IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received May 8, 2020, accepted May 30, 2020, date of publication June 10, 2020, date of current version June 22, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3001278

Stability Analysis for the Modification Method Under the a Priori Strategy of the PTSP

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This work was supported in part by the CRISTAL-GRIFT Laboratory through Project (PACT-PRF), and in part by the SESAME University, Tunisia.

ABSTRACT We propose in this paper a new formulation for the stability of the Traveling Salesman Problem (TSP) compared with its probabilistic version, the Probabilistic Traveling Salesman Problem (PTSP). It is a real extension of the TSP, where the number of customers to be served each time is a random variable. That is only a subset of customers will need its services, moreover this subset varies from day to day. From the literature, several methods of resolution of the TSP have been proposed. In order to use these methods as they are for the PTSP came the idea of the study of stability. It is interested in finding cases where the solution for the TSP is also for the PTSP. First we survey and comment a number of easy TSPs. We also present via the notion of ‘‘Master tour’’ the stable problems in the TSP-PTSP context. An exact Branch and Bound is used for recognizing the TSPs that are not stable. Finally, we propose a new modification method, -different from the usual method of PTSP-, called the method of Taxi Driver, it takes the structure of the tour into consideration.

INDEX TERMS Stability, PTSP, exact algorithm, taxi driver strategy.

I. INTRODUCTION

The probabilistic traveling salesman problem PTSP is a generalization of the classical traveling salesman problem TSP and one of the most important stochastic routing problems, whose formulation explicitly contains probabilistic elements. Several motivations have led researchers to take an interest in studying the effects of the introduction of probabilistic elements into TSP. The most evoked are, first the ability to design and evaluate models that are more appropriate to real problems. Second, the possibility of evaluating the stability of optimal deterministic problems solutions when the instances of problem are disrupted by the absence of certain data. The first work on the PTSP study was initiated by Jaillet [29]. It has been used to model many practical applications since its introduction [7], [8], [11], [21], [31].

The resolution of the PTSP by exact algorithm has proven hard due to the hardness of the problem. A very limited number of works has been devoted to exact methods such as the branch and cut algorithm [33], Branch and Bound algorithm [3], [10], [46]. The PTSP is NP-Hard problem, therefore a number of heuristic and meta-heuristic algorithms have been developed in order to find a suboptimal solutions such as the local search heuristics 2-p-Opt and 1-Shift [12], [13], 3-opt algorithm and some combinations with the last two were investigated in [39]. Several metaheuristic algorithms were proposed such as the simulated annealing [6], [17], ant colony algorithm ACA [13], [14], genetic algorithm [37], [38] and other hybrid methods [5], [36], [40], [51].

In the deterministic version, the problem is defined on a graph $G = (V, E, n)$, of n nodes (customers) whose arcs are valued. Consider a matrix D_{st} of distances between nodes of the graph. In the probabilistic version PTSP, a demand at each time occurs a distribution of probabilities P on the set of all the subsets of V(i.e. each instance $\xi \subseteq V$ has a probability of occurrences $P(\xi)$) [29]. The first idea coming to mind for process this probabilistic version consists in optimally solving the various potential realizations, namely the customers, who have a request for a given day. This approach is called ‘‘re-optimization strategy’’ [29]. The average cost obtained under the re-optimization strategy is defined by the equation 1.

$$E(Long_{opt}) = \sum_{i=1}^{2^n} P(\xi_i) Long_{opt}(\xi_i) \tag{1}$$

The associate editor coordinating the review of this manuscript and approving it for publication was Huiqing Wen.

where $Long_{opt}(\xi)$ is the cost of optimal solution of problem through the subset of customers ξ, V is the set of all customers and $P(\xi)$ is the probability of presence of the ξ.

This strategy is certainly optimal but it presents several disadvantages. Indeed, we should individually solve 2^n NP-Hard problems, which is impossible even with high performance resources. In addition, for many applications, it is necessary to obtain a solution to each realization in real time. It is therefore necessary to adopt another resolution strategy which takes into account the variations in the initial instance and which is less expensive in terms of calculations than the re-optimization strategy. This strategy is called "*a priori* strategy" [29].

Consider an *a priori* tour Γ across the n nodes of V and for each subset of customers ξ, the method of modification μ to generate a solution through ξ. It consists in "skipping" the absent customers from the *a priori* tour Γ. Let $Long_{(\Gamma,\mu)}$ be the real random variable which will assign a given instance $\xi \subseteq V$ to its induced length from *a priori* tour Γ $-Long_{(\Gamma,\mu)}(\xi)-$ using the skipping method μ. The PTSP solution consists of searching an *a priori* tour Γ^* which minimizes the expected cost of $Long_{(\Gamma,\mu)}$ ([6], [29]). Then, the PTSP is expressed as follows according to the equation 2.

$$\min_{\Gamma} E(Long_{(\Gamma,\mu)}) \tag{2}$$

with

$$E(Long_{(\Gamma,\mu)}) = \sum_{i=1}^{2^n} P(\xi_i) Long_{(\Gamma,\mu)}(\xi_i) \tag{3}$$

Jaillet [29] proposed an approach to calculate the exact value of $E(Long_{(\Gamma,\mu)})$ (equation 3) with $O(n^2)$ time. We are concerned here with homogeneous version, it is presented as follows.

Let $d_{st}(i,j)$ be a distance between the customers i, j, and $p = P(i) = P(j)\ \forall i, j \in \Gamma$, $q = 1 - p$, the expected cost is then equal to

$$E(Long_{\Gamma}) = p^2 \sum_{r=0}^{n-2} q^r \sum_{i=1}^{n} d_{st}(i, \Gamma^r(i)) \tag{4}$$

$\Gamma^r(i)$ denotes the successor r of i in the *a priori* tour Γ.

The figure 1 illustrates the principle of the *a priori* strategy. Indeed, the figure 1(a) represents the *a priori* solution that assumes all nodes (A, B, C, D, E, F) are present, the figure 1(b) represents the solution after applying the modification method, which consists in skipping of the *a priori* solution the absent nodes (C and F) in order to obtain a solution restricted to the present data.

In the literature the notion of stability was approached in the combinatorial optimization problems COPs according to different aspects and for each problem separately. The objective is to extract the relationship between a solution of a given COP and its parameters when some data is absent or unavailable. Indeed, several research focussed on the study of stable problems compared with probabilistic version.

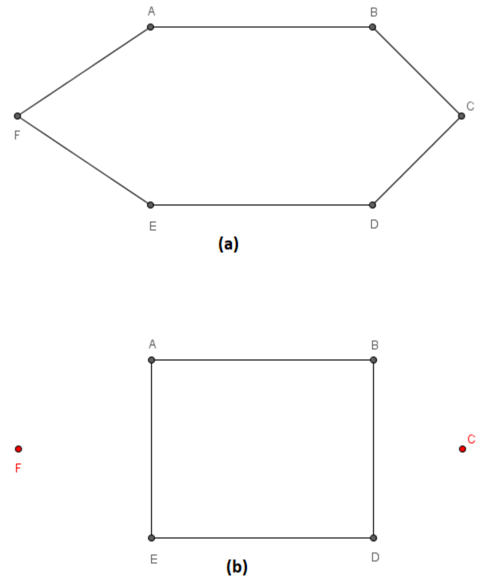

FIGURE 1. An *a priori* tour through 6 customers (a), and the sub-tour solution when the customers C, F are absent, utilizing modification method μ, by keeping the same order (b).

Here, we refer to Boria *et al.* [15], who has interested to the stability in the probabilistic Minimum Spanning Tree Problems for complete graph. In [16] Bouyahia *et al.* studied the stability in the Probabilistic Scheduling Problems.

Our aim is to propose a formulation of the stability of TSP compared with PTSP. In order to conduct it, we carry out an overview of the different easy TSPs. Then, we dedicate two techniques, the search for a tour guaranteeing the stability "master tour" [43], [44] and a branch and bound algorithm developed in [3]. The rest of the paper has the following structure. In the next section we explain the notion of stability. In section 3 we present an overview of easy TSP. Section 4 introduces unstable TSP under the skipping method. The stable TSP under the skipping method will be also proposed in Section 5. Section 6 introduces small stable TSP under the method of Taxi Driver. Finally, Section 7 concludes the paper.

II. NOTION OF STABILITY UNDER TSP

In this section, we begin by giving our definition of the concept of stability. We then discuss the problem of stability as described in the literature. Finally, we unveil the link that exists between the concept of stability and that of the Master tour. Another challenge is to examine the stability of the initial problem. The optimal solution of it can be reconsidered despite the data perturbations of problem.

A. DEFINITION OF STABILITY

For the PTSP one defines the stability such that: one says that the PTSP is stable if the random variable associated with the strategy of re-optimization and this one of the *a priori* strategy have the same real value. More formally, let μ be a modification method for the PTSP, $Long_{opt}(\xi)$ be the optimal length through the instance ξ, the stability can be formulated as follows.

Definition 1: The TSP is called μ-stable if there exists a solution Γ^* whose random variable of the *a priori* strategy according to μ is equal to the random variable of the re-optimization strategy.

$$Long_{(\Gamma^*,\mu)}(\xi) = Long_{opt}(\xi) \quad \forall \xi \subset \{0, \ldots, n\} \tag{5}$$

Based on the the article of Gabrel *et al.* [27], we start and situate this concept of stability. Two aspects of stability are identified. The first is when uncertainty influences the feasibility of a solution, then a robust optimization consists in searching for a solution that will be feasible for each considered realization. The second is when the uncertainty influences the optimality of solutions, and then in this case the robust optimization consist of obtaining a "suboptimal" solution valid for each realization of initial problem. In light of these elements, we want to improve the concept of stability and we designed our own mathematical definition. Other forms of "stabilities" have been mentioned, for example, Montemanni *et al.* [42] which is an extension of the TSP. However, when we compared our concept of stability, we find a concept called "Master tour" [18]. It happens to be almost identical to our vision of stability in certain cases.

B. MASTER TOUR

We mean by the stability the fact that the optimality remains unchanged even if some data of the problem is random. Indeed, a PTSP is stable if the random variables associated with both re-optimization and *a priori* strategies have the same real value, hence the need of the Master Tour.

Definition 2 [18]: A tour M_Γ, for a set V of customers called a master tour if it checks next property. For all $\xi \subset V$, the optimal tour for ξ is obtained by skipping in the tour Γ the absent customers.

The problem of the master tour is to see if V has a Master tour or not. This problem has been formulated for the first time by Papadimitriou [43], [44] and the research work on this axis remains very limited. We present the theorem that characterizes the stability for symmetric matrices.

Theorem 1 [See [18]]: For a symmetric matrix $M = (m_{i,j})$ of dimension $n \times n$ the tour $(1, 2, \ldots, n-1, n)$ represents a master tour if and only if the matrix M is a Kalmanson matrix.

III. LITERATURE REVIEW OF EASY TSP

Here, we present a literature review of easy TSP. In this section M denotes the distance matrix. We begin by recalling the TSP framework. The TSP is a famous standard NP-Hard problem. Formally, the TSP can be presented as follows. Consider a graph (V, E, M, n) where V is the set of nodes $(V = \{1, \ldots, n\})$, E denotes the set of edges and $M = (m_{u,v})$ the edge costs ($M = (m_{uv})_{1\leq u,v\leq n}$ where m_{uv} is the distance between the customers u and v). The problem consists in searching a minimal length tour $\mathrm{T} = (t_1, t_2, \ldots, t_n)$ across a given number n of customers [24], [26] as shown in equation 6.

$$\min_{\mathrm{T}}((\sum_{i=1}^{n-1} m_{t_i,t_{i+1}}) + m_{t_n,t_1}) \tag{6}$$

The TSP is an NP-Hard problem, however the pyramidal tour offers several easy cases. Indeed, the notion of pyramidal tour is very interesting. Let us first define this concept. A pyramidal tour through n customers, consists of visiting a sequence of customers of increasing indices until the city n. Then to visit the remaining customers in descending order of indices. Finally to return to the first city of the cycle. The interest of such a tour reside in the following two points. The first is that the determination of the minimum cost of the pyramidal tours it can be calculated in $O(n^2)$ time, whereas the total number of pyramidal tours of size n customers is exponential as a function of n. Therefore, the pyramidal tours are a subset of the cyclic permutations on which we aim to optimize it in real time. The second is that there are certain structures on cost matrices that assure the existence of an optimal pyramidal tour. In this case (ie where the cost matrix has these combinatorial structures), the TSP consists of finding the shortest pyramidal tour in polynomial time.

The TSP limited to the class of M is said to be pyramidally solvable if the matrix M admits an optimal pyramidal tour. Since the 1970s, several pyramidal TSP have been studied. In this section we examine typical examples on this subject. Indeed, they are special type of matrices of which its structures ensures the existence of an optimal pyramidal tour.

- Monge matrices
- Supnick matrices
- Demidenko matrices
- Kalmanson matrices
- Van der Veen matrices
- Matrices of "general distribution".

Before examining these classes of matrices, we expose a fundamental result which is at the origin of the success of pyramidal TSP.

Theorem 2 [See [32]]: The problem of finding an optimal pyramidal tour with respect to M, can be solved in complexity equals $O(n^2)$ time, and that is for any distance matrix M of dimension $n \times n$.

The first paper that addressed the pyramidal tours in the notion of TSP was that of Aizenshtat and Kravchuk [2]. It researched the TSP on "ordered product matrices" which have the general term.

$M = (m_{u,v}) = (\pi_u \cdot \sigma_v)$ where $0 \leq \pi_1 \leq \pi_2 \leq \ldots \leq \pi_n$ and $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \geq \sigma_0$.

Later, it turned out that this type of matrix was a sub-case of the Monge matrices [22]. A matrix M of dimension $n \times n$ represents a Monge matrix if it checks.

$$m_{uv} + m_{ps} \leq m_{us} + m_{rv} \text{ for all } 1 \leq u < r \leq n, \quad 1 \leq v < s \leq n.$$

Lawler *et al.* [34] note the matrices that verify this last property, matrices of distribution, that indicates the construction of these matrices from certain non-negative matrices called density matrices. The name "Monge Matrix" has been invented by Hoffman [28] who generalizes a suggestion made by Monge [41]. For more information on the matrices of Monge (see the article of Burkard *et al.* [19]).

We present here the result of Aizenshtat and Kravchuk [2]. It is explicitly contained in the theorem 3.

Theorem 3 [See Lawler et al. [34]]: Let M be a Monge matrix, there is certainly an optimal pyramidal tour.

We also present the result of Park [45] that exhibited that the TSP on the Monge matrices is solvable in a complexity of $O(n)$. Indeed, in [45] the authors managed to accelerate the dynamic program of Klyaus based on research techniques treated by Aggarwal *et al.* [1], then later by Eppstein [25]. We recall that our goal in this section is to present easy cases of TSP, precisely the class of "Pyramidal TSP", where we already introduced "Monge matrices". The second case that we present is that of "Van der Veen matrices". It is a symmetric matrix, so for the clarity of our report we divide our inspection of these particular cases into two parts, symmetric matrices and asymmetric matrices.

A. THE SYMMETRIC CASE

We review in the first part the "Van der Veen matrix". It is defined as follows [50], a Van der Veen matrix is a symmetric matrix M of dimension $n * n$ that checks.

$$m_{uv} + m_{v+1,l} \leq m_{ul} + m_{v,v+1} \quad \text{for all} 1 \leq u < v \lneqq v+1 \lneqq l \leq n.$$

A theorem is similar to that of Monge matrix is given as follows.

Theorem 4 [See [50]]: The TSP on the Van der Veen matrices is pyramidally solvable.

The third pyramidal TSP is called TSP on symmetric Demidenko matrices that are a class of symmetric Monge matrices.

Let M be a symmetric matrix of dimension $n * n$. M is a Demidenko matrix if it satisfies.

$$m_{uv} + m_{v+1,l} \leq m_{u,v+1} + m_{v,l} \quad \text{for all } 1 \leq u < v \lneqq v+1 \lneqq l \leq n.$$

From this inequations, there is no order relationship between the Van der Veen matrices class and the symmetric Demidenko matrices class. In addition, by definition, the symmetric Demidenko matrices themselves are included in a general class, namely the class of asymmetric Demidenko matrices. A similar theorem is formulated as follows.

Theorem 5 [See [48]]: The TSP on symmetric matrices of Demidenko is pyramidally solvable.

We continue with the fourth special case, Supnick matrices.

Let $n * n$ M be a symmetric matrix. It is called a Supnick matrix if

$$m_{uv} + m_{v+1,l} \leq c_{u,v+1} + m_{v,l} \leq m_{ul} + c_{v,v+1} \quad \text{for all } 1 \leq u < v \lneqq v+1 \lneqq l \leq n.$$

In the [48] the authors are shown, for an arbitrary diagonal, a Supnick matrices is a subclass of symmetric Monge matrices. In addition, it is also subclass of Van der Veen matrices. It follows that the TSP restricted to Supnick matrices is pyramidally solvable. The result is shown in the theorem 6.

Theorem 6 [See [47]]: The tour $(1, 3, 5, 7, \ldots, 6, 4, 2)$. is a solution of minimal length of TSP on the Supnick matrices.

The fifth and last case, we expose a type of matrix called Kalmanson matrices.

Let M be a symmetric matrix of dimension $n*n$. It is called a matrix Kalmanson if it satisfies the inequalities:

$$m_{uv} + m_{kl} \leq m_{uk} + c_{vl} \text{ for all } 1 \leq u < v \lneqq k \lneqq l \leq n.$$
$$m_{ul} + m_{vk} \leq m_{uk} + m_{vl} \text{ for all } 1 \leq u < v \lneqq k \lneqq l \leq n.$$

From the results presented in [48], the Kalmanson matrices are a subclass of the symmetric Demidenko matrices. Finally, we give the theorem relating to this class of matrices.

Theorem 7 [See [30]]: The tour $(1, 2, \ldots, n-1, n)$. is a solution of the TSP on the Kalmanson matrices.

In the next section, we review the asymmetric cases of the pyramidal TSP, as we have already done for the symmetric case.

B. THE ASYMMETRIC CASE

Many types of asymmetric matrices have been studied in scientific literature. In this section, we review two very specific classes. The class of Demidenko matrices and the class of generalized distribution matrices. We present here some pyramidally solvable cases of these two classes of matrices.

The first case that we approach is that of generalized distribution matrices.

A generalized distribution matrix is a $n * n$ matrix M of type (*) (respectively of type(**), (***), (***)) if the following condition is verified.

$m_{uv} + m_{kp} + m_{pq} \leq m_{up} + m_{pv} + m_{kq}$ for all $u, v, k, p, q \in S_*$ (respectively $S_{**}, S_{***}, S_{****}$) where

$$\begin{aligned} S_* &= \{u, v, k, p, q \in \{1, \ldots, n\} : \\ &\quad u, v, k < p < q, u \neq v, u \neq k\} \\ S_{**} &= \{u, v, k, p, q \in \{1, \ldots, n\} : \\ &\quad u, v, q < p < k, u \neq v, v \neq q\} \\ S_{***} &= \{u, v, k, p, q \in \{1, \ldots, n\} : \\ &\quad k < p < u, v, q, u \neq v, v \neq q\} \\ S_{****} &= \{u, v, k, p, q \in \{1, \ldots, n\} : \\ &\quad q < p < u, v, k, u \neq v, u \neq k\} \end{aligned}$$

In [20] the authors demonstrated that these four classes of generalized distribution matrices include the Monge matrices.

Theorem 8 [See [20]]: The TSP on generalized distribution matrices of type (*) (respectively of type (**), (***) and (****)) is pyramidally solvable.

The second case is the Demidenko matrices class. A matrix M of dimension $n * n$ is called Demidenko if the four conditions are met. For all $u, v, l \in \{1, \ldots, n\}$ and $u < v \lneqq v+1 \lneqq l$ we have:

1) $m_{uv} + m_{vv+1} + m_{v+1,l} \leq m_{uv+1} + m_{v+1v} + m_{vl}$
2) $m_{uv} + m_{v+1v} + m_{l,v+1} \leq m_{v+1u} + m_{vv+1} + m_{lv}$
3) $m_{uv} + m_{l,v+1} \leq m_{uv+1} + m_{lv}$
4) $m_{uv} + m_{v+1,l} \leq m_{v+1u} + m_{vl}$

Demidenko matrix is the subject of the next theorem.

Theorem 9 [See [23]]: The TSP on the Demidenko matrices is pyramidally solvable.

There are a significant number of articles in the literature that were discovered many pyramidally solvable TSP. These results are summarized in the paper of Demidenko [23]. In fact, these classes detected are only subclasses of the class of Demidenko matrices. We cite as example, special class. A matrix M of dimension $n * n$ is called Klyaus matrix if we have, for all $i < j \lneqq k$

- $m_{uv} + m_{uv} \geq 0$
- $m_{uv} + m_{vk} \leq m_{uk}$
- $m_{uv} + m_{kv} \leq m_{ku}$

Finally we can refer to the articles Van der Veen *et al.* [49] and Baki and Kabadi [4], for more results on pyramidally solvable TSP.

C. OTHER EASY TSPs

This list of TSP, is collected in the article [18], composed of non-pyramidal TSPs that have been proven easy. We start by introducing the product matrices.

Let M be a matrix of dimension $n * n$. M is a product matrix if for any two vectors u_1 and u_2 of $\mathbb{R}^n$ we have $m_{u,v} = u_{1u} * u_{2v}$. The TSP restricted on symmetric product matrices has been proven easy [18]. The second TSP that we cite from this list is the TSP on the Brownian matrix. A matrix M of dimension $n * n$ is Brownian matrix if the vectors u_1 and u_2 verifies:

$$m_{u,v} = \begin{cases} u_{1u} & \text{if } u < v, \\ u_{2v} & \text{else.} \end{cases}$$

IV. UNSTABLE TSP UNDER THE SKIPPING METHOD

This section reflects the originality of our work. We have tried to obtain ‘‘counter examples’’ via an exact Branch and Bound algorithm. So we present here, the different TSP detected unstable.

A. PROBABILISTIC BRANCH AND BOUND ALGORITHM

The Branch and Bound algorithm that we use here was proposed by Amar *et al.* [3]. It consist in dividing the initial problem into smaller subproblems based on a lower bound associated with root of problem. This lower bound is based on the expected length (equation 4) of a tour that introduced by Jaillet [29].

Let M be the distance matrix between the customers 1, 2, 3 and 4.

TABLE 1. Matrix example.

M=

	1	2	3	4
1	∞	m_{12}	m_{13}	m_{14}
2	m_{21}	∞	m_{23}	m_{24}
3	m_{31}	m_{32}	∞	m_{34}
4	m_{41}	m_{42}	m_{43}	∞

The lower bound is calculated in equation 7

$$E_{p-Bound\mathbb{R}} = E_{BoundTSP}(n)(p^2 \sum_{r=0}^{n-2} q^r) = E_{BoundTSP}(n)p(1-q^{n-1}) \quad (7)$$

with $E_{BoundTSP}(n) = \sum_{i=1}^{n} \min R_i + \sum_{j=1}^{n} \min C_j$

R_i is the i^{th} row and M_i is the j^{th} column.

Then the same article proposed the two evaluations which are represented by $E_{p-Bound12}$ and $E_{p-Bound\overline{12}}$:

1) Choose the edge 12:

$$E_{p-Bound12} = E_{p-Bound\mathbb{R}} + p^2 \sum_{r=1}^{n-2} q^r [\min^{(r)}_{X\neq 1} d(1,X)] + p^2 E_{BoundTSP_{Next}} \quad (8)$$

with $E_{BoundTSP_{Next}}$ is the evaluation of TSP.

2) Not choose the edge 12:

$$E_{p-Bound\overline{12}} = E_{p-Bound\mathbb{R}} + p^2[\min^{(1)}_{K\neq 2} d(1,K) + \min^{(1)}_{K\neq 1}(d(K,2))] + p^2 \sum_{r=2}^{n-2} q^r [\min^{(r)}_{X\neq K} d(1,X) + \min^{(r)}_{X\neq 2} d(K,X)] \quad (9)$$

$E_{p-Bound\overline{12}}$ is the probabilistic penalty cost of $\overline{12}$, $min^{(i)} d(1,X)$ is the i^{th} minimum of row 1, n is the number of customers.

These evaluations are represented on the distance matrix $M_{Evaluations}$.

TABLE 2. Probabilistic penalties.

$M_{Evaluations} =$

	1	2	3	4
1	∞	$(E_{p-Bound\overline{12}})0(E_{p-Bound12})$	-	-
2	-	∞	-	-
3	-	-	∞	-
4	-	-	-	∞

If these two bounds are equal, then an optimal solution is found, and we stop there. Otherwise, the set of solutions is divided into two or more sub-problems, according to these two probabilistic evaluations $E_{p-Bound12}$ and $E_{p-Bound\overline{12}}$. The method is then applied recursively to these sub-problems by generating a tree structure as shown in figure 2.

First, this algorithm allows us to carry out the relationship between the TSP and its probabilistic version PTSP.

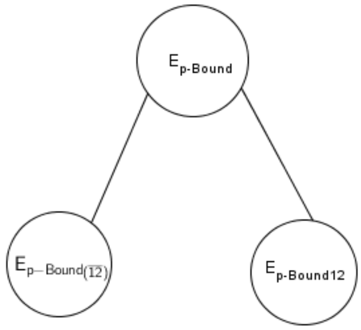

FIGURE 2. Tree for the PTSP.

We selected from literature a bound introduced by Jaillet [29].

$$\frac{E_{Long_{\Gamma_{PTSP}}} - E_{Long_{\Gamma_{TSP}}}}{E_{\Gamma_{TSP}}} \le \frac{1-p}{p^2} \tag{10}$$

where $E_{Long_{\Gamma_{PTSP}}}$ represented the expected length of optimal tour for PTSP and $E_{Long_{\Gamma_{TSP}}}$ expected length of optimal tour for TSP.

We propose in this section numerical results. Experimentally, for each size n varying from 4 to 10, we generate a matrix of order n which represents the distances matrix between the customers. Then for each value of probability p from 0,1 to 1 with an interval of 0,1 the optimal expected length of tour is calculated. We seek an optimal solution and its expected length, the results are represented in the Table 3.

The Table 3 clearly shows that the limit of Jaillet [29] has been verified. For a low probability we have no idea about the relationship between the two versions. But if $p \in [0.9.1]$, in the majority case we found that the TSP solutions and the PTSP solutions are the same.

B. TSP ASSOCIATED WITH MONGE MATRICES

We recall the definition of a Monge matrix, it is a matrix M of dimension $n * n$ which verifies: $m_{uv} + m_{rs} \le m_{us} + m_{rv}$ for all $1 \le u < r \le n$, $1 \le v < s \le n$.

We generated the following Monge matrix:

$$\begin{pmatrix} \infty & 9 & 12 & 14 & 17 \\ 9 & \infty & 15 & 16 & 15 \\ 12 & 15 & \infty & 10 & 9 \\ 14 & 16 & 10 & \infty & 5 \\ 19 & 15 & 9 & 5 & \infty \end{pmatrix}$$

The exact resolution algorithm gives the results as shown in Table 4:

We observe that $\Gamma_{PTSP} \neq \Gamma_{TSP}$, which breaks the equality of the random variables $Long_{(\Gamma,\mu)}$ and $Long_{opt}$, which clearly gives, the non-stability of TSP on the Monge matrices.

C. TSP ASSOCIATED WITH VAN DER VEEN MATRICES

The Van der Veen matrix as we presented, it is a symmetric matrix M of dimension $n * n$ that verifies, $m_{uv} + m_{v+1,l} \le m_{ul} + m_{v,v+1}$ for all $1 \le u < v \lneqq v+1 \lneqq l \le n$.

TABLE 3. Resolution for matrices of size 4-7-8-9-10. *T*opt is optimal tour, p represents the probability, and E is the expected length.

n	p	E	Γopt	p	E	Γopt
4	0.1	0.065	2013	0.6	1.328	2013
	0.2	0.234	2013	0.7	1.596	2013
	0.3	0.474	2013	0.8	1.837	0213
	0.4	0.749	2013	**0.99**	2.0479	**0213**
	0.5	1.041	2013	**1**	2.230	**0213**
7	0.1	0.174	5214630	0.6	2.016	1524360
	0.2	0.547	5214630	0.7	2.277	1524360
	0.3	0.978	5214630	0.8	2.503	1524360
	0.4	1.356	1524603	**0.99**	2.838	**5136042**
	0.5	1.713	1524360	**1**	2.855	**5136042**
8	0.1	0.217	57103426	0.6	2.10	57421036
	0.2	0.465	57103426	0.7	2.451	57421036
	0.3	1.162	57103426	0.8	2.783	57134206
	0.4	1.541	57103426	**0.99**	2.94	**57134206**
	0.5	1.847	57421036	**1**	2.857	**557134206**
9	0.1	0.407	856247301	0.6	2.216	856247301
	0.2	1.113	856247301	0.7	2.5478	856247301
	0.3	1.378	856247301	0.8	2.773	856247301
	0.4	1.556	856247301	**0.99**	2.838	**856247301**
	0.5	1.713	856247301	**1**	2.855	**856247301**
10	0.1	0.407	5721038649	0.6	2.713	5721038649
	0.2	1.113	5721038649	0.7	2.567	5721038649
	0.3	1.768	5721038649	0.8	2.713	5721038649
	0.4	1.956	5721038649	**0.99**	2.938	**5721038649**
	0.5	2.700	5721038649	**1**	3.12	**5721038649**

TABLE 4. Non-stability of TSP on the Monge matrices.

	Optimal tour	Cost
TSP	1 2 4 5 3	51.000000
PTSP (p=0.5)	1 2 5 4 3	27.781250

For the following generated Van der Veen matrix.

$$\begin{pmatrix} \infty & 6 & 13 & 8 & 18 & 9 \\ 6 & \infty & 9 & 2 & 15 & 13 \\ 13 & 9 & \infty & 11 & 4 & 2 \\ 8 & 2 & 11 & \infty & 16 & 7 \\ 18 & 15 & 4 & 16 & \infty & 2 \\ 9 & 13 & 2 & 7 & 2 & \infty \end{pmatrix}$$

The exact resolution algorithm provides the results as shown in Table 5.

TABLE 5. Non-stability of TSP on the Van der Veen matrices.

	Optimal tour	Cost
TSP	1 2 4 3 5 6	34.000000
PTSP (p=0.5)	1 4 2 3 5 6	23.500000

We observe that $\Gamma_{PTSP} \neq \Gamma_{TSP}$, which breaks the equality of the random variables $Long_{(\Gamma,\mu)}$ and $Long_{opt}$, which clearly gives, the non-stability of TSP on the Van der Veen matrices.

D. TSP ASSOCIATED WITH SUPNICK MATRICES

A Supnick matrix is a symmetric matrix M of dimension $n * n$ such that, $m_{uv} + m_{v+1,l} \le c_{u,v+1} + m_{v,l} \le m_{ul} + m_{v,v+1}$ for all $1 \le u < v \lneqq v+1 \lneqq l \le n$.

For the following generated Supnick matrix.

$$\begin{pmatrix} \infty & 2 & 3 & 5 & 4 & 9 \\ 2 & \infty & 6 & 8 & 6 & 8 \\ 3 & 6 & \infty & 9 & 7 & 7 \\ 5 & 8 & 9 & \infty & 9 & 5 \\ 4 & 6 & 7 & 9 & \infty & 0 \\ 9 & 8 & 7 & 5 & 0 & \infty \end{pmatrix}$$

The exact resolution algorithm provides the results as shown in Table 6.

TABLE 6. Non-stability of TSP on the Supnick matrices.

	Optimal tour	Cost
TSP	1 2 3 5 6 4	25.000000
PTSP (p=0.5)	1 2 4 6 5 3	15.734375

We observe that $\Gamma_{PTSP} \neq \Gamma_{TSP}$, which breaks the equality of the random variables $Long_{(\Gamma,\mu)}$ and $Long_{opt}$, which clearly gives, the non-stability of TSP on the Supnick matrices.

E. TSP ASSOCIATED WITH KLYAUS MATRICES

It is a matrix$M$ of dimension $n * n$ that satisfies: for all $u < v \lneqq k$

- $m_{uv} + m_{vu} \geq 0$
- $m_{uv} + m_{vk} \leq m_{uk}$
- $m_{uv} + m_{kv} \leq m_{ku}$

For the following Klyaus matrix:

$$\begin{pmatrix} \infty & 1 & 9 & 12 & 17 \\ 1 & \infty & 0 & 9 & 14 \\ 9 & 0 & \infty & 2 & 4 \\ 13 & 9 & 2 & \infty & 1 \\ 19 & 15 & 4 & 0 & \infty \end{pmatrix}$$

The exact resolution algorithm provides the results as shown in Table 7.

TABLE 7. Non-stability of TSP on the Klyaus matrices.

	Optimal tour	Cost
TSP	1 2 3 5 4	18.000000
PTSP (p=0.5)	1 5 4 3 2	14.843750

We observe that $\Gamma_{PTSP} \neq \Gamma_{TSP}$, which breaks the equality of the random variables $Long_{(\Gamma,\mu)}$ and $Long_{opt}$, which clearly gives, the non-stability of TSP on the Klyaus matrices.

F. TSP ASSOCIATED WITH THE PRODUCT MATRICES

For two vectors u_1 and u_2 of R^n such as $m_{u,v} = u_{1u} * u_{2v}$.

For the next generated product matrix.

$$\begin{pmatrix} \infty & 100 & 92 & 18 & 18 & 3 & 2 \\ 100 & \infty & 117 & 23 & 22 & 3 & 2 \\ 92 & 117 & \infty & 24 & 23 & 3 & 3 \\ 18 & 23 & 24 & \infty & 23 & 4 & 3 \\ 18 & 22 & 23 & 23 & \infty & 4 & 3 \\ 3 & 3 & 3 & 4 & 4 & \infty & 3 \\ 2 & 2 & 3 & 3 & 3 & 3 & \infty \end{pmatrix}$$

The exact resolution algorithm provides the results as shown in Table 8.

TABLE 8. Non-stability of TSP on the product matrices.

	Optimal tour	Cost
TSP	1 5 2 6 3 7 4	70.000000
PTSP (p=0.5)	1 4 2 7 5 3 6	69.210938

We observe that $\Gamma_{PTSP} \neq \Gamma_{TSP}$, which breaks the equality of the random variables $Long_{(\Gamma,\mu)}$ and $Long_{opt}$, which clearly gives: the non-stability of TSP on the product matrices.

G. TSP ASSOCIATED WITH BROWNIAN MATRICES

Recall that the definition of a Brownian matrix is, a matrix M of dimension $n * n$ and for two vectors u_1 and u_2 of R^n such as:

$$m_{u,v} = \begin{cases} u_{1u} & \text{if } u < v, \\ u_{2v} & \text{else.} \end{cases}$$

For the next generated Brownian matrix.

$$\begin{pmatrix} \infty & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & \infty & 7 & 7 & 7 & 7 & 7 \\ 3 & 2 & \infty & 9 & 9 & 9 & 9 \\ 3 & 2 & 10 & \infty & 10 & 10 & 10 \\ 3 & 2 & 10 & 37 & \infty & 11 & 11 \\ 3 & 2 & 10 & 37 & 45 & \infty & 14 \\ 3 & 2 & 10 & 37 & 45 & 10 & \infty \end{pmatrix}$$

The exact resolution algorithm provides the results as shown in Table 9.

TABLE 9. Non-stability of TSP on the Brownian matrices.

	Optimal tour	Cost
TSP	1 5 2 3 4 7 6	44.000000
PTSP (p=0.5)	1 4 5 7 6 2 3	25.117188

We observe that $\Gamma_{PTSP} \neq \Gamma_{TSP}$, which breaks the equality of the random variables $Long_{(\Gamma,\mu)}$ and $Long_{opt}$, which clearly gives, the non-stability of TSP on the Brownian matrices.

H. OTHER UNSTABLE TSPs

We give in this section the non-stable *TSPs* by the Demidenko matrices and the matrices of generalized distributions of type (*) (respectively of type (**), (***), (****)). These matrices have been deduced non-stable by means of the order relations [18] which exist in all the matrices.

The Demidenko matrices are unstable because they contain Supnick matrices as subclass. The matrices of generalized distributions are unstable since they contain for its four types the Klyaus matrices as subclasses.

V. STABLE TSP UNDER THE SKIPPING METHOD

We first expose the TSP that are stable under some conditions or unconditionally by using the skipping method. We present the TSP on the constant matrices and the Kalmanson matrices.

A. CONSTANT MATRICES

The matrices M of the form $m_{u,v} = u_{1u} + u_{2v}$ are called constant matrices. The following theorem, due to Berenguer [9] characterizes the TSP on such a matrix.

Theorem 10: Only the constant matrices check the property, all the tours have the same length.

Proposition 1 [6]: The TSP on constant matrices is stable.

Remark: For the proof of this proposition it suffices to remark that every submatrix of a constant matrix is a constant matrix, and therefore every sub-tour is an optimal.

B. KALMANSON MATRICES

The TSP on the Kalmanson matrices is the second type of TSP that displays the stability without any additional constraint.

Proposition 2: The TSP on Kalmanson matrices is stable.

The definition of a master tour in section II-B is equivalent to our definition of stability. The theorem 1 asserts the existence of a master tour for Kalmanson matrices, which gives the stability for Kalmanson matrices.

C. UPPER-TRIANGULAR MATRICES

We provide here examples of TSP which are stable under the additional conditions. We first introduce the TSP on the upper-triangular matrices. A matrix M is upper-triangular if

$$\forall (u, v)/u \geq v \qquad m_{u,v} = 0.$$

Lawler *et al.* [35] showed that the TSP corresponding to the upper-triangular matrices is easy. Bellalouna [6] showed- in the following propositions- that the TSP on this type of matrix is stable under some conditions.

Proposition 3 [6]: Let M be a positive upper-triangular matrix. We suppose that the nodes 1 and n are present and that the minimal length between 1 and n is $(1, n)$ so the TSP is stable.

Proposition 3 [6]: Let M be a positive upper-triangular matrix. We suppose that for $u < v$ we have $m_{u,v} \leq m_{k,v}$ $\forall$ $u+1 \leq k \leq v-1$, so, the TSP is stable.

D. SMALL MATRICES

In this section we suppose that $a = u_1 = a_1, a_2, \ldots a_n$ and $b = u_2 = b_1, b_2, \ldots b_n$.

Definition 3: A small matrix M can be defined as, for two vectors a and b of $\mathbb{R}^n$ we have

$$c_{uv} = \min \{a_u, b_v\}.$$

We establish the stability for the small TSPs (associated with small matrices) by imposing some conditions.

Proposition 5 [35]: Let M be a small matrix such as $m_{uv} \neq m_{pq} \forall u \neq p$ and v$\neq p$, we suppose that $D_2 = \{1\}, D_0 = \{n\}$ and $D_b = \emptyset$. Without loss of generality, it will be supposed that: $a_1 < a_2 < \ldots < a_{n-1} < a_n$. So:

The small TSP is stable if and only if $a_{n-1} < b_1$ and $a_n < \min\limits_{2 \leq u \leq n-1} \{b_u\}$

In the following proposition, we remove ($a_{n-1} < b_1$ and $a_n < \min\limits_{2 \leq u \leq n-1} \{b_u\}$) and we are looking for the solution Γ_{PTSP}. We suppose that the a_u and b_v are arranged in ascending order and we will note $\Gamma^* = \{1, 2, \ldots, n-1, n\}$ (Γ^* is an optimal tour in the deterministic sense).

Proposition 6: We suppose that $D_2 = \{1\}, D_0 = \{n\}$, $D_b = \{\}$. Without loss of generality, it will be supposed that, $a_1 < a_2 < \ldots < a_{n-1} < a_n$. We suppose more than $b_2 < \ldots < b_{n-1}$, so the small TSP is stable and we have

$$\Gamma_{PTSP} = \Gamma^*$$

Remark: In this study, we limited it to the case $D_2 = \{1\}$, $D_0 = \{n\}$, $D_b = \{\}$. Our attempt of the generalization appears very complex under the method of modification ''skipping method''. It motivates us to look for a new method of modification that is effective without being complex.

VI. SMALL STABLE TSP UNDER THE METHOD OF TAXI DRIVER

This method is interesting when $D_2 \neq \emptyset$. One of the remarkable peculiarities of small TSP is the possibility of grouping all customers into four sub-sets or four ''governorates'', which are none other than D_2, D_0, D_a, D_b.

Having this ''map'' (we designate by map the distribution of customers in governorates) the tour Γ_{TSP} takes the below ''structure''

$$\Gamma_{TSP} = 1_{D_2}, D_a, 1_{D_0}, D_b, 1_{D_2}, 1_{D_0}, 1_{D_2}, \ldots, 1_{D_0}, 1_{D_2}.$$

We define the structure of a tour as the given order of traffic through the customers. For example we say that a tour Γ has the following structure (we note $\Gamma = 1_{D_2}, D_a, 1_{D_0}$) if and only if we start from a node of D_2 then we visit all the nodes of D_a in any order, finally we visit a node of D_0.

Two aspects characterize this method. The first is the fact that the induced tour through ξ of the a *a priori* tour by Taxi driver method adopts the same structure of the Γ_{TSP} (or each induced tour ''try to adopt'' the structure of Hamiltonian tour, Γ_{TSP}). The second is that the traffic through the present customers is no longer systematic as was the case in the skipping method. But rather the traffic is guided according to a ''map'',the merit of such a circulation is that we try every time, to go from one city to another, to cash an absolute minimum cost (ie $m_{uv} \in D$).

Remark: The ideal, of course, is that the $m_{uv} \in D_\xi$. The ''map'' is fixed from the beginning, it is the same for all ξ (the map is the given sets (D_2, D_0, D_a, D_b)). If the map depends of ξ $(D_2^\xi, D_0^\xi, D_a^\xi, D_b^\xi)$ we have therefor 2^n relative maps and our method is transformed into a method of re-optimization.

A. DESCRIPTION OF METHOD

Having any *a priori* tour and a subset of nodes ξ, this method consists of modifying this tour so that it tends, in structure, towards the tour Γ_{TSP}.

Let ξ be present sub-nodes. To generate the induced tour through ξ, we will apply the skipping method as long as the order of passage between the customers conforms to the Γ_{TSP} structure, namely

$$\Gamma_{TSP} = 1_{D_2}, D_a, 1_{D_0}, D_b, 1_{D_2}, 1_{D_0}, 1_{D_2}, \ldots, 1_{D_0}, 1_{D_2}$$

If this order (after skipping the absent customers) is not respected, then we rectify or force the tour in order to adopt the structure above.

- This method can be conceived as the skipping method
- Adjustment of the new tour.

Remark: For the uniqueness of the induced tour, we opt for the following convention, when we find the taxi driver is obliged to rectify the trajectory of the tour (and this happens when, applying the skipping method, we lose the structure of the tour Γ_{TSP} or we lose this order of circulation across customers, namely $1_{D_2}, D_a, 1_{D_0}, D_b, 1_{D_2}, 1_{D_0}, 1_{D_2}, \ldots, 1_{D_0}, 1_{D_2}$), we then choose the node that not yet visited, the smallest in its governorate.

B. PERFORMANCE OF THE METHOD

- Preliminary results lead us to consider that this method is more efficient than the skipping method.
- We find the stability of the PTSP when we use the conditional probability. Indeed, this method coincides with th re-optimization strategy if the probability is conditioned such as, if a node is absent from D_2, then there is a node of D_0 is absent. Also if k nodes are absent from D_2, then there are k nodes of D_0 disappear, and vice versa.

VII. CONCLUSION

In this paper, we proposed a new formulation for the stability of the TSP-PTSP. This concept of stability led us to make the link with the problem of the Master Tour. We first presented a summary of easy TSPs via the notion of pyramidal tour which is likely to contain stable TSPs. Then we showed that the small TSP is a stable problem for the skipping method under certain conditions. And in order to better pinpoint the problem, we removed these last conditions and we submitted another modification method, more appropriate to the small TSP, which we called method of Taxi Driver, for which we reestablished the stability. The proposed notion of stability depends on the modification method. Thus, for the majority of TSP, the study of stability remains difficult and the TSP rather goes to the non-stability. Moreover, several TSPs is unstable, by means of counter examples treated by an algorithm of exact resolution. Therefore, as a future work we plan to design appropriate modification methods that goes to the value of re-optimization strategy $Long_{opt}$ with real time, and can be extended to the case where size of problems is important.

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