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Mean Square Stabilization of Multi-Input Discrete-Time Systems Over Stochastic Multiplicative and Additive White Gaussian Noise Channels

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ABSTRACT This paper studies mean square stabilization for multi-input discrete-time systems over a general fading channel, and the channel is modeled as a cascade of multiplicative noise and white Gaussian additive noise. The main objective is to determine the minimum mean capacity required to enable stabilization. The basic idea of our method is to consider stabilization from the viewpoint of a supply/demand balance. Specifically, for communication resources, each system control input is viewed as the demand side, while the channel is viewed as the supply side, and the supply resource of the channel is characterized by the mean square capacity of each channel. Stabilization of the networked control system requires the balance of supply and demand. Based on whether the channel resources are configurable, two different methods for balancing the supply and the demand are discussed. If the channel resources are configurable, the demand side can be satisfied by adjusting the supply side (channel resources); otherwise, the demand side (a certain transceiver design mechanism) can be adjusted to meet the requirements of the supplier. For both cases, sufficient and necessary conditions for stabilizing discrete-time networked control systems are given.

INDEX TERMS Mean square stabilization, general fading channel, multi-input discrete-time systems.

I. INTRODUCTION

Networked control systems (NCSs) are feedback control systems in which the system and controller communicate through a shared network. These systems are becoming increasingly important and are widely used in mobile sensor networks [1], highway systems [2], multi-agent systems [3], and so forth. Many journals and conferences have focused on this issue; for example, several special issues on NCSs have been published [4] and the network control design and the applications to switched systems [5], [6].

Mean square stabilization under input channel information constraints is a basic problem for NCSs. Many different forms of information constraints are often used in studies, such as data rate constraints [7], [8], quantizations [9], [10], signal-to-noise ratio (SNR) constraints [11], [12], packet loss [13]–[15], quantization and packet loss [16], and delays [17], [18]. Using a logarithmic quantizer to quantize the input signal, the authors in [9] discussed the state feedback mean square stabilization problem for a single-input system, and based on the Lyapunov method, the coarsest quantization density required for quadratic stabilization was obtained, where the requirement can be expressed in terms of the Mahler measure of a plant, i.e., the product of unstable poles. In [13], mean square stabilization for NCSs via state feedback under a multiplicative random input channel was discussed, and a sufficient and necessary condition in terms of topological entropy was provided. These results reveal that

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the Mahler measure or topological entropy of a system plays an important role in system analysis and design and can be regarded as a measure of the open-loop system instability. Reference [19] studied the relationship between observability and optimal control through a data rate constrained channel and topological entropy, and the results further supported the above point.

The stabilization problem for multi-input systems is more complicated than that for single-input systems. It is difficult to obtain the minimum network capacity requirements for networked multi-input systems that can be stabilized by state feedback. Fortunately, [20]–[24] have made good progress on this issue. For example, based on independent parallel channels under SNR constraints, [23], [24] discussed the mean square stabilization problem for discrete-time multi-input multi-output systems and gave a sufficient and necessary condition based on the value of unstable poles and their directions. Recently, many studies have used the idea of channel resource allocation to explore the minimum information requirements for the stabilization of an NCS. With the additional design freedom of channel resource allocation, it has been proven that the minimum total capacity required for stabilizing a multi-input networked system can be characterized by the topological entropy of an open-loop system. Another method is to consider each channel resource as fixed and to treat the network stabilization problem from the perspective of communication theory with the aid of a multi-input and multi-output transceiver design mechanism. Based on a similar idea, a sufficient and necessary condition for the mean square stabilization of continuous-time NCSs under an additive noise channel was given based on the majorization method [25]. Similar methods have been extended to solve the mean square stabilization problem of a continuous-time NCS with multiplicative noise channels and more general channel cases [26], [27].

Inspired by the stabilization of continuous-time NCSs, particularly by the ideas in [25]–[27], this paper discusses mean square stabilization for discrete-time NCSs using a similar approach. In particular, we provide a sufficient condition and a necessary condition for stabilizing discrete-time NCSs when the channel resources cannot be arbitrarily allocated. The main difference between our work and the literature [25]–[27] is that we address the stabilization problem for discrete-time NCSs. Formally, this article is a generalization of [25], [28], [29], but the difficulties in solving the discrete-time case are as follows: 1) Since the open-loop system topological entropy is inconsistent for the definitions of a continuous-time system and a discrete-time system, i.e., the sum of the unstable poles for a continuous-time plant and the logarithm of the product of the modulus of the unstable poles for a discrete-time plant, the proof methods used for the continuous-time system cannot be directly applied to the discrete-time plant, and other methods must be used to address stabilization of NCSs with configurable channel resources. 2) In particular, when discussing the sufficient and necessary conditions for stabilizing the networked system under fixed resources and because the open-loop system topological entropy is inconsistent with the definitions of a continuous-time system and a discrete-time system, a necessary and sufficient condition is given for the continuous-time system. However, this paper gives only a sufficient condition and a necessary condition for the discrete-time system; only in some special cases are the sufficient condition and necessary condition the same. It can be seen that the proof methods used for the continuous system cannot be directly applied in this paper, and other methods are required.

There are three difficulties in solving the mean square stabilization problem of multi-input NCSs under information constraints: 1) how to construct the state feedback gain based on a general channel; 2) how to characterize the information transmission capacity of a memoryless digital channel, where, by using the approach of considering the analog channel in [27], we give the channel capacity description of a digital channel; and 3) how to establish the relationship between the networked system stabilization problem and the channel capacity with both multiplicative and additive noise constraints, where we first convert the stabilization problem with multiplicative channels into the stabilization problem of additive noise channels with SNR constraints. The previous conclusions and methods for dealing with the stabilization problem under additive SNR constraints can be directly applied.

The approach taken in this work is characterized by two features: 1) When the channel resources can be allocated, the stabilization problem is transformed into the channel/controller joint design problem. By joint design, a necessary and sufficient condition for mean square stabilization is obtained; that is, the minimum channel capacity for stabilizing an NCS must be met and is expressed by the open-loop system topological entropy. 2) When channel resources cannot be allocated, the problem is transformed into a coding/controller joint design problem using transceiver/receiver matrices. To achieve mean square stabilization, the coder and controller should be properly designed. We present a necessary condition and a sufficient condition for stabilizing NCSs using majorization theory, and the obtained conclusions indicate that the minimum network requirement for stabilizing an NCS is closely related to the Mahler measure of cyclic decomposition subsystems.

Notably, both the channel/controller joint design and the coder/controller joint design problems can be regarded as a balance of the supply and demand of communication resources. When controlling an NCS, each input requires a certain amount of communication resources. Supply and demand balance can be designed in two different ways. The channel/controller joint design adjusts the supply to meet the demand, while the transceiver/controller joint design takes the opposite approach; that is, the adjustment is carried out to satisfy the supply.

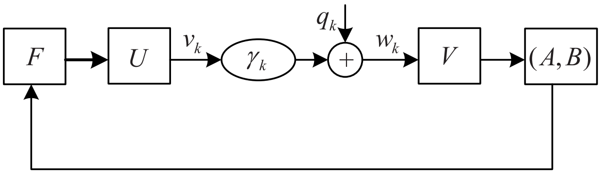

FIGURE 1. Multi-input system over stochastic fading channels.

II. PROBLEM DESCRIPTION

Consider the system shown in Fig. 1, where the discrete-time multi-input linear system is described by

$$x_{k+1} = Ax_k + Bu_k \tag{II.1}$$

where x_k is the state, u_k is the control input and F denotes the state feedback gain. Suppose the system (A, B) is stabilizable.

A. CHANNEL DESCRIPTION

Each communication channel is modeled as the cascade of the transmitter/receiver matrix pair $\{U, V\}$ and a general fading channel with an input-output relationship as follows:

$$w_{ik} = \gamma_{ik} v_{ik} + q_{ik}, \quad i = 1, 2, \cdots, m. \tag{II.2}$$

where γ_{ik} is multiplicative noise modeled as a white Gaussian process with mean μ_i and covariance $\mathrm{E}[(\gamma_{ik}-\mu_i)(\gamma_{jk}-\mu_j)] = \sigma_i^2\delta_{ij}$, and additive noise q_{ik} is a white Gaussian process with zero mean and variance $\mathrm{E}[q_{ik}q_{jk}] = p_i^2\delta_{ij}$. All of the channels together can be described as

$$w_k = \gamma_k v_k + q_k \tag{II.3}$$

where $\gamma_k = \mathrm{diag}\{\gamma_{1k}, \gamma_{2k}, \cdots, \gamma_{mk}\}$ and $q_k = [q_{1k}, q_{2k}, \cdots, q_{mk}]'$. Let $M = \mathrm{diag}\{\mu_1, \mu_2, \cdots, \mu_m\}$, $\Sigma^2 = \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \cdots, \sigma_m^2\}$ and $Q = \mathrm{diag}\{p_1^2, p_2^2, \cdots, p_m^2\}$. For convenience of description, without loss of generality, we assume $\mu_i = 1$, for $i = 1, 2, \cdots, m$.

The channel input signal v_{ik} in Fig. 1 needs to meet the static power constraints;

$$\mathrm{E}\{v_i^2\} < s_i^2$$

that is, the static SNR constraints are actually imposed on the channels, i.e., $\mathrm{E}\{v_i^2\}/p_i^2 < s_i^2/p_i^2$.

B. INFORMATION TRANSMISSION CAPACITY

The next question is how to characterize the information transmission capacity of such a fading channel. Using the idea of describing the analog channel capacity in [27] and the definition of the channel capacity for a single multiplicative or additive digital channel constraint [25], [26], [30], this paper defines the channel signal-to-noise ratio (SNR) as

$$\frac{1}{\mathrm{SNR}_i} = \frac{1}{\mathrm{SNR}_i^+} + \frac{1}{\mathrm{SNR}_i^\times} \tag{II.4}$$

where

$$\mathrm{SNR}_i^+ = \frac{s_i^2}{p_i^2}, \quad \mathrm{SNR}_i^\times = \frac{1}{\sigma_i^2}$$

Defining the channel capacity of the ith channel as

$$\mathcal{C}_i = \frac{1}{2}\log\tilde{\mathcal{C}}_i, \quad \tilde{\mathcal{C}}_i = 1 + \mathrm{SNR}_i$$

the total channel capacity is equal to the sum of the individual channel capacities, i.e.,

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_m \tag{II.5}$$

Clearly, the high SNR constraints and the low randomness of multiplicative noise will increase the channel capacity, thereby enhancing the reliability of information transmission through the channel.

Remark 1: Although the concept of capacity defined in this paper is used to facilitate the problem description and simplification of the results statement, its definition is indeed consistent with our understanding. We note that a general fading channel can be considered a cascade of multiplicative noise and additive noise, and when $\sigma_i = 0$, $\mathrm{SNR}_i = \mathrm{SNR}_i^+$, and the channel capacity is reduced to the additive white Gaussian noise channel capacity [30]; similarly, when $p_i = 0$, $\mathrm{SNR}_i = \mathrm{SNR}_i^\times$, and the channel capacity is converted into the multiplicative random channel capacity [15].

Problem Description: When the network channel is modeled as a general fading channel and the system is a discrete-time system, we need to establish conclusions corresponding to the literature [27]; that is, we seek to determine the requirements of channel capacity $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m$ for stabilizing the NCS shown in Fig. 1, where the communication channel is modeled as a combination of multiplicative noise and additive noise. Specifically, we must consider two situations:

1) Configurable channel resources: the total channel resource $\mathcal{C}$ can be arbitrarily allocated to parallel channels;

2) Fixed channel resources: each channel capacity $\mathcal{C}_i$ is given in advance and cannot be allocated.

The main purpose of this paper is to discuss the relationship between the minimum channel capacity and the mean square stabilization for the discrete-time NCS shown in Fig. 1 for both of these cases.

III. PRELIMINARIES

This section will provide some basic knowledge regarding Wonham decomposition, $\mathcal{H}_2$ complementary sensitivity, cyclic decomposition, and majorization theory.

A. WONHAM DECOMPOSITION

For any stabilizable pair (A, B), where $A \in R^{n\times n}$, and $B \in R^{n\times m}$, based on Wonham decomposition theory [31], [32], a series of similar transformations is used to obtain the Wonham decomposition as follows:

$$\bar{A} = \begin{bmatrix} A_1 & \star & \cdots & \star \\ 0 & A_2 & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} b_1 & \star & \cdots & \star \\ 0 & b_2 & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_m \end{bmatrix} \tag{III.1}$$

where $\star$ represents a matrix element that will not be used in the subsequent derivation, and $\bar{B}$ has no fixed form. $A_i \in R^{n_i \times n_i}$, $b_i \in R^{n_i}$, $\sum_{i=1}^{m} n_i = n$, and each subsystem (A_i, b_i) is stabilizable. In fact, the standard structure (III.1) reveals certain structural properties of each input channel in Fig. 1.

B. CYCLIC DECOMPOSITION

Lemma 2 [33]: For each stabilizable linear system (A, B), where $A \in R^{n\times n}$ and $B \in R^{n\times m}$, there exist nonsingular matrices P and Q such that:

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 & \cdots & 0 \\ 0 & \tilde{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{A}_s \end{bmatrix},$$
$$\tilde{B} = \begin{bmatrix} \tilde{b}_1 & \star & \cdots & \star & \star \\ 0 & \tilde{b}_2 & \cdots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \star \\ 0 & 0 & \cdots & \tilde{b}_s & \star \end{bmatrix} \quad \text{(III.2)}$$

where $\tilde{A}_i(i = 1, 2, \cdots, s)$ is a cyclic subsystem whose minimum polynomial $\alpha_i(\lambda)$ satisfies $\alpha_1(\lambda) = \alpha(\lambda)$ and $\alpha_{i+1}(\lambda)|\alpha_i(\lambda)$ for $i = 1, 2, \cdots, s-1$. $\alpha_{i+1}(\lambda)|\alpha_i(\lambda)$ means that $\alpha_i(\lambda)$ can be divided by $\alpha_{i+1}(\lambda)$. $\tilde{A} = P^{-1}AP, \tilde{B} = P^{-1}BQ$, and $\star$ represents a matrix block that will not be used in the next derivation. A is transformed into a cyclic decomposition form, and the subsystem $(\tilde{A}_i, \tilde{b}_i)$, $(i = 1, 2, \cdots, s)$ is stabilizable.

C. MAJORIZATION

Some useful lemmas are as follows.

Lemma 3 [34]: The majorization inequality $x \preceq^{\omega} y$ ($x \prec^{\omega} y$, respectively) holds if and only if there exists a vector z such that $x \geq z$ ($x > z$, respectively) and $z \preceq y$.

The above lemma characterizes the relationship between majorization and weak majorization.

Lemma 4 [34]: There exists a real symmetric matrix X with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and diagonal elements $d_1, d_2, \cdots, d_n$ if and only if

$$[d_1, d_2, \cdots, d_n]' \preceq [\lambda_1, \lambda_2, \cdots, \lambda_n]'$$

When the conditions in Lemma 4 are met, there are many ways to calculate the expected real matrix X, as described in the literature [35]. More information about majorization theory is provided in [34].

D. OPTIMAL COMPLEMENTARY SENSITIVITY

Considering the NCS shown in Fig. 1, assuming that the channel is temporarily ideal, the complementary sensitivity function (the transfer function from the channel noise q_k to the channel input v_k) is given by:

$$T(z) = F(zI - A - BF)^{-1}B$$

Before proceeding, we now recall the Mahler measure [36] of a matrix $A \in R^{n\times n}$, denoted $M(A)$, which is simply the absolute value of the product of the unstable eigenvalues of matrix A, i.e., $M(A) = \Pi_{i=1}^{n} \max\{1, |\lambda_i(A)|\}$, and the topological entropy [37] of matrix A, denoted $H(A)$, which is the logarithm of $M(A)$, i.e., $H(A) = \log M(A)$. The topological entropy of open-loop plants can be regarded as a measure of the degree of the instability of a linear system.

Lemma 5 [30]: Assuming that (A, B) is stabilizable, the following holds:

$$\inf_{F:A+BF \text{is stable}} \frac{1}{2} \log\det\left\{I + \frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})T(e^{j\omega})^* d\omega\right\} \geq H(A) \quad \text{(III.3)}$$

In particular, when $m = 1$,

$$\inf_{F:A+BF \text{ is stable}} ||T(z)||_2^2 = M(A)^2 - 1 \quad \text{(III.4)}$$

For the single-input system case, i.e., $m = 1$, please refer to the literature [38].

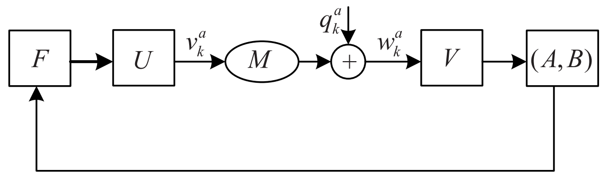

FIGURE 2. Auxiliary system.

IV. EQUIVALENCE RELATION

Next, the NCS shown in Fig. 2 is introduced, in which the channel is modeled as a constant gain M plus additive white Gaussian noise with zero mean and power spectral density $P_{q_k^a} = \Sigma^2 \odot P_{v_k^a} + Q$. By establishing the auxiliary system shown in Fig. 2, we can convert the mean square stabilization problem of the system shown in Fig. 1 into the internal stability problem of the system shown in Fig. 2 under SNR constraints, and then, we only need to consider the auxiliary system, which is easier to handle.

The equivalence of the system in Fig. 1 with the system in Fig. 2 can be given by the following lemma.

Lemma 6 [39]: Considering the two NCSs shown in Figs. 1-2, the following statements are established:

1) The NCS of Fig. 1 is mean square stable if and only if the NCS of Fig. 2 is internally stable and there exists a finite positive and semidetermined $P_{q_k^a} \geq 0$ such that $P_{q_k^a} = \Sigma^2 \odot P_{v_k^a} + Q$ is established, where $P_{v_k^a}$ represents the power spectral density of the steady-state error signal v^a in Fig. 2;

2) If the NCS of Fig. 2 is internally stable and there exists $P_{q_k^a}$ such that $P_{q_k^a} = \Sigma^2 \odot P_{v_k^a} + Q$, then for the same choice of $P_{q_k^a}$, the power spectral density of the steady-state signal v_k^a in Fig. 2 is equal to the value of the steady-state signal v_k in Fig. 1, i.e., $P_{v_k} = P_{v_k^a}$.

This lemma is also the multichannel counterpart of the theorems in the works of Maass and Silva [29] and of Silva and Solis [40], and the corresponding discrete counterpart of Lemma 1 is given in [39]. Thus, the proof is omitted.

V. NETWORKED SYSTEM STABILIZATION WITH CONFIGURABLE CHANNEL RESOURCES

In this section, we assume that the total channel capacity $\mathcal{C}$ is fixed but can be arbitrarily allocated to each channel resource $\mathcal{C}_i$; here, we seek to explore the minimum value of the total channel capacity for stabilizing the NCS shown in Fig. 1.

Notably, the assumption that the channel resources are configurable comes from the actual applications. On the one hand, by providing greater network bandwidth or using high-performance network equipment, we can reduce the network-induced distortion and thus increase the channel capacity; on the other hand, a high channel capacity is inevitably accompanied by high costs; therefore, we limit the total channel capacity and achieve the desired network performance by reasonably allocating resources to each subchannel. Fortunately, the existing resource allocation approaches provide powerful methods and give new design freedom. The networked system stabilization problem discussed here can be transformed into the channel/controller joint design problem, and then, the minimum channel capacity for networked system mean stabilization is solvable.

From the perspective of communication resource supply and demand balance, channel resource allocation can be considered as adjusting the supply to meet different controller input requirements. The following theorem gives a necessary and sufficient condition for the stabilization of NCSs over the SNR constraints when the channel resources are configurable.

Theorem 7: The NCS shown in Fig. 1 over general fading channels with SNR constraints can be mean square stabilized, if and only if $\mathcal{C} > H(A)$.

Proof (Necessity): Assume that the original system (A, B) shown in Fig. 1 is mean square stabilizable. That is, from Lemma 6, the corresponding auxiliary system in Fig. 2 can be internally stable, and its static covariance matrix satisfies $P_{v_k} = P_{v_k^a}$, i.e., the power of the static signal v_k in Fig. 1 is equal to the power of the static signal v_k^a in Fig. 2. Therefore, we only need to discuss the mean square stabilization problem for the auxiliary system.

In the auxiliary system, the complementary sensitivity function (the transfer function from signal q_k^a to signal v_k^a) is expressed as:

$$T(z) = UF(zI - A - BF)^{-1}BV \tag{V.1}$$

Let v_{ik}^a be the ith element of v_k^a, and $\{\cdot\}_{ii}$ represent the ith diagonal element of a matrix; the power spectral density of signal v_{ik}^a can be expressed as $\{T(e^{j\omega})P_{q_k^a}T(e^{j\omega})^*\}_{ii}$, and the power of signal v_{ik}^a is obtained as follows:

$$E[(v_i^a)^2] = \frac{1}{2\pi}\int_0^{2\pi}\{T(e^{j\omega})P_{q_k^a}T(e^{j\omega})^*\}_{ii}d\omega \tag{V.2}$$

Both sides of equation (V.2) are multiplied by $P_{q_k^a}^{-\frac{1}{2}}$, and the SNR of the *ith* channel in (II.4) is obtained as follows:

$$\frac{E[(v_i^a)^2]}{\sigma_i^2 E[(v_i^a)^2] + p_i^2} = \frac{1}{2\pi}\int_0^{2\pi}\{P_{q_k^a}^{-\frac{1}{2}}T(e^{j\omega})P_{q_k^a}T(e^{j\omega})^*P_{q_k^a}^{-\frac{1}{2}}\}_{ii}d\omega$$

Therefore, the channel capacity of the *ith* channel is given by:

$$\mathcal{C}_i = \frac{1}{2}\log\left\{I + \frac{1}{2\pi}\int_0^{2\pi}\{P_{q_k^a}^{-\frac{1}{2}}T(e^{j\omega})P_{q_k^a}T(e^{j\omega})^*P_{q_k^a}^{-\frac{1}{2}}\}d\omega\right\}_{ii}$$

and the total channel capacity is obtained by summing the channel capacities of each channel as follows:

$$\begin{aligned}
\mathcal{C} &= \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_m \\
&= \frac{1}{2}\log\prod_{i=1}^{m}\left\{I + \frac{1}{2\pi}\int_0^{2\pi}\{P_{q_k^a}^{-\frac{1}{2}}T(e^{j\omega})P_{q_k^a}T(e^{j\omega})^*P_{q_k^a}^{-\frac{1}{2}}\}d\omega\right\}_{ii} \\
&\geq \frac{1}{2}\log\det\left\{I + \frac{1}{2\pi}\int_0^{2\pi}\{P_{q_k^a}^{-\frac{1}{2}}T(e^{j\omega})P_{q_k^a}T(e^{j\omega})^*P_{q_k^a}^{-\frac{1}{2}}\}d\omega\right\} \\
&= \frac{1}{2}\log\det\left\{I + \frac{1}{2\pi}\int_0^{2\pi}\{\tilde{T}(e^{j\omega})\tilde{T}(e^{j\omega})^*\}d\omega\right\} \\
&\geq H(A)
\end{aligned}$$

where $\tilde{T}(z) = \tilde{F}(zI - A - \tilde{B}\tilde{F})^{-1}\tilde{B}$, $\tilde{F} = P_{q_k^a}^{-\frac{1}{2}}UF$,and $\tilde{B} = BVP_{q_k^a}^{\frac{1}{2}}$, The first equation is obtained by Hadamard's inequality [41]; i.e., for any $m \times m$ positive definite matrix Q, $\det(W) \leq \Pi_{i=1}^m W_{ii}$ is always true, and the necessary and sufficient condition for the equality is that Q is a diagonal matrix. The second equation is obtained by Lemma 5. In addition, under the SNR constraints, i.e., $E\{v_i^2\} = E\{(v_i^a)^2 < s_i^2\}$,

$$H(A) < \frac{1}{2}\log\prod_{i=1}^{m}\left(1 + \frac{1}{\sigma_i^2 + \frac{p_i^2}{s_i^2}}\right)$$

The necessity proof is completed.

Sufficiency: The sufficiency proof is jointly determined by the channel/controller. Without loss of generality, assuming that (A, B) is already in the Wanham decomposition form (III.1), from Lemma 5, for each stabilizable subsystem (A_i, b_i), the controller gain f_i can be designed such that $\| T_i(z) \|_2^2 = M(A_i)^2 - 1$, where $T_i(z) = f_i(zI - A_i - b_i f_i)^{-1}b_i$. Using these subsystem controller gains to design the entire system controller gain as $F = \mathrm{diag}\{f_1, f_2, \cdots, f_m\}$, the closed system shown in Fig. 2 can be made internally stable.

Since $\mathcal{C} > H(A)$, there exists a positive ϵ such that $\mathcal{C} > H(A) + \epsilon$. Furthermore, let $\mathcal{C}_i > H(A_i) + \epsilon/m$ be used for resource allocation. Choose the scaling matrix as $U = D^{-1}$, where $D = \mathrm{diag}\{1, \eta, \cdots, \eta^{m-1}\}$, and η is a small positive number. Additionally, let $S = \mathrm{diag}\{I_{n_1}, \eta I_{n_2}, \cdots, \eta^{m-1}I_{n_m}\}$. Then

$$T(z) = UF(zI - A - BF)^{-1}BV = \hat{F}(zI - \hat{A} - \hat{B}\hat{F})^{-1}\hat{B}$$

where

$$\hat{F} = D^{-1}FS$$

$$\hat{A} = S^{-1}AS = \begin{bmatrix} A_1 & o(\eta) & \cdots & o(\eta) \\ 0 & A_2 & \cdots & o(\eta) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_m \end{bmatrix},$$

$$\hat{B} = S^{-1}BD = \begin{bmatrix} b_1 & o(\eta) & \cdots & o(\eta) \\ 0 & b_2 & \cdots & o(\eta) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_m \end{bmatrix}$$

When the constant $\eta \to 0$, we have $o(\eta)/\eta \to 0$. Therefore,

$$T(z) = \text{diag}\{T_1(z), T_2(z), \cdots, T_m(z)\} + o(\eta)$$

Since the noise covariance in the auxiliary system is $P_{q_k^a} = \Sigma^2 \odot P_{v_k^a} + Q$, after a series of matrix operations, we have

$$\begin{aligned} &\frac{E[(v_i^a)^2]}{\sigma_i^2 E[(v_i^a)^2] + p_i^2} \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi} \{P_{q_k^a}^{-\frac{1}{2}} T(e^{j\omega}) P_{q_k^a} T(e^{j\omega})^* P_{q_k^a}^{-\frac{1}{2}}\}_{ii} d\omega \\ &\quad = ||T_i(z)||_2^2 + o(\eta) \end{aligned}$$

Since $||T_i(z)||_2^2 = M(A_i)^2 - 1$, when η takes a small enough positive real number, we obatin

$$\frac{E[(v_i^a)^2]}{\sigma_i^2 E[(v_i^a)^2] + p_i^2} < \frac{s_i^2}{\sigma_i^2 s_i^2 + p_i^2}$$

Then

$$E[(v_i^a)^2] < s_i^2, \quad \text{for } i = 1, 2, \cdots, m.$$

By Lemma 6, the system shown in Fig. 1 can be stabilized and $E[(v_i^2] = E[(v_i^a)^2] < s_i^2$; that is, the SNR constraints are satisfied, and the sufficiency is verified.

The above theorem shows that the minimum channel capacity required for networked system mean square stabilization under SNR constraints is given by the open-loop system topological entropy, i.e., $H(A)$. On the other hand, the structural proof of sufficiency indicates how to design the controller gain and resource allocation.

To summarize, the controller and channel joint design procedure is as follows:

Step 1: Decompose the system (A, B) into subsystems (A_i, b_i) with the Wonham form.

Step 2: Design the corresponding controller gain f_i for each subsystem by solving the corresponding H_2 complementary sensitivity problem

$$\inf_{f_i: A_i + b_i f_i \text{is stable}} ||T_i(z)||_2$$

where $T_i(z) = f_i(zI - A_i + b_i f_i)^{-1} b_i$.

Step 3: Take the transmitter/receiver matrices as $V = D$ and $U = D^{-1}$, where $D = \text{diag}\{1, \eta, \cdots, \eta^{m-1}\}$; a small enough η can eliminate the coupling between the subsystems.

Step 4: Allocate the total resource $\mathcal{C}$ to each input channel such that $\mathcal{C}_i > H(A_i)$.

Remark 8: When $\sigma_i = 0$, $\text{SNR}_i = \text{SNR}_i^+$, and the channel capacity is reduced to the additive white Gaussian noise channel capacity, the result is consistent with the conclusion in [30]; and when $p_i = 0$, $\text{SNR}_i = \text{SNR}_i^\times$, the channel capacity is converted into the multiplicative random channel capacity, the result is consistent with the conclusion in [15].

VI. NETWORKED SYSTEM STABILIZATION WITH FIXED CHANNEL RESOURCES

The above method allocates the total resources to each parallel channel, adjusting the supply to meet the demand. However, in some cases, we may encounter situations where network devices are allocated in advance and cannot be arbitrarily assigned. In this case, each channel resource $\mathcal{C}_i$ is given in advance but cannot be allocated. This raises the question of whether there are other design methods that can be used to compensate for the lack of resource allocation. The answer is clear, and by designing the transmitter/receiver matrices U and V, the controller design problem is translated into the controller/transceiver joint design problem.

Before giving the main conclusions, we analyze the transmitter design mechanism from the perspective of the supply and demand balance. Because the channel capacity of each channel is fixed in advance, the supplier cannot be operated. With the appropriate linear transmitter matrix U, each channel transmits a linear combination of all input signals, which can be used to effectively adjust the channel requirements to meet the supply, and is precisely just the opposite of the above-described method. The corresponding conclusions are given below.

Theorem 9: The NCSs shown in Fig. 1 over general fading channels with SNR constraints can achieve mean square stabilization if

$$\begin{aligned} &[\tilde{\mathcal{C}}_1 - 1,\ \tilde{\mathcal{C}}_2 - 1,\ \cdots,\ \tilde{\mathcal{C}}_m - 1]' \\ &\quad \prec^w \underbrace{[M(\tilde{A}_1)^2 - 1,\ \bigwedge,\ \cdots,\ M(\tilde{A}_s)^2 - 1,\ 0,\ \cdots,\ 0]'}_{m} \end{aligned} \tag{VI.1}$$

where $\bigwedge = M(\tilde{A}_2)^2 - 1$ and only if

$$\begin{aligned} &[\tilde{\mathcal{C}}_1,\ \tilde{\mathcal{C}}_2,\ \cdots,\ \tilde{\mathcal{C}}_m]' \\ &\quad \prec^w \underbrace{[M(\tilde{A}_1)^2,\ M(\tilde{A}_2)^2, \cdots, M(\tilde{A}_s)^2, 0, \cdots,\ 0]'}_{m} \end{aligned} \tag{VI.2}$$

Proof (Necessity): Suppose there exist a state feedback gain F and a pair of transmitter/controller matrices $\{U, V\}$ such that the closed-loop system is mean square stable under SNR constraints. By Lemma 6, we only need to consider the auxiliary system shown in Fig. 2.

For convenience of description, let the relationship of each channel capacity be as follows, that is, in nonincreasing order

$$\tilde{\mathcal{C}}_1 \geq \tilde{\mathcal{C}}_2 \geq \cdots \geq \tilde{\mathcal{C}}_m \tag{VI.3}$$

According to the relationship in Lemma 2 $\alpha_{i+1}(\lambda)|\alpha_i(\lambda)$, it is easy to see that the spectrum of $\tilde{A}_{i+1}$ is always included in the spectrum of $\tilde{A}_i$, so the inequality $M(\tilde{A}_1)^2 \geq M(\tilde{A}_2)^2 \geq \cdots \geq M(\tilde{A}_s)^2 > 0$ holds. The majorization (VI.2) is equivalent to the following inequalities:

$$\sum_{i=j}^{m} \tilde{\mathcal{C}}_i > \sum_{i=j}^{k} M(\tilde{A}_i)^2 \tag{VI.4}$$

for $j = 1, 2, \cdots, s$. By Theorem 7, the total channel resources should be greater than open-loop system topological entropy when the system is stabilized under SNR constraints, i.e.

$$\sum_{i=1}^{m} \mathcal{C}_i > \sum_{i=1}^{k} H(\tilde{A}_i) \tag{VI.5}$$

The inequality (VI.5) can be rewritten in the following logarithmic form

$$\frac{1}{2}\log\prod_{i=1}^{m}\tilde{\mathcal{C}}_i > \log\prod_{i=1}^{k} M(\tilde{A}_i) \tag{VI.6}$$

Multiplying both sides of inequality (VI.6) by 2 and then taking the exponential function, we obtain

$$\prod_{i=1}^{m}\tilde{\mathcal{C}}_i > \prod_{i=1}^{k} M(\tilde{A}_i)^2 \Longrightarrow \sum_{i=1}^{m}\tilde{\mathcal{C}}_i > \sum_{i=1}^{k} M(\tilde{A}_i)^2 \tag{VI.7}$$

where (VI.7) is established from $\tilde{\mathcal{C}}_i, i = 1, 2, \cdots, m$, and $M(\tilde{A}_i)^2, i = 1, 2, \cdots, k$ are all positive real numbers. Therefore, when $j = 1$, inequality (VI.4) holds.

To discuss the case for $j = 2$, we first perform a controllable-uncontrollable decomposition to the first column of BV in the system (A, BV). In other words, there exist a nonsingular matrix $P \in R^{n\times n}$ and a state transition $z_k = \bar{P}^{-1}x_k^a$ such that the dynamic equation of the auxiliary system becomes

$$\begin{bmatrix} z_{1,k+1} \\ z_{2,k+1}\end{bmatrix} = \begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22}\end{bmatrix}\begin{bmatrix} z_{1,k} \\ z_{2,k}\end{bmatrix} + \begin{bmatrix}\tilde{B}_{11} & \tilde{B}_{12} \\ 0 & \tilde{B}_{22}\end{bmatrix}\begin{bmatrix} q_{1k}^a + v_{1k}^a \\ \tilde{q}_{2,k}^a + \tilde{v}_{2,k}^a\end{bmatrix} \tag{VI.8}$$

where

$$\begin{aligned} z(k) &= [z_{1,k}' \;\; z_{2,k}']' \\ \tilde{q}_{2,k}^a &= [q_{2k}^a, q_{3k}^a, \cdots, q_{mk}^a]' \\ \tilde{v}_{2,k}^a &= [v_{2k}^a, v_{3k}^a, \cdots, v_{mk}^a]' \end{aligned}$$

Let $\tilde{F} = UF\bar{P}$, and decompose it into the following block matrix

$$\tilde{F} = \begin{bmatrix}\tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22}\end{bmatrix} \tag{VI.9}$$

where $\tilde{F}_{11} \in R^{1\times n_1}$ and $\tilde{F}_{22} \in R^{(m-1)\times(n-n_1)}$. Next, we consider the subsystem $(\tilde{A}_{22}, \tilde{B}_{22})$ whose dynamic system is expressed as:

$$\begin{aligned} z_{2,k+1} &= \tilde{A}_{22}z_{2,k} + \tilde{B}_{22}\tilde{q}_{2,k}^a + \tilde{B}_{22}\tilde{v}_{2,k}^a \\ \tilde{v}_{2,k}^a &= \tilde{F}_{21}z_{1,k} + \tilde{F}_{22}z_{2,k} \end{aligned}$$

Taking the Laplace transform of the above equations, we have

$$\mathcal{L}(\tilde{v}_{2,k}^a) = [\mathcal{T}_{21}(z) \quad \mathcal{T}_{22}(z)]\begin{bmatrix}\mathcal{L}(z_{1,k}) \\ \mathcal{L}(z_{2,k})\end{bmatrix}$$

where

$$\mathcal{T}_{21}(z) = \tilde{F}_{21} + \tilde{F}_{22}(zI - \tilde{A}_{22} - \tilde{F}_{22}\tilde{F}_{22})^{-1}\tilde{B}_{22}\tilde{F}_{21}$$

$$\mathcal{T}_{22}(z) = \tilde{F}_{22}(zI - \tilde{A}_{22} - \tilde{B}_{22}\tilde{F}_{22})^{-1}\tilde{B}_{22}$$

Set $\tilde{P}_{q_k^a} = \tilde{\Sigma}^2 \odot P_{\tilde{v}_k^a} + \tilde{Q}$, where $P_{\tilde{v}^a}$ is the static covariance matrix of signal $\tilde{v}_{2,k}^a$, $\tilde{\Sigma}^2 = \mathrm{diag}(\sigma_2^2, \sigma_3^2, \cdots, \sigma_m^2)$, and $\tilde{Q} = \mathrm{diag}(p_2^2, p_3^2, \cdots, p_m^2)$. Since $z_{1,k}$ is independent of $\tilde{v}_{2,k}^a$, we obtain

$$E[(\tilde{v}_{i+1,k}^a)^2] \geq \frac{1}{2\pi}\int_0^{2\pi}\{\mathcal{T}_{22}(e^{j\omega})\tilde{P}_{q_k^a}\mathcal{T}_{22}(e^{j\omega})^*\}_{ii}d\omega \tag{VI.10}$$

for $i = 1, 2, \cdots, m-1$. By equation (VI.10), we obtain

$$\begin{aligned} &\frac{E[(v_{i+1}^a)^2]}{\sigma_i^2 E[(v_{i+1}^a)^2] + p_i^2} \\ &\geq \frac{1}{2\pi}\int_0^{2\pi}\{\tilde{P}_{q_k^a}^{-\frac{1}{2}}\mathcal{T}(e^{j\omega})\tilde{P}_{q_k^a}\mathcal{T}(e^{j\omega})^*\tilde{P}_{q_k^a}^{-\frac{1}{2}}\}_{ii}d\omega \end{aligned} \tag{VI.11}$$

Summing both sides of the above equation, we obtain

$$\begin{aligned} &\sum_{i=2}^{m}\mathcal{C}_i \\ &\geq \frac{1}{2}\log\det\left\{I + \frac{1}{2\pi}\int_0^{2\pi}\{\tilde{P}_q^{-\frac{1}{2}}\mathcal{T}(e^{j\omega})\tilde{P}_q\mathcal{T}(e^{j\omega})^*\tilde{P}_q^{-\frac{1}{2}}\}d\omega\right\} \\ &= \frac{1}{2}\log\det\left\{I + \frac{1}{2\pi}\int_0^{2\pi}\{\tilde{\mathcal{T}}(e^{j\omega})\tilde{\mathcal{T}}(e^{j\omega})^*\}d\omega\right\} \\ &\geq H(\tilde{A}_{22}) \end{aligned}$$

By Lemma 6, the static power satisfies $E[(v_i^a)^2] = E[(v_i)^2] < s_i^2$, and therefore,

$$\sum_{i=2}^{m}\mathcal{C}_i > H(\tilde{A}_{22}) \tag{VI.12}$$

By the nature of cyclic decomposition, inequality $H(\tilde{A}_{22}) \geq \sum_{i=2}^{k} H(\tilde{A}_i)$ holds; thus,

$$\sum_{i=2}^{m}\mathcal{C}_i > \sum_{i=2}^{k} H(\tilde{A}_i) \Longrightarrow \sum_{i=2}^{m}\tilde{\mathcal{C}}_i > \sum_{i=2}^{k} M(\tilde{A}_i)^2 \tag{VI.13}$$

By imitating the above process, it is easy to verify that when $j = 3, 4, \cdots, s$, formula (VI.4) is also true. The necessity proof is completed.

Sufficiency: To prove the sufficiency, we need to design a transmitter matrix U, a receiver matrix V and a controller gain matrix F such that the closed-loop system is mean square stable under the SNR constraints.

Without loss of generality, suppose (A, B) is already in a cyclic decomposition form, where each cyclic subsystem $(\tilde{A}_i, \tilde{b}_i)$ is a stabilizable subsystem with dimension n_i. Then, controller gain can be designed for each subsystem $(\tilde{A}_i, \tilde{b}_i)$ as $\tilde{f}_i$ such that $||\tilde{T}_i(z)||_2^2 = M(\tilde{A}_i)^2 - 1$, where

$$\tilde{T}_i(z) = \tilde{f}_i(zI - \tilde{A}_i - \tilde{b}_i\tilde{f}_i)^{-1}\tilde{b}_i$$

Let $\tilde{f} = \mathrm{diag}(\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_s)$, and select the controller gain matrix F as follows

$$\begin{bmatrix}\tilde{I} \\ 0_{(m-s)\times n}\end{bmatrix}$$

Thus, $A + BF$ is stable, and the auxiliary networked system in Fig. 2 is internally stable. In particular, choose the transmitter and receiver matrices as

$$U = P_{q_k^a}^{\frac{1}{2}} W D^{-1}, \quad V = D W' P_{q_k^a}^{-\frac{1}{2}} \tag{VI.14}$$

where $W \in R^{m\times m}$ is a unitary matrix to be designed. Let $D = \text{diag}\{1, \eta, \cdots, \eta^{m-1}\}$, where η is a positive small real number, and define

$$S = \text{diag}\{I_{\tilde{n}_1}, \eta I_{\tilde{n}_2}, \cdots, \eta^{m-1} I_{\tilde{n}_m}\} \tag{VI.15}$$

Then, we have

$$\begin{aligned} T(z) &= UF(zI - A - BFC)^{-1}BV \\ &= P_{q_k^a}^{\frac{1}{2}} W (zI - \tilde{A} - \tilde{B}\tilde{F})^{-1} W' P_{q_k^a}^{-\frac{1}{2}} \end{aligned}$$

where

$$\tilde{F} = D^{-1}FS$$

$$\tilde{A} = S^{-1}AS = \begin{bmatrix} \tilde{A}_1 & 0 & \cdots & 0 \\ 0 & \tilde{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{A}_s \end{bmatrix},$$

$$\tilde{B} = S^{-1}BD = \begin{bmatrix} \tilde{b}_1 & o(\eta) & \cdots & o(\eta) & o(\eta) \\ 0 & \tilde{b}_2 & \cdots & o(\eta) & o(\eta) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{b}_s & o(\eta) \end{bmatrix}$$

When $\eta \to 0$, $\frac{o(\eta)}{\eta}$ tends to zero. Therefore, we have

$$\begin{aligned} T(z) &= o(\eta) \\ &+ P_{q_k^a}^{\frac{1}{2}} W \text{diag}\,\underbrace{[\tilde{T}_1(z), \tilde{T}_2(z), \cdots, \tilde{T}_s(z), 0, \cdots, 0]'}_{m} W' P_{q_k^a}^{-\frac{1}{2}} \end{aligned} \tag{VI.16}$$

Then, we obtain

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \{P_{q_k^a}^{-\frac{1}{2}} T(e^{j\omega}) P_{q_k^a} T(e^{j\omega})^* P_q^{-\frac{1}{2}}\} d\omega \\ &= o(\eta) + W(\text{diag}\,\underbrace{[\tilde{T}_1^2(z), \tilde{T}_2^2(z), \cdots, \tilde{T}_s^2(z), 0, \cdots, 0]'}_{m}) W' \end{aligned}$$

Similar to (11), we obtain

$$\begin{aligned} &\frac{E[(v_i^a)^2]}{\sigma_i^2 E[(v_i^a)^2] + p_i^2} \\ &= o(\eta) \\ &\quad + \{W(\text{diag}\,\underbrace{[\tilde{T}_1^2(z), \tilde{T}_2^2(z), \cdots, \tilde{T}_s^2(z), 0, \cdots, 0]'}_{m}) W'\}_{ii} \\ &= o(\eta) \\ &\quad + \{W(\text{diag}\underbrace{[M(\tilde{A}_1)^2 - 1, \bigwedge, \cdots, M(\tilde{A}_s)^2 - 1, 0, \cdots, 0]'}_{m}) W'\}_{ii} \end{aligned}$$

The second equation is established because $||\tilde{T}_i(z)||_2^2 = M(\tilde{A}_i)^2 - 1$. By (VI.1) and Lemma 3, there exists a vector $[\gamma_1, \gamma_2, \cdots, \gamma_m]'$ such that

$$[\tilde{\mathcal{C}}_1 - 1, \ \tilde{\mathcal{C}}_2 - 1, \ \cdots, \ \tilde{\mathcal{C}}_m - 1]' > [\gamma_1, \gamma_2, \cdots, \gamma_m]' \tag{VI.17}$$

and

$$\begin{aligned} &[\gamma_1, \gamma_2, \cdots, \gamma_m]' \\ &\quad \preceq \underbrace{[M(\tilde{A}_1)^2 - 1, \ \bigwedge, \ \cdots, \ M(\tilde{A}_s)^2 - 1, \ 0, \ \cdots, \ 0]'}_{m} \end{aligned} \tag{VI.18}$$

By Lemma 4, a matrix W can always be constructed such that

$$\begin{aligned} &\{W(\text{diag}\,\underbrace{[\tilde{T}_1^2(z), \ \tilde{T}_2^2(z), \ \cdots, \ \tilde{T}_s^2(z), \ 0, \ \cdots, \ 0]'}_{m}) W'\}_{ii} \\ &\qquad + o(\eta) = \gamma_i \end{aligned} \tag{VI.19}$$

for $i = 1, 2, \cdots, m$. Combining the above equations and inequalities (VI.17)-(VI.20), we have

$$\frac{1}{2}\log\left\{I + \frac{1}{2\pi}\int_0^{2\pi} \{\tilde{P}_{q_k^a}^{-\frac{1}{2}} \mathcal{T}(e^{j\omega}) \tilde{P}_{q_k^a} \mathcal{T}(e^{j\omega})^* \tilde{P}_{q_k^a}^{-\frac{1}{2}}\} d\omega\right\}_{ii} < \mathcal{C}_i \tag{VI.20}$$

When η is a small enough real number, we obtain

$$E[(v_{ik}^a)^2] < s_i^2$$

Finally, by Lemma 6, the networked system shown in Fig. 1 is mean square stable and $E[(v_i)^2] = E[(v_i^a)^2] < s_i^2$ is established, that is, the SNR constraints are satisfied. The sufficiency is verified.

The proof for the sufficiency is based on the concept of construction. Importantly, although resource allocation is no longer applicable in this case, we expect to use the transmitter/receiver matrices as an alternative to provide a new controller design degree of freedom. This is because the transmitter/receiver matrices can reorganize the minimum stabilization resource requirements of different control inputs among the subsystem in order to match the given supply.

In particular, the transmitter/controller joint design procedure is as follows:

Step 5: Decompose the system into the subsystems $(\tilde{A}_i, \tilde{b}_i)$.

Step 6: For each subsystem, solve a corresponding H_2 complementary sensitivity problem to obtain the corresponding subsystem gain $\tilde{f}_i$,

$$\inf_{\tilde{f}_i : \tilde{A}_i + \tilde{b}_i\tilde{f}_i \text{is stable}} ||T_i(z)||_2$$

where $T_i(z) = \tilde{f}_i(zI - \tilde{A}_i + \tilde{b}_i\tilde{f}_i)^{-1}\tilde{b}_i$.

Step 7: Choose a sufficiently small positive real number η to construct the matrix $D = \text{diag}\{1, \eta, \cdots, \eta^{m-1}\}$.

Step 8: Select an appropriate vector $[\gamma_1, \gamma_2, \cdots, \gamma_m]$ to satisfy inequalities (VI.17) and (VI.18).

Step 9: Using the technique described in [35], calculate the unitary matrix W that satisfies equation (VI.19), and then calculate the transmitter/receiver matrices in equation (VI.14).

The conditions in Theorem 9 are proposed in terms of a strictly weak majorization inequality, which is stronger than the condition in Theorem 7. In other words, we need the following additional conditions: 1) the total channel capacity should be larger than the system topological entropy, and 2) the SNR of each subsystem should be less dispersed than the Miller measure of the cyclic subsystem.

Remark 10: When $\sigma_i = 0$, $\text{SNR}_i = \text{SNR}_i^+$, and the channel capacity is reduced to the additive white Gaussian noise channel capacity, the result is consistent with the conclusion in [25]; and when $p_i = 0$, $\text{SNR}_i = \text{SNR}_i^\times$, the channel capacity is converted into the multiplicative random channel capacity, the result is consistent with the conclusion in [26].

VII. AN ILLUSTRATIVE EXAMPLE

In this section, we present a numerical example to illustrate the controller and channel joint design problem. Consider the unstable system (A, B) with

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \qquad \text{(VII.1)}$$

where $B_1 = [1\ 1\ 0]'$ and $B_1 = [0\ 1\ 1]'$. It is clearly observed that (A, B) is stabilizable, but (A, B) cannot be transformed into two stabilizable single-input systems by merely combining two input signals from two channels. That is, for any $b \in \text{span}\{B_1, B_2\}$, (A, b) cannot be stabilized since when $\lambda = 2$, the matrix $[\lambda I - A, b]$ is not full row rank. This means that the system cannot be converted to a stabilizable single-input system by only a linear combination of input signals from two channels. Since (A, B) is already in the form of the Wonham/cyclic decomposition, the corresponding subsystems are given by

$$A = \text{diag}\{A_1, A_2\}, \quad b_1 = [1\ \ 1]', \quad b_2 = 1$$

where $A_1 = \text{diag}\{4, 2\}$ and $A_2 = 2$. The topological entropy of the system is calculated as: $H(A) = H(A_1) + H(A_2) = \log_2(4 \times 2) + \log_2 2 = 3 + 1 = 4$.

Let the total channel capacity be $C = 4+2\times10^{-2} > H(A)$. For a general fading channel, assume that the multiplicative noise mean is $\mu_i = 1$, for $i = 1, 2, \cdots, m$, and the covariance matrix of additive noise is $Q =$ Diag$\{0.01, 0.16\}$.

A. CONFIGURABLE CHANNEL RESOURCE

According to Theorem 7, the necessary and sufficient condition for the mean square stabilizability of the NCS under the SNR constraints is that the total channel capacity satisfies $C > H(A_1)+H(A_2) = 4$. Next, we show how to configure the total channel capacity for each subchannel. Assuming that the total channel capacity is $4+2\times10^{-3}$, we allocate the channel capacity into two subsystems, $C_1 = 3 + 10^{-3} > H(A_1)$ and $C_2 = 1 + 10^{-3} > H(A_2)$. This resource allocation can be performed by adjusting the stationary transfer power s_i^2 and the variance σ_i^2 of the multiplicative noise.

In particular, let

$$\Sigma^2 = \text{diag}\{0.001, 0.12\}, \quad s_1^2 = 6.75, \quad s_2^2 = 4.4$$

for the controller design. We can solve the H_2 optimal complementary sensitivity function $T(z)$ of two subsystems (A_1, b_1) and (A_2, b_2). The optimal controller gains are obtained as $f_1 = [-6.5625 \quad 1.3125]$ and $f_2 = -1.5$. Let $F = \text{diag}\{f_1, f_2\}$.

Furthermore, design the scaling matrix Γ as $\Gamma = \text{diag}\{1, 10^{-3}\}$. At this point, the controller/channel joint design is completed. Using MATLAB, it can be seen that the Frobenius norm of the state covariance matrix tends to a finite constant, namely,

$$\lim_{k\to\infty} ||X(k)||_F = 0.1335$$

and the closed-loop evolution of $||X(k)||_F$ is shown in Fig.3, from Fig.3 we can also see that the Frobenius norm of the state covariance matrix tends to a finite constant. Consequently, the closed-loop system is mean square stable and meets the SNR constraints:

$$\text{E}\{v_1^2\} = 6.5853 < s_1^2 = 6.75, \quad \text{E}\{v_2^2\} = 4.3589 < s_2^2 = 4.4$$

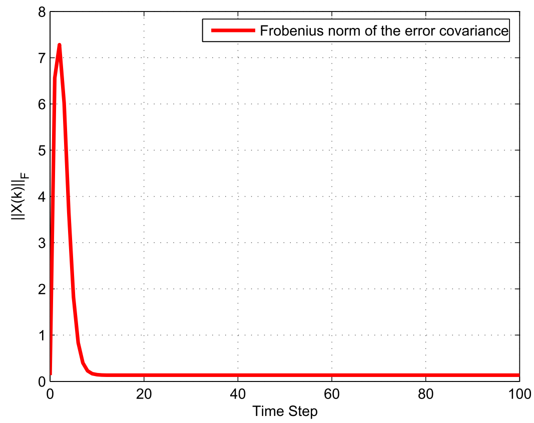

FIGURE 3. Closed-loop evolution of $||X(k)||_F$.

B. FIXED CHANNEL RESOURCES

In contrast to the previous example, the channel resources in this subsection cannot be arbitrarily assigned. The admissible channel capacity is set in advance to $s_1^2 = 8$ and $s_2^2 = 9$, and the corresponding covariance of multiplicative noise is $\Sigma^2 = \text{diag}\{0.01, 0.02\}$. That is, the channel capacities of each subchannel are fixed as $\mathcal{C}_1 = 2.2554$ and $\mathcal{C}_2 = 1.6290$ because, in this case, the cyclic decomposition is the same as the Wonham decomposition form, and we still use the above-described subsystems for analysis and design. The controller is obtained by solving the H_2 optimal problem as in the previous section. In addition, even though $\mathcal{C}_1 < H(A_1)$, the system satisfies the strict weak majorization condition, namely, $[\tilde{\mathcal{C}}_1,\ \tilde{\mathcal{C}}_2]' \prec^\omega [M^2(A_1),\ M^2(A_2)]'$, Therefore, by Theorem 9, the system can still be stabilized by jointly designing the controller/transmitter and choosing $\gamma_1 = 48$ and $\gamma_2 = 20$ such that $[\tilde{\mathcal{C}}_1,\ \tilde{\mathcal{C}}_2]' > [\gamma_1,\ \gamma_2]'$ and $[\gamma_1,\ \gamma_2]' \preceq [M^2(A_1),\ M^2(A_2)]'$. Since $M^2(A_1) < \gamma_1 < \gamma_2 < M^2(A_2)$, matrix W can be constructed as follows:

$$W = \frac{\begin{bmatrix} -\sqrt{\gamma_1 - M^2(A_2)} & \sqrt{M^2(A_1) - \gamma_1} \\ \sqrt{M^2(A_1) - \gamma_1} & \sqrt{\gamma_1 - M^2(A_2)} \end{bmatrix}}{\sqrt{M^2(A_1) - M^2(A_2)}}$$

Thus, we obtain

$$W = \begin{bmatrix} -0.3300 & 0.1160 \\ 0.1022 & 0.3218 \end{bmatrix}$$

Clearly, W is a unitary matrix, and when we choose an appropriate small enough positive real number $\eta = 0.001$, the diagonal elements of the below matrix tend to γ_1 and γ_2

$$\{W(\text{diag}\,\underbrace{[\tilde{T}_1^2(z),\ \tilde{T}_2^2(z),\ \cdots,\ \tilde{T}_s^2(z),\ 0,\ \cdots,\ 0]'}_{m})W'\}_{ii}$$

Furthermore, the transmitter and receiver matrices are given as:

$$U = \begin{bmatrix} -0.5000 & 0.8660 \\ 0.8660 & 0.5000 \end{bmatrix},\quad V = \begin{bmatrix} -0.5000 & 0.8660 \\ 0.8660 & 0.5000 \end{bmatrix}$$

At this point, the design process is completed. Through simulation, the Frobenius norm of the steady-state covariance matrix tends to a finite constant, i.e.,

$$\lim_{k\to\infty} ||X(k)||_F = 0.6166$$

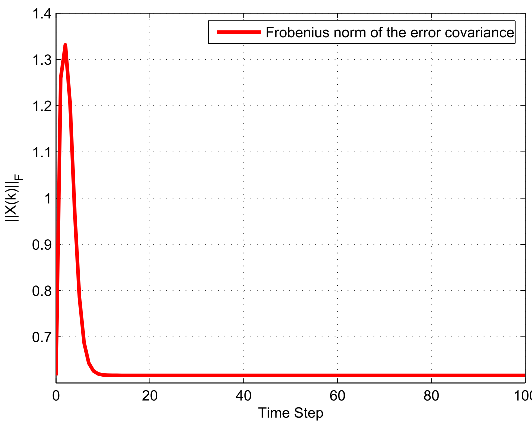

FIGURE 4. Closed-loop evolution of $||X(k)||_F$.

and the closed-loop evolution of $||X(k)||_F$ is shown in Fig.4, from Fig.4 we can also see that the Frobenius norm of the state covariance matrix tends to a finite constant. According to the above results, the closed-loop system is stable. In addition, the SNR constraints are satisfied,

$$E\{v_1^2\} = 7.9514 < s_1^2 = 8,\quad E\{v_2^2\} = 8.9514 < s_2^2 = 9$$

From the above number examples, we can see that the multi-input discrete-time systems over stochastic multiplicative and additive white Gaussian noise channels can be stabilized as long as the conditions in Theorem 5.1 or Theorem 6.1 are satisfied. These results can provide theoretical basis for the controller design of NCSs.

VIII. CONCLUSIONS

This paper discusses the mean square stabilization problem for discrete-time NCSs. A memoryless fading channel is modeled as a cascade of multiplicative noise and additive noise channel. The channel capacity for such a channel is given from the perspective of information theory. Two channel structures are discussed depending on whether the total channel capacity can be allocated. The necessary and sufficient conditions for mean square stabilizability of NCSs under both structures are given by using the channel resources and the majorization conditions, respectively. The conditions are expressed in terms of the open-loop system topological entropy or the Mahler measure. Finally, numerical examples demonstrate the validity of the conclusions.

REFERENCES

[1] P. Ogren, E. Fiorelli, and N. E. Leonard, "Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment," *IEEE Trans. Autom. Control*, vol. 49, no. 8, pp. 1292–1302, Aug. 2004.

[2] P. Seiler and R. Sengupta, "Analysis of communication losses in vehicle control problems," in *Proc. IEEE Amer. Control Conf.*, Arlington, TX, USA, Jun. 2001, pp. 1491–1496.

[3] X. Wang and H. Su, "Consensus of hybrid multi-agent systems by event-triggered/self-triggered strategy," *Appl. Math. Comput.*, vol. 359, pp. 490–501, Oct. 2019.

[4] G. C. Goodwin, E. I. Silva, and D. E. Quevedo, "Analysis and design of networked control systems using the additive noise model methodology," *Asian J. Control*, vol. 12, pp. 443–459, Jul. 2010.

[5] L. Chen, M. Liu, X. Huang, S. Fu, and J. Qiu, "Adaptive fuzzy sliding mode control for network-based nonlinear systems with actuator failures," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 3, pp. 1311–1323, Jun. 2018.

[6] L. Chen, P. Shi, and M. Liu, "Fault reconstruction for Markovian jump systems with iterative adaptive observer," *Automatica*, vol. 105, pp. 254–263, Jul. 2019.

[7] J. Baillieul, "Feedback coding for information-based control: Operating near the data-rate limit," in *Proc. 41rd IEEE Conf. Decis. Control*, Las Vegas, NV, USA, Dec. 2002, pp. 3229–3236.

[8] G. N. Nair and R. J. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, no. 4, pp. 585–593, Apr. 2003.

[9] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Autom. Control*, vol. 46, no. 9, pp. 1384–1400, Sep. 2001.

[10] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1698–1711, Nov. 2005.

[11] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Trans. Autom. Control*, vol. 52, no. 8, pp. 1391–1403, Aug. 2007.

[12] Y. Li, E. Tuncel, J. Chen, and W. Su, "Optimal tracking performance of discrete-time systems over an additive white noise channel," in *Proc. 48th IEEE Conf. Decis. Control (CDC) Held Jointly 28th Chin. Control Conf.*, Dec. 2009, pp. 2070–2075.

[13] N. Elia, "Remote stabilization over fading channels," *Syst. Control Lett.*, vol. 54, no. 3, pp. 237–249, 2005.

[14] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1453–1464, Sep. 2004.

[15] N. Xiao, L. Xie, and L. Qiu, "Feedback stabilization of discrete-time networked systems over fading channels," *IEEE Trans. Autom. Control*, vol. 57, no. 9, pp. 2176–2189, Sep. 2012.

[16] K. Tsumura, H. Ishii, and H. Hoshina, "Tradeoffs between quantization and packet loss in networked control of linear systems," *Automatica*, vol. 45, no. 12, pp. 2963–2970, Dec. 2009.

[17] J. Nilsson, B. Bernhardsson, and B. Wittenmark, "Stochastic analysis and control of real-time systems with random time delays," *Automatica*, vol. 34, no. 1, pp. 57–64, Jan. 1998.

[18] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Control Syst. Mag.*, vol. 21, no. 1, pp. 84–99, Feb. 2001.

[19] A. V. Savkin, "Analysis and synthesis of networked control systems: Topological entropy, observability, robustness and optimal control," *Automatica*, vol. 42, no. 1, pp. 51–62, Jan. 2006.

[20] T. Tanaka, "Semidefinite representation of sequential rate-distortion function for stationary Gauss-Markov processes," in *Proc. IEEE Conf. Control Appl. (CCA)*, Sep. 2015, pp. 1217–1222.

[21] T. Tanaka, K.-K.-K. Kim, P. A. Parrilo, and S. K. Mitter, "Semidefinite programming approach to Gaussian sequential rate-distortion trade-offs," *IEEE Trans. Autom. Control*, vol. 62, no. 4, pp. 1896–1910, Apr. 2017.

[22] F. J. Vargas, J. Chen, and E. I. Silva, "Necessary and sufficient conditions for mean square stabilization over MIMO SNR-constrained channels," in *Proc. IEEE 51st IEEE Conf. Decis. Control (CDC)*, Dec. 2012, pp. 26–31.

[23] F. J. Vargas, E. I. Silva, and J. Chen, "Explicit conditions for stabilization over noisy channels subject to SNR constraints," in *Proc. 9th Asian Control Conf. (ASCC)*, Istanbul, Turkey, Jun. 2013, pp. 1–6.

[24] F. J. Vargas, E. I. Silva, and J. Chen, "Stabilization of two-input two-output systems over SNR-constrained channels," *Automatica*, vol. 49, no. 10, pp. 3133–3140, Oct. 2013.

[25] W. Chen, S. Wang, and L. Qiu, "MIMO control using MIMO communication: A majorization condition for networked stabilizability," in *Proc. Amer. Control Conf. (ACC)*, Jul. 2015, pp. 2205–2210.

[26] W. Chen, S. Wang, and L. Qiu, "A majorization condition for MIMO stabilizability via MIMO transceivers with pure fading subchannels," in *Proc. 34th Chin. Control Conf. (CCC)*, Jul. 2015, pp. 6800–6804.

[27] Q. Liu, W. Chen, Z. Wang, and L. Qiu, "Stabilization of MIMO systems over multiple independent and memoryless fading noisy channels," *IEEE Trans. Autom. Control*, vol. 64, no. 4, pp. 1581–1594, Apr. 2019.

[28] W. Chen and L. Qiu, "Stabilization of networked control systems with multirate sampling," *Automatica*, vol. 49, no. 6, pp. 1528–1537, Jun. 2013.

[29] A. I. Maass and E. I. Silva, "Performance limits in the control of single-input linear time-invariant plants over fading channels," *IET Control Theory Appl.*, vol. 8, no. 14, pp. 1384–1395, Sep. 2014.

[30] L. Qiu, G. Gu, and W. Chen, "Stabilization of networked multi-input systems with channel resource allocation," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 554–568, Mar. 2013.

[31] W. Wonham, "On pole assignment in multi-input controllable linear systems," *IEEE Trans. Autom. Control*, vol. 12, no. 6, pp. 660–665, Dec. 1967.

[32] D. Z. Zheng, *Linear System Theory*. Beijing, China: Tsinghua Univ. Publishing House, 2002.

[33] W. Wonham, *Linear Multivariable Control: A Geometrix Approach*, 3rd ed. New York, NY, USA: Springer-Verlag, 1985.

[34] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications*. New York, NY, USA: Springer-Verlag, 2011.

[35] N. N. Chan and K. H. Li, "Diagonal elements and eigenvalues of a real symmetric matrix," *J. Math. Anal. Appl.*, vol. 91, pp. 562–566, 1983.

[36] K. Mahler, "An application of Jensen's formula to polynomials," *Mathematika*, vol. 7, no. 2, pp. 98–100, Dec. 1960.

[37] R. Bowen, "Entropy for group endomorphisms and homogeneous space," *Trans. Amer. Math. Soc.*, vol. 153, no. 1, pp. 401–414, 1971.

[38] Y. L. Wu, Z. P. Shen, and Y. Liu, "Mean square detectability of multi-output systems over stochastic multiplicative channels," *IET Control Theory Appl.*, vol. 6, no. 6, pp. 796–802, Apr. 2012.

[39] Q. Liu, W. Chen, L. Qiu, and Z. Wang, "Feedback stabilization of networked systems over fading channels with resource allocation," *IFAC-PapersOnLine*, vol. 48, no. 14, pp. 363–367, 2015.

[40] E. I. Silva and S. A. Pulgar, "Control of LTI plants over erasure channels," *Automatica*, vol. 47, no. 8, pp. 1729–1736, Aug. 2011.

[41] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

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