation information with type-1 fuzzy sets (TIFSs) [3], rather than crisp numbers. However, due to the increasing complexity of practical GDM problems, T1FSs are difficult to describe the qualitative evaluation information provided by DMs [4], [5]. For example, for the fog-haze influence factor selection problem, there exists a set of fog-haze weather's influence factors, and we need to determine the importance of fog-haze influence factors. The assessment information for these influence factors is provided by a group of experts via a pairwise comparison method, subsequent to which a judgement matrix can be constructed over these influence factors. However, there are various limitations, including the fact that the experts may not completely know the assessment detailed information, and some influence factors are impacted by fuzziness and hesitancy, etc. Thus, the assessment information provided by experts cannot be described

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with "traditional" T1FSs-based preference relations. Furthermore, in GDM methods without consistency adjustment process, the obtained results may be lack of reliability. Thus, the important and necessary stages are consistency adjustment and alternatives' weights determination. To overcome these limitations, this paper focuses on introducing a new judgement matrix with Type-2 Fuzzy Sets (T2FSs) [6]–[8], and then we propose a new consistency-improving algorithm.

In the process of GDM, it is quite common for experts to elicit their knowledge by using preference relations. The preference relation methods are utilized to express the evaluation information for DMs, subsequent to which a judgement matrix can be constructed over these alternatives [9]–[11]. To this end, preference relations have been introduced and utilized to describe DMs' preference evaluation. In particular, fuzzy preference relations (FPRs) [12], [13] and multiplicative preference relations (MPRs) [14], [15] are two useful preference relations. In the recent past, extensions of FPRs and MPRs have been proposed by various authors [17]–[22].

It is known that one of the necessary steps is to check the quality of preference evaluation information, in which consistency and its measurement play significant roles [23]–[27]. With the interesting consistency properties, Herrera-Viedma *et al.* [28] designed an approach to construct consistent FPRs. Krejčí [29] studied the relationship between multiplicative and additive triangular FPRs. With the help of Abelian linearly ordered group, Xia and Chen [30] established a general method to improve the consistency and consensus levels. Ma *et al.* [31] first introduced several concepts for FPRs, and then they utilize weak transitivity to construct a consistency adjustment approach, which can be applied to increase the consistency level for FPRs. Xu *et al.* [32] proposed two iterative algorithms to improve the additive consistency of FPRs. Xu *et al.* [33] presented some concepts for FPRs, and then proposed a consistency-improving approach for inconsistent LPRs. For incomplete FPRs, Xu *et al.* [34] developed two algorithms for adjusting the inconsistent incomplete FPRs to ones with ordinal consistency. In addition, there also have some efforts to derive the priority weights based on Data Envelopment Analysis (DEA) [35]–[38]. With the help of multiplicative DEA, Liu *et al.* [39] developed a multi-attribute decision making (MADM) method to generate alternatives' priority vector. Under the hesitant multiplicative information environment, Lin and Wang [40] presented the self-weight and cross-weight prioritization approaches for getting the priority vector. Wu *et al.* [41] designed a visual interaction consensus model for social network GDM with trust propagation, which can help DMs to check where can be adjusted to improve the consensus. For large-scale GDM problems, Wu and Xu [42] proposed a minimum adjustment method to research the consensus threshold. Based on a local adjustment strategy, Xu *et al.* [43] investigated a consistency adjustment method to adjust the most inconsistent elements.

The above preference relations use TIFSs to express the DMs' evaluation information, which cannot express the uncertain GDM information comprehensively. To overcome this limitation, we introduce a new judgement matrix with Interval Type-2 Trapezoidal Fuzzy Numbers (IT2TrFNs) [7]. To put this in context, we note that under IT2TrFNs, Qin and Liu [44] constructed a novel multiple attribute GDM method. Chen [45] proposed a GDM method to handle the interval type-2 trapezoidal fuzzy (IT2TrF) multiple criteria decision analysis problems. Based on MULTIMOORA, MOOSRA and TPOP methods, Dorfeshan *et al.* [46] investigated a novel decision methodology with IT2FSs to address uncertain project problems. Ghorabaee *et al.* [47] developed a WASPAS-based integrated approach, and utilized it to solve the interval type-2 fuzzy multi-criteria GDM problems. With the TOPSIS and DEMATEL, Baykasoğlu and Gölcük [48] constructed an IT2TrF model to obtain the ranking of the alternatives.

From above analysis, it is evident that T2FSs are useful and valuable tool to describe the complex evaluation information. Although more and more decision-making methods and theories have been developed on the basis of T2FSs, there are some challenges in getting the reliable GDM results. Lin and Wang [40] have used hesitant MPRs to derive the priority weights. Ma *et al.* [49] directly applied the proposed IT2TrF arithmetic (IT2TrFA) operator to fuse all evaluation information into a collective IT2TrFNs. However, on the one hand, it is hard for DMs to provide the consistent evaluation matrices directly. On the other hand, we know that lack of acceptable consistency leads to inconsistent conclusions. Thus, the decision-making results derived by methods in Lin and Wang [40] and Ma *et al.* [49] may be unreliable. Besides, Wang *et al.* [50] developed a method for interval type-2 fuzzy multiple-attribute GDM problems. For the multiple criteria hierarchical GDM problems, Chen and Lee [51] proposed a new method based on arithmetic operations and FPRs of IT2FSs. However, with the methods in Wang *et al.* [50] and Chen and Lee [51], one must transform the IT2TrF information matrix provided by DMs into a set of ranking value matrices, strength matrices and fuzzy preference matrices, which makes the evaluation process less than transparent and may lead to information loss (see details given in Section V).

The above issues motivate this research to develop some decision-making models for finding the reliable ranking of the alternatives with IT2TrFNs. Therefore, in this paper, we introduce a new concept of Interval Type-2 Trapezoidal Fuzzy Preference Relations (IT2TrFPRs) with IT2TrFNs, which appears to be more reasonable and convenient for handling higher uncertainty information. Then, in order to obtain the acceptably consistent preference relations, we propose a novel consistency-improving algorithm for FPRs to derive the acceptable multiplicative consistent IT2TrFPR, in which the local consistency adjustment strategy (LCAS) is utilized to retain the preference evaluation information of DMs as much as possible while relying on the DM's original preference information to the extent feasible. Furthermore, an IT2TrF DEA model is constructed to determine the IT2TrF priority weight vector of alternatives. Finally, we design a novel interval type-2 trapezoidal fuzzy decision-making (IT2TFDM) method that including order consistency checking process, consistency controlling process, priority weight vector determining process and the desirable alternative selection process.

The rest of the paper is set out according to the following scheme. Section 2 offers some basic knowledge about FPRs and IT2FSs, and then defines the IT2TrFPR and its order consistency. In Section 3, the concepts of multiplicative consistency and consistency index of IT2TrFPRs are presented, and we also propose an algorithm based on LCAS to improve the consistency. Section 4 designs an IT2TFDM method based on IT2TrF DEA model. An illustration of the IT2TFDM method in selecting the most important fog-haze influence factor and a comparative analysis are provided in Section 5. This paper is closed with conclusions in Section 6.

II. PRELIMINARIES

A. FPR AND ITS MULTIPLICATIVE CONSISTENCY

With respect to a decision-making problem [52], assume that $X = \{x_1, x_2, \cdots, x_n\}$ is a set of alternatives, $N = \{1, 2, \cdots, n\}$.

Definition 1 [12]: A real-valued matrix $P = (p_{ij})_{n\times n}$ is FPR, if $p_{ij} \in [0, 1]$, and

$$p_{ij} + p_{ji} = 1, p_{ii} = 0.5, \quad \forall i, j \in N, \tag{1}$$

where p_{ij} indicates the preference intensity of the alternative x_i over x_j.

Definition 2 [10]: Suppose $P = (p_{ij})_{n\times n}$ is an FPR on $X = \{x_1, x_2, \cdots, x_n\}$, then $P = (p_{ij})_{n\times n}$ is multiplicative consistent, if

$$p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji}, \quad i, k, j \in N. \tag{2}$$

Definition 3 [10]: Suppose that $P = (p_{ij})_{n\times n}$ is an FPR on $X = \{x_1, x_2, \cdots, x_n\}$, then $P = (p_{ij})_{n\times n}$ is multiplicative consistent, if there exists a normalized priority weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$, such that

$$p_{ij} = \frac{w_i}{w_i + w_j}, \quad i, j \in N, \tag{3}$$

where $w_i > 0, i \in N$, and $\sum_{i=1}^{n} w_i = 1$.

B. IT2FSs

In order to express the uncertain decision-making information comprehensively, Zadeh [7] first introduced the concept of IT2FSs, which is a generalization of the concept of T1FSs [53].

Definition 4 [54]: Suppose that A^L and A^U are two generalized trapezoidal fuzzy numbers, h_A^L and h_A^U are the heights of A^L and A^U, respectively, and $h_A^L, h_A^U \in [0, 1]$. An T2TrFN A in the universe of discourse E is defined as follows:

$$A = [A^L, A^U] = [(a_1^L, a_2^L, a_3^L, a_4^L; h_A^L), (a_1^U, a_2^U, a_3^U, a_4^U; h_A^U)], \tag{4}$$

where $0 \le a_1^L \le a_2^L \le a_3^L \le a_4^L \le 1, 0 \le a_1^U \le a_2^U \le a_3^U \le a_4^U \le 1, a_1^U \le a_1^L, a_4^L \le a_4^U, 0 \le h_A^L \le h_A^U \le 1$, and the lower membership function $A^L(x)$ and upper membership function $A^U(x)$ of A are denoted as:

$$A^L(x) = \begin{cases} \dfrac{h_A^L(x - a_1^L)}{a_2^L - a_1^L}, & a_1^L \le x \le a_2^L \\ h_A^L, & a_2^L \le x \le a_3^L \\ \dfrac{h_A^L(a_4^L - x)}{a_4^L - a_3^L}, & a_3^L \le x \le a_4^L \\ 0, & otherwise, \end{cases}$$

$$A^U(x) = \begin{cases} \dfrac{h_A^U(x - a_1^U)}{a_2^U - a_1^U}, & a_1^U \le x \le a_2^U \\ h_A^U, & a_2^U \le x \le a_3^U \\ \dfrac{h_A^U(a_4^U - x)}{a_4^U - a_3^U}, & a_3^U \le x \le a_4^U \\ 0, & otherwise. \end{cases}$$

In order to compare the different IT2TrFNs, Qin and Liu [44] introduced the following arithmetic average ranking value function for IT2TrFNs.

Definition 5 [44]: Suppose that A is an IT2TrFN, then the arithmetic average ranking value function of A is defined as:

$$\Delta(A) = \left(\frac{a_1^U + a_4^U}{2} + \frac{h_A^L + h_A^U}{2}\right) \times \frac{\sum_{r=1}^{4}\left(a_r^L + a_r^U\right)}{8}. \tag{5}$$

Definition 6 [44]: Assume that A_1 and A_2 are two IT2TrFNs, then

(1) If $\Delta(A_1) > \Delta(A_2)$, then $A_1 > A_2$, which indicates A_1 is better than A_2;

(2) If $\Delta(A_1) < \Delta(A_2)$, then $A_1 < A_2$, which indicates A_1 is worse than A_2;

(3) If $\Delta(A_1) = \Delta(A_2)$, then $A_1 = A_2$, which indicates A_1 and A_2 is indifferent.

C. IT2TrFPRs

In what follows, motivated by FPRs [10], we introduce the concept of IT2TrFPRs to sufficiently describe original uncertain decision-making information.

Definition 7: Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set of alternatives, an IT2TrFPR $\tilde{A}$ on X is characterized by a compassion matrix $\tilde{A} = (\tilde{A}_{ij})_{n\times n} \subset X \times X$, where $\tilde{A}_{ij} = [(a_{ij(1)}^L, a_{ij(2)}^L, a_{ij(3)}^L, a_{ij(4)}^L; h_{ij}^L), (a_{ij(1)}^U, a_{ij(2)}^U, a_{ij(3)}^U, a_{ij(4)}^U; h_{ij}^U)]$ is an IT2TrFN representing the interval type-2 trapezoidal fuzzy preference degree of alternative x_i over x_j, and $\tilde{A}_{ij}$ should satisfy the following requirements:

$$a_{ii(s)}^L = a_{ii(s)}^U = h_{ii}^L = h_{ii}^U = 0.5,$$
$$a_{ij(s)}^L + a_{ji(5-s)}^L = 1, a_{ij(s)}^U + a_{ji(5-s)}^U = 1, h_{ij}^L + h_{ji}^U = 1, \tag{6}$$

where $0 \le a^L_{ij(1)} \le a^L_{ij(2)} \le a^L_{ij(3)} \le a^L_{ij(4)} \le 1, 0 \le a^U_{ij(1)} \le a^U_{ij(2)} \le a^U_{ij(3)} \le a^U_{ij(4)} \le 1, a^U_{ij(1)} \le a^L_{ij(1)}, a^L_{ij(4)} \le a^U_{ij(4)}, 0 \le h^L_{ij} \le h^U_{ij} \le 1$.

Example 1: Let $\tilde{A}$ be an IT2TrFPR as follows:

$$\tilde{A} = \begin{pmatrix} [(0.5,0.5,0.5,0.5;0.5),(0.5,0.5,0.5,0.5;0.5)] & [(0.1,0.3,0.4,0.6;0.3),(0.1,0.2,0.5,0.7;0.4)] & [(0.3,0.6,0.7,0.9;0.6),(0.3,0.5,0.7,1.0;0.7)] & [(0.2,0.3,0.4,0.5;0.2),(0.1,0.2,0.4,0.6;0.4)] \\ [(0.4,0.6,0.7,0.9;0.6),(0.3,0.5,0.8,0.9;0.7)] & [(0.5,0.5,0.5,0.5;0.5),(0.5,0.5,0.5,0.5;0.5)] & [(0.5,0.6,0.7,0.8;0.7),(0.3,0.5,0.8,0.9;0.9)] & [(0.3,0.5,0.6,0.8;0.5),(0.2,0.4,0.7,0.8;0.7)] \\ [(0.1,0.3,0.4,0.7;0.3),(0.0,0.3,0.5,0.7;0.4)] & [(0.2,0.3,0.4,0.5;0.1),(0.1,0.2,0.5,0.7;0.3)] & [(0.5,0.5,0.5,0.5;0.5),(0.5,0.5,0.5,0.5;0.5)] & [(0.6,0.7,0.8,0.9;0.4),(0.5,0.7,0.9,0.9;0.6)] \\ [(0.5,0.6,0.7,0.8;0.6),(0.4,0.6,0.8,0.9;0.8)] & [(0.2,0.4,0.5,0.7;0.3),(0.2,0.3,0.6,0.8;0.5)] & [(0.1,0.2,0.3,0.4;0.4),(0.1,0.1,0.3,0.5;0.6)] & [(0.5,0.5,0.5,0.5;0.5),(0.5,0.5,0.5,0.5;0.5)] \end{pmatrix}.$$

In the following, we present the concept of order consistency for IT2TrFPRs.

Definition 8: Assume that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR, if there exists a permutation $\sigma : N \to N$, such that $\tilde{A}_{\sigma(1)j} < \tilde{A}_{\sigma(2)j} < \cdots < \tilde{A}_{\sigma(n)j}, \forall j \in N$, then IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is order consistent.

Example 2: Let $X = \{x_1, x_2, \cdots, x_n\}$ be four alternatives, a DM evaluates these alternatives and constructs an IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times 4}$, which is shown as follows:

$$\tilde{A} = \begin{pmatrix} [(0.5,0.5,0.5,0.5;0.5),(0.5,0.5,0.5,0.5;0.5)] & [(0.2,0.3,0.4,0.4;0.3),(0.2,0.3,0.5,0.6;0.5)] & [(0.1,0.2,0.3,0.4;0.3),(0.0,0.2,0.4,0.5;0.4)] & [(0.0,0.1,0.2,0.4;0.1),(0.0,0.1,0.3,0.4;0.3)] \\ [(0.6,0.6,0.7,0.8;0.5),(0.4,0.5,0.7,0.8;0.6)] & [(0.5,0.5,0.5,0.5;0.5),(0.5,0.5,0.5,0.5;0.5)] & [(0.2,0.2,0.4,0.5;0.3),(0.1,0.2,0.4,0.5;0.5)] & [(0.1,0.2,0.3,0.4;0.2),(0.1,0.1,0.3,0.5;0.4)] \\ [(0.6,0.7,0.8,0.9;0.6),(0.5,0.6,0.8,1.0;0.7)] & [(0.5,0.6,0.8,0.8;0.5),(0.5,0.6,0.8,0.9;0.7)] & [(0.5,0.5,0.5,0.5;0.5),(0.5,0.5,0.5,0.5;0.5)] & [(0.1,0.3,0.4,0.4;0.3),(0.1,0.2,0.4,0.5;0.4)] \\ [(0.6,0.8,0.9,1.0;0.7),(0.6,0.7,0.9,1.0;0.9)] & [(0.6,0.7,0.8,0.9;0.6),(0.5,0.7,0.9,0.9;0.8)] & [(0.6,0.6,0.7,0.9;0.6),(0.5,0.6,0.8,0.9;0.7)] & [(0.5,0.5,0.5,0.5;0.5),(0.5,0.5,0.5,0.5;0.5)] \end{pmatrix}.$$

Based on the above IT2TrFP $\tilde{A}$ and Definitions 5 and 6, we have $\tilde{A}_{1j} < \tilde{A}_{2j} < \tilde{A}_{3j} < \tilde{A}_{4j}, j = 1, 2, 3, 4$, i.e., there exists a permutation $\sigma(i) = i$, such that $\tilde{A}_{\sigma(1)j} < \tilde{A}_{\sigma(2)j} < \tilde{A}_{\sigma(3)j} < \tilde{A}_{\sigma(4)j}, j = 1, 2, 3, 4$, it follows that $x_{\sigma(1)} \prec x_{\sigma(2)} \prec x_{\sigma(3)} \prec x_{\sigma(4)}$. Therefore, the ranking order among these four alternatives is $x_1 \prec x_2 \prec x_3 \prec x_4$, we obtain that the desirable alternative is x_4.

According to Example 2, we know that Definition 8 provides us a novel approach with IT2TrFPRs to make decision quickly and efficiently in certain cases.

III. MATH CONSISTENCY ADJUSTMENT APPROACH FOR IT2TrFPRs

In this section, we first present two concepts of multiplicative consistency for IT2TrFPRs, which is followed by a novel method for checking whether an IT2TrFPR is multiplicative consistent. Then, a consistency-improving algorithm for FPRs, derived from an IT2TrFPR, is constructed, in which the LCAS are utilized to retain the preference evaluation of DM as much as possible.

A. EQUATIONS MULTIPLICATIVE CONSISTENCY OF IT2TrFPRs

From Definition 7, we have $a^L_{ij(s)} + a^L_{ji(5-s)} = 1, a^U_{ij(s)} + a^U_{ji(5-s)} = 1, h^L_{ij} + h^U_{ji} = 1, \forall s = 1, 2, 3, 4, i, j \in N$, then we can construct ten preference relations $P^m = (p^m_{ij})_{n\times n}(m = 1, 2, \cdots, 10)$ from an IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$, where $\tilde{A}_{ij} = [(a^L_{ij(1)}, a^L_{ij(2)}, a^L_{ij(3)}, a^L_{ij(4)}; h^L_{ij}), (a^U_{ij(1)}, a^U_{ij(2)}, a^U_{ij(3)}, a^U_{ij(4)}; h^U_{ij})]$ as follows:

$$p^m_{ij} = \begin{cases} a^L_{ij(m)}, & i < j \\ 0.5, & i = j \\ a^L_{ij(5-m)}, & i > j, \end{cases} \quad m = 1, 2, 3, 4, \; p^5_{ij} = \begin{cases} h^L_{ij}, & i < j \\ 0.5, & i = j \\ h^U_{ij}, & i > j, \end{cases} \quad i, j \in N,$$

$$p^m_{ij} = \begin{cases} a^U_{ij(m-5)}, & i < j \\ 0.5, & i = j \\ a^U_{ij(10-m)}, & i > j, \end{cases} \quad m = 6, 7, 8, 9, \; p^{10}_{ij} = \begin{cases} h^U_{ij}, & i < j \\ 0.5, & i = j \\ h^L_{ij}, & i > j, \end{cases} \quad i, j \in N. \tag{7}$$

It is observed that $P^m = (p^m_{ij})_{n\times n}(m = 1, 2, \cdots, 10)$ are ten FPRs. In what follows, several concepts of multiplicative consistent IT2TrFPRs are introduced based on FPRs [10].

Definition 9: Assume that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR on $X = \{x_1, x_2, \cdots, x_n\}$, if FPRs $P^m = (p^m_{ij})_{n\times n}(m = 1, 2, \cdots, 10)$, constructed by Eq. (7), are multiplicative consistent, i.e., $P^m = (p^m_{ij})_{n\times n}(m = 1, 2, \cdots, 10)$ satisfy the following multiplicative transitivity:

$$p^m_{ij} \cdot p^m_{jk} \cdot p^m_{ki} = p^m_{ik} \cdot p^m_{kj} \cdot p^m_{ji}, \quad i, k, j \in N, m = 1, 2, \cdots, 10, \tag{8}$$

then IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is said to be multiplicative consistent.

Remark 1: It is obvious that Definition 9 is independent of alternative labels, therefore, the multiplicative consistency of IT2TrFPRs is robust to permutations of decision-making alternatives.

Theorem 1: Suppose that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR, FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are constructed by Eq. (7), then the following statements are equivalent:

$$(1)\quad p_{ij}^m \cdot p_{jk}^m \cdot p_{ki}^m = p_{ik}^m \cdot p_{kj}^m \cdot p_{ji}^m,\quad i, k, j \in N,\quad m = 1, 2, \cdots, 10;$$
$$(2)\quad p_{ij}^m \cdot p_{jk}^m \cdot p_{ki}^m = p_{ik}^m \cdot p_{kj}^m \cdot p_{ji}^m,\quad i < k < j,\quad m = 1, 2, \cdots, 10.$$

As the uncertainty and complexity in GDM problems, the crisp weight vector may not suitable to describe the importance levels of the alternatives. To overcome this drawback, we introduce the notion of IT2TrF weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$, where $w_i = [w_i^L, w_i^U] = [(w_{i(1)}^L, w_{i(2)}^L, w_{i(3)}^L, w_{i(4)}^L; h_{w_i}^L), (w_{i(1)}^U, w_{i(2)}^U, w_{i(3)}^U, w_{i(4)}^U; h_{w_i}^U)]$ $(i \in N)$ is an IT2TrFN, and $0 \le w_{i(1)}^L \le w_{i(2)}^L \le w_{i(3)}^L \le w_{i(4)}^L \le 1, 0 \le w_{i(1)}^U \le w_{i(2)}^U \le w_{i(3)}^U \le w_{i(4)}^U \le 1, w_{i(1)}^U \le w_{i(1)}^L, w_{i(4)}^L \le w_{i(4)}^U, 0 \le h_{w_i}^L \le h_{w_i}^L \le 1, i \in N$. w_i^L and w_i^U denote the lower bound and upper bound of the membership degree's importance of the alternative $x_i (i \in N)$. In what follows, we present the relationship between IT2TrFPR and IT2TrF weight vector by the aid of ten FPRs that constructed by Eq. (7).

Definition 10: Assume that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR on $X = \{x_1, x_2, \cdots, x_n\}$, FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are constructed by Eq. (7), then IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is multiplicative consistent, if there exists an IT2TrF weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$, such that

$$\begin{cases} p_{ij}^m = \dfrac{w_{i(m)}^L}{w_{i(m)}^L + w_{j(m)}^L}, & m = 1, 2, \cdots, 5, i, j \in N, \\ p_{ij}^m = \dfrac{w_{i(m-5)}^U}{w_{i(m-5)}^U + w_{j(m-5)}^U}, & m = 6, 7, \cdots, 10, i, j \in N, \end{cases} \tag{9}$$

Motivated by Chiclana *et al.* [4], the following result is obtained to check whether an IT2TrFPR is multiplicative consistent.

Theorem 2: Assume that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR on $X = \{x_1, x_2, \cdots, x_n\}$, FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are constructed by Eq. (7), then the following propositions are equivalent:

(i) $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is multiplicative consistent.

(ii) For $\forall i < j, m = 1, 2, \cdots, 10$, we have

$$p_{ij}^m = \frac{1}{1 + \prod_{r=0}^{j-i-1}\left(\frac{1}{p_{i+r,i+r+1}^m} - 1\right)}. \tag{10}$$

Proof: The Proof of Theorem 2 is provided in Appendix.

It is observed that Theorem 2 uses the elements on secondary diagonal to check the multiplicative consistency. If we only utilize secondary diagonal elements to construct a multiplicative consistent FPRs, it may be unreasonable. Therefore, we propose a novel method to construct multiplicative consistent FPRs, in which we use all elements to derive the multiplicative consistent FPRs.

Theorem 3: Assume that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR on $X = \{x_1, x_2, \cdots, x_n\}$, ten FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are constructed by Eq. (7). Let

$$\bar{p}_{ij}^m = \left(\sum\nolimits_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}\right) \Big/ \sum\nolimits_{l=1}^n \frac{p_{il}^m + p_{jl}^m}{\sum_{h=1}^n p_{hl}^m},\quad m = 1, 2, \cdots, 10 \tag{11}$$

then $\bar{P}^m = (\bar{p}_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are multiplicative consistent FPRs, and for $i < j$, $\bar{p}_{ij}^m$ is an increasing function with respect to p_{ij}^m.

Proof: The Proof of Theorem 3 is provided in Appendix.

Example 3: Suppose that there are four alternatives $X = \{x_1, x_2, x_3, x_4\}$, a DM constructs an IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times 4}$ shown in Example 1. By using Theorem 3, one can obtain ten multiplicative consistent FPRs $\bar{P}^m = (\bar{p}_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$:

$$\bar{P}^1 = \begin{pmatrix} 0.5 & 0.3311 & 0.3082 & 0.3039 \\ 0.6689 & 0.5 & 0.4738 & 0.4687 \\ 0.6918 & 0.5262 & 0.5 & 0.4949 \\ 0.6961 & 0.5313 & 0.5051 & 0.5 \end{pmatrix},$$

$$\bar{P}^2 = \begin{pmatrix} 0.5 & 0.4236 & 0.4554 & 0.4580 \\ 0.5764 & 0.5 & 0.5323 & 0.5349 \\ 0.5446 & 0.4677 & 0.5 & 0.5027 \\ 0.5420 & 0.4651 & 0.4973 & 0.5 \end{pmatrix},$$

$$\bar{P}^3 = \begin{pmatrix} 0.5 & 0.4553 & 0.5204 & 0.5087 \\ 0.5447 & 0.5 & 0.5649 & 0.5406 \\ 0.4796 & 0.4351 & 0.5 & 0.5159 \\ 0.4913 & 0.4594 & 0.4841 & 0.5 \end{pmatrix},$$

$$\bar{P}^4 = \begin{pmatrix} 0.5 & 0.5099 & 0.6265 & 0.5591 \\ 0.4901 & 0.5 & 0.6172 & 0.5492 \\ 0.3735 & 0.3828 & 0.5 & 0.5304 \\ 0.4409 & 0.4508 & 0.4696 & 0.5 \end{pmatrix},$$

$$\bar{P}^5 = \begin{pmatrix} 0.5 & 0.3895 & 0.4868 & 0.3895 \\ 0.6105 & 0.5 & 0.5979 & 0.5000 \\ 0.5132 & 0.4021 & 0.5 & 0.4021 \\ 0.6105 & 0.5000 & 0.5979 & 0.5 \end{pmatrix},$$

$$\bar{P}^6 = \begin{pmatrix} 0.5 & 0.3525 & 0.2747 & 0.2578 \\ 0.6475 & 0.5 & 0.4103 & 0.3896 \\ 0.7253 & 0.5897 & 0.5 & 0.4784 \\ 0.7422 & 0.6104 & 0.5216 & 0.5 \end{pmatrix},$$

$$\bar{P}^7 = \begin{pmatrix} 0.5 & 0.3894 & 0.3785 & 0.3894 \\ 0.6106 & 0.5 & 0.4885 & 0.5000 \\ 0.6215 & 0.5115 & 0.5 & 0.4915 \\ 0.6106 & 0.5000 & 0.5085 & 0.5 \end{pmatrix},$$

$$\bar{P}^8 = \begin{pmatrix} 0.5 & 0.4643 & 0.5507 & 0.5881 \\ 0.5357 & 0.5 & 0.5857 & 0.6222 \\ 0.4493 & 0.4143 & 0.5 & 0.5381 \\ 0.4119 & 0.3778 & 0.4619 & 0.5 \end{pmatrix},$$

$$\bar{P}^9 = \begin{pmatrix} 0.5 & 0.5492 & 0.7180 & 0.6861 \\ 0.4508 & 0.5 & 0.6764 & 0.6420 \\ 0.2820 & 0.3236 & 0.5 & 0.5619 \\ 0.3139 & 0.3580 & 0.4381 & 0.5 \end{pmatrix},$$

$$\bar{P}^{10} = \begin{pmatrix} 0.5 & 0.4280 & 0.5930 & 0.5262 \\ 0.5720 & 0.5 & 0.6608 & 0.5975 \\ 0.4070 & 0.3392 & 0.5 & 0.4325 \\ 0.4738 & 0.4025 & 0.5675 & 0.5 \end{pmatrix}$$

Based on Theorem 3, we can get the following corollary.

Corollary 1: Suppose that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR, ten FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are constructed by Eq. (7), their multiplicative consistent FPRs are $\bar{P}^m = (\bar{p}_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$, then $\tilde{A}$ is multiplicative consistent if and only if $P^m = \bar{P}^m(m = 1, 2, \cdots, 10)$.

B. CONSISTENCY-IMPROVING APPROACH FOR IT2TrFPRs WITH LCAS

With the development of social economy and technology, DMs can hardly give a completely consistent IT2TrFPR. Thus, Corollary 1 cannot hold, and then there must exist $m \in \{1, 2, \cdots, 10\}$, such that $\bar{P}^m \neq P^m$. Therefore, we utilize $\frac{1}{n(n-1)}\sum_{i\neq j}\left|\bar{p}_{ij}^m - p_{ij}^m\right|$ to measure the deviation between P^m and $\bar{P}^m$, and the consistency degree of IT2TrFPR $\tilde{A}$ can be calculate as $\frac{1}{10n(n-1)}\sum_{m=1}^{10}\sum_{i\neq j}\left|\bar{p}_{ij}^m - p_{ij}^m\right|$. While $\left|\bar{p}_{ji}^m - p_{ji}^m\right| = \left|(1-\bar{p}_{ij}^m) - (1-p_{ij}^m)\right| = \left|\bar{p}_{ij}^m - p_{ij}^m\right|$, then we have

$$\begin{aligned} &\frac{1}{n(n-1)}\sum_{i\neq j}\left|\bar{p}_{ij}^m - p_{ij}^m\right| \\ &= \frac{1}{n(n-1)}\sum_{i<j}\left(\left|\bar{p}_{ij}^m - p_{ij}^m\right| + \left|\bar{p}_{ji}^m - p_{ji}^m\right|\right) \\ &= \frac{1}{n(n-1)}\sum_{i<j}\left(\left|\bar{p}_{ij}^m - p_{ij}^m\right| + \left|\bar{p}_{ij}^m - p_{ij}^m\right|\right) \\ &= \frac{2}{n(n-1)}\sum_{i<j}\left|\bar{p}_{ij}^m - p_{ij}^m\right|, \end{aligned}$$

thus $\frac{1}{10n(n-1)}\sum_{m=1}^{10}\sum_{i\neq j}\left|\bar{p}_{ij}^m - p_{ij}^m\right| = \frac{1}{5n(n-1)}\sum_{m=1}^{10}\sum_{i<j}\left|\bar{p}_{ij}^m - p_{ij}^m\right|$.

Therefore, one can utilize $\frac{2}{n(n-1)}\sum_{i<j}\left|\bar{p}_{ij}^m - p_{ij}^m\right|$ and $\frac{1}{5n(n-1)}\sum_{m=1}^{10}\sum_{i<j}\left|\bar{p}_{ij}^m - p_{ij}^m\right|$ to measure the consistency of P^m and $\tilde{A}$, respectively.

Definition 11: Let $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ be an IT2TrFPR, FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are constructed by Eq. (7), and their multiplicative consistent FPRs are $\bar{P}^m = (\bar{p}_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$, then

$$CI(P^m) = \frac{2}{n(n-1)}\sum_{i<j}\left|\bar{p}_{ij}^m - p_{ij}^m\right| \tag{12}$$

is called the consistency index of FPR P^m,

$$CI(\tilde{A}) = \frac{1}{10}\sum_{m=1}^{10}CI(P^m) = \frac{1}{5n(n-1)}\sum_{m=1}^{10}\sum_{i<j}\left|\bar{p}_{ij}^m - p_{ij}^m\right| \tag{13}$$

is called the consistency index of $\tilde{A}$.

According to Eq. (13), we have $CI(\tilde{A}) \in [0, 1]$. If $CI(\tilde{A}) = 0$, then $\tilde{A}$ is a completely multiplicative consistent IT2TrFPR.

Definition 12: Let $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ be an IT2TrFPR, $\overline{CI}$ be a threshold value of acceptable consistency, if $CI(A) \leq \overline{CI}$, then IT2TrFPR $\tilde{A}$ is said to be acceptable multiplicative consistent.

Algorithm I

Input: An IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$, consistency threshold $\overline{CI}$, adjusted parameter $\theta(0 < \theta < 1)$.

Output: Ten FPRs $\widehat{P}^m = (\widehat{p}_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ with acceptable multiplicative consistency.

Step 1: Apply Eq. (7) to establish ten FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$.

Step 2: Initialize FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ and iteration t. Let $P_{(t)}^m = (p_{ij(t)}^m)_{n\times n} = P^m$ and $t = 0$.

Step 3: By utilizing Theorem 3, we can use all the related elements to construct the multiplicative consistent FPRs $\bar{P}_{(t)}^m = (\bar{p}_{ij(t)}^m)_{n\times n}(m = 1, 2, \cdots, 10)$.

Step 4: Calculate $CI(P_{(t)}^m)$ for $P_{(t)}^m = (p_{ij(t)}^m)_{n\times n}$, where

$$CI(P_{(t)}^m) = \frac{2}{n(n-1)}\sum_{i<j}\left|\bar{p}_{ij(t)}^m - p_{ij(t)}^m\right|. \tag{14}$$

Step 5: Check the multiplicative consistency level. If $\frac{1}{10}\sum_{m=1}^{10}CI(P_{(t)}^m) \leq \overline{CI}$, then implement Step 7; otherwise, implement Step 6.

Step 6: Apply LCAS to find out the element $p_{i^*,j^*(t)}^{m^*}$ with highest level of inconsistency, where $\left|\bar{p}_{i^*,j^*(t)}^{m^*} - p_{i^*,j^*(t)}^{m^*}\right| = \max_{1\leq m\leq 10, i<j}\left|\bar{p}_{ij(t)}^m - p_{ij(t)}^m\right|$. Construct the adjusted FPRs $P_{(t+1)}^m = (p_{ij(t+1)}^m)_{n\times n}$, where

$$p_{ij(t+1)}^m = \begin{cases} (1-\theta)\cdot p_{i^*,j^*(t)}^{m^*} + \theta\cdot\bar{p}_{i^*,j^*(t)}^{m^*}, & i=i^*, j=j^*, m=m^* \\ p_{ij(t)}^m, & otherwise \\ 1-p_{ji(t+1)}^m, & i=j^*,\ j=i^*,\ m=m^*, \end{cases} \tag{15}$$

Let $t = t + 1$, and return to Step 2.

Step 7: Let $\widehat{P}^m = P_{(t)}^m(m = 1, 2, \cdots, 10)$. Output the acceptable multiplicative consistent FPRs $\widehat{P}^m = (\widehat{p}_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$.

Step 8: End.

If a DM provides an unacceptable IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$, then some of the FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are unacceptable. Now, we design a consistency-improving algorithm to adjust the consistency level of the obtained ten FPRs.

It is obvious that one of the most important goals of consistency-adjustment algorithm is to retain as much of the original evaluation information as possible in the modified FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$, i.e., at each iteration, we check and adjust the most inconsistent element $p_{i^*j^*}^{m^*}$ in ten FPRs, where $\left|\bar{p}_{i^*,j^*}^{m^*} - p_{i^*,j^*}^{m^*}\right| = \max_{1\leq m\leq 10, i<j} \left|\bar{p}_{ij}^m - p_{ij}^m\right|$.

Next, we discuss the convergence of Algorithm I.

Theorem 4: Suppose that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR, $\theta(0 < \theta < 1)$ is the iterative adjusted parameter, $\{P_{(t)}^m, m = 1, 2, \cdots, 10\}$ is the FPRs sequence in Algorithm I, $\{\bar{P}_{(t)}^m, m = 1, 2, \cdots, 10\}$ is the multiplicative consistent FPRs sequence in Algorithm I, $CI(P_{(t)}^m)$ is the consistency index of $P_{(t)}^m$, then we have

$$\frac{1}{10}\sum_{m=1}^{10} CI(P_{(t+1)}^m) \leq \frac{1}{10}\sum_{m=1}^{10} CI(P_{(t)}^m) \quad \text{for each } t. \quad (16)$$

Proof: The Proof of Theorem 4 is provided in Appendix.

IV. IT2TFDM METHOD WITH IT2TRF DEA

In the following, we first present an interval type-2 fuzzy DEA model to computer the efficiency score values and priority weights of alternatives, which is followed by the construction of IT2TFDM method.

A. IT2TRF DEA MODEL

Assume that $X = \{x_1, x_2, \cdots, x_n\}$ is a set of alternatives, $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR, ten FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are constructed by Eq. (7). Let each alternative $x_i(i \in N)$ be considered as a decision-making unit (DMU), thus i-th column of the FPRs $P^m = (p_{ij}^m)_{n\times n}$ can be regarded as the output of x_i [37]. With the help of the Charnes-Cooper-Rhodes DEA model [55], [56], an IT2TrF DEA model is proposed to calculate the relative efficiency of alternatives $x_i(i \in N)$ as follows:

$$\max\ \beta_i^m$$
$$s.t. \begin{cases} \sum_{p=1}^n u_p^m \cdot p_{pk}^m \geq \beta_i^m \cdot p_{ik}^m, & k \in N, \\ \sum_{p=1}^n u_p^m \leq 1, & \\ \beta_i^m\ free, u_p \geq 0, & p \in N. \end{cases} \quad (17)$$

where u_p^m represents the proportion for constructing composite units with alternative x_p.

Theorem 5: Suppose that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is an IT2TrFPR, ten FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are constructed by Eq. (7), their multiplicative consistent FPRs $\bar{P}^m = (\bar{p}_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are derived by Eq. (11). Let $\beta_i^*(i \in N)$ be the optimal solution of model (17) and $\sigma : N \to N$ be a permutation, such $\beta_{\sigma(1)}^{m*} \geq \beta_{\sigma(2)}^{m*} \geq \cdots \geq \beta_{\sigma(n)}^{m*}$ for each multiplicative consistent FPR $P^m = (p_{ij}^m)_{n\times n}$, then we have

$$(1) \quad w_{\sigma(1),(m)}^L = 1 \Big/ \sum_{i=1}^n \frac{\beta_{\sigma(1)}^{m*}}{2\beta_{\sigma(i)}^{m*} - \beta_{\sigma(1)}^{m*}}, w_{\sigma(i),(m)}^L = \frac{w_{\sigma(i),(m)}^L \cdot \beta_{\sigma(1)}^{m*}}{2\beta_{\sigma(i)}^{m*} - \beta_{\sigma(1)}^{m*}},$$
$$i = 2, 3, \cdots, n, m = 1, 2, \cdots, 5,$$

$$(2) \quad w_{\sigma(1),(m-5)}^U = 1 \Big/ \sum_{i=1}^n \frac{\beta_{\sigma(1)}^{m*}}{2\beta_{\sigma(i)}^{m*} - \beta_{\sigma(1)}^{m*}}, w_{\sigma(i),(m-5)}^U = \frac{w_{\sigma(i),(m-5)}^U \cdot \beta_{\sigma(1)}^{m*}}{2\beta_{\sigma(i)}^{m*} - \beta_{\sigma(1)}^{m*}}$$
$$i = 2, 3, \cdots, n, m = 6, 7, \cdots, 10.$$

Proof: The Proof of Theorem 5 is provided in Appendix.

In the process of determining alternatives' weight vector, the proposed IT2TrF DEA model fully utilizes the obtained acceptable multiplicative consistent FPRs to derive alternatives' weight vector, which makes GDM process more transparent and avoids decision-making randomness. Therefore, the proposed model is more reasonable and systematic than some of the existing subjective models.

B. IT2TFDM METHOD

This subsection addresses the IT2TrF evaluation decision-making problem, we investigate a novel IT2TFDM method to rank a set of alternatives/objectives $X = \{x_1, x_2, \cdots, x_n\}$ and select the desirable alternative/objective.

V. NUMERICAL EXAMPLE AND COMPARATIVE ANALYSIS

A. CASE DESCRIPTION AND SOLUTION BY IT2TFDM METHOD

Hefei, as the capital city of Anhui province in China, is increasingly polluted by fog-haze weather accompany with the development of economy in recent years. In general, the important influence factors of fog-haze weather are PM10 concentration x_1, PM2.5 concentration x_2, geographical conditions x_3 and meteorological condition x_4. To assess the most critical influence factor of fog-haze weather, the city's environmental protection department invites a group of related experts to evaluate these influence factors. Because there exists too much uncertainty, the evaluation information given by experts or DMs is presented with IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times 4}$, where $\tilde{A}_{ij} = [(a_{ij(1)}^L, a_{ij(2)}^L, a_{ij(3)}^L, a_{ij(4)}^L; h_{ij}^L), (a_{ij(1)}^U, a_{ij(2)}^U, a_{ij(3)}^U, a_{ij(4)}^U; h_{ij}^U)]$. The threshold value of acceptable consistency of $\tilde{A} = (\tilde{A}_{ij})_{4\times 4}$ is set at $\overline{CI} = 0.01$. Here, we use Algorithm II to work out the most important influence factor.

$$\tilde{A} = \begin{pmatrix} [(0.5, 0.5, 0.5, 0.5; 0.5), (0.5, 0.5, 0.5, 0.5; 0.5)] \\ [(0.5, 0.6, 0.7, 0.8; 0.5), (0.4, 0.6, 0.8, 0.9; 0.7)] \\ [(0.2, 0.2, 0.3, 0.4; 0.3), (0.1, 0.2, 0.4, 0.6; 0.4)] \\ [(0.4, 0.7, 0.8, 0.9; 0.3), (0.4, 0.6, 0.8, 0.9; 0.5)] \end{pmatrix}$$

Algorithm II

Input: An IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$.

Output: The best alternative/objective $\tilde{x}$.

Stage A: Order consistency checking process

Use Definition 8 to check the order consistency for original IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$. If IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is order consistent, then go to Stage E; otherwise, go to Stage B.

Stage B: Multiplicative consistency checking process

Apply Theorem 2 to check the multiplicative consistency for IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$. If IT2TrFPR $\tilde{A}$ is multiplicative consistent, then go to Stage D; otherwise, go to Stage C.

Stage C: Multiplicative consistency improving process

By utilizing Algorithm I, one can improve the multiplicative consistency level of ten FPRs constructed by Eq. (7), and ten acceptable multiplicative consistent FPRs $\overset{\frown}{P}^m = (\overset{\frown}{p}{}^m_{ij})_{n\times n} (m = 1, 2, \cdots, 10)$ are obtained.

Stage D: Priority weight vector determining process

According to model (17) and Theorem 5, one can obtain the IT2TrF weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$.

Stage E: Selecting the best alternative/objective process

Applying Definitions 5 and 6, we derive the ranking of alternatives $x_i (i \in N)$ in accordance with $w_i (i \in N)$, and the best alternative $\tilde{x}$ is selected.

The flow chart of IT2TFDM method can be shown in Fig. 1.

$$
\begin{array}{l}
[(0.2, 0.3, 0.4, 0.5; 0.3), (0.1, 0.2, 0.4, 0.6; 0.5)]\\
[(0.5, 0.5, 0.5, 0.5; 0.5), (0.5, 0.5, 0.5, 0.5; 0.5)]\\
[(0.3, 0.5, 0.5, 0.6; 0.4), (0.2, 0.4, 0.5, 0.6; 0.5)]\\
[(0.2, 0.4, 0.4, 0.5; 0.2), (0.2, 0.3, 0.5, 0.6; 0.8)]\\
[(0.6, 0.7, 0.8, 0.8; 0.6), (0.4, 0.6, 0.8, 0.9; 0.7)]\\
[(0.4, 0.5, 0.5, 0.7; 0.5), (0.4, 0.5, 0.5, 0.8; 0.6)]\\
[(0.5, 0.5, 0.5, 0.5; 0.5), (0.5, 0.5, 0.5, 0.5; 0.5)]\\
[(0.1, 0.2, 0.3, 0.4; 0.4), (0.1, 0.1, 0.3, 0.5; 0.6)]\\
[(0.1, 0.2, 0.3, 0.6; 0.5), (0.1, 0.2, 0.4, 0.6; 0.7)]\\
[(0.5, 0.6, 0.6, 0.8; 0.2), (0.4, 0.5, 0.7, 0.8; 0.8)]\\
[(0.6, 0.7, 0.8, 0.9; 0.4), (0.5, 0.7, 0.9, 0.9; 0.6)]\\
[(0.5, 0.5, 0.5, 0.5; 0.5), (0.5, 0.5, 0.5, 0.5; 0.5)]
\end{array}\Bigg).
$$

Stage A: By utillizing Definitions 5 and 6, we have $a_{11} > a_{31}$ and $a_{14} < a_{34}$. According to Definition 8, $\tilde{A}$ is a non-order consistent IT2TrFPR, then go to Stage B.

Stage B: Using Theorem 2 to check the multiplicative consistency for IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times 4}$. Owing to $\frac{1}{1+\prod_{r=0}^{3-1-1}\left(\frac{1}{p^5_{1+r,1+r+1}}-1\right)} = \frac{1}{1+\left(\frac{1}{p^5_{12}}-1\right)\left(\frac{1}{p^5_{23}}-1\right)} = 0.3 \neq 0.5 = p^5_{13}$, it indicates IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times 4}$ is non-multiplicative consistent, then go to Stage C.

Stage C: Let adjusted parameter $\theta = 0.2$. By using Algorithm I, we get ten acceptable multiplicative consistent FPRs $\overset{\frown}{P}^m = (\overset{\frown}{p}{}^m_{ij})_{n\times n} (m = 1, 2, \cdots, 10)$ as follows:

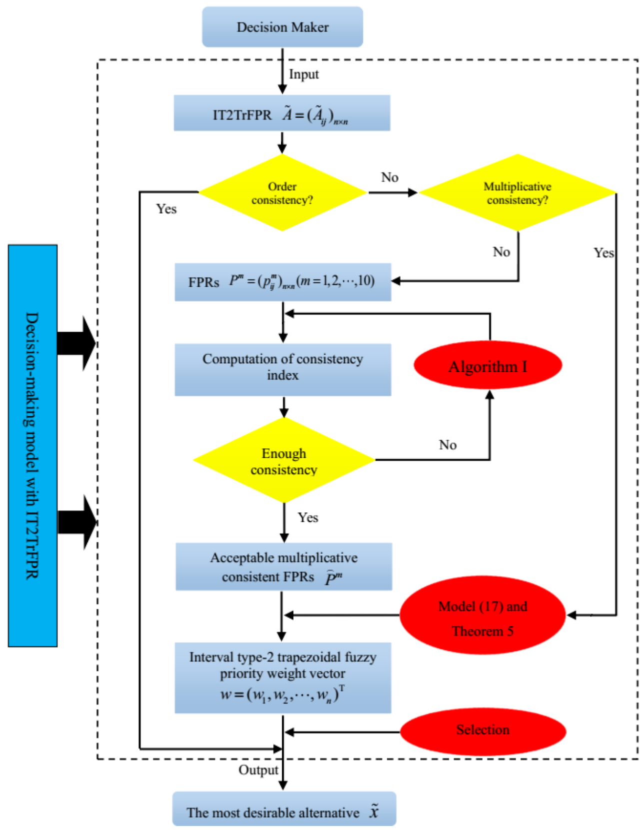

FIGURE 1. The IT2TFDM method.

$$\overset{\frown}{P}^1 = \begin{pmatrix} 0.5 & 0.3535 & 0.4106 & 0.3353 \\ 0.6465 & 0.5 & 0.4470 & 0.4913 \\ 0.5894 & 0.5530 & 0.5 & 0.4943 \\ 0.6647 & 0.5087 & 0.5057 & 0.5 \end{pmatrix},$$

$$\overset{\frown}{P}^2 = \begin{pmatrix} 0.5 & 0.4036 & 0.4516 & 0.4419 \\ 0.5964 & 0.5 & 0.5185 & 0.5388 \\ 0.5484 & 0.4815 & 0.5 & 0.5104 \\ 0.5581 & 0.4612 & 0.4896 & 0.5 \end{pmatrix},$$

$$\overset{\frown}{P}^3 = \begin{pmatrix} 0.5 & 0.4682 & 0.5243 & 0.5086 \\ 0.5318 & 0.5 & 0.5261 & 0.5503 \\ 0.4757 & 0.4739 & 0.5 & 0.5243 \\ 0.4914 & 0.4497 & 0.4757 & 0.5 \end{pmatrix},$$

$$\overset{\frown}{P}^4 = \begin{pmatrix} 0.5 & 0.4952 & 0.5939 & 0.6406 \\ 0.5048 & 0.5 & 0.5875 & 0.6548 \\ 0.4061 & 0.4125 & 0.5 & 0.5820 \\ 0.3594 & 0.3452 & 0.4180 & 0.5 \end{pmatrix},$$

$$\overset{\frown}{P}^5 = \begin{pmatrix} 0.5 & 0.3912 & 0.5261 & 0.4452 \\ 0.6088 & 0.5 & 0.4949 & 0.4041 \\ 0.4739 & 0.5051 & 0.5 & 0.4243 \\ 0.5548 & 0.5959 & 0.5757 & 0.5 \end{pmatrix},$$

$$\overset{\frown}{P}^6 = \begin{pmatrix} 0.5 & 0.2843 & 0.3198 & 0.2917 \\ 0.7157 & 0.5 & 0.4322 & 0.4519 \\ 0.6802 & 0.5678 & 0.5 & 0.4785 \\ 0.7083 & 0.5481 & 0.5215 & 0.5 \end{pmatrix},$$

$$\widehat{P}^{7}=\begin{pmatrix}0.5 & 0.3731 & 0.4249 & 0.4164\\ 0.6269 & 0.5 & 0.5023 & 0.5142\\ 0.5751 & 0.4977 & 0.5 & 0.5019\\ 0.5836 & 0.4858 & 0.4981 & 0.5\end{pmatrix},$$

$$\widehat{P}^{8}=\begin{pmatrix}0.5 & 0.4835 & 0.5336 & 0.5917\\ 0.5165 & 0.5 & 0.5305 & 0.6175\\ 0.4664 & 0.4695 & 0.5 & 0.5785\\ 0.4083 & 0.3825 & 0.4215 & 0.5\end{pmatrix},$$

$$\widehat{P}^{9}=\begin{pmatrix}0.5 & 0.5311 & 0.6470 & 0.6801\\ 0.4689 & 0.5 & 0.6275 & 0.6715\\ 0.3530 & 0.3725 & 0.5 & 0.5978\\ 0.2199 & 0.3285 & 0.4022 & 0.5\end{pmatrix},$$

$$\widehat{P}^{10}=\begin{pmatrix}0.5 & 0.4755 & 0.5607 & 0.6367\\ 0.5245 & 0.5 & 0.5743 & 0.6414\\ 0.4393 & 0.4257 & 0.5 & 0.5785\\ 0.3633 & 0.3586 & 0.4215 & 0.5\end{pmatrix}.$$

Stage D: Utilizing the optimal solution of model (17) and Theorem 5, the IT2TrF priority weights are determined as follows:

$$\begin{aligned}w_1 &= [(0.2729, 0.2888, 0.3205, 0.3557; 0.4401),\\ &\quad (0.2618, 0.2733, 0.3575, 0.3949; 0.4665)],\\ w_2 &= [(0.3536, 0.3611, 0.4005, 0.4413; 0.5485),\\ &\quad (0.3229, 0.3400, 0.4288, 0.4666; 0.6769)],\\ w_3 &= [(0.3020, 0.3227, 0.3545, 0.3894; 0.4466),\\ &\quad (0.2511, 0.2860, 0.3689, 0.4017; 0.5244)],\\ w_4 &= [(0.2210, 0.2433, 0.2774, 0.3008; 0.3555),\\ &\quad (0.1664, 0.1965, 0.3119, 0.3346; 0.3967)].\end{aligned}$$

Stage E: According to Definition 5, we have $\Delta(w_1) = 0.2467$, $\Delta(w_2) = 0.3923$, $\Delta(w_3) = 0.2716$, $\Delta(w_4) = 0.1607$. Since $\Delta(w_2) > \Delta(w_3) > \Delta(w_1) > \Delta(w_4)$, then $x_2 \succ x_3 \succ x_1 \succ x_4$, and it is show that the most important influence factor is x_2.

B. COMPARISON WITH OTHER APPROACHES

With respect to the interval type-2 fuzzy MADM problems, Ma *et al.* [49] proposed an IT2TrFA operator and developed a MADM method. Now, we utilize the method in Ma *et al.* [49] to handle the aforementioned problem.

Step 1: With IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times4}$, we apply the IT2TrFA operator [49]:

$$\begin{aligned}\tilde{A}_i &= \frac{1}{n}\left(\tilde{A}_{i1}\oplus\tilde{A}_{i2}\oplus\tilde{A}_{i3}\oplus\tilde{A}_{i4}\right)\\ &= \Bigg[\Bigg(\frac{1}{4}\sum_{j=1}^{4} a^{L}_{ij(1)}, \frac{1}{4}\sum_{j=1}^{4} a^{L}_{ij(2)}, \frac{1}{4}\sum_{j=1}^{4} a^{L}_{ij(3)},\\ &\qquad \frac{1}{4}\sum_{j=1}^{4} a^{L}_{ij(4)}; \min_{j=1,2,3,4} h^{L}_{ij}\Bigg),\\ &\qquad \Bigg(\frac{1}{4}\sum_{j=1}^{4} a^{U}_{ij(1)}, \frac{1}{4}\sum_{j=1}^{4} a^{U}_{ij(2)}, \frac{1}{4}\sum_{j=1}^{4} a^{U}_{ij(3)},\\ &\qquad \frac{1}{4}\sum_{j=1}^{4} a^{U}_{ij(4)}; \min_{j=1,2,3,4} h^{U}_{ij}\Bigg)\Bigg] \qquad (18)\end{aligned}$$

to integrate $\tilde{A}_{ij}(j = 1, 2, 3, 4)$ into the overall IT2TrFN $\tilde{A}_i$ of the influence factor $x_i(i = 1, 2, 3, 4)$ as follows:

$$\begin{aligned}\tilde{A}_1 &= [(0.3500, 0.4250, 0.5000, 0.6000; 0.3000),\\ &\quad (0.2750, 0.3750, 0.5250, 0.6500; 0.5000)],\\ \tilde{A}_2 &= [(0.4500, 0.5500, 0.5750, 0.6750; 0.2000),\\ &\quad (0.3750, 0.5000, 0.6250, 0.7000; 0.5000)],\\ \tilde{A}_3 &= [(0.4250, 0.4750, 0.5250, 0.6250; 0.3000),\\ &\quad (0.3750, 0.4750, 0.5750, 0.7000; 0.4000)],\\ \tilde{A}_4 &= [(0.3000, 0.4500, 0.5000, 0.5750; 0.2000),\\ &\quad (0.3000, 0.3750, 0.5250, 0.6250; 0.5000)].\end{aligned}$$

Step 2: By using Definition 7, one can obtain that

$$\Delta(\tilde{A}_1) = 0.3989, \quad \Delta(\tilde{A}_2) = 0.4937,$$
$$\Delta(\tilde{A}_3) = 0.4632, \quad \Delta(\tilde{A}_4) = 0.3707.$$

Step 3: Because $\Delta(\tilde{A}_2) > \Delta(\tilde{A}_3) > \Delta(\tilde{A}_1) > \Delta(\tilde{A}_4)$, then we have $x_2 \succ x_3 \succ x_1 \succ x_4$, and the fog-haze weather's most important influence factor is x_2.

Next, we use the ranking values of IT2FSs-based method in Wang *et al.* [50] to cope with the above influence factors evaluation problem. The mainly steps are included:

Step 1': Based on the IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times4}$ provided by city environmental department committee, we calculate the ranking value R_{ij} of $\tilde{A}_{ij}$ as follows:

$$\begin{aligned}R_{ij} &= M_1(\tilde{A}^L_{ij}) + M_1(\tilde{A}^U_{ij}) + M_2(\tilde{A}^L_{ij}) + M_2(\tilde{A}^U_{ij})\\ &\quad + M_3(\tilde{A}^L_{ij}) + M_3(\tilde{A}^U_{ij})\\ &\quad - \frac{1}{4}\begin{pmatrix}\left(S_1(\tilde{A}^L_{ij}) + S_1(\tilde{A}^U_{ij}) + S_2(\tilde{A}^L_{ij}) + S_2(\tilde{A}^U_{ij})\right.\\ \left.+S_3(\tilde{A}^L_{ij}) + S_3(\tilde{A}^U_{ij}) + S_4(\tilde{A}^L_{ij}) + S_4(\tilde{A}^U_{ij})\right)\end{pmatrix}\\ &\quad + h^L_{ij} + h^U_{ij}, \qquad (19)\end{aligned}$$

where

$$M_s(\tilde{A}^T_{ij}) = \frac{a^T_{ij(s)} + a^T_{ij(s+1)}}{2},$$

$$S_s(\tilde{A}^T_{ij}) = \sqrt{\frac{1}{2}\sum_{l=s}^{s+1}\left(a^T_{ij(l)} - M_s(\tilde{A}^T_{ij})\right)^2}, \quad s = 1, 2, 3,$$

$$S_4(\tilde{A}^T_{ij}) = \sqrt{\frac{1}{4}\sum_{l=1}^{4}\left(a^T_{ij(l)} - \frac{1}{4}\sum_{l=1}^{4} a^T_{ij(l)}\right)^2}, \quad T = L, U.$$

Then, the ranking value matrix $R = (R_{ij})_{4\times4}$ can be obtained:

$$R=\begin{pmatrix}4.000 & 2.6240 & 5.3938 & 2.7802\\ 4.8240 & 4.000 & 3.4983 & 4.4957\\ 2.3938 & 4.0978 & 4.000 & 5.3931\\ 4.7802 & 3.5957 & 2.4931 & 4.000\end{pmatrix}.$$

Step 2': Applying the IT2TrFA operator [50] to fuse $R_{ij}(j = 1, 2, 3, 4)$ into the overall ranking values $R_i(i = 1, 2, 3, 4)$ of influence factors $x_i(i = 1, 2, 3, 4)$ as follows:

$$R_1 = 3.6995, \quad R_2 = 4.2045, \quad R_3 = 3.9712, \quad R_4 = 3.7173.$$

Step 3': It can be seen that $R_2 > R_3 > R_4 > R_1$, then we have $x_2 \succ x_3 \succ x_4 \succ x_1$, and the most important influence factor is x_2.

In Ref. [51], based on the FPRs of IT2FSs, Chen and Lee presented a new method to cope with fuzzy multiple criteria hierarchical GDM problems. Utilizing Chen and Lee [51]'s approach to address the fog-haze influence factors selection problem, the steps are listed as follows:

Step 1'': See Step 1.

Step 2'': According to Eqs. (7) and (8) in [51], two strength matrices $E^L = (E_{ij}^L)_{4\times 4}$ and $E^U = (E_{ij}^U)_{4\times 4}$ can be constructed as follows:

$$E^L = \begin{pmatrix} 0.5 & 0.7368 & 0.7200 & 0.3448 \\ 0.2632 & 0.5 & 0.3793 & 0.1351 \\ 0.2800 & 0.6207 & 0.5 & 0.1875 \\ 0.6552 & 0.8649 & 0.8125 & 0.5 \end{pmatrix},$$

$$E^U = \begin{pmatrix} 0.5 & 0.7442 & 0.6291 & 0.4571 \\ 0.2558 & 0.5 & 0.3939 & 0.2391 \\ 0.3709 & 0.6061 & 0.5 & 0.3191 \\ 0.5429 & 0.7609 & 0.6809 & 0.5 \end{pmatrix}.$$

Step 3'': Utilize Eqs. (9)-(11) and (13) in [51], one can generate the following lower FPR $P^L = \left(p(\tilde{A}_i^L \geq \tilde{A}_j^L)\right)_{4\times 4}$ and upper FPR $P^U = \left(p(\tilde{A}_i^U \geq \tilde{A}_j^U)\right)_{4\times 4}$:

$$P^L = \begin{pmatrix} 0.5 & 0.2632 & 0.2800 & 0.6552 \\ 0.7368 & 0.5 & 0.6207 & 0.8649 \\ 0.7200 & 0.3793 & 0.5 & 0.8125 \\ 0.3448 & 0.1351 & 0.1875 & 0.5 \end{pmatrix},$$

$$P^U = \begin{pmatrix} 0.5 & 0.2558 & 0.3709 & 0.5429 \\ 0.7442 & 0.5 & 0.6061 & 0.7609 \\ 0.6291 & 0.3939 & 0.5 & 0.6809 \\ 0.4571 & 0.2391 & 0.3191 & 0.5 \end{pmatrix}.$$

Step 4'': Apply Eqs. (12) and (14) in [51] to calculate the ranking values $Rank(\tilde{A}_i^L)$ and $Rank(\tilde{A}_i^U)$ of $\tilde{A}_i^L (i = 1, 2, 3, 4)$ and $\tilde{A}_i^U (i = 1, 2, 3, 4)$:

$$Rank(\tilde{A}_1^L) = 0.2249, \quad Rank(\tilde{A}_2^L) = 0.3102,$$
$$Rank(\tilde{A}_3^L) = 0.2843, \quad Rank(\tilde{A}_4^L) = 0.1806,$$
$$Rank(\tilde{A}_1^U) = 0.2225, \quad Rank(\tilde{A}_2^U) = 0.3009,$$
$$Rank(\tilde{A}_3^U) = 0.3670, \quad Rank(\tilde{A}_4^U) = 0.2096.$$

Step 5'': Based on Eq. (15) in [51], the ranking values $Rank(\tilde{A}_i)$ of $\tilde{A}_i (i = 1, 2, 3, 4)$ can be obtained: $Rank(\tilde{A}_1) = 0.2236$, $Rank(\tilde{A}_2) = 0.3056$, $Rank(\tilde{A}_3) = 0.2757$, $Rank(\tilde{A}_4) = 0.1951$.

Step 6'': Obviously, $Rank(\tilde{A}_2) > Rank(\tilde{A}_3) > Rank(\tilde{A}_1) > Rank(\tilde{A}_4)$. Therefore, the four influence factors are ranked as $x_2 \succ x_3 \succ x_1 \succ x_4$, and the most important influence factor is x_2.

Under the IT2TrF information environment, Qin and Liu [44] presented a new method with the combined ranking value. By using method in [44], the following steps are given to get the most important influence factor:

Step 1''': By using Eqs. (16)-(18) in [44], the ranking value matrices $R_k (k = 1, 2, 3)$ are obtained as follows:

$$R_{(1)} = \begin{pmatrix} 0.5 & 0.2531 & 0.9100 & 0.2969 \\ 0.7619 & 0.5 & 0.6181 & 0.6737 \\ 0.2250 & 0.3825 & 0.5 & 0.9000 \\ 0.7219 & 0.3488 & 0.2000 & 0.5 \end{pmatrix},$$

$$R_{(2)} = \begin{pmatrix} 0.5 & 0.1866 & 0.8504 & 0.2124 \\ 0.7067 & 0.5 & 0.5823 & 0.5765 \\ 0.1698 & 0.3379 & 0.5 & 0.8538 \\ 0.6489 & 0.2694 & 0.1489 & 0.5 \end{pmatrix},$$

$$R_{(3)} = \begin{pmatrix} 0.5 & 0.1366 & 0.7919 & 0.1542 \\ 0.6549 & 0.5 & 0.5505 & 0.4961 \\ 0.1306 & 0.2953 & 0.5 & 0.8087 \\ 0.5797 & 0.2067 & 0.1121 & 0.5 \end{pmatrix}.$$

Step 2''': One can obtain the IT2TrF entropy matrix E based on Eq. (26), as shown at the bottom of page 13, in [44]:

$$E = \begin{pmatrix} 1 & 0.1915 & 0.7894 & 0.1330 \\ 0.8543 & 1 & 0.5502 & 0.7799 \\ 0.2337 & 0.8746 & 1 & 0.6050 \\ 0.5538 & 0.3867 & 0.1006 & 1 \end{pmatrix}$$

Step 3''': According to Eqs. (28), (42) and (47) in [44], the combined ranking values $R(x_i)(i = 1, 2, 3, 4)$ of influence factors $x_i (i = 1, 2, 3, 4)$ can be derived:

$$R(x_1) = 1.3880, \quad R(x_2) = 1.4219,$$
$$R(x_3) = 1.4044, \quad R(x_4) = 1.3812.$$

Step 4''': It is obvious that $R(x_2) > R(x_3) > R(x_1) > R(x_4)$, then these influence factors are ranked as $x_2 \succ x_3 \succ x_4 \succ x_1$. Therefore, the most important influence factor of fog-haze weather is x_2.

According to the fuzzy multi-criteria GDM (FMCGDM) method which was investigated by Kundu *et al.* [57], first of all, we use Eqs. (5) and (6) in [57] to derive the upper relative preference matrix and lower relative preference matrix, respectively, which is followed by the calculation of relative preference matrix. Then, by using Eq. (7) in [57], one can obtain the collective relative preference for each influence factor x_i, and the final upper preference index and final lower preference index of x_i are determined by using Eqs. (8) and (9) in [57]. Finally, based on the obtained final preference index, we utilize Eq. (10) in [57] to derive preference weights as follows:

$$W_1 = 0.1880, \quad W_2 = 0.3497, \quad W_3 = 0.3504, \quad W_4 = 0.1119.$$

As $W_3 > W_2 > W_1 > W_4$, therefore, the ranking order of these influence factors is $x_3 \succ x_2 \succ x_1 \succ x_4$, and the most important influence factor is x_3.

According to above analysis, decision-making results that derived by different methods are listed in Table 1.

C. DISCUSSION

From the above numerical example and comparison with other methods, the proposed IT2TFDM method has the following characteristics:

TABLE 1. The decision-making results by different methods.

Approaches	The ranking results of influence factors	The most important factor
IT2TFDM method	$x_2 \succ x_3 \succ x_1 \succ x_4$	x_2
Ma et al. [49]'s method	$x_2 \succ x_3 \succ x_1 \succ x_4$	x_2
Wang et al. [50]'s method	$x_2 \succ x_3 \succ x_4 \succ x_1$	x_2
Chen and Lee [51]'s method	$x_2 \succ x_3 \succ x_1 \succ x_4$	x_2
Qin and Liu [44]'s method	$x_2 \succ x_3 \succ x_1 \succ x_4$	x_2
Kundu et al. [57]' method	$x_3 \succ x_2 \succ x_1 \succ x_4$	x_3

(1) It is known that the consistency adjustment process is important to avoid obtaining the inconsistent results. However, the method in Ma *et al.* [49] does not check the acceptable consistency of IT2TrFPRs, and it directly uses the IT2TrFA operator to fuse decision-making information into the collective IT2TrFNs. By contrast, with the proposed IT2TFDM method, we first check the consistency for original IT2TrFPRs and improve their consistency level, and then we derive the IT2TrF priority weights. Therefore, the proposed IT2TFDM method is much more reasonable and systematic than Ma *et al.* [49]'s method.

(2) From TABLE 1, it is observed that the ranking results of four influence factors that derived by IT2TFDM method and Wang *et al.* [50]' method are different. With Wang *et al.* [50]' method, one must apply single ranking value measure approach to transfer the original IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times4}$ into its corresponding ranking value matrix $R = (R_{ij})_{4\times4}$. However, with the proposed IT2TFDM method, we directly use the original evaluation information to derive the priority weights for influence factors, in which the evaluation preference information of DMs can be retained as much as possible. Therefore, the proposed IT2TFDM method is more efficient than the method in Wang *et al.* [50].

(3) According to the above decision-making process, it is observed that the methods in our paper, Chen and Lee [51] and Qin and Liu [44] produce the same ranking results of these influence factors for fog-haze. In the process of IT2TFDM method, we utilize the LCAS-driven Algorithm I to derive the acceptable consistent FPRs, in which the preference evaluation of DM can be retained as much as possible. However, with the approach in Chen and Lee [51], one must transfer the original IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times4}$ into its corresponding strength matrices and fuzzy preference matrix, and then calculate the ranking values of four influence factors. In addition, in the process of fog-haze influence factor selection with Qin and Liu [44]'s method, we also need transfer IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times4}$ into three ranking value matrices and its IT2TrF entropy matrix, which is followed by the construction of optimal model, and then the combined ranking values of four influence factors are obtained. Thus, the methods in Chen and Lee [51] and Qin and Liu [44] are indirect and the elements cannot describe original decision-making information. Therefore, the proposed IT2TFDM method is more reliable than the approaches in Chen and Lee [51] and Qin and Liu [44].

(4) Compared with FMCGDM method investigated by Kundu *et al.* [57], IT2TFDM method and FMCGDM method in [57] generate the different decision-making results. In fact, based on the original IT2TrFPR $\tilde{A} = (\tilde{A}_{ij})_{4\times4}$, we obtain the arithmetic average ranking value function is $\Delta(\tilde{A}_{23}) = 0.6181 > 0.5$, which indicates that the influence factor x_2 is more important than x_3, i.e., $x_2 \succ x_3$. Thus, the proposed IT2TFDM method can output more accurate result.

(5) Convenient and efficient decision-making process, i.e., checking the order consistency of original IT2TrFPRs. For instance, in Example 2, the ranking of the alternatives can be obtained directly. Therefore, our method offers a certain level of convenience in some cases.

VI. CONCLUSION

For the GDM problems with IT2TrFNs, this paper first presents several new notions, such as IT2TrFPRs, order consistency of IT2TrFPRs and multiplicative consistency of IT2TrFPRs. Then, we present a new method to check the multiplicative consistency of IT2TrFPRs, which is followed by the introduction of consistency index for IT2TrFPRs. Based on LCAS, a convergent consistency adjustment algorithm is developed to determine the acceptable multiplicative consistent IT2TrFPRs. Subsequently, we construct an IT2TrF DEA model, which is utilized to develop the IT2TFDM method. Finally, we gave a case study on the fog-haze influence factors evaluation problem, and the result of the comparative analysis showed that advantages of the proposed IT2TFDM method.

The main advantages of the proposed IT2TFDM method are summarized as follows:

a. Due to the IT2TrFNs are useful and valuable tool to describe the qualitative evaluation information provided by DMs, thus, the practical implication of the proposed IT2TFDM method is that it can be utilized to address the fuzziness and uncertainty characteristics in complex GDM problems, for example, the fog-haze influence factors evaluation problem.

b. In the process of consistency improvement, it is inevitable to modify the original evaluation information provided by DMs with the aim of increasing the consistency degree of IT2TrFPR. The proposed IT2TFDM method utilized LCAS to reach the acceptable consistency by retaining the DMs' preferences as much as possible.

c. The IT2TrF weight vector of alternatives is determined with the proposed IT2TrF DEA model, and then we can derive the ranking order of alternatives.

d. The application of the proposed IT2TFDM method has significant managerial implications. By using the proposed IT2TFDM method to deal with the fog-haze influence factors evaluation problem, we can effectively evaluate the importance of haze influencing factors in a city. On the one hand, the evaluation results provide scientific basis for meteorological department to select appropriate alternatives, which can be applied to prevent the haze weather. On the other hand, the evaluation results provide management support for the investment optimization of enterprises, including adjustment of energy industrial structure, upgrading of energy-saving industrial, etc.

e. The proposed IT2TFDM method can be employed in other fields, such as supplier selection, risk evaluation, hotel location selection, and so on.

However, there are some limitations of the proposed IT2TFDM method. On the one hand, the proposed IT2TFDM method does not consider the cooperation consensus among DMs, and assumes that the provided IT2TrFPR is complete. On the other hand, the alternatives can only be evaluated as efficient or inefficient with the constructed IT2TrF DEA model, and we cannot get the complete ranking order of alternatives in some cases.

Therefore, in the future, based on the trust relationship among DMs, we will investigate the consensus-reaching models for IT2TrFPRs. Besides, future research is to develop the IT2TrF cross-efficiency model for evaluating the alternatives.

APPENDIX

proof of Theorem 2:

(i) ⇒ (ii) According to (ii), for $\forall i < j, m = 1, 2, \cdots, 10$, we have

$$p^m_{i,i+k} = \frac{\prod_{r=0}^{k-1} p^m_{i+r,i+r+1}}{\prod_{r=0}^{k-1} p^m_{i+r,i+r+1} + \prod_{r=0}^{k-1}\left(1 - p^m_{i+r,i+r+1}\right)}. \quad (20)$$

Therefore, we only need to prove that if $\tilde{A}$ is multiplicative consistent, then Eq. (20) is holds.

As $i, j \in N$, $i < j$, then let $k = j - i$. Thus, Eq. (11) can be rewritten as

$$p^m_{i,i+k} = \frac{\prod_{r=0}^{k-1} p^m_{i+r,i+r+1}}{\prod_{r=0}^{k-1} p^m_{i+r,i+r+1} + \prod_{r=0}^{k-1}\left(1 - p^m_{i+r,i+r+1}\right)}, \quad m = 1, 2, \cdots, 10. \quad (21)$$

If $k = 1$, then we have $p^m_{i,i+1} = \frac{p^m_{i+0,i+0+1}}{p^m_{i+0,i+0+1} + \left(1 - p^m_{i+0,i+0+1}\right)}$, $m = 1, 2, \cdots, 10$, which obviously holds.

Assume that Eq. (21) holds for $k = l$, that is

$$p^m_{i,i+l} = \frac{\prod_{r=0}^{l-1} p^m_{i+r,i+r+1}}{\prod_{r=0}^{l-1} p^m_{i+r,i+r+1} + \prod_{r=0}^{l-1}\left(1 - p^m_{i+r,i+r+1}\right)}, \quad m = 1, 2, \cdots, 10. \quad (22)$$

Assume that $k = l+1$. Owing to IT2TrFPR $\tilde{A}$ is multiplicative consistent, from Definition 9 and Theorem 1, we have

$$p^m_{ij} \cdot p^m_{jk} \cdot p^m_{ki} = p^m_{ik} \cdot p^m_{kj} \cdot p^m_{ji}, \quad \forall 1 \le i < k < j \le n, m = 1, 2, \cdots, 10.$$

Because $i < i + l < i + l + 1$, one can obtain

$$p^m_{i,i+l+1} \cdot \left(1 - p^m_{i+l,i+l+1}\right) \cdot \left(1 - p^m_{i,i+l}\right) = p^m_{i,i+l} \cdot p^m_{i+l,i+l+1} \cdot \left(1 - p^m_{i,i+l+1}\right), \quad m = 1, 2, \cdots, 10,$$

it is followed which indicates Eq. (23), as shown at the bottom of the next page, holds for $k = l + 1$. Thus, if IT2TrFPR $\tilde{A}$ is multiplicative consistent, then Eq. (20) is holds.

(ii) ⇒ (i) On the converse, Proposition (ii) can be written as Eq. (11). Let $i < k < j$, and then for $\forall m = 1, 2, \cdots, 10$, we have

$$p^m_{ik(s)} = \frac{\prod_{r=0}^{k-i-1} p^m_{i+r,i+r+1}}{\prod_{r=0}^{k-i-1} p^m_{i+r,i+r+1} + \prod_{r=0}^{k-i-1}\left(1 - p^m_{i+r,i+r+1}\right)}, \quad i < k, \quad (24)$$

$$p^m_{kj(s)} = \frac{\prod_{r=0}^{j-k-1} p^m_{i+r,i+r+1}}{\prod_{r=0}^{j-k-1} p^m_{k+r,k+r+1} + \prod_{r=0}^{j-k-1}\left(1 - p^m_{k+r,k+r+1}\right)}, \quad k < j. \quad (25)$$

Suppose that

$$\Phi_{ij} = \prod_{r=0}^{j-i-1} p^m_{i+r,i+r+1} + \prod_{r=0}^{j-i-1}\left(1 - p^m_{i+r,i+r+1}\right),$$
$$\Phi_{kj} = \prod_{r=0}^{j-k-1} p^m_{k+r,k+r+1} + \prod_{r=0}^{j-k-1}\left(1 - p^m_{k+r,k+r+1}\right),$$
$$\Phi_{ik} = \prod_{r=0}^{k-i-1} p^m_{i+r,i+r+1} + \prod_{r=0}^{k-i-1}\left(1 - p^m_{i+r,i+r+1}\right),$$

then for $i < k < j, m = 1, 2, \cdots, 10$, we have (26) and

$$\begin{aligned} &p^m_{ik} \cdot p^m_{kj} \cdot p^m_{ji} \\ &= p^m_{ik} \cdot p^m_{kj} \cdot \left(1 - p^m_{ij}\right) \\ &= \frac{\prod_{r=0}^{k-i-1} p^m_{i+r,i+r+1}}{\Phi_{ik}} \times \frac{\prod_{r=0}^{j-k-1} p^m_{k+r,k+r+1}}{\Phi_{kj}} \\ &\quad \times \left(1 - \frac{\prod_{r=0}^{j-i-1} p^m_{i+r,i+r+1}}{\Phi_{ij}}\right) \\ &= \frac{\prod_{r=0}^{j-i-1} p^m_{i+r,i+r+1} \cdot \prod_{r=0}^{j-i-1}\left(1 - p^m_{i+r,i+r+1}\right)}{\Phi_{ij}\Phi_{kj}\Phi_{ik}}. \end{aligned} \quad (27)$$

Hence,

$$p^m_{ij} \cdot p^m_{jk} \cdot p^m_{ki} = p^m_{ik} \cdot p^m_{kj} \cdot p^m_{ji}, \quad \forall 1 \le i < k < j \le n, m = 1, 2, \cdots, 10.$$

According to Definition 9 and Theorem 1, one can obtain that $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is multiplicative consistent. This completes the proof of Theorem 2. ■

Proof of Theorem 3:

(1) First, we prove that $\bar{P}^m = (\bar{p}^m_{ij})_{n\times n} (m = 1, 2, \cdots, 10)$ are multiplicative consistent FPRs.

It is obvious that for all $\forall i, j \in N, m = 1, 2, \cdots, 10$, we have $\bar{p}_{ii}^m = \left(\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}\right) \Big/ \left(2\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}\right) = 0.5$, and

$$0 \le \bar{p}_{ij}^m = \frac{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{il}^m + p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}} = \frac{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \left(\frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m} + \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}\right)} \le \frac{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}} = 1, \tag{28}$$

$$\bar{p}_{ij}^m + \bar{p}_{ji}^m = \frac{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{il}^m + p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}} + \frac{\sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{il}^m + p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}} = \frac{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m} + \sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m} + \sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}} = 1. \tag{29}$$

According to Definition 1, $\bar{P}^m = (\bar{p}_{ij}^m)_{n\times n} (m = 1, 2, \cdots, 10)$ are FPRs.

On the other hand, for $\forall i, j \in N, m = 1, 2, \cdots, 10$, we have

$$\begin{aligned} \frac{\bar{p}_{ik}^m}{\bar{p}_{ki}^m} \cdot \frac{\bar{p}_{kj}^m}{\bar{p}_{jk}^m} &= \frac{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m} \Big/ \sum_{l=1}^n \frac{p_{il}^m + p_{kl}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{kl}^m}{\sum_{h=1}^n p_{hl}^m} \Big/ \sum_{l=1}^n \frac{p_{kl}^m + p_{il}^m}{\sum_{h=1}^n p_{hl}^m}} \times \frac{\sum_{l=1}^n \frac{p_{kl}^m}{\sum_{h=1}^n p_{hl}^m} \Big/ \sum_{l=1}^n \frac{p_{kl}^m + p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m} \Big/ \sum_{l=1}^n \frac{p_{kl}^m + p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}} \\ &= \frac{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{kl}^m}{\sum_{h=1}^n p_{hl}^m}} \cdot \frac{\sum_{l=1}^n \frac{p_{kl}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}} = \frac{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}} \\ &= \frac{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m} \Big/ \sum_{l=1}^n \frac{p_{il}^m + p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m} \Big/ \sum_{l=1}^n \frac{p_{jl}^m + p_{il}^m}{\sum_{h=1}^n p_{hl}^m}} = \frac{\bar{p}_{ij}^m}{\bar{p}_{ji}^m}, \end{aligned} \tag{30}$$

i.e.,

$$\bar{p}_{ij}^m \cdot \bar{p}_{jk}^m \cdot \bar{p}_{ki}^m = \bar{p}_{ik}^m \cdot \bar{p}_{kj}^m \cdot \bar{p}_{ji}^m, \quad i, k, j \in N, m = 1, 2, \cdots, 10.$$

$$\begin{aligned} p_{i,i+l+1}^m &= \frac{p_{i,i+l}^m \cdot p_{i+l,i+l+1}^m}{p_{i,i+l}^m \cdot p_{i+l,i+l+1}^m + \left(1 - p_{i,i+l}^m\right)\left(1 - p_{i+l,i+l+1}^m\right)} \\ &= \frac{p_{i+l,i+l+1}^m}{p_{i+l,i+l+1}^m + \left(\frac{1}{p_{i,i+l}^m} - 1\right)\left(1 - p_{i+l,i+l+1}^m\right)} \\ &= \frac{p_{i+l,i+l+1}^m}{p_{i+l,i+l+1}^m + \left(\frac{\prod_{r=0}^{l-1} p_{i+r,i+r+1}^m + \prod_{r=0}^{l-1}\left(1 - p_{i+r,i+r+1}^m\right)}{\prod_{r=0}^{l-1} p_{i+r,i+r+1}^m} - 1\right)\left(1 - p_{i+l,i+l+1}^m\right)} \\ &= \frac{p_{i+l,i+l+1}^m}{p_{i+l,i+l+1}^m + \frac{\prod_{r=0}^{l-1}\left(1 - p_{i+r,i+r+1}^m\right)}{\prod_{r=0}^{l-1} p_{i+r,i+r+1}^m} \cdot \left(1 - p_{i+l,i+l+1}^m\right)} \\ &= \frac{\prod_{r=0}^{l+1-1} p_{i+r,i+r+1}^m}{\prod_{r=0}^{l+1-1} p_{i+r,i+r+1}^m + \prod_{r=0}^{l+1-1}\left(1 - p_{i+r,i+r+1}^m\right)}, \end{aligned} \tag{23}$$

$$\begin{aligned} p_{ij}^m \cdot p_{jk}^m \cdot p_{ki}^m &= p_{ij}^m \cdot \left(1 - p_{kj}^m\right) \cdot \left(1 - p_{ik}^m\right) \\ &= \frac{\prod_{r=0}^{j-i-1} p_{i+r,i+r+1}^m}{\Phi_{ij}} \times \left(1 - \frac{\prod_{r=0}^{j-k-1} p_{k+r,k+r+1}^m}{\Phi_{kj}}\right) \times \left(1 - \frac{\prod_{r=0}^{k-i-1} p_{i+r,i+r+1}^m}{\Phi_{ik}}\right) \\ &= \frac{\prod_{r=0}^{j-i-1} p_{i+r,i+r+1}^m \times \left(\prod_{r=0}^{j-k-1}\left(1 - p_{k+r,k+r+1}^m\right)\right) \times \left(\prod_{r=0}^{k-i-1}\left(1 - p_{i+r,i+r+1}^m\right)\right)}{\Phi_{ij}\Phi_{kj}\Phi_{ik}} \\ &= \frac{\prod_{r=0}^{j-i-1} p_{i+r,i+r+1}^m \cdot \prod_{r=0}^{j-i-1}\left(1 - p_{i+r,i+r+1}^m\right)}{\Phi_{ij}\Phi_{kj}\Phi_{ik}}, \end{aligned} \tag{26}$$

From Definition 2, it is certified that FPRs $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are multiplicative consistent.

(2) Now, we prove that for $i < j$, $\bar{p}_{ij}^m$ is an increasing function with respect to p_{ij}^m. As

$$\bar{p}_{ij}^m = \left(\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}\right) \Big/ \left(\sum_{l=1}^n \frac{p_{il}^m + p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}\right)$$
$$= \left(\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}\right) \Big/ \left(\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m} + \sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}\right)$$
$$= \left(1 + \left(\sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}\right) \Big/ \left(\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}\right)\right)^{-1}, \quad i < j,$$

then let $\bar{b}_{ij}^m = \frac{\sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}}$, $i < j$. In what follows, we prove that $\bar{b}_{ij}^m(i < j)$ is a decreasing function with respect to p_{ij}^m. Since

$$b_{ij}^m = \frac{\sum_{l=1}^n \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}}{\sum_{l=1}^n \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}} = \frac{\frac{p_{ji}^m}{\sum_{h=1}^n p_{hi}^m} + \sum_{l\neq i} \frac{p_{jl}^m}{\sum_{h=1}^n p_{hl}^m}}{\frac{p_{ij}^m}{\sum_{h=1}^n p_{hj}^m} + \sum_{l\neq j} \frac{p_{il}^m}{\sum_{h=1}^n p_{hl}^m}}$$
$$= \frac{\frac{1-p_{ij}^m}{1-p_{ij}^m+\sum_{h\neq j}^n p_{hi}^m} + \frac{p_{jj}^m}{p_{ij}^m+\sum_{h\neq i} p_{hl}^m} + \sum_{l\neq i,j} \frac{p_{jl}^m}{\sum_{h\neq i} p_{hl}^m}}{\frac{p_{ij}^m}{p_{ij}^m+\sum_{h\neq i} p_{hj}^m} + \frac{p_{ii}^m}{1-p_{ij}^m+\sum_{h\neq j} p_{hi}^m} + \sum_{l\neq i,j} \frac{p_{il}^m}{\sum_{h\neq j} p_{hl}^m}},$$

let

$$g(p_{ij}^m) = \frac{1-p_{ij}^m}{1-p_{ij}^m+\sum_{h\neq j}^n p_{hi}^m} + \frac{p_{jj}^m}{p_{ij}^m+\sum_{h\neq i} p_{hl}^m} + \sum_{l\neq i,j} \frac{p_{jl}^m}{\sum_{h\neq i} p_{hl}^m},$$
$$h(p_{ij}^m) = \frac{p_{ij}^m}{p_{ij}^m+\sum_{h\neq i} p_{hj}^m} + \frac{p_{ii}^m}{1-p_{ij}^m+\sum_{h\neq j} p_{hi}^m} + \sum_{l\neq i,j} \frac{p_{il}^m}{\sum_{h\neq j}^n p_{hl}^m},$$

thus $b_{ij}^m = \frac{g(p_{ij}^m)}{h(p_{ij}^m)}$, and

$$\frac{\partial g(p_{ij}^m)}{\partial p_{ij}^m} = -\left(\frac{\sum_{h\neq j}^n p_{hi}^m}{\left(1-p_{ij}^m+\sum_{h\neq j}^n p_{hi}^m\right)^2} + \frac{p_{jj}^m}{\left(p_{ij}^m+\sum_{h\neq i} p_{hj}^m\right)^2}\right), \tag{31}$$

$$\frac{\partial h(p_{ij}^m)}{\partial p_{ij}^m} = \frac{p_{ii}^m}{\left(1-p_{ij}^m+\sum_{h\neq j}^n p_{hi}^m\right)^2} + \frac{\sum_{h\neq i} p_{hj}^m}{\left(p_{ij}^m+\sum_{h\neq i} p_{hj}^m\right)^2}. \tag{32}$$

It is obvious that $g(p_{ij}^m) \geq 0$, $h(p_{ij}^m) \geq 0$, $\frac{\partial g(p_{ij}^m)}{\partial p_{ij}^m} \leq 0$ and $\frac{\partial h(p_{ij}^m)}{\partial p_{ij}^m} \geq 0$, then we have

$$\frac{\partial b_{ij}^m}{\partial p_{ij}^m} = \frac{1}{\left(h(p_{ij}^m)\right)^2} \cdot \left(\frac{\partial g(p_{ij}^m)}{\partial p_{ij}^m} \cdot h(p_{ij}^m) - \frac{\partial h(p_{ij}^m)}{\partial p_{ij}^m} \cdot g(p_{ij}^m)\right) \leq 0. \tag{33}$$

which implies $\bar{b}_{ij}^m(i < j)$ is a decreasing function with respect to p_{ij}^m. Therefore, $\bar{p}_{ij}^m(i < j)$ is an increasing function with respect to p_{ij}^m. This completes the proof of Theorem 3. ■

Proof of Theorem 4:

From Eq. (15), for each t, we have $p_{i^*,j^*(t+1)}^{m^*} = (1-\theta)\cdot p_{i^*,j^*(t)}^{m^*} + \theta \cdot \bar{p}_{i^*,j^*(t)}^{m^*}$ and $p_{ij(t+1)}^m = p_{ij(t)}^m$ $(i, j, m) \neq (i^*, j^*, m^*)$, $i, j \in N$, $m = 1, 2, \cdots, 10$. Thus, by using Eq. (14), for each t, we have (34), as shown at the bottom of the next page.

Therefore, the proof of Theorem 4 is completed. ■

Proof of Theorem 5:

As $\tilde{A} = (\tilde{A}_{ij})_{n\times n}$ is a multiplicative consistent IT2TrFPR, then $P^m = (p_{ij}^m)_{n\times n}(m = 1, 2, \cdots, 10)$ are multiplicative consistent FPRs.

From Definition 12, if $m = 1, 2, \cdots, 5$, then there exists an IT2TrF weight vector $w = (w_1, w_2, \cdots, w_n)^T$, such that $p_{ij}^m = \frac{w_{i(m)}^L}{w_{i(m)}^L + w_{j(m)}^L}$. Thus, model (17) can be rewritten as:

$$\max \beta_i^m$$
$$s.t. \begin{cases} \sum_{p=1}^n u_p^m \cdot \frac{w_{p(m)}^L}{w_{p(m)}^L + w_{k(m)}^L} \geq \beta_i^m \cdot \frac{w_{i(m)}^L}{w_{i(m)}^L + w_{k(m)}^L}, \\ \quad k \in N, \\ \sum_{p=1}^n u_p^m \leq 1, \quad \beta_i^m \ free, u_p \geq 0, p \in N. \end{cases} \tag{35}$$

Without loss of generality, let $0 \leq w_{1(m)}^L \leq w_{2(m)}^L \leq \cdots \leq w_{n(m)}^L \leq 1$, then $\frac{w_{p(m)}^L}{w_{p(m)}^L + w_{k(m)}^L} \geq 0$ for all $p = 1, 2, \cdots, n$. In this situation, when $\sum_{p=1}^n u_p^m \leq 1$ convert to $\sum_{p=1}^n u_p^m = 1$, then β_i^m can achieve the maximum.

In addition, because $0 \leq \frac{w_{1(m)}^L}{w_{1(m)}^L + w_{k(m)}^L} \leq \frac{w_{2(m)}^L}{w_{2(m)}^L + w_{k(m)}^L} \leq \cdots \leq \frac{w_{n(m)}^L}{w_{n(m)}^L + w_{k(m)}^L}$ for all $k \in N$, then one can obtain that the optimal solution is $u_p^{m*} = 0$, $p = 1, 2, \cdots, n-1$, $u_n^{m*} = 1$, it follows that $\frac{w_{n(m)}^L}{w_{n(m)}^L + w_{k(m)}^L} \geq \beta_i^m \cdot \frac{w_{i(m)}^L}{w_{i(m)}^L + w_{k(m)}^L}$, $k \in N$ i.e., $\beta_i^m \leq \frac{w_{n(m)}^L(w_{i(m)}^L + w_{k(m)}^L)}{w_{i(m)}^L(w_{n(m)}^L + w_{k(m)}^L)}$, $k \in N$. Therefore, the optimal objective value of model (35) is

$$\beta_i^{m*} = \min_{k\in N}\left\{\frac{w_{n(m)}^L(w_{i(m)}^L + w_{k(m)}^L)}{w_{i(m)}^L(w_{n(m)}^L + w_{k(m)}^L)}\right\} = \frac{w_{n(m)}^L(w_{i(m)}^L + w_{1(m)}^L)}{w_{i(m)}^L(w_{n(m)}^L + w_{1(m)}^L)}. \tag{36}$$

As $0 \le w_{1(m)}^L \le w_{2(m)}^L \le \cdots \le w_{n(m)}^L \le 1$, then $\beta_1^{m*} \ge \beta_2^{m*} \ge \cdots \ge \beta_n^{m*}$, it follows that

$$\max_{i\in N}\{\beta_i^{m*}\} = \beta_1^{m*} = \frac{w_{n(m)}^L(w_{1(m)}^L + w_{1(m)}^L)}{w_{1(m)}^L(w_{n(m)}^L + w_{1(m)}^L)} = \frac{2w_{n(m)}^L}{w_{n(m)}^L + w_{1(m)}^L}. \tag{37}$$

Combining Eqs. (36) and (37), we have

$$2\beta_i^{m*} = \frac{2w_{n(m)}^L(w_{i(m)}^L + w_{1(m)}^L)}{w_{i(m)}^L(w_{n(m)}^L + w_{1(m)}^L)} = \frac{2w_{n(m)}^L}{w_{n(m)}^L + w_{1(m)}^L} \cdot \frac{w_{i(m)}^L + w_{1(m)}^L}{w_{i(m)}^L} = \max_{i\in N}\{\beta_i^{m*}\} \cdot \frac{w_{i(m)}^L + w_{1(m)}^L}{w_{i(m)}^L}, \tag{38}$$

then, one can obtain $w_{i(m)}^L = \frac{w_{1(m)}^L \cdot \max_{i\in N}\{\beta_i^{m*}\}}{2\beta_i^{m*} - \max_{i\in N}\{\beta_i^{m*}\}}$, $i \in N$, thus

$$1 = \sum_{i=1}^{n} w_{i(m)}^L = \sum_{i=1}^{n} \frac{w_{1(m)}^L \cdot \max_{i\in N}\{\beta_i^{m*}\}}{2\beta_i^{m*} - \max_{i\in N}\{\beta_i^{m*}\}} = w_{1(m)}^L \cdot \sum_{i=1}^{n} \frac{\max_{i\in N}\{\beta_i^{m*}\}}{2\beta_i^{m*} - \max_{i\in N}\{\beta_i^{m*}\}},$$

it is followed that

$$w_{1(m)}^L = 1 \Big/ \sum_{i=1}^{n} \frac{\max_{i\in N}\{\beta_i^{m*}\}}{2\beta_i^{m*} - \max_{i\in N}\{\beta_i^{m*}\}}. \tag{39}$$

$$\begin{aligned}
\frac{1}{10}\sum_{m=1}^{10} CI(P_{(t+1)}^m) &= \frac{1}{5n(n-1)}\sum_{m=1}^{10}\sum_{i<j}\left|\bar{p}_{ij(t+1)}^m - p_{ij(t+1)}^m\right| \\
&= \frac{1}{5n(n-1)}\left(\left|\bar{p}_{i^*,j^*(t+1)}^{m^*} - p_{i^*,j^*(t+1)}^{m^*}\right| + \sum_{m\neq m^*}\sum_{\substack{(i,j)\neq(i^*,j^*)\\ i<j}}\left|\bar{p}_{ij(t+1)}^m - p_{ij(t+1)}^m\right|\right) \\
&= \frac{1}{5n(n-1)}\left(\left|\bar{p}_{i^*,j^*(t+1)}^{m^*} - \left((1-\theta)\cdot p_{i^*,j^*(t)}^{m^*} + \theta\cdot\bar{p}_{i^*,j^*(t)}^{m^*}\right)\right| + \sum_{m\neq m^*}\sum_{\substack{(i,j)\neq(i^*,j^*)\\ i<j}}\left|\bar{p}_{ij(t)}^m - p_{ij(t)}^m\right|\right) \\
&= \frac{1}{5n(n-1)}\left(\left|(1-\theta)\left(\bar{p}_{i^*,j^*(t+1)}^{m^*} - p_{i^*,j^*(t)}^{m^*}\right) + \theta\left(\bar{p}_{i^*,j^*(t+1)}^{m^*} - \bar{p}_{i^*,j^*(t)}^{m^*}\right)\right| + \sum_{m\neq m^*}\sum_{\substack{(i,j)\neq(i^*,j^*)\\ i<j}}\left|\bar{p}_{ij(t)}^m - p_{ij(t)}^m\right|\right) \\
&\le \frac{1}{5n(n-1)}\left((1-\theta)\left|\bar{p}_{i^*,j^*(t+1)}^{m^*} - p_{i^*,j^*(t)}^{m^*}\right| + \theta\left|\bar{p}_{i^*,j^*(t+1)}^{m^*} - \bar{p}_{i^*,j^*(t)}^{m^*}\right| + \sum_{m\neq m^*}\sum_{\substack{(i,j)\neq(i^*,j^*)\\ i<j}}\left|\bar{p}_{ij(t)}^m - p_{ij(t)}^m\right|\right) \\
&\le \frac{1}{5n(n-1)}\left((1-\theta)\left|\bar{p}_{i^*,j^*(t)}^{m^*} - p_{i^*,j^*(t)}^{m^*}\right| + \theta\left|p_{i^*,j^*(t)}^{m^*} - \bar{p}_{i^*,j^*(t)}^{m^*}\right| + \sum_{m\neq m^*}\sum_{\substack{(i,j)\neq(i^*,j^*)\\ i<j}}\left|\bar{p}_{ij(t)}^m - p_{ij(t)}^m\right|\right) \\
&< \frac{1}{5n(n-1)}\left(\left|\bar{p}_{i^*,j^*(t)}^{m^*} - p_{i^*,j^*(t)}^{m^*}\right| + \sum_{m\neq m^*}\sum_{\substack{(i,j)\neq(i^*,j^*)\\ i<j}}\left|\bar{p}_{ij(t)}^m - p_{ij(t)}^m\right|\right) \\
&= \frac{1}{5n(n-1)}\sum_{m=1}^{10}\sum_{i<j}\left|\bar{p}_{ij(t)}^m - p_{ij(t)}^m\right| = \frac{1}{10}\sum_{m=1}^{10} CI(P_{(t)}^m).
\end{aligned} \tag{34}$$

By utilizing Eq. (38), we have

$$w^L_{i(m)} = \frac{w^L_{1(m)} \cdot \max_{i\in N}\{\beta_i^{m*}\}}{2\beta_i^{m*} - \max_{i\in N}\{\beta_i^{m*}\}}, \quad i = 2, 3, \cdots, n. \tag{40}$$

Similarly, if $m = 6, 7, \cdots, 10$, one can obtain

$$w^U_{\sigma(1),(m-5)} = 1 \Big/ \sum_{i=1}^{n} \frac{\beta^{m*}_{\sigma(1)}}{2\beta^{m*}_{\sigma(i)} - \beta^{m*}_{\sigma(1)}},$$

$$w^U_{\sigma(i),(m-5)} = \frac{w^U_{\sigma(i),(m-5)} \cdot \beta^{m*}_{\sigma(1)}}{2\beta^{m*}_{\sigma(i)} - \beta^{m*}_{\sigma(1)}}, \quad i = 2, 3, \cdots, n.$$

Therefore, the proof of Theorem 5 is completed. ■

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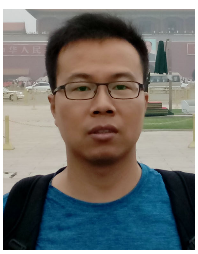

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