IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received May 26, 2020, accepted June 8, 2020, date of publication June 15, 2020, date of current version June 24, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3002397

Radio Labeling for Strong Product $K_3 \boxtimes P_n$

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ABSTRACT Many variations of graph labeling has been defined in the literature. e.g., graceful, harmonious and radio labeling etc. In information technology and in data sciences, we need secrecy of data, different channel assignment and accuracy of transmission of the data. This make the use of graph terminologies indispensable for the computer programs. In this paper, we will discuss multi-distance radio labeling used for channel assignment problem over a wireless communication. A *radio (multi-distance) labeling* of a graph G is a function h from $V(G)$ to the set of non-negative integers such that $|h(u) - h(v)| \geq diam(G) + 1 - d_G(u, v)$,Where $diam(G)$ and $d_G(u, v)$ are diameter and distance between u and v in graph G respectively. The *span* of a radio labeling h is the maximum integer assigned by h and *radio number* of a graph G is the minimum span taken over all radio labeling of G. In this article, we will find relations for radio number of a strong product $K_3 \boxtimes P_n$, $n \geq 3$.

INDEX TERMS Channel assignment, radio labeling, radio number, strong product, complete graph, path, complete graph.

I. INTRODUCTION AND PRELIMINARIES

Use of graph theory, graph labeling in handling secrecy of data, coding of messages, channel assignment and algorithms development introduce a new chapter in information technology and in data sciences. Many real life problems: radar codes, missile guidance coding, crystallography, signal processing, finding a molecular structure and channel assignment over a wireless network have been solved by the aid of graph theory and graph labeling. Different types of labeling and their results have been formulated and designed: graceful, harmonious, magic, VAT, EAT, cordial, prime, co-prime, radio, H-antimagic and complex addition labeling are some of these [1]–[8].

Channel assignment problems when wireless are used to transfers the information over a distance sometimes leaves an unpleasant experience. This annoyance is given by interferences caused by unconstrained simultaneous transmissions. Two good enough channels can be interfere or damage the communications. This interference can be overcome by assigning a certain number to a channel represented as a node in a graph where the edges between two nodes (two channels) shows the connection between two channels. This makes the necessity of use of graph theory and graph labeling in data sciences and communication networks. Hale in [9] exhibited the idea for radio frequency assignment problems. Chartrand *et al.* [10] introduced the graph labeling with limitations related to the radio labeling.

In graph labeling, each station is a vertex and an edge exists when two stations are closer to each other and connected for transmission of data. A survey on radio labeling (multilevel distance labeling) is given in [11]. The radio numbers for cycle, path, square of path, square of a cycle, generalized prism, etc are calculated in [12]–[16]. In [17], the radio number for generalized Petersen graphs $P(n, 2)$ was computed. For more details about radio labeling and radio numbers for different families of graphs, we refer [18]–[22] to the researcher and for applications in systems administration and in media transmission, one can consult [23].

In this paper, we examine the radio labeling of strong product $K_3 \boxtimes P_{2k+1}$ where K_3 is a complete graph on three vertices, P_{2k+1} a path graph on $2k + 1$ vertices and k is any positive integer

The associate editor coordinating the review of this manuscript and approving it for publication was Mugen Peng.

II. MAIN RESULT

A radio labeling of a graph G is a function

$$h : V(G) \rightarrow \mathbb{Z}^+ \cup \{0\}$$

with

$$|h(u) - h(v)| \geq diam(G) + 1 - d(u, v) \quad (1)$$

holds for any pair of vertices u, v. The span of h is

$$< h >= \max_{u,v \in V(G)} \{h(v) - h(u)\}.$$

The radio number of G is denoted by rn(G) and is characterized as the smallest span of radio labeling of G.

Definition 1: [24] The strong product $G \boxtimes H$ is defined as the union of *cartesian product* and *tensor product* of two graph G and H. In other words, a strong product $G \boxtimes H$ is a graph with

1) the vertex set of the cartesian product $V(G) \times V(H)$, and
2) distinct vertices (u, v) and (u', v') are adjacent in $G \boxtimes H$ iff:
   a) $u = u'$ and $vv' \in E(H)$, or
   b) $uu' \in E(G)$ and $v = v'$, or
   c) $uu' \in E(G)$ and $vv' \in E(H)$.

The *center vertices* are all vertices of minimum *eccentricity*, that is the set of all vertices u where the greatest distance $d(u, v)$ to other vertices v is minimal.

In this paper, we consider the strong product $K_3 \boxtimes P_{2k+1}$ where k is any positive integer, let v_0, v'_0 and v''_0 be the centers. Let $v_{L_1}, v_{L_2}, v_{L_3}, \ldots, v_{L_k}$ be the vertices on the left side and $v_{R_1}, v_{R_2}, v_{R_3}, \ldots, v_{R_k}$ be the vertices on the right side with respect to center v_0, $v'_{L_1}, v'_{L_2}, v'_{L_3}, \ldots, v'_{L_k}$ be the vertices on the left side and $v'_{R_1}, v'_{R_2}, v'_{R_3}, \ldots, v'_{R_k}$ be the vertices on the right side with respect to center v'_0 and $v''_{L_1}, v''_{L2}, v''_{L3}, \ldots, v''_{Lk}$ be the vertices on the left side and $v''_{R_1}, v''_{R_2}, v''_{R_3}, \ldots, v''_{R_k}$ be the vertices on the right side with respect to center v''_0.

The strong product of complete graph K_3 and even path P_{2k} is denoted by $K_3 \boxtimes P_{2k}$, let v_{L_0} and v_{R_0}, v'_{L_0} and v'_{R_0}, v''_{L0} and v''_{R0} be the centers. Let $v_{L_1}, v_{L_2}, v_{L_3}, \ldots, v_{L_{(k-1)}}$ be the vertices on the left side and $v_{R_1}, v_{R_2}, v_{R_3}, \ldots, v_{R_{(k-1)}}$ be the vertices on the right side with respect to centers v_{L_0} and v_{R_0}. Let $v'_{L1}, v'_{L_2}, v'_{L_3}, \ldots, v'_{L_{(k-1)}}$ be the vertices on the left side and $v'_{R_1}, v'_{R_2}, v'_{R_3}, \ldots, v'_{R_{(k-1)}}$ be the vertices on the right side with respect to centers v'_{L_0} and v'_{R_0}. Similarly $v''_{L_1}, v''_{L_2}, v''_{L_3}, \ldots, v''_{L_{(k-1)}}$ be the vertices on the left side and $v''_{R_1}, v''_{R_2}, v''_{R_3}, \ldots, v''_{R_{(k-1)}}$ be the vertices on the right side with respect to centers v''_{L_0} and v''_{R_0}.

For $K_3 \boxtimes P_{2k+1}$, let $V(K_3 \boxtimes P_{2k+1}) = V_L \cup V_R \cup V'_L \cup V'_R \cup V''_L \cup V''_R$.

$$V_L = \{v_0, v_{L_1}, \ldots, v_{L_k}\}, \quad V_R = \{v_0, v_{R_1}, \ldots, v_{R_k}\},$$
$$V'_L = \{v'_0, v'_{L_1}, \ldots, v'_{L_k}\}, \quad V'_R = \{v'_0, v'_{R_1}, \ldots, v'_{R_k}\},$$
$$V''_L = \{v''_0, v''_{L_1}, \ldots, v''_{Lk}\}, \quad V''_R = \{v''_0, v''_{R1}, \ldots, v''_{Rk}\}$$

Let for $K_3 \boxtimes P_{2k}$, $V(K_3 \boxtimes P_{2k}) = V_L \cup V_R \cup V'_L \cup V'_R \cup V''_L \cup V''_R$

$$V_L = \{v_{L_0}, v_{L_1}, \ldots, v_{L_{(k-1)}}\}, V_R = \{v_{R_0}, v_{R_1}, \ldots, v_{R_{(k-1)}}\},$$
$$V'_L = \{v'_{L0}, v'_{L1}, \ldots, v'_{L_{(k-1)}}\}, V'_R = \{v'_{R_0}, v'_{R_1}, \ldots, v'_{R_{(k-1)}}\},$$
$$V''_L = \{v''_{L0}, v''_{L1}, \ldots, v''_{L_{(k-1)}}, V''_R = \{v''_{R0}, v''_{R1}, \ldots, v''_{R_{(k-1)}}\}$$

In $K_3 \boxtimes P_n$ two vertices u and v are on *opposite side* if $u \in V_L$ or V'_L or $''V_L$ and $v \in V_R$ or V'_R or V''_R.

Now, we define the *level function* on $V(K_3 \boxtimes P_n)$ to the set of whole numbers W from a center vertex w by

$$L(u) = \{d(u, v) : w \text{ is a center vertex}\},$$

for any $u \in V(K_3 \boxtimes P_n)$.

In $K_3 \boxtimes P_n$, the maximum level is k if $n = 2k + 1$ and $k - 1$ if $n = 2k$.

Through out our discussion in this paper, the order of $K_3 \boxtimes P_n$ is p.

Observation 1: For strong product $K_3 \boxtimes P_n$, we have the following observation:

$$(1) \quad |V(K_3 \boxtimes P_n)| = \begin{cases} 6k + 3, & \text{if } n = 2k + 1; \\ 6k, & \text{if } n = 2k. \end{cases}$$

$$(2) \quad d(u, v) \leq \begin{cases} L(u) + L(v), & \text{if } n = 2k + 1 \text{ and } u, v \text{ are not center vertices}; \\ 1, & \text{if } n = 2k + 1 \text{ and } u, v \text{ are center vertices}; \\ L(u) + L(v) + 1 & \text{if } n = 2k. \end{cases}$$

(3) *If* $u_i, u_{i+1} \in V(K_3 \boxtimes P_n)$, $1 \leq i \leq p - 1$

are on opposite side and

$$d(u_i, u_{i+1}) = d(u_{i+1}, u_{i+2})$$

or

$$d(u_i, u_{i+1}) = d(u_{i+1}, u_{i+2}) \pm 1$$

then

$$d(u_i, u_{i+2}) = 1.$$

Theorem 1: Let $K_3 \boxtimes P_n$ be a strong product and $k = \lfloor \frac{n}{2} \rfloor$ then

$$\text{rn}(K_3 \boxtimes P_n) \geq \begin{cases} 2k(3k + 2) + 1, & \text{if } n = 2k + 1; \\ 2k(3k - 1) + 1, & \text{if } n = 2k. \end{cases}$$

Moreover, the equality holds if and only if there exists a radio labeling h with ordering $\{u_1, u_2, \ldots, u_p\}$ of vertices of $K_3 \boxtimes P_n$ such that $h(u_1) = 0 < h(u_2) < \cdots < h(u_p)$ where all the following holds (for all $1 \leq i \leq p - 1$)

(1) u_i and u_{i+1} are on opposite sides.

(2) $\{u_1, u_p\} = \{w_1, w_p\}$ where w_1, w_p are center vertices.

Proof: Let h be an optimal radio labeling for $K_3 \boxtimes P_n$, where

$$h(u_1) = 0 < h(u_2) < \cdots < h(u_p),$$

then

$$h(u_{i+1}) - h(u_i) \geq (d+1) - d(u_i, u_{i+1}) \quad \text{for all } 1 \leq i \leq p-1.$$

Adding these $p-1$ inequalities, we get

$$rn(K_3 \boxtimes P_n) = h(u_p) \geq (p-1)(d+1) - \sum d(u_i, u_{i+1}), \tag{2}$$

where $i = 1, \ldots, p-1$.

Case 1: when n is odd.

$$\sum d(u_i, u_{i+1}) = \sum d(u_i, u_{i+1}) + d(u_{p-1}, u_p),$$

for $i = 1, 2, \ldots, p-1$.

Using observation 1(2), as u_{p-1} and u_p are center vertices so,

$$\sum d(u_i, u_{i+1}) + d(u_{p-1}, u_p) \leq \sum [L(u_i) + L(u_{i+1})] + 1,$$

where $i = 1, 2, \ldots, p-2$.

$$\begin{aligned} \sum d(u_i, u_{i+1}) &\leq \sum [L(u_i) + L(u_{i+1})] + 1 \\ &= 2 \sum L(u) - L(u_1) - L(u_p) + 1 \; u \in V(G). \end{aligned} \tag{3}$$

Since u_1 and u_p are center vertices so $L(u_1) = L(u_p) = 0$. Now using observation 1(2), we have

$$\sum d(u_i, u_{i+1}) \leq 2 \sum L(u) + 1. \tag{4}$$

By substituting (4) in (2), we get

$$rn(K_3 \boxtimes P_n) = h(u_p) \geq (p-1)(d+1) - 2 \sum L(u) - 1.$$

For $K_3 \boxtimes P_{2k+1}$, for $p = 6k+3, d = 2k$, we have

$$\sum L(u) = 3k(k+1)$$

so,

$$\begin{aligned} rn(K_3 \boxtimes P_n) = h(u_p) &\geq (6\,k + 3 - 1)(2\,k + 1) \\ &\quad - 2(3\,k(k+1)) - 1 \\ &= (6k+2)(2k+1) - 6k(k+1) - 1 \\ &= 12\,k^2 + 6k + 4k + 2 - 6k^2 - 6k - 1 \\ &= 6k^2 + 4k + 1 \\ &= 2k(3k+2) + 1. \end{aligned}$$

Hence, $rn(K_3 \boxtimes P_n) \geq 2k(3k+2) + 1$.

Case 2: when n is even.

For $K_3 \boxtimes P_{2k}$, $i = 1, 2, \ldots, p-1$, we have

$$\begin{aligned} \sum d(u_i, u_{i+1}) &\leq \sum [L(u_i) + L(u_{i+1}) + 1] \\ &= 2 \sum L(u) - L(u_1) - L(u_p) + (p-1). \end{aligned}$$

Since u_1 and u_p are both center vertices, so

$$L(u_1) = L(u_p) = 0$$

and

$$\sum d(u_i, u_{i+1}) \leq 2 \sum L(u) + (p-1). \tag{5}$$

By substituting (5) in (2), we get

$$rn(K_3 \boxtimes P_n) = h(u_p) \geq (p-1)(d+1) - 2 \sum L(u) - (p-1).$$

For $K_3 \boxtimes P_{2k}$, $p = 6k, d = 2k-1$, we have

$$\sum L(u) = 3k(k-1)$$

, so

$$\begin{aligned} rn(K_3 \boxtimes P_n) = h(u_p) &\geq (6k-1)(2k) \\ &\quad - 2(3k(k-1)) - (6k-1) \\ &= 12k^2 - 2k - 6k^2 + 6k - 6k + 1 \\ &= 6k^2 - 2k + 1 \\ &= 2k(3k-1) + 1. \end{aligned}$$

Hence $rn(K_3 \times P_n) \geq 6k^2 - 2k + 1$.

The above two cases gives us:

$$\mathrm{rn}(K_3 \boxtimes P_n) \geq \begin{cases} 2k(3k+2)+1, & \text{if n} = 2k+1; \\ 2k(3k-1)+1, & \text{if n} = 2k. \end{cases}$$

■

Theorem 2: Let h be an assignment of distinct non-negative integers to $V(K_3 \boxtimes P_n)$ and $\{u_1, u_2, \ldots, u_p\}$ be the ordering of $V(K_3 \boxtimes P_n)$ such that $h(u_i) < h(u_{i+1})$ defined by $h(u_1) = 0$ and $h(u_{i+1}) = h(u_i) + d + 1 - d(u_i, u_{i+1})$. Then h is a radio labeling if for any $1 \leq i \leq p-1$ and $k = \lfloor \frac{n}{2} \rfloor$ the following holds:

1) *$d(u_i, u_{i+1}) \leq k+1$ if n is odd.*
2) *$d(u_i, u_{i+1}) \leq k+1$ and $d(u_i, u_{i+1}) \neq d(u_{i+1}, u_{i+2})$ if n is even.*

Proof: Let $h(u_1) = 0$ and $h(u_{i+1}) = h(u_i) + d + 1 - d(u_i, u_{i+1})$ for any $1 \leq i \leq p-1$ and $k = \lfloor \frac{n}{2} \rfloor$. For each $1 \leq i \leq p-1$, let $h_i = h(u_{i+1}) - h(u_i)$. We have to prove that h is a radio labeling if both above relations hold. i.e, we have to show for any $i \neq j$, $|h(u_i) - h(u_j)| \geq d + 1 - d(u_i, u_j)$.

Case 1: when n is odd.

If $n = 2k+1$, then $d = 2k$ and let (1) holds.

Suppose that $j \geq i$, then $h(u_j) - h(u_i) = h_i + h_{i+1} + \cdots + h_{j-1}$.

Applying definition 1,

$$\begin{aligned} h(u_j) - h(u_i) &= (j-i)(d+1) - d(u_i, u_{i+1}) \\ &\quad - d(u_{i+1}, u_{i+2}) - \cdots - d(u_{j-1}, u_j) \\ &\geq (j-i)(d+1) - (j-i)(k+1) \\ &= (j-i)(2k+1) - (j-i)(k+1) \\ &= (j-i)(2k+1-k-1) \\ &= (j-i)(k) \\ &\geq k \\ &= d + 1 - d(u_i, u_j). \end{aligned}$$

Case 2: when n is even.

If $n = 2k$, then $d = 2k - 1$ and let (2) holds: for $j > i$,

$$\begin{aligned} h(u_j) - h(u_i) &= h_i + h_{i+1} + \cdots + h_{j-1} \\ &= (j-i)(d+1) - d(u_i, u_{i+1}) \\ &\quad -d(u_{i+1}, u_{i+2}) - \cdots - d(u_{j-1}, u_j). \end{aligned}$$

If $j{-}i =$ even, then

$$\begin{aligned} h(u_j) - h(u_i) &\geq (j-i)(d+1) - \frac{j-i}{2}(k+1) - \frac{j-i}{2}(k) \\ &= (j-i)(2k) - \frac{j-i}{2}(k+1) - \frac{j-i}{2}(k) \\ &= \frac{j-i}{2}(4k - 2k - 1) \\ &= \frac{j-i}{2}(2k-1) \\ &= (j-i)\left(k - \frac{1}{2}\right) \\ &> k - \frac{1}{2} \\ &> k - 1 \\ &= d + 1 - d(u_i, u_j). \end{aligned}$$

If $j{-}i =$ odd, then

$$\begin{aligned} h\left(u_j\right) - h\left(u_i\right) &\geq (j-i)\left(d+1\right) - \frac{j-i+1}{2}\left(k+1\right) \\ &\quad - \frac{j-i-1}{2}\left(k\right) \\ &= (j-i)\left(2k-1+1\right) - k\left[\frac{(j-i-1)}{2}\right] \\ &\quad - (k+1)\left[\frac{(j-i+1)}{2}\right] \\ &= \frac{j-i}{2}\left(4k - 2k - 1\right) - \left[\frac{k+1}{2} - \frac{k}{2}\right] \\ &= (j-i)\left(k - \frac{1}{2}\right) - \frac{1}{2} \\ &\geq \left(k - \frac{1}{2}\right) - \frac{1}{2} \\ &= k - 1 \\ &= d + 1 - d\left(u_i, u_j\right) \\ &= d\left(u_i, u_j\right). \end{aligned}$$

Thus in both cases h is a radio labeling and hence the result. ■

Theorem 3: Let $K_3 \boxtimes P_n$ be a Strong product of K_3 and P_n and $k = \lfloor \frac{n}{2} \rfloor$, then

$$\mathrm{rn}(K_3 \boxtimes P_n) \leq \begin{cases} 2k(3k+2)+1, & \text{if } n = 2k+1; \\ 2k(3k-1)+1, & \text{if } n = 2k. \end{cases}$$

Proof: To prove the required result we consider two cases:

Case 1: when n is odd.

For $K_3 \boxtimes P_{2k+1}$, let

$$h : V(K_3 \boxtimes P_{2k+1}) \rightarrow \{0, 1, 2, \ldots, 2k(3k+2)+1\}$$

is defined by

$$h(u_{i+1}) = h(u_i) + d + 1 - L(u_i) - L(u_{i+1})$$

for $i = 1, 2, \ldots, p-2$ and

$$h(u_p) = h(u_{p-1}) + d - L(u_{p-1}) - L(u_p)$$

for $i = p{-}1$, as u_{p-1} and u_p are center vertices as per ordering of vertices shown below:

$$v_0 \xrightarrow{k} v_{R_k} \xrightarrow{k+1} v'_{L_1} \xrightarrow{k+1} v''_{R_k} \xrightarrow{k+1} v_{L_1} \xrightarrow{k+1} v'_{R_k} \xrightarrow{k+1} v''_{L_1} \cdots \xrightarrow{k+1} v_{R_1} \xrightarrow{k+1} v'_{Lk} \xrightarrow{k+1} v''_{R_1} \xrightarrow{k+1} v_{L_k} \xrightarrow{k+1} v'_{R_1} \xrightarrow{k+1} v''_{L_k} \xrightarrow{k} v'_0 \xrightarrow{1} v''_0.$$

Case 2: when n is even.

For $K_3 \boxtimes P_{2k}$, let

$$h : V(K_3 \boxtimes P_{2k}) \rightarrow \{0, 1, 2, \ldots, 2k(3k-1)+1\}$$

is defined by

$$h(u_{i+1}) = h(u_i) + d - L(u_i) - L(u_{i+1}),$$

as per ordering of vertices shown below:

$$v_{L_0} \xrightarrow{k} v_{R(k-1)} \xrightarrow{k+1} v_{L_1} \xrightarrow{k} v_{R(k-2)} \xrightarrow{k+1} v_{L_2} \xrightarrow{k} \cdots \xrightarrow{k} v_{R_1} \xrightarrow{k+1} v_{L(k-1)} \xrightarrow{k} v_{R_0} \xrightarrow{1} v'_{L_0} \xrightarrow{k} v_{R(k-1)} \xrightarrow{k+1} v'_{L_1} \xrightarrow{k} v'_{R(k-2)} \xrightarrow{k+1} v'_{L_2} \xrightarrow{k} \cdots \xrightarrow{k} v'_{R_1} \xrightarrow{k+1} v'_{L(k-1)} \xrightarrow{k} v'_{R_0} \xrightarrow{1} v''_{L_0} \xrightarrow{k} v''_{R(k-3)} \xrightarrow{k+1} v''_{L_1} \xrightarrow{k} v''_{R(k-2)} \xrightarrow{k+1} v''_{L_2} \xrightarrow{k} \cdots \xrightarrow{k} v''_{R_1} \xrightarrow{k+1} v''_{L(k-1)} \xrightarrow{k} v''_{R_0}.$$

Thus in both cases (1) and (2), it is possible to assign labeling to the vertices of $K_3 \boxtimes P_n$ with span equal to the lower bound satisfying the condition of theorem 2.

Hence h is a radio labeling. ■

Theorem 4: Let $K_3 \boxtimes P_n$ be a strong product of K_3 and P_n and $k = \lfloor \frac{n}{2} \rfloor$ then,

$$\mathrm{rn}(K_3 \boxtimes P_n) = \begin{cases} 2k(3k+2)+1, & \text{if } n = 2k+1; \\ 2k(3k-1)+1, & \text{if } n = 2k. \end{cases}$$

Proof: The proof follows from Theorem 2 and Theorem 3. ■

Example 1: In Figure 1, the ordering of the vertices and optimal radio labeling of $K_3 \boxtimes P_7$ is shown.

$$\begin{aligned} &v_0 \rightarrow v_{R3} \rightarrow v'_{L_1} \rightarrow v''_{R_3} \rightarrow v_{L_1} \rightarrow v'_{R_3} \rightarrow v''_{L_1} \\ &\rightarrow v_{R_2} \rightarrow v'_{L_2} \rightarrow v''_{R_2} \rightarrow v_{L_2} \rightarrow v'_{R2} \rightarrow v''_{L2} \rightarrow v_{R_1} \\ &\rightarrow v'_{L3} \rightarrow v''_{R_1} \rightarrow v_{L_3} \rightarrow v'_{R1} \rightarrow v''_{L_3} \rightarrow v'_0 \rightarrow v''_0. \end{aligned}$$

Example 2: In Figure 2, the ordering of the vertices and optimal radio labeling of $K_3 \boxtimes P_6$ is shown.

$$\begin{aligned} &v_{L_0} \rightarrow v_{R_2} \rightarrow v_{L_1} \rightarrow v_{R_1} \rightarrow v_{L_2} \rightarrow v_{R_0} \\ &\rightarrow v'_{L_0} \rightarrow v'_{R_2} \rightarrow v'_{L_1} \rightarrow v'_{R_1} \rightarrow v'_{L_2} \rightarrow v'_{R_0} \rightarrow v''_{L_0} \\ &\rightarrow v''_{R_2} \rightarrow v''_{L_1} \rightarrow v''_{R_1} \rightarrow v''_{L_2} \rightarrow v''_{R_0}. \end{aligned}$$

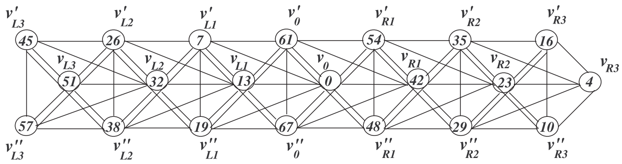

FIGURE 1. $\mathrm{rn}(K_3 \boxtimes P_7) = 67$.

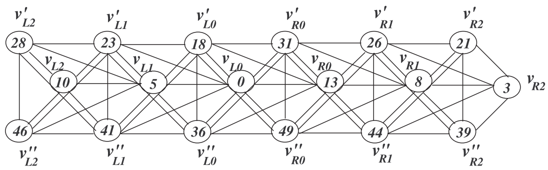

FIGURE 2. $\mathrm{rn}(K_3 \boxtimes P_6) = 49$.

III. CONCLUSION

Channel assignment problems when wireless are used to transfers the information over a distance sometimes leaves an unpleasant experience. This annoyance is given by interferences caused by unconstrained simultaneous transmissions. Two good enough channels can be interfere or damage the communications. This interference can be overcome by assigning a certain number to a channel represented as a node in a graph where the edges between two nodes (two channels) shows the connection between two channels. Radio labeling is used for channel assignment when on interested in assigning the distinct channels to different wireless medium. This article elucidates how radio labeling is used in computer science more particularly in channel assignment over wireless networks and in network analysis when network has a geometrical structure like strong product of $K_3 \boxtimes P_{2k+1}$.

IV. FUTURE DIGESTIONS

For the interesting readers, computing radio numbers for the families of graphs studied in [25], [26] are open problems.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

ACKNOWLEDGEMENTS

The authors are grateful for the valuable comments of the anonymous referees.

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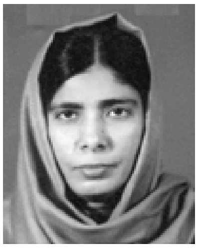

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