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Asymptotically Unbiased Estimation of Mean and Standard Deviation in the Presence of Outlying Errors

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ABSTRACT In this work, we develop a new method for estimating the mean and standard deviation of normally distributed populations when errors exist in the sample. The proposed estimators are asymptotically unbiased if all errors are outlying errors (i.e. all errors lie outside a region defined by the parameters of normal distribution). However, the proposed method requires having an upper bound for the percentage of errors in the sample. The proposed method is compared with the already existing and commonly used estimators in terms of bias, mean square error, and Pitman closeness criterion. The proposed method is found to be superior to the others when the sample size is not too small. A simulation study is designed to test the performance of proposed estimators when most but not all errors are outlying errors. The findings of the simulation study also indicate the superiority of the proposed method when the sample size is moderately large. As an application, we use our method in Phase I of designing a control chart to improve its performance. We apply the method on a dataset where the robustness of our proposed method is tested (and compared with the other estimators) against the presence of outlying errors in Phase I data. In the findings of the application, we notice that proposed estimators were the only ones that identified out of control data points in Phase II when Phase I samples are contaminated with the errors.

INDEX TERMS Normal distribution, parameter estimation, robust estimators, control charts, process monitoring.

I. INTRODUCTION

The sample mean ($\bar{x}$) and standard deviation (s) are usually used as estimators for the population mean (μ) and standard deviation (σ), respectively, in various applications of statistical inference. For instance, these estimators are used in hypotheses testing, statistical process control, and outlier detection. However, the sample mean and standard deviation are very sensitive to outliers, thus the presence of errors in the sample can significantly and negatively affect the bias and mean square error of these estimators. Robust estimators are the current tools in modern statistics for estimating the parameters of a distribution in the presence of errors. [1], [2], [3], [4], [5], [6] and [7] presented methods for applying robust estimators in statistical inference and its applications. In particular, [3] developed a modified version of the Z-score method that depends on robust estimators for outlier detection in normally distributed samples. These estimators are the median and the median absolute deviation from the median (or as defined by [8] $MAD = \mathrm{k} \times$ median$(|x_i - \tilde{x}|)$ where $\tilde{x}$ is the sample median, $k = \frac{1}{\Phi^{-1}\left(\frac{3}{4}\right)}$ and Φ^{-1} (.) is the inverse cumulative distribution function of standard normal distribution). Note that median and *MAD* are usually used in robust estimation since they have the highest breakdown point (50%) and the sharpest influence function's bonds. However, they have low Gaussian efficiency (64% for median and 37% for *MAD*) as indicated by [8]. In statistical process control, robust estimators are used in determining the control limits of control charts (i.e. in Phase I). [9] presented a comparison of robust estimators of standard deviation in normal distributions within the context of quality control. References [10] and [11] used the median for designing control charts to reduce the influence of outliers.

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References [12] and [13] used *MAD* for designing control charts when the assumption of normality is violated. Reference [14] used Tukey's outlier detector to reduce the effect of errors in Phase I data on determining the control limits.

The bias of robust estimators is usually negligible when the percentage of errors in the sample is relatively small. However, for a large percentage of errors, robust estimators can be significantly biased. In this research, we propose new asymptotically unbiased estimators for the true mean and standard deviation of normally distributed populations when outlying errors exist in the sample. In addition, we use our estimators in determining the control limits of control charts when errors exist in Phase I data.

We compare our estimators with the already existing and commonly used estimators in terms of bias, mean square error, and Pitman closeness criterion. Pitman closeness criterion, presented by [15], is defined as follows: $\widehat{x_1}$ is a better estimator for x than $\widehat{x_2}$ if $P(|\widehat{x_1} - x| < |\widehat{x_2} - x|) > 0.5$.

The paper is organized as follows:

- Section II explains the notations and provides definitions related to this work.
- Section III presents the main results and is organized as follows. In subsection III-A, we derive upper and lower bounds for the true mean and standard deviation of normal distribution in the presence of errors. The formulas we derive in subsection III-A are essential in the development of most of our later results. In subsection III-B, we develop our procedure that estimates the true mean and standard deviation of normal distribution in the presence of outlying errors. In subsection III-C, we illustrate our method by a numerical example where we apply the procedure on a simulated sample. In subsection III-D, we compare our estimators to other common estimators by their bias, mean square error, and Pitman closeness criterion explained in [15], [16] and [17].
- In section IV we compare our estimators to other estimators in determining the control limits of control charts before and after contaminating Phase I data by errors.

II. DEFINITIONS AND NOTATIONS

A. TRUE PARAMETERS

When a sample is selected from multiple populations in which one of them is the population of interest, then the true parameter is the value of the parameter computed only from the population of interest. For example, if we are interested in men heights but we have a sample of heights were 95% of the sample is from men while the rest is from women, then the true mean is the mean of men heights (i.e. the mean of the population of interest).

B. OUTLYING ERRORS

An outlying error in a sample is an error that exists outside a region defined by the true parameters of the population's assumed distribution. For example, one can define outlying errors to be the errors that lie outside the region $[\mu-3\sigma, \mu+3\sigma]$ where μ and σ are true parameters of the normal distribution.

C. NOTATIONS

The following table provides explanations of the main notations used in this research.

Notation	Explanation
M	The median.
MAD	The median absolute deviation adjusted for bias ($MAD = k \times \text{median}(\lvert x_i - \tilde{x}\rvert)$ where $\tilde{x}$ is the sample median, $k = \frac{1}{\Phi^{-1}\left(\frac{3}{4}\right)}$ and $\Phi^{-1}(.)$ Is the inverse cumulative distribution function of standard normal distribution).
$MADM$	The median absolute deviation without adjusting for bias ($MADM = \text{median}(\lvert x_i - \tilde{x}\rvert)$ where $\tilde{x}$ is the sample median).
U	The upper bound for the region that defines the outlying errors.
L	The lower bound for the region that defines the outlying errors.
α	The percentage of errors in the data set.

We follow the following general rules in our notations:

- Any statistic computed in the presence of errors will be superscripted by "$*$". For example, M^* denotes the median of the entire sample (including errors).
- We use the subscript "U" to denote the upper bound for a variable or an unknown parameter. For example, α_U denotes an upper bound for the percentage of errors in the data.
- We use the subscript "L" to denote the lower bound for a variable or an unknown parameter. For example, U_L denotes a lower bound for the upper bound of the region that defines the outlying errors.

III. MAIN RESULTS

A. UPPER AND LOWER BOUNDS FOR THE TRUE MEAN AND STANDARD DEVIATION

In this subsection, we derive upper and lower bounds for the true mean and standard deviation of normal distribution in the presence of errors. i.e. we find $\mu_U, \mu_L, \sigma_U, \sigma_L$ such that $\mu_L \leq \mu \leq \mu_U$ and $\sigma_L \leq \sigma \leq \sigma_U$ given that $0 \leq \alpha \leq \alpha_U < 0.5$. These bounds will be used extensively in the following sections of this paper.

Let X be a random variable such that:

$$X \sim (1-\alpha) \times N\left(\mu, \sigma^2\right) + \alpha \times D\left(\bar{\theta}\right)$$

where $N(\mu, \sigma^2)$ is the normal distribution and $D\left(\bar{\theta}\right)$ is an unknown distribution (the distribution of errors) and $0 \leq \alpha \leq \alpha_U < 0.5$.

Deriving μ_L and μ_U for X with known σ and unknown μ: From the definition of the median we obtain:

$$\int_{-\infty}^{M^*} (1-\alpha) \times N\left(\mu, \sigma^2\right) + \alpha \times D\left(\bar{\theta}\right) \mathrm{dx} = 0.5$$

Let $y = \int_{-\infty}^{M^*} D\left(\bar{\theta}\right) dx, 0 \leq y \leq 1$ then:

$$\int_{-\infty}^{M^*} N\left(\mu, \sigma^2\right) \mathrm{dx} = \frac{0.5-\alpha y}{1-\alpha}$$

Using the CDF of the normal distribution, we get:

$$0.5 + 0.5 \times \mathrm{erf}\left(\frac{M^* - \mu}{\sqrt{2}\sigma}\right) = \frac{0.5-\alpha y}{1-\alpha}$$

$$\mu = M^* - \sqrt{2}\sigma \times \mathrm{erf}^{-1}\left(\frac{\alpha - 2\alpha y}{1-\alpha}\right) \quad (1)$$

By maximizing and minimizing μ under the constrains $0 \leq y \leq 1$ and $0 \leq \alpha \leq \alpha_U < 0.5$ we get:

$$\mu_U = M^* + \sqrt{2}\sigma \times \mathrm{erf}^{-1}\left(\frac{\alpha_U}{1-\alpha_U}\right) \quad (2)$$

$$\mu_L = M^* - \sqrt{2}\sigma \times \mathrm{erf}^{-1}\left(\frac{\alpha_U}{1-\alpha_U}\right) \quad (3)$$

Deriving σ_L and σ_U for X with unknown σ and μ:

Let the absolute divination from M^*(AD) of the errors follow the unknown distribution $D_2(\theta)$. Then: $AD(X) \sim (1-\alpha) \times \left(N\left(\mu - M^*, \sigma^2\right) + N\left(M^* - \mu, \sigma^2\right)\right) + \alpha \times D_2\left(\bar{\theta}\right)$ Therefore:

$$\int_0^{MADM^*} (1-\alpha) \times \left(N\left(\mu - M^*, \sigma^2\right) + N\left(M^* - \mu, \sigma^2\right)\right) + \alpha \times D_2\left(\bar{\theta}\right) \mathrm{dx} = 0.5$$

Let $w = \int_0^{MADM^*} D_2\left(\bar{\theta}\right) dx, 0 \leq w \leq 1$ then:

$$\int_0^{MADM^*} N\left(\mu - M^*, \sigma^2\right) + N\left(M^* - \mu, \sigma^2\right) \mathrm{dx} = \frac{0.5-\alpha w}{1-\alpha}$$

Using the CDF of the normal distribution, we get:

$$\left(0.5 + 0.5 \times \mathrm{erf}\left(\frac{MADM^* + M^* - \mu}{\sqrt{2}\sigma}\right)\right) + \left(0.5 + 0.5 \times \mathrm{erf}\left(\frac{MADM^* - M^* + \mu}{\sqrt{2}\sigma}\right)\right) - \left(0.5 + 0.5 \times \mathrm{erf}\left(\frac{\mu - M^*}{\sqrt{2}\sigma}\right)\right) - \left(0.5 + 0.5 \times \mathrm{erf}\left(\frac{M^* - \mu}{\sqrt{2}\sigma}\right)\right) = \frac{0.5-\alpha w}{1-\alpha}$$

Which simplifies to:

$$\mathrm{erf}\left(\frac{MADM^* + M^* - \mu}{\sqrt{2}\sigma}\right) + \mathrm{erf}\left(\frac{MADM^* - M^* + \mu}{\sqrt{2}\sigma}\right) = \frac{1-2\alpha w}{1-\alpha}$$

When replacing μ by its value in equation (1) we get:

$$\mathrm{erf}\left(\frac{MADM^*}{\sqrt{2}\sigma} - \mathrm{erf}^{-1}\left(\frac{\alpha - 2\alpha y}{1-\alpha}\right)\right) + \mathrm{erf}\left(\frac{MADM^*}{\sqrt{2}\sigma} + \mathrm{erf}^{-1}\left(\frac{\alpha - 2\alpha y}{1-\alpha}\right)\right) = \frac{1-2\alpha w}{1-\alpha}$$

By using the KKT conditions we can maximize and minimize the function $\mathrm{f}(\sigma, \mathrm{y}, \mathrm{w}, \alpha) = \sigma$ under the constrains $0 \leq \mathrm{y} \leq 1, 0 \leq \mathrm{w} \leq 1, 0 \leq \alpha \leq \alpha_{\mathrm{U}} < 0.5$, and

$$g(\sigma, y, w, \alpha) = \mathrm{erf}\left(\frac{MADM^*}{\sqrt{2}\sigma} - \mathrm{erf}^{-1}\left(\frac{\alpha - 2\alpha y}{1-\alpha}\right)\right) + \mathrm{erf}\left(\frac{MADM^*}{\sqrt{2}\sigma} + \mathrm{erf}^{-1}\left(\frac{\alpha - 2\alpha y}{1-\alpha}\right)\right) - \frac{1-2\alpha w}{1-\alpha} = 0 \quad (4)$$

We find the minimum at $|y - 0.5| = 0.5, w = 0, \alpha = \alpha_U$ and the maximum at $y = 0.5, w = 1, \alpha = \alpha_U$. Therefore, σ_L is the solution of $g(\sigma_L, 0, 0, \alpha_U) = 0$ or $g(\sigma_L, 1, 0, \alpha_U) = 0$ and σ_U is the solution of $g(\sigma_U, 0.5, 1, \alpha_U) = 0$. By solving the equation $g(\sigma_U, 0.5, 1, \alpha_U) = 0$ we get:

$$\sigma_U = \frac{MADM^*}{\sqrt{2}} \div \mathrm{erf}^{-1}\left(\frac{0.5 - \alpha_U}{1-\alpha_U}\right) \quad (5)$$

However, solving the equation $g(\sigma_L, 0, 0, \alpha_U) = 0$ requires using numerical methods. In our analysis we solve the equation $g(\sigma_U, 0, 0, \alpha_U) = 0$ by Newton's method with an initial guess:

$$\sigma_{L_0} = \frac{MADM^*}{\sqrt{2}} \times \frac{\mathrm{erf}^{-1}\left(\frac{0.5-\alpha_U}{1-\alpha_U}\right)}{\left(\mathrm{erf}^{-1}(0.5)\right)^2} \quad (6)$$

Deriving μ_L and μ_U for X with unknown σ and μ:

In equations (2) and (3), if we consider both μ_U and μ_L as functions of σ, maximizing σ will result in an upper bound as well as a lower bound for μ (i.e. μ_U and μ_L). Therefore:

$$\mu_U = M^* + \sqrt{2}\sigma_U \times \mathrm{erf}^{-1}\left(\frac{\alpha_U}{1-\alpha_U}\right) \quad (7)$$

$$\mu_L = M^* - \sqrt{2}\sigma_U \times \mathrm{erf}^{-1}\left(\frac{\alpha_U}{1-\alpha_U}\right) \quad (8)$$

Note: the new values of μ_L and μ_U that we derived are not the maximum and minimum of μ. However, they do satisfy the condition $\mu_L \leq \mu \leq \mu_U$.

B. ESTIMATION PROCEDURE

In this subsection, we develop a procedure that estimates the true mean and standard deviation of normal distribution in the presence of outlying errors. The asymptotic unbiasedness of our estimators requires the following conditions to be satisfied in the given sample:

- $(1-\alpha) \times 100\%$ of the sample follows a normal distribution. Where $0 \leq \alpha \leq \alpha_{\mathrm{U}} < 0.5$ and α_{U} is known.

TABLE 1. Comparison of the bias and mean square error for several estimators of μ using set A of samples.

		α = 0.15		α = 0.1		α = 0.05	
n	Estimator	Bias	MSE	Bias	MSE	Bias	MSE
5000	μ_0	-0.000011	0.000232	-0.000077	0.000221	0.000053	0.000210
	μ_1	1.049993	1.102683	0.699942	0.490118	0.350059	0.122741
	μ_2	0.224560	0.050795	0.139162	0.019715	0.066780	0.004786
	μ_3	-0.000528	0.000307	0.000251	0.000287	0.000091	0.000265
	μ_4	-0.000005	0.000253	-0.000130	0.000233	-0.000107	0.000218
1000	μ_0	0.000107	0.001172	-0.000222	0.001102	0.000086	0.001054
	μ_1	1.050067	1.103637	0.699782	0.490685	0.350059	0.123543
	μ_2	0.222155	0.051210	0.139624	0.021229	0.065592	0.005933
	μ_3	0.000250	0.001570	-0.000078	0.001423	-0.000130	0.001342
	μ_4	0.000369	0.001277	-0.000339	0.001164	0.000058	0.001094
500	μ_0	0.000087	0.002357	-0.000057	0.002217	0.000056	0.002090
	μ_1	1.050006	1.104512	0.699991	0.491985	0.350007	0.124489
	μ_2	0.222839	0.053413	0.139564	0.023016	0.067144	0.007801
	μ_3	0.000713	0.003145	0.000012	0.002863	-0.000076	0.002669
	μ_4	0.000562	0.002582	0.000078	0.002350	-0.000026	0.002173
100	μ_0	-0.000641	0.011848	0.000163	0.011054	0.000108	0.010589
	μ_1	1.049469	1.111460	0.700186	0.500239	0.350149	0.132654
	μ_2	0.221725	0.067705	0.140534	0.037047	0.064051	0.020516
	μ_3	0.000062	0.015799	0.001136	0.014342	0.001089	0.013488
	μ_4	0.001040	0.013115	0.001081	0.011822	0.000168	0.011134

TABLE 2. Comparison of the bias and mean square error for several estimators of σ using set A of samples.

		α = 0.15		α = 0.1		α = 0.05	
n	Estimator	Bias	MSE	Bias	MSE	Bias	MSE
5000	σ_0	-0.000005	0.000116	0.000058	0.000111	-0.000010	0.000104
	σ_1	1.692460	2.864608	1.326281	1.759202	0.824332	0.679693
	σ_2	0.248458	0.062146	0.144571	0.021252	0.064977	0.004524
	σ_3	0.001084	0.000377	0.000877	0.000341	0.001012	0.000307
	σ_4	0.000391	0.000162	0.000639	0.000140	-0.000456	0.000125
1000	σ_0	0.000004	0.000589	-0.000095	0.000555	0.000096	0.000522
	σ_1	1.693712	2.869587	1.327251	1.762508	0.824967	0.681422
	σ_2	0.246542	0.062817	0.145194	0.022799	0.063363	0.005547
	σ_3	-0.000272	0.001903	0.000660	0.001679	-0.000900	0.001554
	σ_4	-0.001468	0.000804	-0.000714	0.000699	-0.000768	0.000629
500	σ_0	-0.000127	0.001181	-0.000053	0.001114	-0.000187	0.001062
	σ_1	1.695266	2.875792	1.328732	1.767351	0.825602	0.683325
	σ_2	0.245595	0.064430	0.143285	0.024020	0.063663	0.007089
	σ_3	-0.001112	0.003821	-0.001178	0.003355	-0.000128	0.003086
	σ_4	-0.002568	0.001652	-0.002379	0.001407	-0.002194	0.001263
100	σ_0	-0.000532	0.005952	-0.000166	0.005626	-0.000097	0.005351
	σ_1	1.709114	2.930522	1.339545	1.803668	0.833009	0.702609
	σ_2	0.238870	0.077299	0.136673	0.035824	0.057236	0.018295
	σ_3	-0.004229	0.018977	-0.004452	0.016963	-0.003633	0.015515
	σ_4	-0.014238	0.008642	-0.012389	0.007386	-0.011205	0.006668

- The $\alpha \times 100\%$ of the sample that doesn't follow the distribution of the rest of the sample lies outside the region $[L, U]$ where $\mathrm{L} = \mu + \mathrm{Z_l} \times \sigma$, $\mathrm{U} = \mu + \mathrm{Z_u} \times \sigma$, $\mathrm{Z_u} > \mathrm{Z_l}$ and $\mathrm{Z_l}$, $\mathrm{Z_u}$ are known.
- The size of the sample is not too small. The acceptable size required depends on the values of α_{U}, $\mathrm{Z_l}$, $\mathrm{Z_u}$. The simulation provided in the Appendix presents the amount of bias for various sizes under different values of α_{U}.

TABLE 3. Comparison of the bias and mean square error for several estimators of μ using set B of samples.

		α = 0.15		α = 0.1		α = 0.05	
n	Estimator	Bias	MSE	Bias	MSE	Bias	MSE
5000	μ_0	0.000004	0.000235	-0.000089	0.000220	-0.000004	0.000209
	μ_1	0.599994	0.360192	0.399915	0.160130	0.199992	0.040195
	μ_2	0.223699	0.050419	0.138545	0.019537	0.066620	0.004768
	μ_3	0.036161	0.001659	0.026220	0.000996	0.013111	0.000450
	μ_4	0.052197	0.003064	0.039354	0.001836	0.020862	0.000679
1000	μ_0	0.000010	0.001162	-0.000125	0.001108	0.000132	0.001048
	μ_1	0.600006	0.360999	0.399828	0.160861	0.200134	0.041050
	μ_2	0.222532	0.051368	0.139467	0.021217	0.065566	0.005940
	μ_3	0.036857	0.003106	0.026643	0.002279	0.013626	0.001590
	μ_4	0.052566	0.004449	0.040479	0.003101	0.021351	0.001677
500	μ_0	0.000207	0.002345	0.000105	0.002222	0.000010	0.002106
	μ_1	0.600044	0.362045	0.400089	0.162073	0.200028	0.042012
	μ_2	0.222881	0.053421	0.138947	0.022777	0.065441	0.007542
	μ_3	0.037863	0.004938	0.026364	0.003802	0.013595	0.003022
	μ_4	0.053912	0.006332	0.040711	0.004589	0.020707	0.002883
100	μ_0	-0.000345	0.011764	-0.000029	0.011104	0.000888	0.010408
	μ_1	0.599734	0.369699	0.399826	0.169863	0.200707	0.050176
	μ_2	0.222167	0.067859	0.139793	0.036752	0.067463	0.020860
	μ_3	0.041031	0.019642	0.030033	0.016691	0.015511	0.014200
	μ_4	0.059385	0.021080	0.045093	0.016894	0.023939	0.012842

TABLE 4. Comparison of the bias and mean square error for several estimators of σ using set B of samples.

		α = 0.15		α = 0.1		α = 0.05	
n	Estimator	Bias	MSE	Bias	MSE	Bias	MSE
5000	σ_0	-0.000017	0.000117	-0.000008	0.000110	-0.000040	0.000105
	σ_1	0.743686	0.553236	0.562170	0.316194	0.326678	0.106862
	σ_2	0.247190	0.061513	0.143928	0.021058	0.065483	0.004591
	σ_3	0.030288	0.001399	0.019805	0.000792	0.010551	0.000451
	σ_4	0.072021	0.005560	0.050511	0.002817	0.025774	0.000844
1000	σ_0	0.000000	0.000587	-0.000042	0.000553	0.000168	0.000530
	σ_1	0.744354	0.554893	0.562544	0.317255	0.327143	0.107741
	σ_2	0.245310	0.062223	0.144365	0.022578	0.065436	0.005818
	σ_3	0.029244	0.003298	0.020617	0.002456	0.011541	0.001856
	σ_4	0.071102	0.006952	0.050412	0.003884	0.025058	0.001534
500	σ_0	-0.000076	0.001181	-0.000043	0.001113	0.000113	0.001051
	σ_1	0.744865	0.556501	0.563211	0.318797	0.327505	0.108688
	σ_2	0.245782	0.064504	0.142110	0.023643	0.063557	0.007094
	σ_3	0.031514	0.005909	0.019821	0.004406	0.009806	0.003556
	σ_4	0.072086	0.009032	0.048830	0.005061	0.024544	0.002422
100	σ_0	-0.000050	0.005947	-0.000275	0.005637	-0.000171	0.005317
	σ_1	0.751739	0.573662	0.567756	0.330420	0.329616	0.115949
	σ_2	0.238897	0.077285	0.137209	0.035934	0.057614	0.018374
	σ_3	0.030591	0.026021	0.020450	0.020852	0.008310	0.017138
	σ_4	0.065896	0.024127	0.043811	0.015919	0.017409	0.009723

When a sample satisfies the conditions presented above, we can conclude that no errors exist in the region $[L, U]$. Therefore, any subset of the sample that lies inside the region $[L, U]$ will follow a truncated normal distribution. Hence, after finding any region that lies inside the region $[L, U]$, all that remains is to find asymptotically unbiased estimators for the parameters μ and σ in a truncated normal distribution.

Finding a region that lies inside $[L, U]$:

We find a region $[L_U, U_L]$ such that it lies inside the region $[L, U]$ (i.e. $L_U > L$ and $U_L \leq U$). By definition, $L = \mu + Z_l \times \sigma$, $U = \mu + Z_u \times \sigma$. We can easily find an upper bound for L and a lower bound for U as follows:

$$L_U = \mu_U + Z_l \times \sigma_U, Z_l \geq 0 \tag{9}$$

$$L_U = \mu_U + Z_l \times \sigma_L, Z_l < 0 \tag{10}$$

$$U_L = \mu_L + Z_u \times \sigma_L, Z_u \geq 0 \tag{11}$$

$$U_L = \mu_L + Z_u \times \sigma_U, Z_u < 0 \tag{12}$$

where the values of μ_U, μ_L, σ_U, and σ_L can be found using the equations (4), (5), (6), (7), and (8) in subsection III-A. Note: the equations in subsection III-A require the values of M^* and $MADM^*$ of the population. We use the median of the sample ($\widehat{M^*}$) as an estimator for M^* and the median absolute deviation from the median of the sample ($\widehat{MADM^*}$) as an estimator for $MADM^*$. Both estimators are asymptotically unbiased.

Estimating μ and σ in the truncated normal distribution: [18] derived the maximum likelihood estimators for μ and σ of the truncated normal distribution and proved their consistency. Therefore, the first estimators we propose in this paper are the truncated maximum likelihood estimators (TMLEs) of the normal distribution. However, we will also derive robust estimators for μ and σ of the truncated normal distribution to obtain good estimators when the region $[L, U]$ is not entirely free of errors. Let Y be a random variable that follows a truncated normal distribution with parameters μ, σ, L_U, U_L where μ and σ are unknown and L_U and U_L are known then:

$$Y \sim \frac{2 \times N(\mu, \sigma^2) \times I_{(L_U, U_L)}(y)}{\operatorname{erf}\left(\frac{U_L - \mu}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{L_U - \mu}{\sqrt{2}\sigma}\right)}$$

Let $L_U = \mu + Z_1 \times \sigma$ and $U_L = \mu + Z_2 \times \sigma$ then:

$$Y \sim \frac{2 \times N(\mu, \sigma^2) \times I_{(L_U, U_L)}(y)}{\operatorname{erf}\left(\frac{Z_2}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)}$$

First, we find the first and third quartiles of Y (i.e. Q_1^T and Q_3^T respectively):

From the definition of the first quartile we obtain:

$$\int_{\mu + Z_1 \times \sigma}^{Q_1^T} \frac{2 \times N(\mu, \sigma^2)}{\operatorname{erf}\left(\frac{Z_2}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)} = \frac{1}{4}$$

Using CDF of the normal distribution, we get:

$$\frac{\operatorname{erf}\left(\frac{Q_1^T - \mu}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)}{\operatorname{erf}\left(\frac{Z_2}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)} = \frac{1}{4}$$

By solving for Q_1^T we get:

$$Q_1^T = \mu + \sqrt{2}\sigma \times \operatorname{erf}^{-1}\left(\frac{1}{4}\operatorname{erf}\left(\frac{Z_2}{\sqrt{2}}\right) + \frac{3}{4}\operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)\right) \tag{13}$$

Using the same approach we find Q_3^T:

$$Q_3^T = \mu + \sqrt{2}\sigma \times \operatorname{erf}^{-1}\left(\frac{3}{4}\operatorname{erf}\left(\frac{Z_2}{\sqrt{2}}\right) + \frac{1}{4}\operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)\right) \tag{14}$$

We observe that we can find Z_1 and Z_2 by solving the following system of equations:

$$\begin{aligned}\frac{Q_3^T - Q_1^T}{U_L - L_U} &= \frac{\sqrt{2} \times \operatorname{erf}^{-1}\left(\frac{3}{4}\operatorname{erf}\left(\frac{Z_2}{\sqrt{2}}\right) + \frac{1}{4}\operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)\right)}{Z_2 - Z_1} \\ &\quad - \frac{\sqrt{2} \times \operatorname{erf}^{-1}\left(\frac{1}{4}\operatorname{erf}\left(\frac{Z_2}{\sqrt{2}}\right) + \frac{3}{4}\operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)\right)}{Z_2 - Z_1}\end{aligned} \tag{15a}$$

$$\begin{aligned}\frac{U_L + L_U - Q_3^T - Q_1^T}{U_L - L_U} &= \frac{Z_2 + Z_1}{Z_2 - Z_1} - \frac{\sqrt{2} \times \operatorname{erf}^{-1}\left(\frac{3}{4}\operatorname{erf}\left(\frac{Z_2}{\sqrt{2}}\right) + \frac{1}{4}\operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)\right)}{Z_2 - Z_1} \\ &\quad - \frac{\sqrt{2} \times \operatorname{erf}^{-1}\left(\frac{1}{4}\operatorname{erf}\left(\frac{Z_2}{\sqrt{2}}\right) + \frac{3}{4}\operatorname{erf}\left(\frac{Z_1}{\sqrt{2}}\right)\right)}{Z_2 - Z_1}\end{aligned} \tag{15b}$$

However, since Q_1^T and Q_3^T are unknown, we use the first and third quartiles of the sample ($\widehat{Q_1^T}$ and $\widehat{Q_3^T}$ respectively) as asymptotically unbiased estimators for Q_1^T and Q_3^T. We use Newton's method to solve for $\widehat{Z_1}$ and $\widehat{Z_2}$ with initial guess $Z_{1_0} = Z_l$ and $Z_{2_0} = Z_u$. After solving for $\widehat{Z_1}$ and $\widehat{Z_2}$ we use the definitions of U_L and L_U to solve for $\widehat{\mu}$ and $\widehat{\sigma}$ and we get:

$$\widehat{\sigma} = \frac{U_L - L_U}{\widehat{Z_2} - \widehat{Z_1}}, \widehat{\mu} = L_U - \widehat{Z_1} \times \widehat{\sigma} \tag{16}$$

C. NUMERICAL EXAMPLE

In this subsection, our method is applied to a simulated sample (provided in appendix B), where 80% of the sample (240 values) is randomly generated from a normal distribution with ($\mu = 20$, $\sigma = 2$). The remaining 20% (60 values) is randomly generated from a normal distribution with ($\mu_e = 40, \sigma_e = 4$). Note that the expected percentage of errors in the region $[\mu - 2\sigma, \mu + 2\sigma]$ is almost zero. Therefore, our method should result in good estimates if we use $Z_l = -2$, $Z_u = 2$, and $\alpha_U = 0.2$.

- We compute the median and the median absolute deviation from the median of the sample: $\widehat{M^*} = 20.5640$ and $\widehat{MADM^*} = 1.8836$.
- Using equation (5) we find $\sigma_U = 3.8537$.
- When solving the equation $g(\sigma_U, 0, 0, \alpha_U) = 0$ in (4) by the Newton's method with the initial guess in (6) we find that $\sigma_L = 2.0195$.
- Using equations (7) and (8) we find that $\mu_L = 19.3361$ and $\mu_U = 21.7919$.
- Using $Z_l = -2$, $Z_u = 2$ in the equations (10) and (11) we find that $U_L = 23.3751$ and $L_U = 17.7529$.

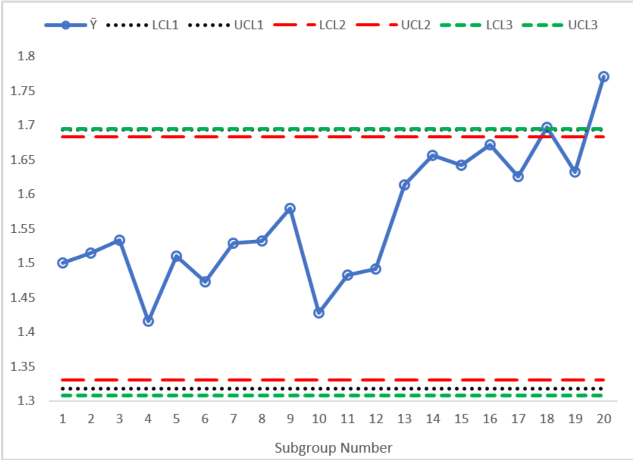

FIGURE 1. Output chart of Phase II data using uncontaminated Phase I data.

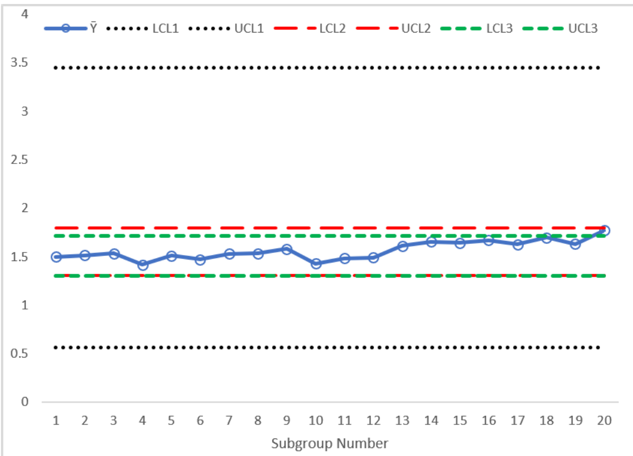

FIGURE 2. Output chart of Phase II data using contaminated Phase I data.

- We exclude the values less than L_U or greater than U_L from the sample. The new truncated sample contains 195 values.
- By applying the maximum likelihood method, we estimate the parameters of the truncated normal distribution using the new truncated sample. We find that $\widehat{\mu_1} = 19.8793$ and $\widehat{\sigma_1} = 2.1411$, which are the first estimators proposed in this paper.
- We compute the first and third quartiles in the new truncated sample: $\widehat{Q_1^T} = 19.1115$ and $\widehat{Q_3^T} = 21.3558$.
- By solving the system of equations in (15) we find $\widehat{Z_1} = -1.0052$ and $\widehat{Z_2} = 1.6123$.
- Using the equations in (16) we find that $\widehat{\mu_2} = 19.9120$ and $\widehat{\sigma_2} = 2.1479$.

D. SIMULATION

In this subsection, we test our estimators and compare them with other common estimators by simulation. We used 2 sets of samples. Set A contains 10^5 simulated samples such that $(1-\alpha)\times$ 100% of the values were generated from a standard normal distribution and the remaining $\alpha\times$ 100% of the values were generated from a normal distribution with $\mu = 7$, $\sigma = 1$. Set B contains 10^5 simulated samples such that $(1-\alpha)\times$ 100% of the values were generated from a standard normal distribution and the remaining $\alpha\times$ 100% of the values were generated from a normal distribution with $\mu = 4$, $\sigma = 1$.

We compare the estimators based on three measures which are: bias, mean square error, and Pitman closeness criterion in Pitman (1937). The ten estimators that we compare are explained in the following table:

Estimator	Explanation
μ_0	The mean of the sample after excluding errors (requires knowing the errors).
μ_1	The mean of the sample (including errors).
μ_2	The median of the sample (including errors).
μ_3	The proposed robust estimator for μ using $\alpha_U = \alpha$, $-Z_l = Z_u = 3$.
μ_4	The proposed truncated maximum likelihood estimator for μ using $\alpha_U = \alpha$, $-Z_l = Z_u = 3$.
σ_0	The mean of the sample after excluding errors divided by c4 (bias adjusting factor).
σ_1	The mean of the sample after (including errors) divided by c4 (bias adjusting factor).
σ_2	$MAD = k \times \text{median}(\lvert x_i - \tilde{x}\rvert)$ where $\tilde{x}$ is the sample median, $k = \frac{1}{\Phi^{-1}\left(\frac{3}{4}\right)}$ and $\Phi^{-1}(.)$ Is the inverse cumulative distribution function of standard normal distribution
σ_3	The proposed robust estimator for μ using $\alpha_U = \alpha$, $-Z_l = Z_u = 3$.
σ_4	The proposed truncated maximum likelihood estimator for σ using $\alpha_U = \alpha$, $-Z_l = Z_u = 3$.

Note: in set A of samples the percentage of errors inside the region $[\mu - 3\sigma, \mu + 3\sigma]$ is almost 0 (i.e. the conditions are satisfied). Therefore, our estimators are expected to have a relatively small bias. However, in the set B of samples, the expected percentage of errors inside the region $[\mu-3\sigma, \mu+3\sigma]$ is not 0 but it is considerably less than α.

In the first four tables of Appendix A, we compare the bias and mean square error for the estimators using two sets of samples. Set A is used for the first two tables while set B is used for the third and fourth tables. From the results we observe the following:

- In both sets (A and B), the absolute mean bias of the proposed four estimators $(\mu_3, \sigma_3, \mu_4, \sigma_4)$ is significantly less than the absolute mean bias of (μ_1, σ_1) and (μ_2, σ_2) for all selected sizes and percentages of errors.
- In both sets (A and B), the *MSE* of the proposed four estimators $(\mu_3,\ \sigma_3,\ \mu_4,\ \sigma_4)$ is less than the *MSE* of

TABLE 5. Breakdown of the probability that our estimator μ_3 is closer to μ than the other estimators (in set A).

n	Estimator	$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$
5000	μ_0	41.32%	41.18%	42.07%
	μ_1	100.00%	100.00%	100.00%
	μ_2	100.00%	100.00%	98.20%
	μ_4	42.73%	42.20%	42.58%
1000	μ_0	40.70%	41.71%	41.68%
	μ_1	100.00%	100.00%	100.00%
	μ_2	99.81%	97.14%	81.53%
	μ_4	42.38%	42.76%	42.46%
500	μ_0	41.17%	41.49%	41.74%
	μ_1	100.00%	100.00%	100.00%
	μ_2	98.08%	90.91%	73.16%
	μ_4	42.71%	42.77%	42.27%
100	μ_0	40.80%	41.62%	41.82%
	μ_1	100.00%	99.96%	95.35%
	μ_2	81.34%	71.04%	59.01%
	μ_4	42.57%	42.91%	42.93%

TABLE 6. Breakdown of the probability that our estimator σ_3 is closer to σ than the other estimators (in set A).

n	Estimator	$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$
5000	σ_0	29.60%	30.19%	30.60%
	σ_1	100.00%	100.00%	100.00%
	σ_2	100.00%	100.00%	97.38%
	σ_4	33.42%	33.01%	32.81%
1000	σ_0	29.67%	30.41%	30.15%
	σ_1	100.00%	100.00%	100.00%
	σ_2	99.85%	96.73%	79.07%
	σ_4	32.93%	33.11%	32.51%
500	σ_0	29.64%	30.20%	30.64%
	σ_1	100.00%	100.00%	100.00%
	σ_2	98.05%	89.74%	71.61%
	σ_4	33.38%	32.82%	32.79%
100	σ_0	29.72%	30.56%	30.98%
	σ_1	100.00%	100.00%	100.00%
	σ_2	80.45%	67.89%	54.90%
	σ_4	34.08%	33.63%	33.45%

TABLE 7. Breakdown of the probability that our estimator μ_3 is closer to μ than the other estimators (in set B).

n	Estimator	$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$
5000	μ_0	13.47%	20.15%	32.58%
	μ_1	100.00%	100.00%	100.00%
	μ_2	100.00%	100.00%	99.29%
	μ_4	96.52%	92.90%	74.25%
1000	μ_0	29.95%	33.84%	38.48%
	μ_1	100.00%	100.00%	99.94%
	μ_2	99.95%	98.65%	83.87%
	μ_4	70.92%	66.07%	52.67%
500	μ_0	33.55%	36.56%	39.53%
	μ_1	100.00%	100.00%	98.84%
	μ_2	99.07%	93.73%	72.94%
	μ_4	61.77%	58.38%	48.45%
100	μ_0	36.79%	38.70%	40.13%
	μ_1	99.83%	97.82%	84.69%
	μ_2	82.81%	70.29%	59.09%
	μ_4	51.38%	49.72%	46.01%

TABLE 8. Breakdown of the probability that our estimator σ_3 is closer to σ than the other estimators (in set B).

n	Estimator	$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$
5000	σ_0	13.51%	19.70%	26.32%
	σ_1	100.00%	100.00%	100.00%
	σ_2	100.00%	100.00%	98.48%
	σ_4	99.52%	97.12%	77.76%
1000	σ_0	23.53%	26.00%	28.81%
	σ_1	100.00%	100.00%	100.00%
	σ_2	99.90%	97.62%	83.45%
	σ_4	79.08%	68.53%	47.05%
500	σ_0	24.89%	27.39%	28.89%
	σ_1	100.00%	100.00%	100.00%
	σ_2	98.64%	91.61%	73.44%
	σ_4	65.04%	55.66%	41.65%
100	σ_0	26.56%	27.91%	29.64%
	σ_1	99.99%	99.92%	96.70%
	σ_2	81.53%	68.95%	53.65%
	σ_4	45.88%	41.78%	36.64%

(μ_1, σ_1) and (μ_2, σ_2) for all selected sizes and percentages of errors.

* In set A, the absolute mean bias of our four estimators $(\mu_3, \sigma_3, \mu_4, \sigma_4)$ is close to the absolute mean bias of (μ_0, σ_0) for all selected sizes and percentages of errors. This indicates that our estimators are almost unbiased even for a sample size as small as 100 and a percentage of errors as high as 15%.
* In set A, the *MSE* of the proposed TMLEs (μ_4, σ_4) is less than the *MSE* of the proposed robust estimators (μ_3, σ_3) for all selected sizes and percentages of errors. This indicates that the TMLEs (μ_4, σ_4) are more appropriate when all the conditions of asymptotic unbiasedness are met.
* In set B, the *MSE* of the proposed TMLEs (μ_4, σ_4) is more than the *MSE* of the proposed robust estimators (μ_3, σ_3) for large and moderate sample sizes and percentages of errors. This indicates that the proposed robust estimators (μ_3, σ_3) are more appropriate when the sample contains non-outlying errors.

We also used the Pitman closeness criterion in [15] to compare our estimator by the others. The remaining tables of Appendix A show the probability that the proposed robust estimators (μ_3, σ_3) are closer to the true parameters than

TABLE 9. A sample of size 300 where 80% of the sample (240 values) were randomly generated from a normal distribution with ($\mu = 20$, $\sigma = 2$). The renaming 20% (60 values) were randomly generated from a normal distribution with ($\mu_e = 40$, $\sigma_e = 4$).

18.0800	21.5940	18.5713	22.3377	17.3819	20.2861	19.5720	20.0587	20.2771	23.7052	19.5498	18.4085
21.2292	22.0552	18.2978	22.3401	18.0933	21.0438	17.5521	20.2078	17.9035	18.0644	18.0265	20.3753
20.0894	19.0214	18.2918	22.1109	19.7149	17.7008	19.5055	18.0722	19.5091	19.5460	21.4000	20.5847
17.4700	20.9885	20.0834	16.4387	18.9290	17.7025	19.3709	19.3446	20.1295	21.4234	20.9372	18.6175
19.2377	20.2139	18.9519	18.9618	21.4472	21.7980	17.8737	16.2649	20.8452	22.4583	19.0206	19.8506
23.7425	20.6672	20.7201	19.5347	18.6912	22.2249	22.8601	18.1208	22.1632	23.0121	20.4139	19.7648
17.1989	16.2941	18.0653	22.7820	21.9606	17.9102	17.9992	17.3376	16.2388	22.0367	17.6296	20.8977
21.0368	21.5768	15.8997	20.0094	19.0337	18.0134	22.0489	19.6538	21.9012	19.2541	23.3334	19.8750
15.6837	22.3024	21.7441	21.3722	15.9579	22.0343	18.6623	18.8501	22.1292	17.2587	16.3707	22.3360
21.3631	19.9652	20.3234	21.7236	23.8299	17.8300	20.0580	17.7180	20.1821	20.3977	19.1967	20.3484
19.7831	18.4279	23.6687	19.5863	17.6121	22.1914	18.8381	21.8977	20.7104	20.2865	18.5520	20.0576
23.8044	20.6208	21.3338	19.9178	16.1985	19.2109	19.9144	22.1101	19.4152	17.7763	18.0142	24.4686
20.6470	20.3256	19.0859	21.2847	22.6767	18.9361	17.2401	20.6356	21.2115	22.6251	16.0795	23.0823
20.8835	22.0031	17.5906	20.6138	17.4243	19.2627	18.8912	21.2034	20.8875	18.7064	18.8114	17.1449
23.1286	20.1607	16.1444	20.6195	18.1010	23.6491	20.5416	21.4672	21.6340	20.4388	22.8869	20.0199
17.4006	19.4096	20.8833	19.9352	17.0072	17.2411	34.9753	40.1817	43.9187	41.2165	38.2287	39.3551
20.7140	17.6214	20.6432	20.6581	19.1110	20.5902	39.0376	38.3066	48.1552	41.4139	41.6122	38.2860
17.3626	18.2589	19.3284	18.0586	16.0169	16.6378	41.4083	34.2338	41.9248	46.5773	41.4851	41.3431
16.4272	17.8322	20.0932	21.6805	21.4006	18.7463	35.4831	43.0510	42.1487	36.4155	43.8706	45.5521
16.8016	20.5974	21.1617	23.0795	20.8275	21.0335	44.8198	37.1739	42.3617	37.4516	45.9078	38.2370
20.0830	20.7464	17.9129	20.5433	19.7223	21.8710	38.1158	38.2060	41.9788	41.7556	42.3742	39.7578
18.4032	21.5268	19.2916	18.3823	19.2509	18.2842	42.7703	45.8991	30.9983	35.2010	41.7229	41.7797
23.0855	18.9157	17.1485	19.7657	20.1820	22.7533	32.8884	37.0836	38.1119	40.9497	43.4016	33.9285
22.3196	20.1954	19.6936	20.3938	20.4397	22.2888	39.9417	39.9373	44.4867	38.3396	38.1017	45.5481
16.4521	21.0460	17.2648	19.1128	20.6414	18.3743	42.8518	38.3097	33.1210	36.0093	41.4217	44.9295

each of the other estimators. From the tables we observe the following:

- In both sets (A and B) the proposed four estimators (μ_3, σ_3, μ_4, σ_4) are better than (μ_1, σ_1) and (μ_2, σ_2) for all selected sizes and percentages of errors.
  - The proposed TMLEs (μ_4, σ_4) are better than the proposed robust estimators (μ_3, σ_3) in set A of samples but worse in set B of samples. This indicates that the proposed robust estimators (μ_3, σ_3) perform better when the sample contains non-outlying errors.

IV. APPLICATION

In this section, we apply our method on a data set from [19]. We aim to develop statistical control charts for the flow width of the resist in a hard-bake process used in semiconductor manufacturing (cf. [20] for more details on the control chart implementation). We compare the performance of three pairs of estimators in identifying out of control subgroups of data before and after the corruption of Phase I data. The three pairs of estimators are (μ_1, σ_1), (μ_2, σ_2), and (μ_3, σ_3) as defined in subsection III-D.

Designing control charts before the corruption of Phase I data:

- We compute the values of all estimators. $\mu_1 = 1.5056$, $\sigma_1 = 0.13995$, $\mu_2 = 1.5064$, $\sigma_2 = 0.13151$, and $\mu_3 = 1.501$, $\sigma_3 = 0.14437$ (using $\alpha_U = 0.2$ and $-Z_l = Z_u = 3$)
- We compute the control limits based on each pair of estimators. $LCL_1 = 1.3178$, $UCL_1 = 1.6934$, $LCL_2 = 1.33$, $UCL_2 = 1.6828$, and $LCL_3 = 1.3073$, $UCL_3 = 1.6947$.
- We plot the means of phase 2 subgroups with the control limits.

Note that all estimators perform equally likely and detect two out of control subgroups (cf. Figure 1).

Designing control charts after corruption of Phase I data: We randomly corrupted 19 data points (15.2%) of Phase I sample by adding the random variable $2\chi_1^2$.

- We compute the values of all estimators. $\mu_1 = 2.0073, \sigma_1 = 1.0768$, $\mu_2 = 1.5519, \sigma_2 = 0.1831$, and $\mu_3 = 1.5094$, $\sigma_3 = 0.1528$ (using $\alpha_U = 0.2$ and $-Z_l = Z_u = 3$)
- We compute the control limits based on each pair of estimators. $LCL_1 = 0.56257$, $UCL_1 = 3.452$, $LCL_2 = 1.3062$, $UCL_2 = 1.7976$, and $LCL_3 = 1.3044$, $UCL_3 = 1.7144$.
- We plot the means of phase 2 subgroups with the control limits.

Note that our estimators perform better than the others and detect one out of control subgroup while the others did not (cf. Figure 2).

V. CONCLUSION

This study proposes new robust estimators for the mean and standard deviation of normally distributed populations when errors exist in the sample. The proposed estimators are asymptotically unbiased if all errors are outlying errors.

A simulation study is used to test and compare the performance of our estimators with the already existing and commonly used robust estimators by three criteria: bias, mean square error, and Pitman closeness criterion. The proposed method is found to be superior to the others for moderate and large sample sizes. The findings of the simulation study also indicate the superiority of the proposed method even when most but not all errors are outlying errors. As an application, our method is used in Phase I of designing a control chart to improve its performance. A real-life dataset is used to test the robustness of our proposed method, in comparison to other estimators, against the presence of outlying errors in Phase I data. In the findings, it is noted that the proposed estimators outperformed the others in terms of identifying out of control data points in Phase II when Phase I samples are contaminated with outliers.

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APPENDIX

A. SIMULATION RESULTS

See Tables 1–8.

B. SIMULATED SAMPLE

See Table 9.

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