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Local Connectivity of Uncertain Random Graphs

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ABSTRACT As the system becomes more and more complex, we are usually in the state of indeterminacy. In the real world, the states of uncertainty and randomness are the two most common types of indeterminacy. An uncertain random graph is applied to describe a graph model with uncertainty and randomness simultaneously. This paper mainly focuses on the connectivity of two vertices in an uncertain random graph. Firstly, a local connectivity index is proposed to unveil the chance measure that two special vertices are connected in an uncertain random graph. Furthermore, a method for calculating the local connectivity index is formulated. In addition, some simplified forms of the method are developed, and an algorithm is designed to obtain the local connectivity index. Finally, the information relevant to the relationship between the local connectivity index and the connectivity index is discussed.

INDEX TERMS Connectivity, chance theory, graph theory, uncertain random graph, uncertainty theory.

I. INTRODUCTION

In real life, graph theory is widely applied to solve a variety of optimization problems, such as the traveling salesman problem (Li *et al.* [21]), transportation problem (Lv *et al.* [31]), network flow problem (Asadi and Kia [2]), and complex network systems (Cheng *et al.* [5], Li and Daniels [22], Li *et al.* [23]). In classical graph theory, these problems are often considered in a determinacy environment, in which all the edges and vertices can be completely determined. Among more research on the theoretical problems and applications of classical graph theory, the interested readers may refer to (Bondy and Murty [3], Bu *et al.* [4], Hu *et al.* [15], Li *et al.* [24], Li *et al.* [25]).

However, we are usually surrounded in the state of indeterminacy. In order to model randomness in a graph, probability theory (Kolmogorov [17]) is introduced into the graph theory and random graph is defined by Erdös and Rényi [7], [8] and Gilbert [12] at nearly the same time.

The use of probability theory is on the premise that the obtained probability is very closed to the real frequency. However, sometimes we have no access to get sufficient data due to the technological or economical difficulties. In this case, we have no choice but to obtain belief degrees from some experts about the occurrence of each event. Some people may regard belief degree as subjective probability. However, it follows the conclusion made by Kahneman and Tversky [16] that the belief degree usually has a much larger range than the real frequency. Additionally, Liu [27] declares that probability theory is not applicable to model belief degree, and presents a counterexample. In fact, fuzzy sets theory (Zadeh [37]) is introduced into the field of graph theory to deal with indeterminacy information. Subsequently, Rosenfeld [34] illustrates and shows much attention to fuzzy graphs.

In order to distinguish from randomness, such indeterminacy caused by personal belief degrees is named uncertainty by Liu [27]. In addition, an uncertainty theory was founded in 2007 by Liu [26] to deal with belief degree. Sometimes, randomness and uncertainty may coexist in a complex system. Moreover, a chance theory is proposed by Liu [29] to handle such complex phenomenon. To deal with complex systems that uncertainty and randomness coexist in graphs, uncertain random graph is defined by Liu [28]. As we can see, connectivity is one of the basic topics of the graph theory, especially for uncertain random graphs. In an uncertain random graph, it is note that the existence of an edge is in the state of indeterminacy. Therefore, it is necessary to develop a deeper understanding of the connectivity of uncertain random graphs.

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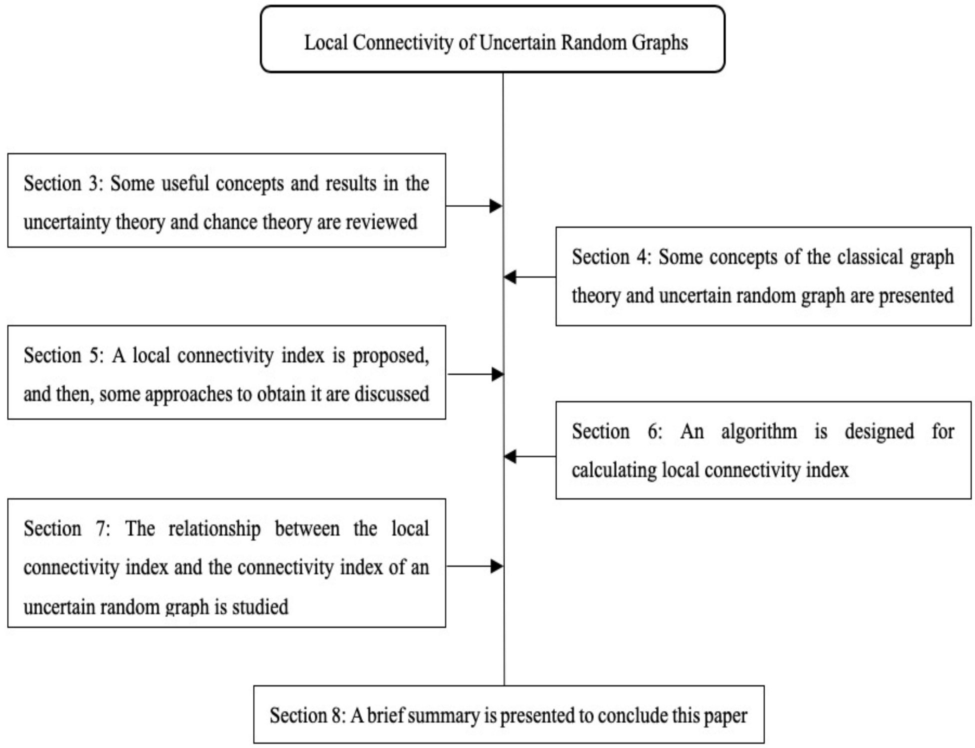

FIGURE 1. **The flowchart of the framework of this paper.**

Inspired by the above discussion, this paper mainly considers the following two issues: First, how likely are the two vertices in an uncertain random graph connected? Second, what is the relationship between the connectivity of two vertices and that of the uncertain random graph? To answer these questions, a local connectivity index is proposed to measure how likely it is for two vertices of an uncertain random graph to be connected. Apart from that, some approaches for calculating the local connectivity index are developed and an algorithm is designed to be used as well. Additionally, this paper also analyzes the relationship between the connectivity of two vertices and that of the uncertain random graph.

This paper first introduces some basic concepts and properties on the uncertainty theory and chance theory in Section 3. Then, in generally, Section 4 presents some concepts of the classical graph theory and uncertain random graph. The main result of this paper is shown in Section 5, where a fundamental method for calculating local connectivity index is firstly given in detail, and then, some simplified forms of the method are discussed. In Section 6, an algorithm is designed for calculating local connectivity index and is illustrated by a numerical example to make it sense. Section 7 is devoted to show the relationship between the local connectivity index and the connectivity index of an uncertain random graph. Section 8 concludes the paper with a brief summary. To give readers a quick understanding of the framework of this paper, a flowchart is presented in Fig. 1.

II. LITERATURE REVIEW

Generally speaking, our research is mainly related to the following four aspects of literature: 1) random graphs, 2) fuzzy graphs, 3) uncertain graphs, and 4) uncertain random graphs. Then, it is expected to review the relevant issues from these aspects.

A random graph is firstly defined by Erdös and Rényi [7], and generated by some random process. At nearly the same time, Gilbert [12] points out the conception of the connectivity of random graphs. After that, many researchers throw themselves into the field of random graphs. For instance, Walkup [35] investigates how much probability at which a random directed bipartite graph contains a matching. Cooper *et al.* [6] study Hamilton cycles in random regular digraphs. Gu and Li [13] analyze the conflict-free connection number of random graphs.

The idea of fuzzy graphs was introduced by Rosenfeld [34] in 1975. Even since then, fuzzy graphs have been received much attention and a variety of works have been carried out. For instance, Mathew and Sunitha [33] classified arcs of a fuzzy graph into three types by considering the strength of an arc. A further study has been reported by Yang *et al.* [36] who investigated some properties of bipolar fuzzy graphs. Apart from that, Akram and Dudek [1] put forward the concepts of regular and totally regular bipolar fuzzy graphs. Recently, on the basis of existing research, M *et al.* [32] discussed Wiener index of various fuzzy graph structures such as fuzzy trees and fuzzy cycles.

An uncertain graph was proposed by Gao and Gao [9] to describe a graph with uncertain edges in 2013. Since then, a wide range of significant topics of uncertain graphs have been investigated by many scholars. For instance, Zhang and Peng [38] developed a knowledge of the connectivity of two vertices in an uncertain graph. Gao *et al.* [11] investigated the distribution functions of the diameter of an uncertain graph. Gao and Qin [10] put their sight at the edge-connectivity of an uncertain graph. Li *et al.* [20] studied the properties of the matching number in an uncertain graph.

A concept of uncertain random graph is presented by Liu [28] to model a type of graph in which some edges exist with some degree in uncertain measure and others exist with some degrees in probability measure. In addition, Zhang *et al.* [39], [40] offer a new insight into the Euler circuit problem and matching problem in an uncertain random graph. Recently, Li and Zhang [18] consider the connectivity of an uncertain random graph with respect to edges. Nearly the same time, one study by Li and Gao [19] describes the vertex-connectivity of an uncertain random graph.

In the real world, someone may have a preference to learn whether two specific vertices are connected or not. To our knowledge, there is currently no related research on the connectivity of two specific vertices in an uncertain random graph. In view of this fact, this paper focuses on investigating the connectivity of two specific vertices of an uncertain random graph. As a result, a local connectivity index of an uncertain random graph is proposed with a comprehensive analysis and some methods are set out to calculate the local connectivity index. The relationship between the local connectivity index and connectivity index is also considered.

In comparison with the existing articles, the main contributions of this work can be summarized as follows. First, compared to the use of other approaches, such as probability theory, fuzzy set theory, and uncertain theory, for graph models with indeterminacy information, this paper provides an opportunity to advance chance theory to deal with indeterminacy information in a complex system, in which randomness and uncertainty may coexist. Second, compared to the existing works related to uncertain random graphs, this paper demonstrates completely the study of the connectivity of two vertices of an uncertain random graph. As a result, this paper contributes to complement existing literatures, and it will certainly enhance the understanding of the connectivity of uncertain random graphs.

III. PRELIMINARIES

In the following, we will introduce some useful concepts and results in the uncertainty theory and chance theory, which will be used to help model connectivity of a graph with uncertain random factors.

Definition 1: (Liu [26]) Let $\mathcal{L}$ be a σ-algebra over a nonempty set Γ. A set function $\mathcal{M}$ defined on $\mathcal{L}$ is called an uncertain measure if it satisfies normality, duality, subadditivity and product axioms.

A function f is called Boolean if it is a mapping from $\{0,1\}^n$ to $\{0,1\}$. Also, an uncertain variable ξ is called Boolean if it takes value either 0 or 1 with an uncertain measure.

Theorem 1: (Gao and Gao [9]) Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,

$$\xi_i = \begin{cases} 1, & \textit{with uncertain measure } \alpha_i \\ 0, & \textit{with uncertain measure } 1-\alpha_i \end{cases}$$

for $i = 1, 2, \cdots, n$. If f is an increasing Boolean function, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(B_1, B_2, \cdots, B_n)=1} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\},$$

where B_i are subsets of $\{0,1\}$, $i = 1, 2, \cdots, n$.

The chance theory produced by Liu [29] is given with regard to model complex systems related to uncertainty and randomness. As a basic concept in chance theory, chance space is refer to the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$, where $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertain space, and $(\Omega, \mathcal{A}, \Pr)$ is a probability space.

Definition 2 (Liu [29]): Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ be a chance space, and $\Theta \in \mathcal{L} \times \mathcal{A}$ be an event. Then the chance measure of Θ is defined by

$$Ch\{\Theta\} = \int_0^1 Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \ge x\} dx.$$

In addition, Liu [29] and Hou [14] illustrate that the chance measure satisfies the characteristic of normality, duality, monotonicity and subadditivity.

Definition 3 (Liu [29]): An uncertain random variable is a measurable function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ to the set of real numbers, i.e.,

$$\{\xi \in B\} = \{(\gamma, \omega) | \xi(\gamma, \omega) \in B\}$$

is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B.

For Boolean random variables and Boolean uncertain variables, we have the following result.

Theorem 2 (Liu [30]): Assume that $\eta_1, \eta_2, \cdots, \eta_m$ are independent Boolean random variables, i.e.,

$$\eta_i = \begin{cases} 1, & \textit{with probability measure } a_i \\ 0, & \textit{with probability measure } 1-a_i \end{cases}$$

for $i = 1, 2, \cdots, m$, and $\tau_1, \tau_2, \cdots, \tau_n$ are independent Boolean uncertain variables, i.e.,

$$\tau_j = \begin{cases} 1, & \textit{with uncertain measure } b_j \\ 0, & \textit{with uncertain measure } 1-b_j \end{cases}$$

for $j = 1, 2, \cdots, n$. If f is a Boolean function, then $\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is a Boolean uncertain random variable such that

$$Ch\{\xi = 1\} = \sum_{(x_1,\cdots,x_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(x_i)\right) f^*(x_1,\cdots,x_m)$$

where

$$f^*(x_1,\cdots,x_m) = \begin{cases} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \textit{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=0} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \textit{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) \ge 0.5, \end{cases}$$

$$\mu_i(x_i) = \begin{cases} a_i, & \textit{if } x_i = 1 \\ 1 - a_i, & \textit{if } x_i = 0 \end{cases} \quad (i = 1, 2, \cdots, m),$$

$$\nu_j(y_j) = \begin{cases} b_j, & \textit{if } y_j = 1 \\ 1 - b_j, & \textit{if } y_j = 0 \end{cases} \quad (j = 1, 2, \cdots, n).$$

IV. NOTATIONS AND CONCEPTS

Generally speaking, a graph can be denoted by $G = (V, E)$, where V is the set of vertices and E is the set of edges. An edge with identical vertices is called a loop, and two edges joining the same pair of vertices are called parallel edges. In this paper, all graphs are assumed to be simple graphs, which contain neither loops nor parallel edges. For more research of graph theory, we may consult Bondy and Murty [3].

For a graph of order (i.e., the number of vertices) n, it usually can be described by a matrix as follows:

$$X = \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ x_{21} & 0 & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & 0 \end{pmatrix}$$

where

$$x_{ij} = \begin{cases} 1, & \text{if there exists an edge between vertices } i \text{ and } j \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $x_{ii} = 0$ and $x_{ij} = x_{ji}$ for $i, j = 1, 2, \cdots, n$, since the graphs in this paper are simple graphs.

As the system becomes more and more complex, it is no doubt that different types of indeterminacy are frequently encountered in practical application of graph theory. Attempts to model indeterminacy in graphs result in the development of random graph defined by Erdös and Rényi [7], [8] and Gilbert [12], and uncertain graph defined by Gao and Gao [9]. In real life, uncertainty and randomness may simultaneously appear in many cases. To deal with this complex system, an uncertain random graph is defined by Liu [28].

Roughly speaking, an uncertain random graph is a graph including all independent edges that some edges exist with some degrees in uncertain measure while others exist with some degrees in probability measure. For a graph of order n, in order to present how much degree of an edge exists, an uncertain random adjacency matrix is proposed by Liu [28] as follows,

$$\mathcal{T} = \begin{pmatrix} 0 & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & 0 & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & 0 \end{pmatrix}$$

where α_{ij} represent that the edges between vertices i and j exist with uncertain measures α_{ij} or probability measure α_{ij}, $i, j = 1, 2, \cdots, n$, respectively.

Definition 4 (Liu [28]): An uncertain random graph is a quartette consisting of a vertex set $\mathcal{V}$, an uncertain edge set $\mathcal{U}$, a random edge set $\mathcal{R}$, and an uncertain random adjacency matrix $\mathcal{T}$, denoted by $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$.

Definition 5 (Zhang et al. [39]): Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. The underlying graph of $\mathbb{G}$, denoted by $\mathbb{G}^*$, is a graph obtained from $\mathbb{G}$ by replacing $0 < \alpha_{ij} < 1$ with $\alpha_{ij} = 1$, $i, j = 1, 2, \cdots, n$, respectively.

According to Definition 5, we know that the underlying graph is a such graph that all the vertices and edges of the original uncertain random graph are exist.

In a classical graph, a sequence $W = v_0e_1v_1e_2\cdots e_kv_k$ is called a (v_0, v_k)-walk, whose terms are alternately vertices and edges. If the edges of the walk W are distinct, then W is said to be a trail. In addition, if the vertices of the walk are distinct, then W is said to be a path.

Recently, Liu [28] has defined a connectivity index of an uncertain random graph as the chance measure that the uncertain random graph is connected. However, someone may show the interests in whether two specific vertices are connected or not. In classical graph theory, two vertices i and j are said to be connected if there exists an (i, j)-path in the graph. In order to show how likely two specific vertices are connected in an uncertain random graph, a local connectivity index is proposed as follows.

Definition 6: Let $\mathbb{G}$ be an uncertain random graph, the local connectivity index of two vertices i and j is the chance measure that i and j are connected in the graph $\mathbb{G}$.

V. LOCAL CONNECTIVITY INDEX

To find a way to calculate the local connectivity index of two special vertices on an uncertain random graph, we will have to introduce some useful symbols, which are from Liu [28]. Firstly, we assume that

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

and

$$\mathbb{X} = \left\{ X \left| \begin{array}{l} x_{ij} = 0 \text{ or } 1, \text{ if } (i,j) \in \mathcal{R} \\ x_{ij} = 0, \text{ if } (i,j) \in \mathcal{U} \\ x_{ij} = x_{ji}, \ i,j = 1,2,\cdots,n \\ x_{ii} = 0, \ i = 1,2,\cdots,n \end{array} \right. \right\}.$$

Next, given a matrix

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix},$$

we define the extension class of Y as

$$Y^* = \left\{ X \left| \begin{array}{l} x_{ij} = y_{ij}, \text{ if } (i,j) \in \mathcal{R} \\ x_{ij} = 0 \text{ or } 1, \text{ if } (i,j) \in \mathcal{U} \\ x_{ij} = x_{ji}, \ i,j = 1,2,\cdots,n \\ x_{ii} = 0, \ i = 1,2,\cdots,n \end{array} \right. \right\}.$$

Based on the notations mentioned above, the following result can be obtained immediately.

Theorem 3: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph of order n. If all edges are independent, then the local connectivity index of u and v is

$$\rho_{\mathbb{G}}(u,v) = \sum_{Y \in \mathbb{X}} \left(\prod_{(i,j)\in\mathcal{R}} \nu_{ij}(Y) \right) g^*(Y)$$

where

$$g^*(Y) = \begin{cases} \sup\limits_{X\in Y^*, g(X)=1} \min\limits_{(i,j)\in\mathcal{U}} \nu_{ij}(X), \\ \quad \text{if} \sup\limits_{X\in Y^*, g(X)=1} \min\limits_{(i,j)\in\mathcal{U}} \nu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{X\in Y^*, g(X)=0} \min\limits_{(i,j)\in\mathcal{U}} \nu_{ij}(X), \\ \quad \text{if} \sup\limits_{X\in Y^*, g(X)=1} \min\limits_{(i,j)\in\mathcal{U}} \nu_{ij}(X) \geq 0.5, \end{cases}$$

$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases} \quad (i,j) \in \mathcal{U},$$

$$\nu_{ij}(Y) = \begin{cases} \alpha_{ij}, & \text{if } y_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } y_{ij} = 0 \end{cases} \quad (i,j) \in \mathcal{R},$$

$$g(X) = \begin{cases} 1, & \text{if there is a positive integer } t \text{ such that} \\ & (X^t)_{uv} > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof: In accordance with the definition of uncertain random graph, the uncertain edges can be represented by Boolean uncertain variables, i.e.,

$$\xi_{ij} = \begin{cases} 1, & \text{with uncertain measure } \alpha_{ij} \\ 0, & \text{with uncertain measure } 1 - \alpha_{ij} \end{cases} \quad (i,j) \in \mathcal{U},$$

and the random edges can be represented by Boolean random variables, i.e.,

$$\eta_{ij} = \begin{cases} 1, & \text{with probability measure } \alpha_{ij} \\ 0, & \text{with probability measure } 1 - \alpha_{ij} \end{cases} \quad (i,j) \in \mathcal{R}.$$

Let G be a classical graph of order n with adjacency matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}.$$

It is universally acknowledged that the vertices u and v are connected if and only if there is a positive integer t such that $(X^t)_{uv} > 0$. It follows from the definition of local connectivity index and Theorem 2 that the result holds.

Theorem 3 furnishes a fundamental method for calculation the local connectivity index of an uncertain random graph. In more detail, we can formulate the method in a much simpler form.

Theorem 4: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. If all edges are independent, then the local connectivity index of u and v is

$$\rho_{\mathbb{G}}(u,v) = \sum_{Y \in \mathbb{X}} \left(\prod_{(i,j)\in\mathcal{R}} \nu_{ij}(Y) \right) g^*(Y)$$

where

$$g^*(Y) = \sup_{X\in Y^*, g(X)=1} \min_{(i,j)\in\mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\},$$

B_{ij} are subsets of $\{0,1\}$.

Proof: Let $X \in Y^*$. If there is an element $x_{ij} \in X$ such that $x_{ij} = 0$, then a new matrix $X' \in Y^*$ can be obtained by replacing $x_{ij} = 0$ with $x_{ij} = 1$. Obviously, $g(X') \geq g(X)$, i.e., $g(X)$ is an increasing Boolean function. It follows from Theorem 1 that the theorem holds immediately. ■

Remark 1: When the uncertain variables disappear, the uncertain random graph becomes a random graph, and the local connectivity index of u and v is

$$\rho_{\mathbb{G}}(u,v) = \sum_{X \in \mathbb{X}} \left(\prod_{1\leq i<j\leq n} \nu_{ij}(X) \right) g(X)$$

where

$$\mathbb{X} = \left\{ X \left| \begin{array}{l} x_{ij} = 0 \text{ or } 1, \ i,j = 1,2,\cdots,n \\ x_{ij} = x_{ji}, \ i,j = 1,2,\cdots,n \\ x_{ii} = 0, \ i = 1,2,\cdots,n \end{array} \right. \right\}.$$

Remark 2: When the random variables disappear, the uncertain random graph becomes an uncertain graph, and the local connectivity index of u and v is

$$\rho_{\mathbb{G}}(u,v) = \sup_{X\in\mathbb{X}, g(X)=1} \min_{1\leq i<j\leq n} \mathcal{M}\{\xi_{ij} \in B_{ij}\}$$

where

$$\mathbb{X} = \left\{ X \left| \begin{array}{l} x_{ij} = 0 \text{ or } 1, \ i,j = 1,2,\cdots,n \\ x_{ij} = x_{ji}, \ i,j = 1,2,\cdots,n \\ x_{ii} = 0, \ i = 1,2,\cdots,n \end{array} \right. \right\}$$

which is the same result that presented in Zhang and Peng [38].

Next, we will present another method for calculating local connectivity index on an uncertain random graph.

Theorem 5: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. The local connectivity index of two vertices u and v is

$$\rho_{\mathbb{G}}(u, v) = \sum_{Y \in \mathbb{X}} \left(\prod_{(i,j) \in \mathcal{R}} \nu_{ij}(Y) \right) g^*(Y)$$

where

$$g^*(Y) = \sup_{X \in Y^*, P \in P(X)} \min_{(i,j) \in \mathcal{U} \cap P} \alpha_{ij},$$

$P(X)$ is the set of (u, v)-paths that with respect to X.

Proof: In accordance with Theorem 4, we will corroborate the result by the following two steps.

Firstly, we prove that

$$\sup_{X \in Y^*, g(X)=1} \min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\} \le \sup_{X \in Y^*, P \in P(X)} \min_{(i,j) \in \mathcal{U} \cap P} \alpha_{ij}.$$

Since $g(X)$ is an increasing Boolean function. Thus, there must exist a series of $\{B'_{ij}\}$, for each edge $(i, j) \in \mathcal{U}$, taking values of $\{1\}$ or $\{0, 1\}$ such that

$$\sup_{X \in Y^*, g(X)=1} \min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\} = \min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B'_{ij}\}.$$

In fact, we can choose $\{B'_{ij}\}$ such that the edges (i, j) with respect to the sets that taking values of $\{1\}$ are all in an (u, v)-path P'.

It is easy to verify that

$$\min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B'_{ij}\} = \min_{(i,j) \in \mathcal{U} \cap P'} \mathcal{M}\{\xi_{ij} = 1\}.$$

Furthermore, we have

$$\begin{aligned} \min_{(i,j) \in \mathcal{U} \cap P'} \alpha_{ij} &= \min_{(i,j) \in \mathcal{U} \cap P'} \mathcal{M}\{\xi_{ij} = 1\} \\ &= \sup_{X \in Y^*, g(X)=1} \min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\}. \end{aligned} \quad (1)$$

It is clear that the following formula usually holds

$$\min_{(i,j) \in \mathcal{U} \cap P'} \alpha_{ij} \le \sup_{X \in Y^*, P \in P(X)} \min_{(i,j) \in \mathcal{U} \cap P} \alpha_{ij}. \quad (2)$$

Combining with (1) and (2), we have

$$\sup_{X \in Y^*, g(X)=1} \min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\} \le \sup_{X \in Y^*, P \in P(X)} \min_{(i,j) \in \mathcal{U} \cap P} \alpha_{ij}.$$

Secondly, we will prove that the following inequality holds

$$\sup_{X \in Y^*, P \in P(X)} \min_{(i,j) \in \mathcal{U} \cap P} \alpha_{ij} \le \sup_{X \in Y^*, g(X)=1} \min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\}.$$

Obviously, there must exist an (u, v)-path P' such that

$$\min_{(i,j) \in \mathcal{U} \cap P'} \alpha_{ij} = \sup_{X \in Y^*, P \in P(X)} \min_{(i,j) \in \mathcal{U} \cap P} \alpha_{ij} = \alpha.$$

Based on P', for each edge $(i, j) \in \mathcal{U}$, a series of B'_{ij} can be obtained by the following way: $B'_{ij} = \{1\}$, if $\alpha_{ij} \ge \alpha$; otherwise, $B'_{ij} = \{0, 1\}$. It is clear that

$$\min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B'_{ij}\} = \alpha = \sup_{X \in Y^*, P \in P(X)} \min_{(i,j) \in \mathcal{U} \cap P} \alpha_{ij}. \quad (3)$$

Additionally,

$$\min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B'_{ij}\} \le \sup_{X \in Y^*, g(X)=1} \min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\}. \quad (4)$$

Combining with (3) and (4), we can obtain the following result immediately

$$\sup_{X \in Y^*, P \in P(X)} \min_{(i,j) \in \mathcal{U} \cap P} \alpha_{ij} \le \sup_{X \in Y^*, g(X)=1} \min_{(i,j) \in \mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\}.$$

The proof is completed.

Corollary 1: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. If its underlying graph $\mathbb{G}^*$ is an (u, v)-path, then the local connectivity index of u and v is

$$\rho_{\mathbb{G}}(u, v) = \left(\prod_{(i,j) \in \mathcal{R}} \alpha_{ij} \right) \left(\min_{(i,j) \in \mathcal{U}} \alpha_{ij} \right).$$

VI. ALGORITHM AND EXAMPLE

On the basis of Theorem 5, it unfolds that the function $g^*(Y)$ takes value in the set of $\{\alpha_{ij} | (i, j) \in \mathcal{U}\}$. This enlightens us to design a much simple algorithm to obtain local connectivity index on an uncertain random graph.

Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph, denote $H = \{(i, j) | (i, j) \in \mathcal{R}, \alpha_{ij} = 1\}$, $F = \{(i, j) | (i, j) \in \mathcal{R}, 0 < \alpha_{ij} < 1\}$. Suppose that $|F| = m$, there are 2^m realizations for random edges of the uncertain random graph $\mathbb{G}$. Denote the subsets of F as $S_1, S_2, \cdots, S_{2^m}$. For each pair of vertices u and v, an algorithm to obtain the value of local connectivity index $\rho_{\mathbb{G}}(u, v)$ will be designed (see Algorithm 1).

In Step 3 and Step 5, the Depth-first algorithm or Breadth-first algorithm can be used to judge whether there is an (u, v)-path in the graph G_k or not. The complexity of the Depth-first algorithm is $O(n^2)$, where n is the number of vertices. Thus, it is easy to verify that the complexity of Algorithm 1 is $O(2^m n^2)$, where n is the number of vertices and m is the number of random edges. For large size of uncertain random graphs, it is clear that the efficiency may be low. Thus, some more efficient algorithms may be investigated for improving the efficiency.

Proposition 1: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. For the pair of vertices u and v, the result obtained by Algorithm 1 is exactly equal to the value of the local connectivity index of vertices u and v, i.e., $\rho_{\mathbb{G}}(u, v)$.

Proof: According to Algorithm 1, all the elements Y of $\mathbb{X}$ are considered. For each matrix Y, the graph G_k contains an (u, v)-path if $U_k \neq 0$. It follows from Theorem 5 that, $g^*(Y) \ge U_k = \alpha$.

If $g^*(Y) > \alpha$, without loss of generality, let us assume that $g^*(Y) = \alpha'$. That is, there exists an (u, v)-path $P' \in P(X)$ such that

$$g^*(Y) = \sup_{X \in Y^*, P \in P(X)} \min_{(i,j) \in \mathcal{U} \cap P} \alpha_{ij} = \min_{(i,j) \in \mathcal{U} \cap P'} \alpha_{ij} = \alpha'.$$

Obviously, for each edge $(i, j) \in \mathcal{U} \cap P'$, we have $\alpha_{ij} \ge \alpha'$. For the matrix Y, algorithm 1 tells us that after choosing the uncertain edges that satisfy $\alpha_{ij} \ge \alpha'$, an (u, v)-path can

Algorithm 1 Algorithm for Calculating the Value of Local Connectivity of Two Vertices u and v.

Step 1: Set $\rho_{\mathbb{G}}(u, v) = 0$. Let $k = 1$.

Step 2: Let $E_R \leftarrow S_k, E \leftarrow E_R \cup H$. For an edge $(i,j) \in F$, if $(i,j) \in E_R$, $\nu_{ij} = \alpha_{ij}$; otherwise $\nu_{ij} = 1 - \alpha_{ij}$. Let $R_k = \prod_{(i,j)\in F} \nu_{ij}$, $E_U = \mathcal{U}$.

Step 3: Constructing a new graph G_k with edge set E and vertex set $\mathcal{V}$. If there is an (u, v)-path in the graph G_k, then $U_k = 1$, go to Step 6. Otherwise, if $E_U = \emptyset$, $U_k = 0$, go to Step 6; if $E_U \neq \emptyset$, go to Step 4.

Step 4: For all edges $(i,j) \in E_U$, choose the edge with the biggest truth value $\alpha_{ij} = \alpha$, such as (s, t). Reset $E \leftarrow E \cup \{(s, t)\}$, $E_U \leftarrow E_U - \{(s, t)\}$.

Step 5: Constructing a new graph G_k with edge set E and vertex set $\mathcal{V}$. If there is an (u, v)-path in the graph G_k, then $U_k = \alpha$. Otherwise, if $E_U = \emptyset$, $U_k = 0$; if $E_U \neq \emptyset$, turn back to Step 4.

Step 6: Let $\rho_{\mathbb{G}}(u, v) = \rho_{\mathbb{G}}(u, v) + R_k U_k$. If $k < 2^m$, let $k \leftarrow k + 1$, turn back to Step 2.

be found in the new graph G_k, and the iteration must be terminated. In other words, we can not obtain α since $\alpha' > \alpha$, which is contradiction to the fact that we have obtained the value of α as U_k. Thus,

$$g^*(Y) = U_k = \alpha.$$

In addition, it follows from Step 1 and Step 2 that

$$\prod_{(i,j)\in\mathcal{R}} \nu_{ij}(Y) = R_k.$$

On the basis of Theorem 5, after running of the Algorithm 1 that the result obtained is exactly equal to the value of the local connectivity index of vertices u and v, i.e., $\rho_{\mathbb{G}}(u, v)$. This completes the proof.

Next, an example will be presented to illustrate the proposed algorithm.

Example 1: Consider an uncertain random graph $\mathbb{G}$ as shown in Fig. 2, in which $\mathcal{R} = \{(2, 3), (5, 6)\}$. Algorithm 1 is employed to calculate the local connectivity index of vertices 2 and 6, i.e., $\rho_{\mathbb{G}}(2, 6)$.

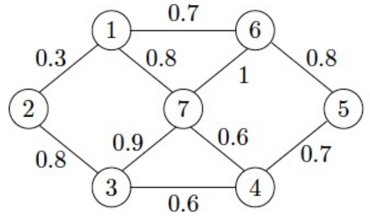

FIGURE 2. Uncertain random graph $\mathbb{G}$ for Example 1.

Since $F = \mathcal{R}$ and $|F| = 2$, then the value of $\rho_{\mathbb{G}}(2, 6)$ can be obtained by the following four iterations in accordance with Algorithm 1.

In the first iteration, when $E_R = \emptyset$, $R_1 = 0.2 \times 0.2 = 0.04$. After choosing the uncertain edges (i,j) that satisfy $\alpha_{ij} \geq 0.3$, a graph G_1 is obtained as shown in Fig. 3. Obviously, there exists a (2, 6)-path P in the graph G_1, e.g., $P : 2 \to 1 \to 6$. Then the iteration is terminated, and $U_1 = 0.3$.

In the second iteration, when $E_R = \{(2, 3)\}$, $R_2 = 0.8 \times 0.2 = 0.16$. After choosing the uncertain edges (i,j) that satisfy $\alpha_{ij} \geq 0.9$, a new graph G_2 is obtained that is shown in Fig. 4. Clearly, there is a (2, 6)-path $P : 2 \to 3 \to 7 \to 6$ in the graph G_2. Then the iteration is terminated, and $U_2 = 0.9$.

In the third iteration, when $E_R = \{(5, 6)\}$, $R_3 = 0.2 \times 0.8 = 0.16$. After choosing the uncertain edges (i,j) that satisfy $\alpha_{ij} \geq 0.3$, a new graph G_3 is obtained as shown in Fig. 5. It is easy to verify that there exists a (2, 6)-path $P : 2 \to 1 \to 6$. Then $U_3 = 0.3$.

In the fourth iteration, when $E_R = \{(2, 3), (5, 6)\}$, $R_4 = 0.8 \times 0.8 = 0.64$. A new graph G_4 can be obtained as shown in Fig. 6 after choosing the uncertain edges (i,j) that satisfy $\alpha_{ij} \geq 0.9$. In the graph G_4, a (2, 6)-path $P : 2 \to 3 \to 7 \to 6$ is obtained. Then the iteration is terminated, and $U_4 = 0.9$. In accordance with Algorithm 1, the local connectivity index $\rho_{\mathbb{G}}(u, v) = 0.04 \times 0.3 + 0.16 \times 0.9 + 0.16 \times 0.3 + 0.64 \times 0.9 = 0.78$.

VII. CONNECTIVITY INDEX

As a matter of fact, connectivity index is proposed by Liu [28] to indicate the chance measure that an uncertain random graph is connected. In this section, we will discuss the relationship between the local connectivity index and the connectivity index on an uncertain random graph.

Theorem 6: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. If all edges are independent, then the connectivity index of $\mathbb{G}$ is

$$\rho(\mathbb{G}) = \sum_{Y\in\mathbb{X}} \left(\prod_{(i,j)\in\mathcal{R}} \nu_{ij}(Y) \right) f^*(Y)$$

where

$$f^*(Y) = \sup_{X\in Y^*, f(X)=1} \min_{(i,j)\in\mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\},$$

B_{ij} are subsets of $\{0, 1\}$,

$$f(X) = \begin{cases} 1, & if\, I + X + X^2 + \cdots + X^{n-1} > 0 \\ 0, & otherwise. \end{cases}$$

Proof: For an adjacency matrix X, it is regularly endorsed that the graph is connected if and only if $f(X) = 1$. Since $f(X)$ is an increasing function, it follows from Theorem 1 that the Theorem is proved.

Theorem 7: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. Then the connectivity index $\rho(\mathbb{G})$ is less then or equal to the local connectivity index of two vertices of $\mathbb{G}$, i.e.,

$$\rho(\mathbb{G}) \leq \rho_{\mathbb{G}}(i,j) \tag{5}$$

holds for any pair of vertices (i,j).

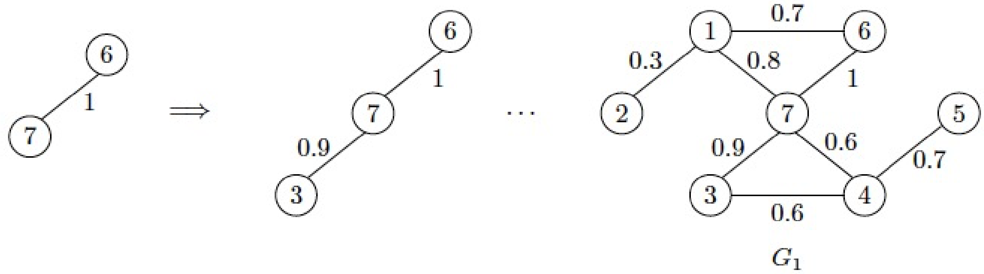

FIGURE 3. The first iteration, i.e., $k = 1$.

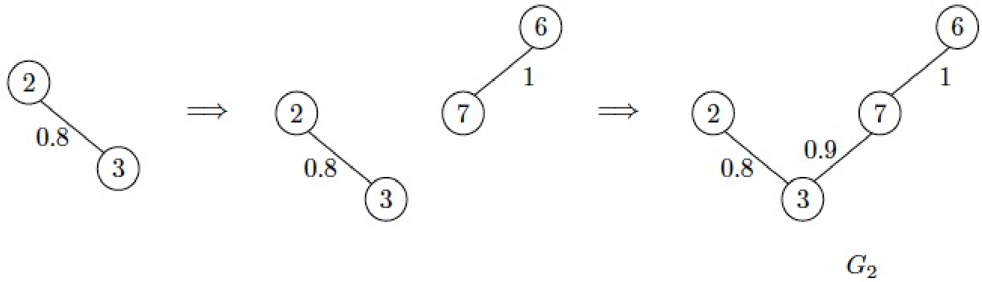

FIGURE 4. The second iteration, i.e., $k = 2$.

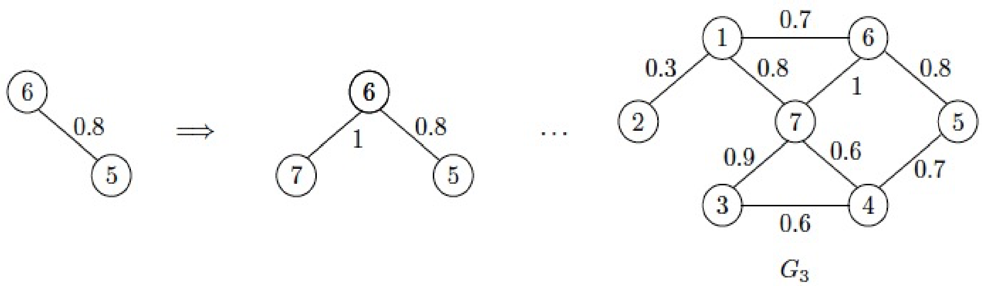

FIGURE 5. The third iteration, i.e., $k = 3$.

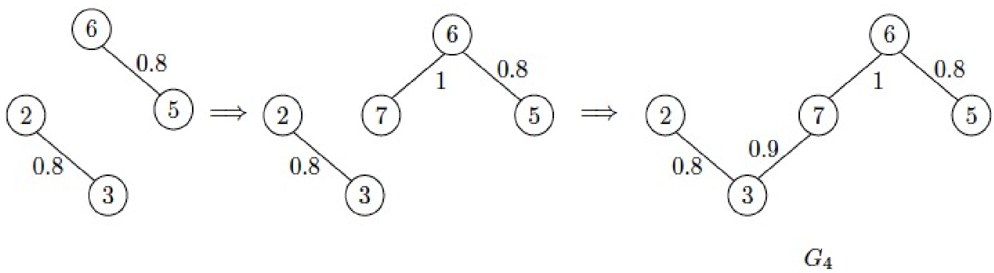

FIGURE 6. The fourth iteration, i.e., $k = 4$.

Proof: Clearly, a graph is connected if and only if every two vertices of it are connected. For each matrix Y, given a $X \in Y^*$, if $f(X) = 1$, then $g(X) = 1$. It follows from Theorems 4 and 6 that

$$f^*(Y) \leq g^*(Y).$$

Then the result holds immediately.

Note that, the inequality of (5) can occur. Let us consider the following example.

Example 2: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph, where

$$\mathcal{R} = \{(1,2), (1,4)\}, \qquad \mathcal{T} = \begin{pmatrix} 0 & 0.3 & 0.2 & 0.6 \\ 0.3 & 0 & 0.8 & 0.3 \\ 0.2 & 0.8 & 0 & 0.6 \\ 0.6 & 0.3 & 0.6 & 0 \end{pmatrix}.$$

Then we can obtain the local connectivity index for each pair of vertices, which is listed in Table 1. And the connectivity index of the graph is 0.524.

TABLE 1. List of the local connectivity indexes.

pair of vertices (i,j)	$(1,2)$	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,4)$
$\rho_{\mathbb{G}}(i,j)$	0.608	0.548	0.728	0.8	0.672	0.636

Also, the equality of (5) can hold. For convenience, denote $g^*_{ij}(Y)$ as the function $g^*(Y)$ that with respect to the pair of vertices i and j.

Theorem 8: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. For each Y, if the inequality $g^*_{uv}(Y) \le g^*_{ij}(Y)$ holds for any pair of vertices (i,j), then

$$\rho(\mathbb{G}) = \rho_{\mathbb{G}}(u, v).$$

Proof: Obviously, $\rho(\mathbb{G}) \le \rho_{\mathbb{G}}(u, v)$, and

$$\rho_{\mathbb{G}}(u, v) = \min_{1\le i<j\le n} \rho_{\mathbb{G}}(i,j).$$

We only need to prove that $\rho(\mathbb{G}) \ge \rho_{\mathbb{G}}(u, v)$ when $\rho_{\mathbb{G}}(u, v) > 0$.

For a matrix Y, if $g^*_{uv}(Y) = 0$, then $f^*(Y) = 0$. Otherwise, if $g^*_{uv}(Y) > 0$, there must exist a matrix X' with a series of $\{B'_{ij}\}$ such that

$$\begin{aligned} g^*_{uv}(Y) &= \min_{(i,j)\in\mathcal{U}} \mathcal{M}\{\xi_{ij} \in B'_{ij}\} \\ &= \sup_{X\in Y^*, g_{uv}(X)=1} \min_{(i,j)\in\mathcal{U}} \mathcal{M}\{\xi_{ij} \in B_{ij}\}, \end{aligned}$$

where $g_{uv}(X)$ denotes the function $g(X)$ that with respect to the pair of vertices u and v. If $f(X') = 1$, $f^*(Y) \ge g^*_{uv}(Y)$ holds immediately; otherwise there exist some pairs of vertices are not connected. In this case, a new matrix X'' with a series of $\{B''_{ij}\}$ can be obtained by taking some uncertain edges (i,j) with $\xi_{ij} = 1$ such that the uncertain random graph is connected, i.e., $f(X'') = 1$. In fact, we can choose X'' such that

$$\min_{(i,j)\in\mathcal{U}} \mathcal{M}\{\xi_{ij} \in B''_{ij}\} = g^*_{uv}(Y),$$

since $g^*_{uv}(Y) \le g^*_{ij}(Y)$ holds for any pair of vertices (i,j). Thus, $f^*(Y) \ge g^*_{uv}(Y)$ holds.

Finally, based on Theorems 4 and 6, we have

$$\rho(\mathbb{G}) \ge \rho_{\mathbb{G}}(u, v)$$

and the theorem is proved.

Example 3: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph, where

$$\mathcal{R} = \{(1,2), (1,4)\}, \quad \mathcal{T} = \begin{pmatrix} 0 & 0.3 & 0.2 & 0.6 \\ 0.3 & 0 & 0.6 & 0.3 \\ 0.2 & 0.6 & 0 & 0.6 \\ 0.6 & 0.3 & 0.6 & 0 \end{pmatrix}.$$

For each matrix Y, Table 2 illustrates the values of $g^*_{ij}(Y)$. It is clear that the inequality

$$g^*_{13}(Y) \le g^*_{ij}(Y)$$

holds for any pair of vertices (i,j). Additionally, we have $\rho(\mathbb{G}) = \rho_{\mathbb{G}}(1,3) = 0.488$.

TABLE 2. The values of $g^*_{ij}(Y)$ for any pair of vertices (i,j).

Edges of Y \ $g^*_{ij}(Y)$	$g^*_{12}(Y)$	$g^*_{13}(Y)$	$g^*_{14}(Y)$
$\emptyset$	0.2	0.2	0.2
$(1,2)$	1	0.6	0.6
$(1,4)$	0.6	0.6	1
$(1,2),(1,4)$	1	0.6	1
Edges of Y \ $g^*_{ij}(Y)$	**$g^*_{23}(Y)$**	**$g^*_{24}(Y)$**	**$g^*_{34}(Y)$**
$\emptyset$	0.6	0.6	0.6
$(1,2)$	0.6	0.6	0.6
$(1,4)$	0.6	0.6	0.6
$(1,2),(1,4)$	0.6	1	0.6

We will then evaluate the local connectivity index and the connectivity index of a communication system.

Generally, a communication system can be described by means of a graph consisting of a set of vertices together with edges. To be exact, the vertices represent communication centers, and edges represent communication channels. In a complex system, the communication channel between any two centers may be carried out to be destroyed. The probability distribution can be used to estimate the reliability of communication channel by a large sample number, which is considered as a probability measure. Otherwise, we can only evaluate the reliability by experts, which is regarded as an uncertain measure.

Example 4: Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph with 4 vertices (communication centers) and 6 edges (communication channels), where

$$\mathcal{R} = \{(1,3), (2,4)\},$$
$$\mathcal{T} = (\alpha_{ij})_{4\times 4} = \begin{pmatrix} 0 & 0.9 & 0.7 & 0.5 \\ 0.9 & 0 & 0.8 & 0.2 \\ 0.7 & 0.8 & 0 & 0.8 \\ 0.5 & 0.2 & 0.8 & 0 \end{pmatrix},$$

α_{ij} represent the reliability of communication channel between communication centers i and j.

For a communication system, the information accessibility between communication centers exert great influences on the reliability of the whole system. On the basis of that, we always want to know 1) the chance that two communication centers are connected; and 2) the chance that the communication system is connected. In the corresponding graph model, they correspond to the local connectivity index and connectivity index problems. In accordance with Algorithm 1, we have $\rho_{\mathbb{G}}(1,4) = 0.82$, that is, communication centers 1 and 4 are connected with possibility 82%. Additionally, we have $\rho(\mathbb{G}) = 0.814$, that is, the communication system is connected with possibility 81.4%.

VIII. CONCLUSION

The importance and originality of this study are that it explores the connectivity of two vertices in uncertain random graphs. The main contributions can be summarized as follows. Firstly, this paper explores the definition of local connectivity index to show the chance measure that two special vertices are connected. Subsequently, this paper contributes to a deeper understanding of a fundamental method for calculating the local connectivity index. Finally, discussing the link between the local connectivity index and the connectivity index helps to understand the connectivity of uncertain random graphs.

It is worth pointing out that related efficient algorithms may be developed to improve the efficiency in the future. Hopefully, some other classical topics, such as regularity and diameter, can be further studied in the future for uncertain random graphs.

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