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Estimating Robustness Through Kirchhoff Index in Mesh Graphs

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ABSTRACT The Kirchhoff index is a new measure of network robustness. In this paper, we study the robustness of $n \times m$ mesh graphes (denoted by $M_{n\times m}$) by determining the most important edges and the least important edges. In other words, we aim to find the edges (denoted by $edge_{max}$) which have the biggest impact on the Kirchhoff index after the edge is deleted and the edges (denoted by $edge_{min}$) which have the least impact on Kirchhoff index after the edge is deleted. The distributions of $edge_{max}$ and $edge_{min}$ are fully characterized. Consequently, we propose a new strategy called modified resistance distance strategy to locate $edge_{max}$ and $edge_{min}$ of $M_{n\times m}$. The applicability and rationality of the modified resistance distance strategy in mesh graphs is proved by comparing with other known strategies, such as the semi-random strategy, the degree product strategy and the resistance distance strategy. Moreover, the modified resistance distance strategy is still applicable in mesh graphs when we use the algebraic connectivity as the measure of graph robustness.

INDEX TERMS Kirchhoff index, network robustness, mesh graphs, Laplacian matrix.

I. INTRODUCTION

Abundant entities can be abstracted into graphs, such as traffic network, warehouse storage network, YouTube social network and so on. Thus, increasing attention is paid to graph theory. The measures of graphs is one of the most vital research directions in graph theory. The graph robustness is the ability of a graph to preserve its connectivity after the loss of nodes and edges [19]. Scholars propose graphs robustness measures based on different methods of calculating the robustness of graphs [1]: algebraic connectivity, Kirchhoff index, Wiener index, etc. (see [2]–[13] for different graph measures or graph indices). As a matter of fact, Kirchhoff index equals the sum of resistance distance of all the vertex pairs in the graph. We are able to measure the robustness of graphs through Kirchhoff index: the higher the Kirchhoff index of the graph is, the less stable the graph is; on the contrary, the lower the Kirchhoff index of the graph is, the more stable the graph is. Moreover, the existence of loads of realistic graph necessities the study of measuring the graph robustness. We are capable of measuring the change of graph robustness by observing the increment or decrement of Kirchhoff index. In addition to the Kirchhoff index, algebraic connectivity is also a widely used graph robustness measurement, nevertheless, algebraic connectivity is different from Kirchhoff index: the higher the algebraic connectivity is, the more robustness the graph is; the lower the algebraic connectivity is, the less the robustness the graph is. In other words, algebraic connectivity and graph robustness are positively related, and Kirchhoff index and graph robustness are negatively related. We will apply algebraic connectivity and Kirchhoff index to measure the robustness of networks in this paper as a comparison of different robustness measurements. We focus on selecting the edges have the most or least influence on graphs Kirchhoff index and we will compare those results with the one based on algebraic connectivity in order to verify the rationality and applicability of our strategy. And all the definitions and preliminary knowledge will be presented in Section 2.

The mesh graphes or mesh networks are widely used in researches: integrating or sharing geographically distributed resources in biology, informatics and the management of computer resources. Predecessors also harnessed heuristic algorithm to generate the most intuitively robust and

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connected mesh graphs and compared it with algebraic connectivity, Kirchhoff index and average edge betweenness to demonstrate the rationality of regarding these measurements as the graphs robustness measurements in [10]. And we focus on the mesh graphs in this paper.

Besides, we calculate relative numeric and mark special edges of representative mesh graphs, including the edges maximize the Kirchhoff index of graphs after removal and the edges have least impact on the Kirchhoff index of graphs after removal. We conclude that the distribution of mesh graphs' special edges by observing edges removal from representative mesh graphs and verify the rationality of the distribution on other mesh graphs. Representative mesh graphs are relatively simple, but the distribution of special edges can be sufficiently illustrated through representative mesh graphs. The discussion of these most important edges and the least importance edges of the mesh graphs will be presented in Section 3.

Moreover, we apply the edges removal strategies proposed by Wang *et al.* [4] to select special edges on mesh graphs, observing whether the strategies which are applicable and rational on complex random network are still feasible on mesh graphs. A new edge selecting strategy, called the modified resistance distance strategy is proposed, after comparing the edges selected by the edges removal strategies proposed by scholars and our empirical results. The rationality and applicability of this new edge selecting strategy will be verified in Section 4. We also apply this new strategy to different mesh graphs on the purpose of validating its applicability. Finally, we apply the modified resistance distance strategy to measure different mesh graphs robustness based on different graphs robustness measurements, such as the algebraic connectivity.

A conclusion addressing also further research directions form the arguments of the last section.

II. PRELIMINARY KNOWLEDGE

All the graphs considered in this paper is undirected connected graphs. The vertex set and edge set of a graph G are denoted as $V(G)$ and $E(G)$, respectively..

Definition 1: The Laplacian matrix L of a graph G with n vertices is defined as

$$L = D - A,$$

where D is the diagonal degree matrix and A is the adjacency matrix of G, respectively.

Definition 2: The resistance distance between vertices v_i and v_j in G, denoted by $r_{ij}(G)$, is the effective resistance between vertices v_i and v_j of the electrical network for which each edge of G is replaced by a resistor of unit resistance.

The resistance distance was first introduced by Klein and Randić [21]. Actually, the resistance distance can be computed by using the Laplacian matrix of G.

Theorem 1 [22]: Let i and j are two arbitrary vertices of graph G. Then

$$r_{ij} = \frac{detL(i,j)}{detL(i)}. \quad (1)$$

where $L(i,j)$ is the submatrix after removing the i^{th} column, the i^{th} row and the j^{th} column, the j^{th} row from the Laplacian matrix of G and $L(i)$ is the submatrix after removing the i^{th} column and the i^{th} row from the Laplacian matrix of G.

Klein and Randić [21] also defined the Kirchhoff index which is often used as a measure to investigate the robustness of graphs.

Definition 3: The Kirchhoff index

$$Kf(G) = \sum_{1\le i<j\le n} r_{ij}(G)$$

is the sum of resistance distances between all pairs of vertices.

Since Kirchhoff index was proposed, there are plenty of works focused on calculating the Kirchhoff index value ([14]–[19]) and minimizing or maximizing the Kirchhoff index in some special graphs ([25]–[27]). Kirchhoff index can also be computed by the eigenvalues of the Laplacian matrix of G.

Theorem 2 [23], [24]: For any connected graph G with order $n (n \ge 2)$,

$$Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\lambda_i}.$$

Here, $\lambda_i (1 \le i \le n-1)$ are all the non-zero Laplacian eigenvalues of G.

There is another useful graph measure which is called the algebra connectivity.

Definition 4: Let $\{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ be the eigenvalue set of the Laplacian matrix of a connected graph G with n vertices. In addition, the arrangement of the eigenvalues satisfies $\lambda_0 = 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1}$. The algebraic connectivity of G is equal to λ_1, which is the second smallest eigenvalue of the Laplacian matrix.

Mesh graphs are graphs whose drawing, embedded in some Euclidean space.

Definition 5: Let $Z_n = \{0, 1, 2 \ldots n-1\}$. An $n \times m$ mesh graph, denoted $M_{n\times m}$, is a graph of vertex set $Z_n \times Z_m$. And two arbitrary vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if either $i_1 = i_2$ and $j_1 = j_2 \pm 1$ or $j_1 = j_2$ and $i_1 = i_2 \pm 1$.

Figure 1 depicts the mesh graph $M_{6\times4}$.

III. THE MOST AND THE LEAST IMPORTANT EDGES IN MESH GRAPHS

Kirchhoff index is hired to measure the robustness of plenty of graph families or networks in recent years. In this section, we propose a traversal algorithm which is applied to search the edges maximize the Kirchhoff index after removing from the graph (denoted by $edge_{max}$) and the edges have the least impact on the Kirchhoff index after removing from the graph (denoted by $edge_{min}$). But we need to mention

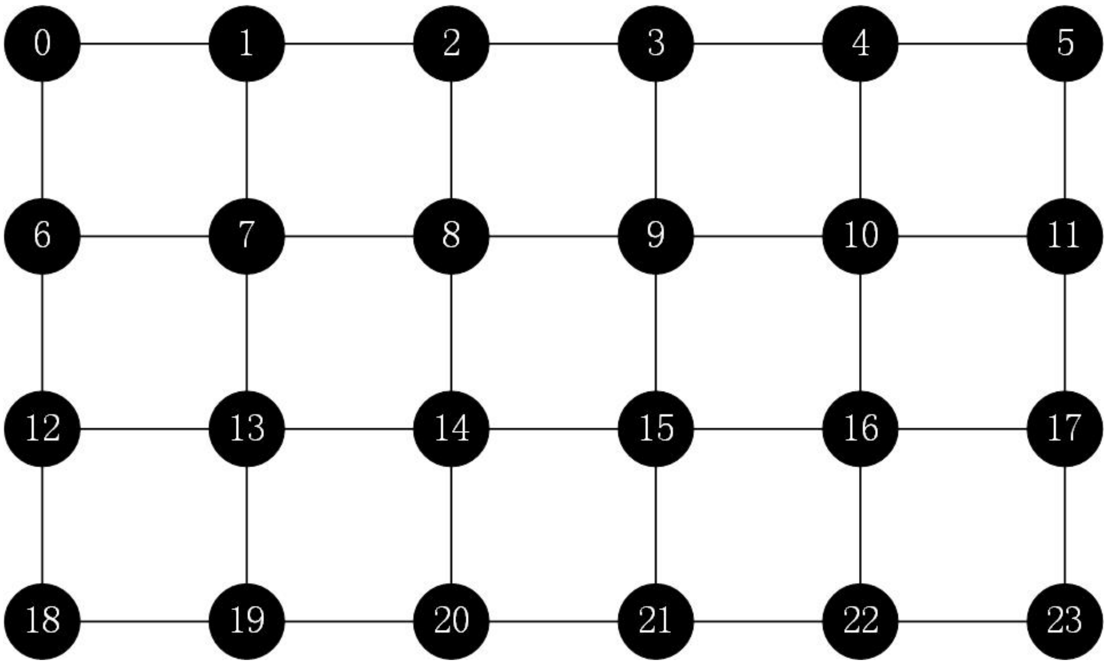


FIGURE 1. $M_{6\times 4}$.

that the traversal algorithm is effective when applied to relatively small graphs, however, it is hardly utilized in quite large graphs because of the exponential increment of time complexity.

A. THE TRAVERSAL ALGORITHM

We propose the following traversal algorithm (see Algorithm 1) to compute the Kirchhoff index of each subgraph of $M_{n\times m}$ when each edge is deleted and determine the $edge_{max}$ and $edge_{min}$. Since $M_{n\times m}$ is symmetric, we only need to traverse about a quarter of the edges.

B. $M_{n\times n}$

The first non-trivial mesh graph $M_{n\times n}$, when n is odd, is $M_{5\times 5}$. Therefore, $M_{5\times 5}$ is established as in Figure 2 and we calculate that $Kf(M_{5\times 5}) = 338.030$. Then the traversal algorithm is applied in $M_{5\times 5}$ and the variations of Kirchhoff index after edges are removed are recorded. As in Figure 2, the $edge_{max}$ are marked the red color and the $edge_{min}$ are marked blue. In addition, the changes of Kirchhoff index with removing edges are also depicted in Table 1, where $Kf(M_{5\times 5} - e)$ stands for the resulting Kirchhoff index after edges are removed and r_{ij} represents the resistance distance between specific two vertices according to the specific removed edges. Due to the symmetry of the mesh graph, we only present partial cases which represent all.

$M_{6\times 6}$ is established as in Figure 3 and we calculate that $Kf(M_{6\times 6}) = 748.435$. Similarly, the traversal algorithm is also utilized in locating $edge_{max}$ and $edge_{min}$ of $M_{6\times 6}$. Besides, $edge_{max}$ are marked red and $edge_{min}$ are marked blue in Figure 3 and the variation of Kirchhoff index is showed in Table 2.

Through other similar calculations for $M_{n\times n}$, we draw a conclusion: the $edge_{max}$ lie on the center of the boundary of $M_{n\times n}$ and the $edge_{min}$ are adjacent and perpendicular to the $edge_{max}$.

Algorithm 1 The Traversal algorithm

Input :
The mesh graph $M_{n\times m}$;
Output:
$edge_{max}$;
$edge_{min}$;

Select an edge e_0 of $M_{n\times m}$;
Initialize $Kf_{max} = Kf_{min} = Kf(M_{n\times m} - e_0)$;
Initialize $edge_{max} = edge_{min} = \{e_0\}$;
for *each edge* $e \in E(M_{n\times m}) - e_0$ **do**
  compute the $Kf(M_{n\times m} - e)$
  If $Kf(M_{n\times m} - e) < Kf_{min}$
    set $Kf_{min} = Kf(M_{n\times m} - e)$
      $edge_{min} = \{e\}$
  Else If $Kf(M_{n\times m} - e) = Kf_{min}$
      set $edge_{min} = edge_{min} \cup \{e\}$
  If $Kf(M_{n\times m} - e) > Kf_{max}$
    set $Kf_{max} = Kf(M_{n\times m} - e)$
      $edge_{max} = \{e\}$
  Else If $Kf(M_{n\times m} - e) = Kf_{max}$
      set $edge_{max} = edge_{max} \cup \{e\}$
end

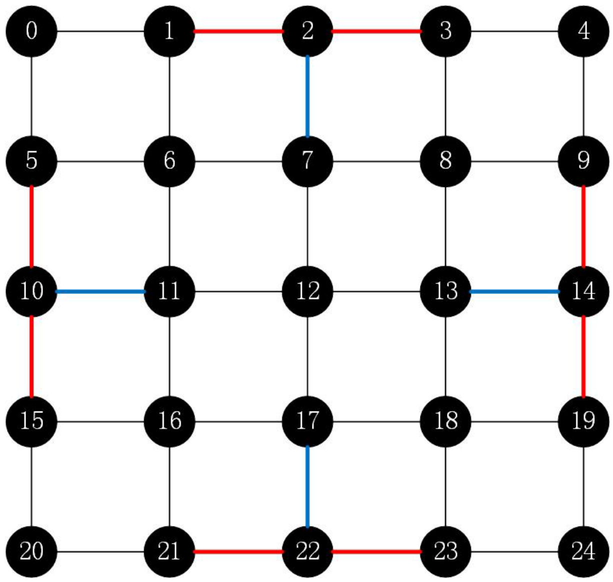


FIGURE 2. The distribution of $edge_{max}$ and $edge_{min}$ of $M_{5\times5}$.

TABLE 1. The changes of Kirchhoff index after removing edges from $M_{5\times5}$.

Removed Edge	$Kf(M_{5\times5}-e)$	r_{ij}
$(0,1)$	367.628	0.699
$(1,2)$	370.618	0.661
$(5,6)$	353.246	0.564
$(6,7)$	356.603	0.537
$(10,11)$	351.866	0.553
$(11,12)$	354.750	0.525

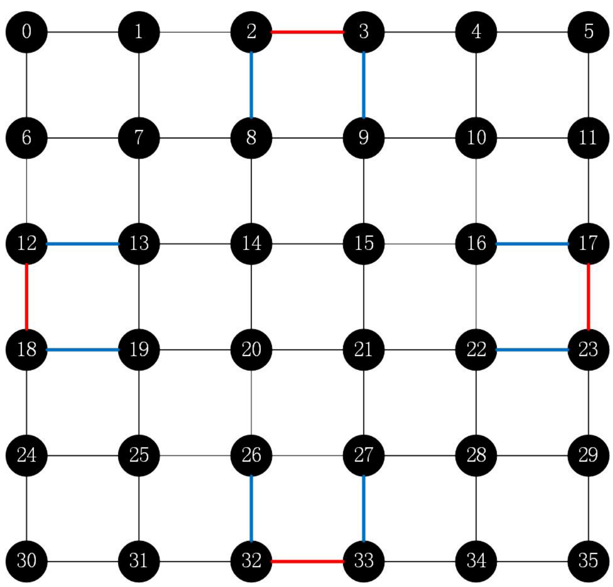


FIGURE 3. The distribution of $edge_{max}$ and $edge_{min}$ of $M_{6\times6}$.

C. $M_{n\times m}$ WHEN $n \neq m$

Now we consider the mesh graphs $M_{n\times m}$ when $n \neq m$. And we always assume that $n > m$.

TABLE 2. The changes of Kirchhoff index after removing edges from $M_{6\times6}$.

Removed Edge	$Kf(M_{6\times6}-e)$	r_{ij}
$(0,1)$	791.212	0.698
$(1,2)$	796.252	0.658
$(2,3)$	798.034	0.652
$(6,7)$	770.393	0.563
$(7,8)$	775.853	0.535
$(8,9)$	777.762	0.532
$(12,13)$	767.954	0.550
$(13,14)$	772.342	0.520
$(14,15)$	774.093	0.517

We take $M_{6\times4}$ as an starting example in order to represent the $M_{n\times m}$ when n is even and $n \geq m$. We calculate that $Kf(M_{6\times4}) = 321.836$. Figure 4 and Table 3 delineate the locations of the $edge_{max}$ and $edge_{min}$ and the changes of Kirchhoff index, respectively.

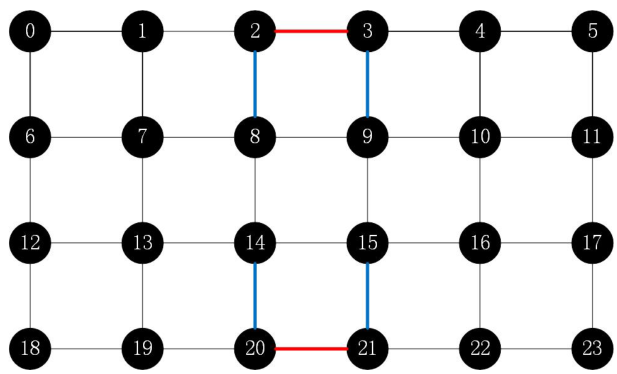


FIGURE 4. The distribution of $edge_{max}$ and $edge_{min}$ of $M_{6\times4}$.

TABLE 3. The changes of Kirchhoff index after removing edges from $M_{6\times4}$.

Removed Edge	$Kf(M_{6\times4}-e)$	r_{ij}
$(0,1)$	351.520	0.700
$(1,2)$	356.657	0.662
$(2,3)$	358.669	0.657
$(6,7)$	337.636	0.567
$(7,8)$	342.985	0.542
$(8,9)$	344.995	0.541
$(0,6)$	349.582	0.700
$(1,7)$	335.792	0.564
$(2,8)$	334.154	0.551
$(6,12)$	351.105	0.668
$(7,13)$	337.540	0.540
$(8,14)$	335.286	0.523

When n is odd, analogous analysis is applied to $M_{5\times3}$ and to calculate the Kirchhoff index of $M_{5\times3}$. We have $Kf(M_{5\times3}) = 117.340$. The relative results of $M_{5\times3}$ as in Figure 5 and Table 4.

Now for this case, by similarly using the traversal algorithm, we conclude that the $edge_{max}$ lie on the center of the longer boundary of the mesh graph and the $edge_{min}$ are adjacent and perpendicular to the $edge_{max}$. Description of the

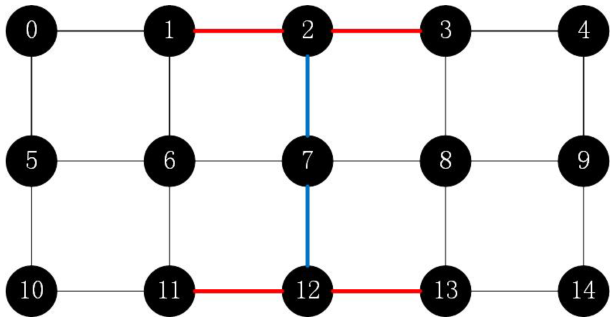


FIGURE 5. The distribution of $edge_{max}$ and $edge_{min}$ of $M_{5\times3}$.

TABLE 4. The changes of Kirchhoff index after removing edges from $M_{5\times3}$.

Removed Edge	$Kf(M_{5\times3}-e)$	r_{ij}
(0, 1)	136.872	0.705
(1, 2)	140.642	0.673
(5, 6)	128.650	0.583
(6, 7)	132.570	0.564
(0, 5)	134.261	0.705
(1, 6)	125.652	0.567

properties of $edge_{max}$ and $edge_{min}$ will be further discussed in the next section.

IV. THE MODIFIED RESISTANCE DISTANCE STRATEGY

Wang *et al.* [4] introduced three strategies to locate the $edge_{max}$ in graphs. We will introduce these strategies below and use them to locate the $edge_{max}$ in mesh graphs. Our results show that these three strategies are not effective in mesh graphs. Therefore, based on the results in Section 3, we propose another quite effective strategy which is called modified resistance distance strategy.

A. SEMI-RANDOM STRATEGY

Let the degree of vertex i be the smallest in $M_{n\times m}$, while the other vertex j is randomly chosen in the neighbour of i. Then (i, j) is the edge selected by the semi-random strategy.

B. DEGREE PRODUCT STRATEGY

Let the degree product of two adjacent vertices i and j is the smallest in $M_{n\times m}$. Then (i, j) is the edge selected by the degree product strategy.

C. RESISTANCE DISTANCE STRATEGY

We choose the edge (i, j) such that the resistance distance of r_{ij} is the largest among the resistance distances of all edges.

It is easy to check that all the proposed strategies discussed above have the same results of $edge_{max}$ distributions. We show the selected edges for $M_{6\times6}$ in Figure 6. Nevertheless, the results of $edge_{max}$ distribution is totally different from that the conclusion in Section 3, which implies the inapplicability of proposed complex network edges selecting strategies when they are applied to the mesh graphs.

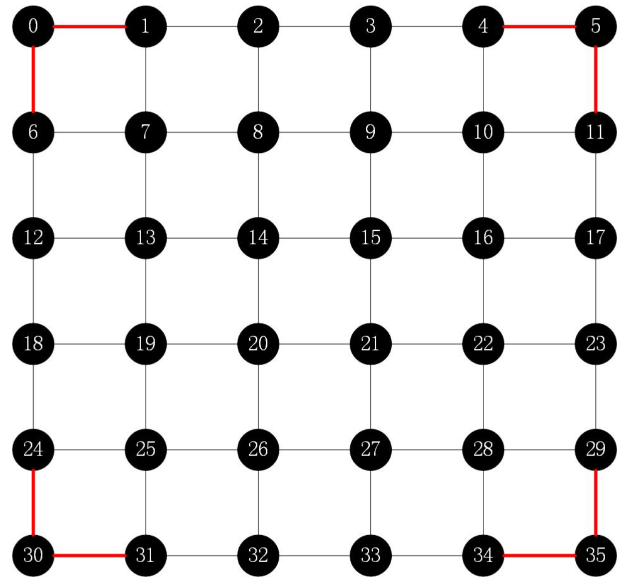


FIGURE 6. The results of proposed $edge_{max}$ selecting strategies of $M_{6\times6}$.

Consequently, based on the results in Section 3, a novel strategy should be presented to locate the $edge_{max}$ of $M_{n\times m}$.

D. MODIFIED RESISTANCE DISTANCE STRATEGY

We rank all the edges of $M_{n,n}$ from the largest resistance distance to the smallest resistance distance and the edges with the same resistance distance have the same ranking. Then we find that the $edge_{max}$ ranks always $\lfloor \frac{n}{2} \rfloor$. We show the rank results for $M_{n\times n}$ in Tabel 5 and the fully rank result for $M_{8\times8}$ in Table 6.

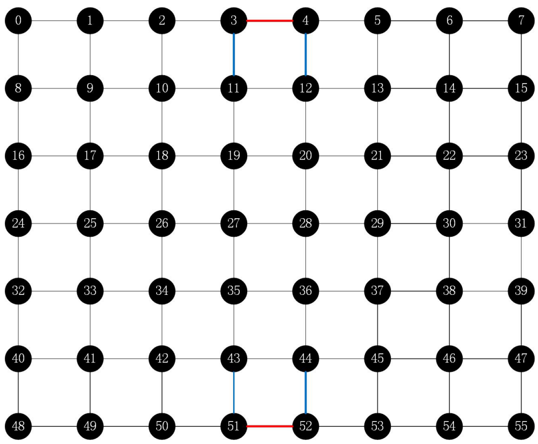
Therefore to locate the $edge_{max}$, we propose the modified resistance distance strategy for $M_{n,n}$, which is to choose the edge of rank $\lfloor \frac{n}{2} \rfloor$ in the ranking of resistance distance of all the edges. Moreover, we choose the edge of rank $n-1$ as the $edge_{min}$ in the ranking of resistance distance of all the edges.

TABLE 5. The resistance distance ranking of $edge_{max}$ and $edge_{min}$ in $M_{n\times n}$.

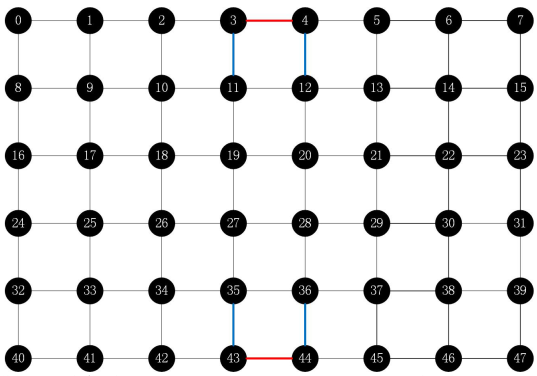
n	$edge_{max}$ ranking	$edge_{min}$ ranking
4	2	3
5	2	4
6	3	5
7	3	6
8	4	7
9	4	8
10	5	9
11	5	10
12	6	11
13	6	12

In addition, the modified resistance distance strategy is also applicable for $M_{n\times m}$ when $n > m$. The result shows the $edge_{max}$ and $edge_{min}$ only depend on n. We show the result for $M_{8\times m}$ in Figure 7.

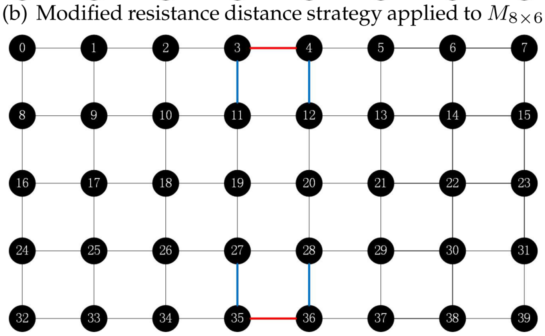
We have confirmed that the modified resistance distance strategy is applicable for $M_{n\times m}$ when the robustness measure



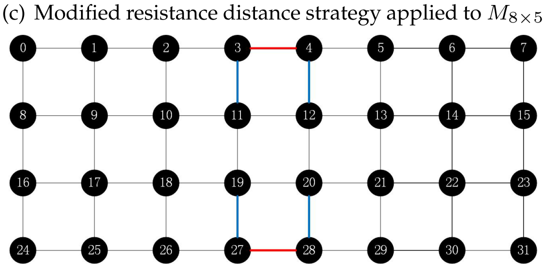
(a) Modified resistance distance strategy applied to $M_{8\times7}$



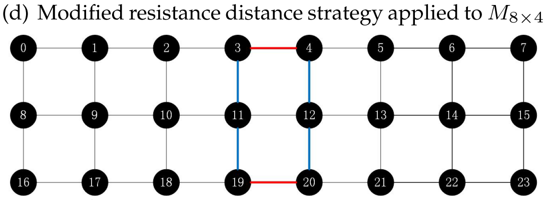
(b) Modified resistance distance strategy applied to $M_{8\times6}$



(c) Modified resistance distance strategy applied to $M_{8\times5}$



(d) Modified resistance distance strategy applied to $M_{8\times4}$



(e) Modified resistance distance strategy applied to $M_{8\times3}$

FIGURE 7. **Modified resistance distance strategy applied to $M_{8\times m}$.**

is Kirchhoff index. Now we change the robustness measure to another well-known graph measure, the algebraic connectivity, on the purpose of verifying the applicability of the modified resistance distance strategy when a different measure is applied. The $edge_{max}$ (resp. $edge_{min}$) is the edge whose removal minimize (resp. maximize) the algebraic connectivity of $M_{n\times m}$. We present the result for $M_{8\times8}$ in Figure 8.

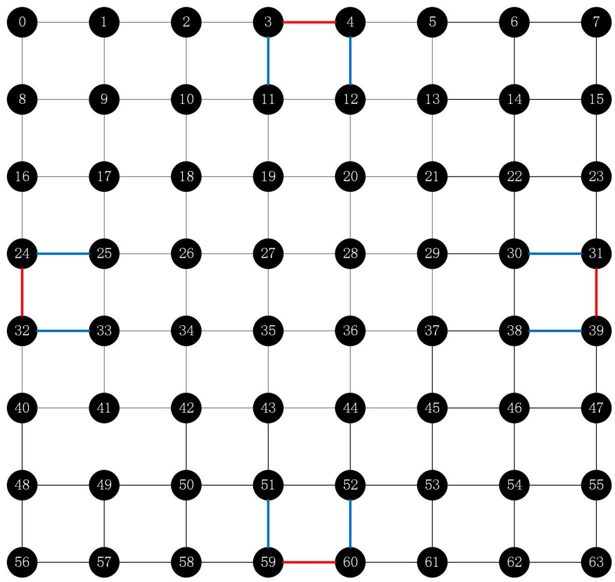


FIGURE 8. **The distribution of $edge_{max}$ and $edge_{min}$ for algebraic connectivity.**

TABLE 6. **The resistance distance ranking of $edge_{max}$ and $edge_{min}$ of $M_{8\times8}$.**

$Edge$	r_{ij}	ranking	$Kf(M_{8\times8}-e)$
(0, 1)	0.698	1	2666.611
(1, 2)	0.656	2	2676.816
(2, 3)	0.647	3	2682.963
(3, 4)	0.645	4	2684.985
(1, 9)	0.563	5	2629.442
(2, 10)	0.550	6	2624.755
(3, 11)	0.548	7	2623.626
(9, 17)	0.533	8	2640.120
(10, 11)	0.528	9	2646.251
(11, 12)	0.527	10	2648.186
(10, 18)	0.518	11	2633.036
(17, 18)	0.514	12	2633.036
(11, 19)	0.513	13	2630.708
(18, 19)	0.513	14	2638.666
(19, 27)	0.510	15	2635.837
(27, 28)	0.509	16	2637.644

We discover that the distribution of $edge_{max}$ and $edge_{min}$ when algebraic connectivity is used as graph robustness measure is the same as the distribution when Kirchhoff index is applied. And the modified resistance distance strategy is also effective to determine the $edge_{max}$ and $edge_{min}$ for the algebraic connectivity.

V. CONCLUSION

We apply the traversal algorithm to the mesh graphs in order to discover the distribution pattern of $edge_{max}$ and

$edge_{min}$ in $M_{n\times m}$. We conclude that the $edge_{max}$ locate on the center of the longer boundary of $M_{n\times m}$, while $edge_{min}$ is adjacent and perpendicular to $edge_{max}$. Besides, $edge_{max}$ and $edge_{min}$ are relevant to their resistance distance ranking: the $edge_{min}$ ranks $n-1(n \geq 4)$, while the $edge_{max}$ ranks $\lfloor \frac{n}{2} \rfloor$. Hence, the modified resistance distance strategy based on the discovery of resistance distance ranking is proposed to seek for the $edge_{max}$ and $edge_{min}$ of $M_{n\times m}$. In addition, the rationality and the applicability of the modified resistance distance strategy in mesh graphs is also verified. The modified resistance distance strategy permits us to locate $edge_{max}$ and $edge_{min}$ of $M_{n\times m}$ with relatively low time complexity and also applicable when the graph robustness measure is changed to the algebraic connectivity.

It is very interesting to test the modified resistance distance strategy in other graphs and compare it with more other graph robustness measures in the future.

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